First Name: ______________ Last Name: ______________

Student-No: ______________ Section: ______________

Grade:

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Indefinite Integrals

1. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the indefinite integral $\int \arctan\left(\frac{1}{x}\right) dx$ for $x > 0$.

   Answer: $x \arctan \frac{1}{x} + \frac{1}{2}\ln(1+x^2) + C$

   Solution: We do integration by parts with:

   $$u(x) = \arctan\left(\frac{1}{x}\right) \Rightarrow u'(x) = -\frac{1}{1+x^2},$$
   $$v'(x) = 1 \Rightarrow v(x) = x.$$

   $$\int \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right) - \int -\frac{x}{1+x^2}\, dx$$

   The second term on the rhs can be easily calculated by substituting $u(x) = 1+x^2$, and we finally get:

   $$\int \arctan\left(\frac{1}{x}\right) dx = x \arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(1+x^2) + C$$

   (b) Calculate the indefinite integral $\int -3x\sqrt{3+3x}\, dx$ for $x < 1$.

   Answer: $\frac{2}{5}(3+3x)^{3/2}(\frac{2}{3} - x) + C$

   Solution: We take $u(x) = 3+3x$, then we have $u'(x) = 3$ and we replace $-3x$ by $3 - u(x)$, such that we write

   $$\int -3x\sqrt{3+3x}\, dx = \frac{1}{3}\int 3(-3x)\sqrt{3+3x}\, dx = \frac{1}{3}\int (3-u)u^{1/2}u'\, dx$$

   and apply substitution rule as:

   $$\int (3-u)u^{1/2}u'\, dx = \left(\int 3u^{1/2} - u^{3/2}\, du\right)_{u=3+3x}$$

   Anti-differentiating the simple polynomial function $3u^{1/2} - u^{3/2}$ and eventually substituting $u(x) = 3+3x$, we finally get:

   $$\int -3x\sqrt{3+3x}\, dx = \frac{2}{3}\left((3+3x)^{3/2} - \frac{1}{5}(3+3x)^{5/2}\right) + C = \frac{2}{5}(3+3x)^{3/2}(\frac{2}{3} - x) + C$$

Note that this problem can also be solved by IBP (but more challenging) with:

$$u(x) = -3x \Rightarrow u'(x) = -3,$$
$$v'(x) = (3+3x)^{1/2} \Rightarrow v(x) = \frac{2}{3}\left(\frac{1}{3}\right)(3+3x)^{3/2} = \frac{2}{9}(3+3x)^{3/2}.$$

such that

$$\begin{aligned} I &= -3x\left(\frac{2}{9}\right)(3+3x)^{3/2} - \int -3\frac{2}{9}(3+3x)^{3/2}\,dx \\ &= \left(-\frac{2}{3}x\right)(3+3x)^{3/2} + \frac{2}{3}\int (3+3x)^{3/2}\,dx \end{aligned}$$

Given that the anti-derivative of $\int (3+3x)^{3/2}\,dx$ is $\frac{2}{5}\left(\frac{1}{3}\right)(3+3x)^{5/2} + C = \frac{2}{15}(3+3x)^{5/2} + C$, we get:

$$\begin{aligned} I = (3+3x)^{3/2}\left(-\frac{2}{3}x + \frac{4}{45}(3+3x)\right) + C &= (3+3x)^{3/2}\left(\frac{4}{15} - \frac{2}{5}x\right) + C \\ &= \frac{2}{5}(3+3x)^{3/2}\left(\frac{2}{3} - x\right) + C \end{aligned}$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{x^2+x+4}{x^3+3x+x^2+3}\,dx$.

Answer: $\ln|x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$

Solution: This is a partial fraction indefinite integral. We first recognize that $x = -1$ is a root of the denominator, such that we can write

$$x^3 + 3x + x^2 + 3 = (x+1)(ax^2 + bx + c) = ax^3 + (b+a)x^2 + (b+c)x + c$$

which gives $a = 1$, $b = 0$ and $c = 3$. Next, we decompose the fraction as:

$$\frac{x^2+x+4}{x^3+3x+x^2+3} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+(Bx+C)(x+1)}{x^3+3x+x^2+3}$$

which gives $A = 1$, $B = 0$ and $C = 1$. We can now calculate the indefinite integral as:

$$\int \frac{x^2+x+4}{x^3+3x+x^2+3}\,dx = \int \frac{1}{x+1} + \frac{1}{x^2+3}\,dx = \ln|x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

Definite Integrals

2. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_1^e \frac{1-\ln(x)}{x}\,dx$.

