First Name: ____________________ Last Name: ____________________

Student-No: ____________________ Section: ____________________

Grade:

The remainder of this page has been left blank for your workings.

Riemann Sum and FTC

1. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the infinite sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6i^2}{n^3(\frac{i^3}{n^3}+1)}$$

by first writing it as a definite integral and then evaluating it.

Answer: $2\ln(2) = \ln(4)$

Solution: We identify $a = 0$, $b = 1$, $\Delta(x) = \frac{1}{n}$, $x_i = \frac{i}{n}$, and

$$f(x_i) = \frac{6x_i^2}{x_i^3+1}.$$

This yields,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6i^2}{n^3(\frac{i^3}{n^3}+1)} = \int_0^1 \frac{6x^2}{x^3+1}dx.$$

To calculate the integral, let $u = x^3 + 1$. Then $du = 3x^2dx$, $u(0) = 1$, and $u(1) = 2$. Then

$$\int_0^1 \frac{6x^2}{x^3+4}dx = 2\int_1^2 \frac{1}{u}du = 2[\ln(u)]_1^2 = 2\ln(2).$$

(b) Define $F(x)$ and $g(x)$ by $F(x) = \int_{2\sqrt{\pi}}^{x} t\cos(t^2)\,dt$ and $g(x) = x^3F(x)$. Calculate $g'(\sqrt{\pi})$.

Answer: $-\pi^2$

Solution: We use the product rule to get: $g'(x) = 3x^2F(x) + x^3F'(x)$. From FTC I, we get:

$$F'(x) = x\cos(x^2)$$

and we calculate $F(x)$ using the substitution $u = t^2$, $u' = 2t$ and the fact that $\sin(4\pi) = 0$, such that:

$$F(x) = \int_{2\sqrt{\pi}}^{x} t\cos(t^2)\,dt = \frac{1}{2}[\sin(t^2)]_{2\sqrt{\pi}}^{x} = \frac{1}{2}\sin(x^2)$$

Bringing pieces together, we can write:

$$g'(x) = \frac{3}{2}x^2\sin(x^2) + x^4\cos(x^2)$$

Taking $x = \sqrt{\pi}$, we finally get:

$$g'(\sqrt{\pi}) = \frac{3}{2}\pi \sin(\pi) + \pi^2 \cos(\pi) = -\pi^2$$

Indefinite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int 4(x-2)^7 \sin\left((x-2)^4\right)\,\mathrm{d}x$.

Answer:
$-(x-2)^4\cos((x-2)^4)+\sin((x-2)^4)+C$

Solution: We start by substituting $u=(x-2)^4$. This gives:

$$\int u\sin(u)\,\mathrm{d}u.$$

We now do integration by parts and obtain:

$$\left(u(-\cos(u)) - \int -\cos(u)\,\mathrm{d}u\right)$$

which simplifies to

$$u(-\cos(u))+\sin(u)+C.$$

Now resubstitute u to get the final answer:

$$(x-2)^4(-\cos\left((x-2)^4\right))+\sin\left((x-2)^4\right)+C.$$

(b) Calculate the indefinite integral $\int (2+3\sin\theta)^{\frac{9}{2}}\cos\theta\,d\theta$.

Answer: $\frac{2}{33}(2+3\sin\theta)^{\frac{11}{2}}+C$

Solution: By substitution, with

$$u(\theta)=2+3\sin\theta$$
$$u'(\theta)=3\cos\theta$$

Then

$$\int (2+3\sin\theta)^{\frac{9}{2}}\cos\theta\,d\theta = \int \frac{1}{3}u^{\frac{9}{2}}\,du$$

so that

$$\frac{1}{3}\frac{2}{11}(2+3\sin\theta)^{\frac{11}{2}}+C$$

(c) (A Little Harder): Calculate the indefinite integral $\int 2x^5\sqrt{1-x^2}\,dx$ for $|x| \le 1$.

Answer: $-\frac{2}{7}(1-x^2)^{7/2} + \frac{4}{5}(1-x^2)^{5/2} - \frac{2}{3}(1-x^2)^{3/2} + C$

Solution: We use the substitution $u(x) = 1 - x^2$, $u'(x) = -2x$, and then use $x^2 = 1 - u$ to replace the remaining x^4 term. This gives

$$I = \int x^4\sqrt{1-x^2}2x\,dx = -\int (1-u)^2 u^{1/2}\,du$$
$$= -\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du = -\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C\,.$$

Putting in $u = 1 - x^2$ gives the final answer

$$I = -\frac{2}{7}(1-x^2)^{7/2} + \frac{4}{5}(1-x^2)^{5/2} - \frac{2}{3}(1-x^2)^{3/2}\,.$$

Definite Integrals

3. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan^{3/2}(x)}\,dx$.

Answer: $2 - \frac{2}{3^{1/4}}$

Solution: We let $u = \tan(x)$ and obtain:

$$I = \int_1^{\sqrt{3}} \frac{du}{u^{3/2}} = -2u^{-1/2}|_1^{\sqrt{3}} = 2 - \frac{2}{3^{1/4}}$$

(b) Calculate $\displaystyle\int_0^1 \arctan(4x)\,dx$.

