First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

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Riemann Sum and FTC

1. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2\sin(\frac{i^3}{n^3}+2)}{n^3}$$

by first writing it as a definite integral and then evaluating it.

Answer: $-\cos(3)+\cos(2)$

Solution: We identify $a=0$, $b=1$, $\Delta(x)=\frac{1}{n}$, $x_i=\frac{i}{n}$, and

$$f(x_i)=3x_i^2\sin(x_i^3+2).$$

This yields,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2\sin(\frac{i^3}{n^3}+2)}{n^3}=\int_0^1 3x^2\sin(x^3+2)dx.$$

To calculate the integral, let $u=x^3+2$. Then $du=3x^2dx$, $u(0)=2$, and $u(1)=3$. Then

$$\int_0^1 3x^2\sin(x^3+2)dx=\int_2^3\sin(u)du=[-\cos(u)]_2^3=-\cos(3)+\cos(2).$$

   (b) Define $F(x)$ and $g(x)$ by $F(x)=\int_2^x\frac{t}{2t^2+1}\,dt$ and $g(x)=F(2x)+xF(x)$. Calculate $g'(0)$.

Answer: $-\frac{\ln 9}{4}$

Solution: We use the product rule and the chain rule to get: $g'(x)=2F'(2x)+F(x)+x\,F'(x)$. From FTC I, we get:

$$F'(x)=\frac{x}{2x^2+1}$$
$$F'(2x)=\frac{2x}{8x^2+1}$$

Taking $x=0$, the two contributions above vanish, so we are left with calculating $F(0)$. We use a standard substitution $u=2t^2+1$, $u'=4t$ such that:

$$F(x)=\frac{1}{4}\int_2^x\frac{4t}{2t^2+1}\,dt=\frac{1}{4}[\ln(2t^2+1)]_2^x=\frac{1}{4}(\ln(2x^2+1)-\ln 9)$$

Taking $x = 0$, we finally get:

$$g'(0) = \frac{1}{4}(\ln 1 - \ln 9) = -\frac{\ln 9}{4}$$

Indefinite Integrals

2. [12 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int x(x-2)^5\,dx$.

Answer: $\frac{1}{7}(x-2)^7 + \frac{1}{3}(x-2)^6 + C$
OR $\frac{x}{6}(x-2)^6 - \frac{1}{42}(x-2)^7 + C$

Solution: Method 1: Using the substitution $u = x - 2$, $u' = 1$ and writing $x = u + 2$, we get:

$$\int x(x-2)^5\,dx = \int (u+2)u^5\,du = \int (u^6 + 2u^5)\,du = \frac{1}{7}u^7 + \frac{1}{3}u^6 + C$$

Substituting back $u = x - 2$, we get:

$$\int x(x-2)^5\,dx = \frac{1}{7}(x-2)^7 + \frac{1}{3}(x-2)^6 + C$$

Method 2: We use integration by parts with $u = x$ and $v' = (x-2)^5$. We get $u' = 1$ and $v = \frac{1}{6}(x-2)^6$. This gives

$$\int x(x-2)^5\,dx = \frac{x}{6}(x-2)^6 - \int \frac{1}{6}(x-2)^6\,dx = \frac{x}{6}(x-2)^6 - \frac{1}{42}(x-2)^7 + C$$

(b) Calculate the indefinite integral $\int (5 + 3\sin\theta)^{\frac{7}{2}}\cos\theta\,d\theta$.

Answer: $\frac{2}{27}(5 + 3\sin\theta)^{\frac{9}{2}} + C$

Solution: By substitution, with

$$u(\theta) = 5 + 3\sin\theta$$
$$u'(\theta) = 3\cos\theta$$

Then

$$\int (5 + 3\sin\theta)^{\frac{7}{2}}\cos\theta\,d\theta = \int \frac{1}{3}u^{\frac{7}{2}}\,du$$

so that

$$\frac{1}{3}\frac{2}{9}(5 + 3\sin\theta)^{\frac{9}{2}} + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int x\,(\ln x)^2\ dx$ for $x > 0$.

Answer: $\frac{x^2}{2}[\ln(x)]^2 - \frac{x^2}{2}\ln(x) + x^2/4 + C$

Solution: We use integration by parts with $u = (\ln x)^2$ and $dv/dx = x$. We get $du/dx = 2\ln(x)/x$ and $v = x^2/2$. This gives

$$I = \int x\,(\ln x)^2\ dx = \frac{x^2}{2}[\ln x]^2 - \int x \ln x\, dx\,.$$

Now do integration by parts again on the second integral. Put $u = \ln x$ and $dv/dx = x$ so that $du/dx = 1/x$ and $v = x^2/2$. This gives

$$I = \frac{x^2}{2}[\ln x]^2 - \left(\frac{x^2}{2}\ln x - \frac{1}{2}\int x\, dx\right)\,.$$

Performing the last integral and putting in the constant gives

$$I = \frac{x^2}{2}[\ln(x)]^2 - \frac{x^2}{2}\ln(x) + x^2/4 + C\,.$$

Definite Integrals

3. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{\pi/2}^{\pi} \cos^3(x)\sin^2(x)\,dx$.

Answer: $-\frac{2}{15}$

Solution: This is a trigonometric integral that is calculated as:

$$I = \int_{\pi/2}^{\pi} \cos^3(x)\sin^2(x)\,dx = \int_{\pi/2}^{\pi} \cos^2(x)\sin^2(x)\cos(x)\,dx$$
$$= \int_{\pi/2}^{\pi} (1-\sin^2(x))\sin^2(x)\cos(x)\,dx$$
$$= \int_{\pi/2}^{\pi} (\sin^2(x)-\sin^4(x))\cos(x)\,dx$$

which gives, upon substituting $u = \sin(x)$ and $du = \cos(x)dx$:

$$I = \left[\frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x)\right]_{\pi/2}^{\pi} = -\frac{1}{3} + \frac{1}{5} = -\frac{2}{15}$$

(b) Calculate $\displaystyle\int_0^1 \arctan(2x)\,dx$.