Answer: $\frac{1}{2}$

Solution: We split the integral into 2 integrals and calculate the 2nd integral by substitution $u(x) = \ln x$, $u'(x) = \frac{1}{x}$ and $f(u) = u$ as:

$$\int_1^e \frac{1-\ln(x)}{x}\,dx = \int_1^e \frac{1}{x}\,dx - \int_1^e \frac{\ln(x)}{x}\,dx = [\ln(x)]_1^e - \frac{1}{2}[\ln(x))^2]_1^e = 1 - \frac{1}{2} = \frac{1}{2}$$

(b) Calculate $\int_4^5 \frac{x-4}{\sqrt{8x-14-x^2}}\,dx$.

Answer: $\sqrt{2}-1$

Solution: We can rewrite $8x - 14 - x^2$ as $2 - (x-4)^2$ and use a trigonometric substitution as

$$x - 4 = \sqrt{2}\sin\theta \quad , \quad x'(\theta) = \frac{dx}{d\theta} = \sqrt{2}\cos\theta,$$
$$x = 4 \Rightarrow \theta = 0 \quad , \quad x = 5 \Rightarrow \theta = \pi/4$$

to get:

$$I = \int_4^5 \frac{x-4}{\sqrt{8x-14-x^2}}\,dx = \int_0^{\pi/4} \frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\sqrt{2}\cos\theta\,d\theta$$

Now we replace $\sqrt{1-\sin^2\theta}$ by $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ as $\cos\theta$ is positive on $[0, \pi/4]$ and finally calculate:

$$I = \sqrt{2}\int_0^{\pi/4} \sin\theta\,d\theta = \sqrt{2}[-\cos\theta]_0^{\pi/4} = \sqrt{2} - 1$$

Note that this problem can also be solved by standard substitution: $u(x) = 8x - 14 - x^2$, $u'(x) = 8 - 2x = -2(x-4)$, $u(4) = 2$, $u(5) = 1$ as

$$\int_4^5 \frac{x-4}{\sqrt{8x-14-x^2}}\,dx = -\frac{1}{2}\int_4^5 \frac{-2(x-4)}{\sqrt{8x-14-x^2}}\,dx = -\frac{1}{2}\int_4^5 u'u^{-1/2}\,dx$$

and then

$$-\frac{1}{2}\int_4^5 u'u^{-1/2}\,dx = -\frac{1}{2}\int_2^1 u^{-1/2}\,du = [-u^{1/2}]_2^1 = -1 + 2^{1/2}$$

Riemann Sum and FTC

3. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Which definite integral corresponds to $\lim_{n\to\infty}\sum_{i=1}^{n}\ln(\frac{3i}{n}-\frac{3}{n}+1)\sin(\frac{6i}{n}-\frac{6}{n})\frac{3}{n}$?

(A) $\int_0^3 \ln(x+1)\sin(2x)dx$

(B) $3\int_0^1 \ln(x+1)\sin(2x)dx$

(C) $\frac{1}{2}\int_0^6 \ln(x+1)\sin(2x)dx$

(D) $\int_0^6 \ln(\frac{x}{2}+1)\sin(x)dx$

(E) $2\int_0^3 \ln(\frac{x}{2}+1)\sin(x)dx$

Answer: A

Solution: Pick $x_i = \frac{3i}{n}$, so $x_0 = 0$, $x_n = 3$ and $\Delta x = \frac{3}{n}$. Then we can rewrite the summation as:

$$\sum_{i=1}^{n} \ln(x_{i-1}+1)\sin(2x_{i-1})\Delta x$$

which corresponds to the Left Riemann Sum for option (A).

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_{-1}^{x} t^2\,dt$ and $g(x) = (F(x^2))^4$. Calculate $g'(1)$.