Answer: $\arctan(4) - \frac{1}{8}\ln(17)$

Solution: We use integration by parts with $u = \arctan(4x)$ and $v' = 1$. We get $u = \frac{4}{1+16x^2}$ and $v = x$. This gives

$$I = \int_0^1 \arctan(4x)\,dx = [x\arctan(4x)]_0^1 - \int_0^1 \frac{4x}{1+16x^2}\,dx$$

$$= \arctan(4) - \frac{1}{8}\int_0^1 \frac{32x}{1+16x^2}\,dx$$

Using the substitution $u = 1 + 16x^2$, $u' = 32x$, we get:

$$I = \arctan(4) - \frac{1}{8}\left[\ln(u)\right]_1^{17} = \arctan(4) - \frac{1}{8}\ln(17)$$

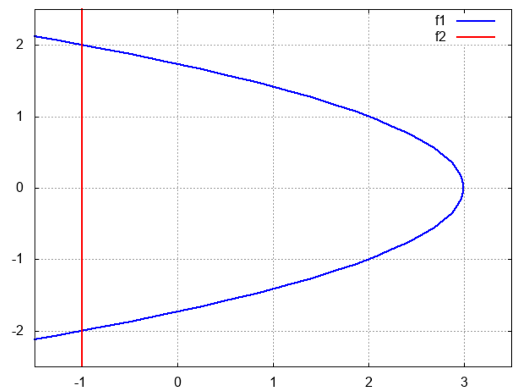
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) 2 marks Sketch by hand the finite area enclosed between the curves defined by the functions $y^2 - 3 = -x$ and $x = -1$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) 4 marks Write the definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{2} (4 - y^2)\ dy$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$3 - y^2 = -1 \Leftrightarrow y^2 = 4.$$

The intersection points are therefore $(-1, -2)$ and $(-1, 2)$. We then label the curve $x_B = 3 - y^2$ and $x_R = -1$ and notice that $x_B \geq x_R$ for $-2 \leq y \leq 2$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{2} \left(3 - y^2 + 1\right)\ dy = \int_{-2}^{2} \left(4 - y^2\right)\ dy$$

(c) 2 marks Evaluate the integral.

Answer: $\frac{32}{3}$

Solution:

$$A = \int_{-2}^{2} \left(4 - y^2\right)\, dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = 16 - \frac{16}{3} = \frac{32}{3}$$

5. 4 marks Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $y = 5\sqrt{x} - 2$ and $y = x + 2$ about the vertical line $x = -3$. **Do not evaluate the integral.**

Answer: $\pi \int_3^{18} (y+1)^2 - \left(\frac{(y+2)^2}{25} + 3\right)^2 \, dy$

Solution: Intersection points are given by $5\sqrt{x} - 2 = x + 2$.

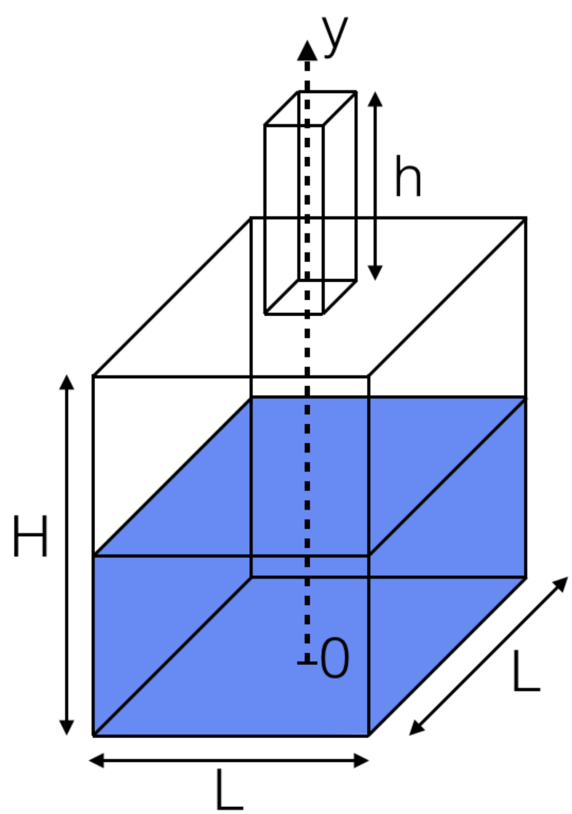
Solving for x, we determine the 2 intersection points

$$I_1 = (1, 3) \quad , \quad I_2 = (16, 18).$$

We integrate in y, hence we write x as a function of y for the 2 curves and apply a shift of $+3$, we finally establish:

$$\pi \int_3^{18} (y+1)^2 - \left(\frac{(y+2)^2}{25} + 3\right)^2 \, dy.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000 kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10 m/s^2$.
We take $H = 4m$, $L = 6m$ and $h = 7m$.



(a) 2 marks Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $3.6 \cdot 10^5 \int_0^2 (11 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (11 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 4/2 = 2$ as

$$W = \int_0^2 g\rho L^2(11 - y)\,dy = 3.6 \cdot 10^5 \int_0^2 (11 - y)\,dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $7.2 \cdot 10^6 J$

Solution:

$$W = 3.6 \cdot 10^5 \int_0^2 (11 - y)\,dy = 3.6 \cdot 10^5 \left[11y - \frac{y^2}{2}\right]_0^2 = 3.6 \cdot 10^5 \left(11 \cdot 2 - \frac{2^2}{2}\right)$$
$$= 3.6 \cdot 10^5 \cdot 20 = 7.2 \cdot 10^6 J$$