Answer: $\arctan(2) - \frac{1}{4}\ln(5)$

Solution: We use integration by parts with $u = \arctan(2x)$ and $v' = 1$. We get $u = \frac{2}{1+4x^2}$ and $v = x$. This gives

$$I = \int_0^1 \arctan(2x)\,dx = [x\arctan(2x)]_0^1 - \int_0^1 \frac{2x}{1+4x^2}\,dx$$
$$= \arctan(2) - \frac{1}{4}\int_0^1 \frac{8x}{1+4x^2}\,dx$$

Using the substitution $u = 1+4x^2$, $u' = 8x$, we get:

$$I = \arctan(2) - \frac{1}{4}\left[\ln(u)\right]_1^5 = \arctan(2) - \frac{1}{4}\ln(5)$$

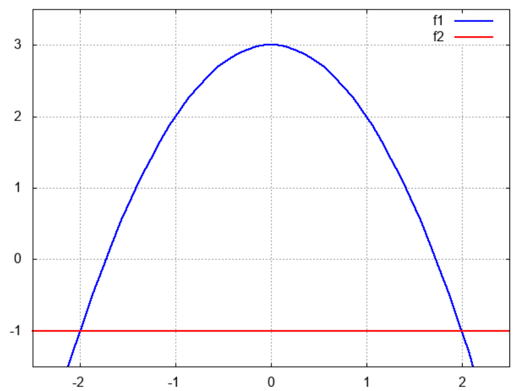
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) 2 marks Sketch by hand the finite area enclosed between the curves defined by the functions $y + x^2 - 3 = 0$ and $y = -1$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) 4 marks Write the definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{2} (4 - x^2)\ dx$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$3 - x^2 = -1 \Leftrightarrow x^2 = 4.$$

The intersection points are therefore $(-2, -1)$ and $(2, -1)$. We then label the curve $y_B = 3 - x^2$ and $y_R = -1$ and notice that $y_B \geq y_R$ for $-2 \leq x \leq 2$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{2} \left(3 - x^2 + 1\right)\ dx = \int_{-2}^{2} \left(4 - x^2\right)\ dx$$

(c) [2 marks] Evaluate the integral.

Answer: $\frac{32}{3}$

Solution:

$$A = \int_{-2}^{2} \left(4 - x^2\right)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 16 - \frac{16}{3} = \frac{32}{3}$$

5. [4 marks] Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = \frac{(y-1)^2}{9}$ and $x = y - 3$ about the horizontal line $y = 2$. **Do not evaluate the integral.**

Answer: $\pi \int_1^4 (3\sqrt{x} - 1)^2 - (x+1)^2 \, dx$

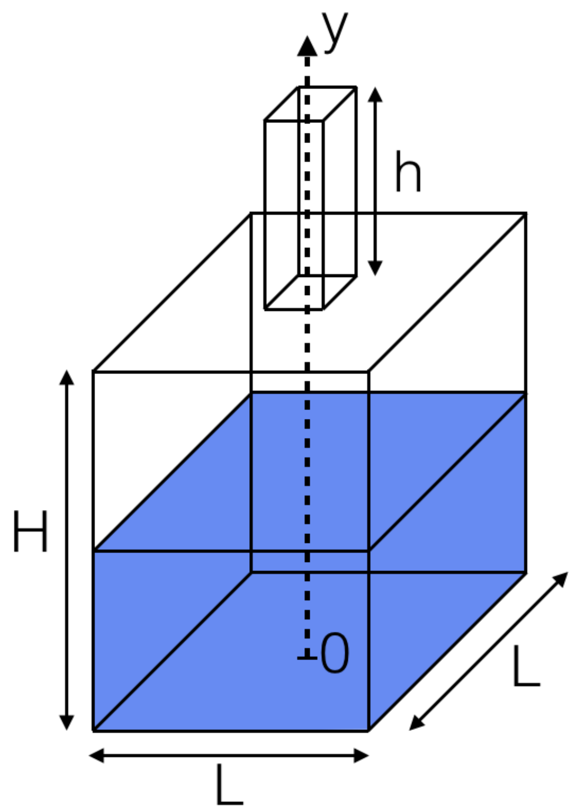
Solution: Intersection points are given by $\frac{(y-1)^2}{9} = y - 3$.
Solving for y, we determine the 2 intersection points

$$I_1 = (1,4) \quad , \quad I_2 = (4,7).$$

We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of -2, we finally establish:

$$\pi \int_1^4 (3\sqrt{x} - 1)^2 - (x+1)^2 \, dx.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10m/s^2$.
We take $H = 8m$, $L = 5m$ and $h = 2m$.



(a) [2 marks] Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $10^6 \int_0^4 (10 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (10 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 8/2 = 4$ as

$$W = \int_0^4 g\rho L^2 (10 - y)\, dy = 2.5 \cdot 10^5 \int_0^4 (10 - y)\, dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $8 \cdot 10^6 J$

Solution:

$$W = 2.5 \cdot 10^5 \int_0^4 (10 - y)\, dy = 2.5 \cdot 10^5 \left[10y - \frac{y^2}{2}\right]_0^4 = 2.5 \cdot 10^5 \left(10 \cdot 4 - \frac{4^2}{2}\right)$$
$$= 2.5 \cdot 10^5 \cdot 32 = 8 \cdot 10^6 J$$