Answer: $\frac{64}{27}$

Solution: We use the derivative of the power of a function rule to get $g'(x) = 4F'(x^2)(F(x^2))^3$, and the chain rule and FTC I to calculate $F'(x^2) = F'(y)y'(x) = x^4 \cdot 2x = 2x^5$ with $y(x) = x^2$. Hence we have:

$$g'(x) = 8x^5\,(F(x^2))^3$$

and

$$g'(1) = 8 \cdot 1^5 F(1)^3$$

By simple integration, we calculate:

$$F(1) = \int_{-1}^{1} t^2\,dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$$

and get $g'(1) = 8\left(\frac{2}{3}\right)^3 = \frac{64}{27}$.

(c) Let $F(x) = \int_{x^2}^{x^3} 9e^{t^2}\,dt$. Find the equation of the tangent line to the graph of $y = F(x)$ at $x = 1$. Tip: recall that the tangent line to the graph of $y = F(x)$ at $x = x_0$ is given by the equation $y = F(x_0) + F'(x_0)(x - x_0)$.

Answer: $y = 9e(x - 1)$

Solution: We first write $F(x)$ for any real number c as:

$$F(x) = -\int_c^{x^2} 9e^{t^2}\,dt + \int_c^{x^3} 9e^{t^2}\,dt$$

Then use FTC I and the chain rule to get:

$$F'(x) = -9e^{x^4}2x + 9e^{x^6}3x^2$$

Then we calculate $F(1)$ and $F'(1)$, we get $F(1) = \int_1^1 9e^{t^2}\,dt = 0$ and $F'(1) = 9e$, and finally the equation of the tangent $y - F(1) = F'(1)(x - 1)$ becomes

$$y = 9e(x - 1)$$

Areas and volumes

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. [4 marks] Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = \sqrt{x-1}$ and $x = 1 + \sqrt{y}$ about the horizontal line $y = -2$. **Do not evaluate the integral.**

Answer: $\pi \int_1^2 (2+\sqrt{x-1})^2 - (2+(x-1)^2)^2 \, dx$

Solution: Intersection points are given by $\sqrt{x-1} = (x-1)^2$.

Solving for x, we determine the 2 intersection points

$$I_1 = (1,0) \quad , \quad I_2 = (2,1).$$

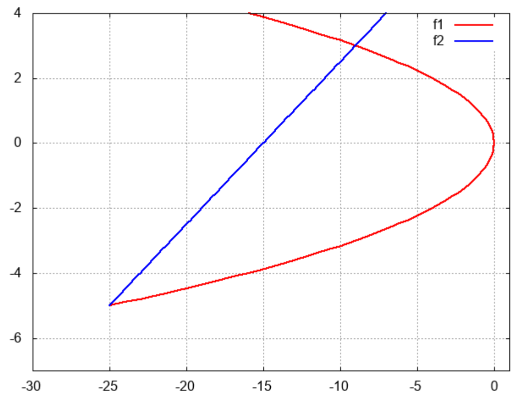
We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of $+2$, we finally establish:

$$\pi \int_1^2 (2+\sqrt{x-1})^2 - (2+(x-1)^2)^2 \, dx.$$

5. (a) [2 marks] Sketch by hand the finite area enclosed by $y^2 + x = 0$ and $2y - x = 15$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) [4 marks] Write a definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-5}^{3} \left(-y^2 - 2y + 15\right)\, dy$

Solution: We first find the intersection between the two curves, given by the solution of:

$$-y^2 = 2y - 15 \Leftrightarrow (y + 5)(y - 3) = 0.$$

We then label the curve $x_R = -y^2$ and $x_B = 2y - 15$ and notice that $x_B \leq x_R$ for $-5 \leq y \leq 3$. The area is therefore given by the following definite integral:

$$A = \int_{-5}^{3} \left(-y^2 - 2y + 15\right)\, dy$$

(c) 2 marks Evaluate the integral to compute the area enclosed.

Answer: $\frac{256}{3}$

Solution:

$$A = \left[-\frac{y^3}{3} - \frac{2y^2}{2} + 15y\right]_{-5}^{3} = \frac{256}{3}$$