First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Grade:

The remainder of this page has been left blank for your workings.

Riemann Sum and FTC

1. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3}\sqrt{\frac{i^3}{n^3}+2}$$

by first writing it as a definite integral and then evaluating it.

Answer: $2\sqrt{3}-\frac{4}{3}\sqrt{2}$.

Solution: We identify $a=0$, $b=1$, $\Delta(x)=\frac{1}{n}$, $x_i=\frac{i}{n}$, and

$$f(x_i)=3x_i^2\sqrt{x_i^3+2}.$$

This yields,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2}{n^3}\sqrt{\frac{i^3}{n^3}+2}=\int_0^1 3x^2\sqrt{x^3+2}dx.$$

To calculate the integral, let $u=x^3+2$. Then $du=3x^2dx$, $u(0)=2$, and $u(1)=3$. Then

$$\int_0^1 3x^2\sqrt{x^3+2}dx=\int_2^3\sqrt{u}du=[\frac{2}{3}u^{3/2}]_2^3=2\sqrt{3}-\frac{4}{3}\sqrt{2}.$$

(b) Define $F(x)$ and $g(x)$ by $F(x)=\displaystyle\int_0^{x^2}e^{-t^2}\,dt$ and $g(x)=F(\sin x)$. Calculate $g'(\pi/4)$.

Answer: $e^{-1/4}$

Solution: We use the product rule to get: $g'(x)=F(u(x))'=F'(u)u'(x)$ with $u(x)=\sin^2 x$.
By FTC I, we get $F'(u)=e^{-u^2}$, such that:

$$g'(x)=e^{-\sin^4(x)}2\sin(x)\cos(x)$$

Taking $x=\pi/4$, we get:

$$g'(\pi/4)=e^{-1/4}2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}$$

Indefinite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int 3(x+1)^5 \sin\left((x+1)^3\right)\,\mathrm{d}x$.

Answer:
$-(x+1)^3 \cos\left((x+1)^3\right) + \sin\left((x+1)^3\right) + C$

Solution: We start by substituting $u = (x+1)^3$. This gives:

$$\int u \sin(u)\,\mathrm{d}u.$$

We now do integration by parts and obtain:

$$\left(u(-\cos(u)) - \int -\cos(u)\,\mathrm{d}u\right)$$

which simplifies to

$$u(-\cos(u)) + \sin(u) + C.$$

Now resubstitute u to get the final answer:

$$(x+1)^3(-\cos\left((x+1)^3\right)) + \sin\left((x+1)^3\right) + C.$$

(b) Calculate the indefinite integral $\int (1+3\sin\theta)^{\frac{11}{2}} \cos\theta\, d\theta$.

Answer: $\frac{2}{39}(1+3\sin\theta)^{\frac{13}{2}} + C$

Solution: By substitution, with

$$u(\theta) = 1 + 3\sin\theta$$
$$u'(\theta) = 3\cos\theta$$

Then

$$\int (1+3\sin\theta)^{\frac{11}{2}} \cos\theta\, d\theta = \int \frac{1}{3} u^{\frac{11}{2}}\, du$$

so that

$$\frac{1}{3}\frac{2}{13}(1+3\sin\theta)^{\frac{13}{2}} + C$$

(c) (A Little Harder): Calculate the indefinite integral $\int \frac{\ln(9+x^2)}{x^2}\, dx$.

Answer: $-\frac{\ln(9+x^2)}{x} + \frac{2}{3}\arctan(\frac{x}{3}) + C$

Solution: We use integration by parts with $u = \ln(9 + x^2)$ and $v' = \frac{1}{x^2}$, such that $u' = \frac{2x}{9+x^2}$ and $v = -\frac{1}{x}$. This gives:

$$\int \frac{\ln(9+x^2)}{x^2}\, dx = -\frac{1}{x}\ln(9+x^2) - \int -\frac{1}{x}\frac{2x}{9+x^2}\, dx = -\frac{1}{x}\ln(9+x^2) + 2\int \frac{1}{9+x^2}\, dx$$

We calculate the second term on the rhs by first rewriting it as:

$$\int \frac{1}{9+x^2}\, dx = \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\, dx = \frac{1}{3}\int \frac{1/3}{1+\left(\frac{x}{3}\right)^2}\, dx$$

followed by a substitution with $u = \frac{x}{3}$, $u' = \frac{1}{3}$ such that

$$\frac{1}{3}\int \frac{1/3}{1+\left(\frac{x}{3}\right)^2}\, dx = \frac{1}{3}\int \frac{1}{1+u^2}\, du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

The final result is

$$\int \frac{\ln(9+x^2)}{x^2}\, dx = -\frac{\ln(9+x^2)}{x} + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + C$$

Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{-\pi/2}^{\pi/2}(3+x^3)\cos(x)\,dx$.

Answer: 6

Solution: Upon splitting the integral, the second integral vanishes because the function $x^3\cos(x)$ is odd and domain is symmetric, and we only need to compute

$$I = \int_{-\pi/2}^{\pi/2} 3\cos(x)\,dx = 3\left[\sin(x)\right]_{-\pi/2}^{\pi/2} = 6$$

(b) Calculate $\displaystyle\int_0^1 \frac{3x^2}{2x^2+2}\,dx$.

Answer: $\frac{3}{2}\left(1-\frac{\pi}{4}\right)$

Solution: We first rewrite the definite integral as

$$I = \int_0^1 \frac{3x^2}{2x^2+2}\,dx = \frac{3}{2}\int_0^1 \frac{x^2}{x^2+1}\,dx = \frac{3}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$
$$= \frac{3}{2}\int_0^1 \left(1-\frac{1}{x^2+1}\right)dx$$

In this form, the integrand is very easy to anti-differentiate and we finally get:

$$I = \frac{3}{2}\left[x-\arctan(x)\right]_0^1 = \frac{3}{2}\left(1-\frac{\pi}{4}\right)$$

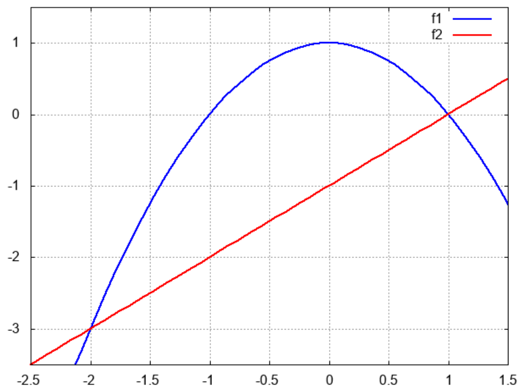
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) [2 marks] Sketch by hand the finite area enclosed between the curves defined by the functions $y = 1 - x^2$ and $2y + 2 = 2x$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) [4 marks] Write the definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{1} (2 - x - x^2)\ dx$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$1 - x^2 = x - 1 \Leftrightarrow (x-1)(x+2) = 0.$$

The intersection points are therefore $(-2, -3)$ and $(1, 0)$. We then label the curve $y_B = 1 - x^2$ and $y_R = x - 1$ and notice that $y_B \geq y_R$ for $-2 \leq x \leq 1$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{1} \left(1 - x^2 - x + 1\right)\ dx = \int_{-2}^{1} \left(2 - x - x^2\right)\ dx$$

(c) 2 marks Evaluate the integral.

Answer: $\frac{9}{2} = 4.5$

Solution:

$$A = \int_{-2}^{1} \left(2 - x - x^2\right)\, dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = 6 + \frac{3}{2} - 3 = \frac{9}{2}$$

5. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = 4\sqrt{x} - 2$ and $y = x + 1$ about the vertical line $x = -1$. **Do not evaluate the integral.**

Answer: $\pi \int_2^{10} y^2 - \left(\frac{(y+2)^2}{16} + 1\right)^2 \, dy$

Solution: Intersection points are given by $4\sqrt{x} - 2 = x + 1$.

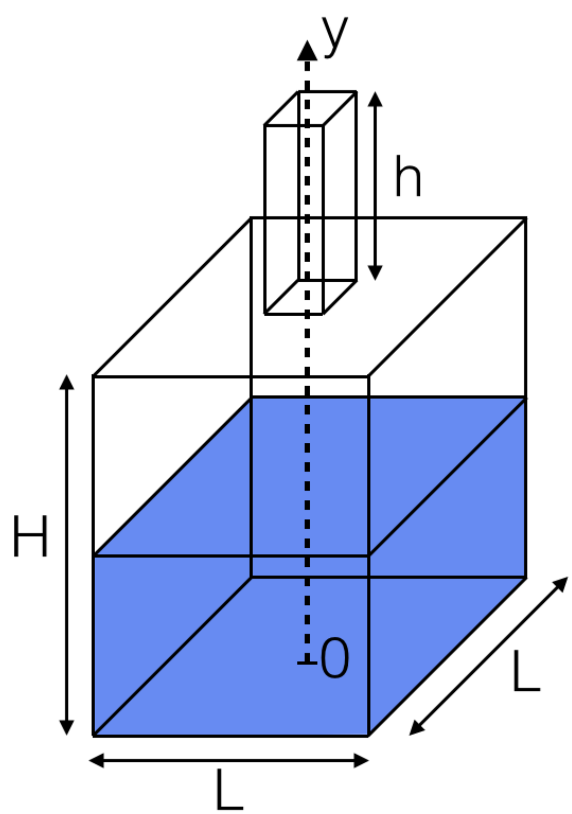
Solving for x, we determine the 2 intersection points

$$I_1 = (1, 2) \quad , \quad I_2 = (9, 10).$$

We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of $+1$, we finally establish:

$$\pi \int_2^{10} y^2 - \left(\frac{(y+2)^2}{16} + 1\right)^2 \, dy.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000 kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10 m/s^2$.
We take $H = 4m$, $L = 4m$ and $h = 2m$.



(a) [2 marks] Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $1.6 \cdot 10^5 \int_0^2 (6 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (6 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 4/2 = 2$ as

$$W = \int_0^2 g\rho L^2(6-y)\,dy = 1.6\cdot 10^5 \int_0^2 (6-y)\,dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $1.6\cdot 10^6 J$

Solution:

$$W = 1.6\cdot 10^5 \int_0^2 (6-y)\,dy = 1.6\cdot 10^5 \left[6y - \frac{y^2}{2}\right]_0^2 = 1.6\cdot 10^5\left(6\cdot 2 - \frac{2^2}{2}\right)$$
$$= 1.6\cdot 10^5 \cdot 10 = 1.6\cdot 10^6 J$$