First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Grade:

The remainder of this page has been left blank for your workings.

Riemann Sum and FTC

1. [8 marks] Each part is worth 4 marks. Please write your answers in the boxes.

   (a) Calculate the infinite sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2e^{\frac{i^3}{n^3}+2}}{n^3}$$

   by first writing it as a definite integral and then evaluating it.

Answer: $e^3 - e^2$

Solution: We identify $a = 0$, $b = 1$, $\Delta(x) = \frac{1}{n}$, $x_i = \frac{i}{n}$, and

$$f(x_i) = 3x_i^2\exp(x_i^3 + 2).$$

This yields,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3i^2\exp(\frac{i^3}{n^3}+2)}{n^3} = \int_0^1 3x^2\exp(x^3+2)dx.$$

To calculate the integral, let $u = x^3 + 2$. Then $du = 3x^2dx$, $u(0) = 2$, and $u(1) = 3$. Then

$$\int_0^1 3x^2\exp(x^3+2)dx = \int_2^3 \exp(u)du = [\exp(u)]_2^3 = e^3 - e^2.$$

   (b) Define $F(x)$ and $g(x)$ by $F(x) = \int_0^x (2t-1)e^t\,dt$ and $g(x) = x^2F(x)$. Calculate $g'(1/2)$.

Answer: $3 - 2e^{1/2}$

Solution: We use the product rule to get: $g'(x) = 2xF(x) + x^2F'(x)$. From FTC I, we get:

$$F'(x) = (2x-1)e^x$$

and we calculate $F(x)$ using integration by parts with $u = 2t - 1$, $v' = e^t$, such that:

$$F(x) = \int_0^x (2t-1)e^t\,dt = [(2t-1)e^t]_0^x - \int_0^x 2e^t\,dt = [(2t-3)e^t]_0^x = (2x-3)e^x + 3$$

Bringing pieces together, we can write:

$$g'(x) = 2x((2x-3)e^x + 3) + x^2(2x-1)e^x$$

Taking $x = 1/2$, we finally get:

$$g'(1/2) = 2\frac{1}{2}\left(\left(2\frac{1}{2} - 3\right)e^{1/2} + 3\right) = 3 - 2e^{1/2}$$

Indefinite Integrals

2. 12 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate the indefinite integral $\int (x+2)(x-7)^4\,dx$.

Answer: $\frac{1}{6}(x-7)^6+\frac{9}{5}(x-7)^5+C$

Solution: Using the substitution $u = x-7$, $u' = 1$ and writing $x = u+7$, we get:

$$\int (x+2)(x-7)^4\,dx = \int (u+9)u^4\,du = \int (u^5+9u^4)\,du = \frac{1}{6}u^6 + \frac{9}{5}u^5 + C$$

Substituting back $u = x-2$, we get:

$$\int (x+2)(x-7)^4\,dx = \frac{1}{6}(x-7)^6 + \frac{9}{5}(x-7)^5 + C$$

(b) Calculate the indefinite integral $\int (8+2\sin\theta)^{\frac{3}{2}}\cos\theta\,d\theta$.

Answer: $\frac{1}{5}(8+2\sin\theta)^{\frac{5}{2}}+C$

Solution: By substitution, with

$$u(\theta) = 8+2\sin\theta$$
$$u'(\theta) = 2\cos\theta$$

Then

$$\int (8+2\sin\theta)^{\frac{3}{2}}\cos\theta\,d\theta = \int \frac{1}{2}u^{\frac{3}{2}}\,du$$

so that

$$\frac{1}{2}\frac{2}{5}(8+2\sin\theta)^{\frac{5}{2}}+C$$

(c) (A Little Harder): Calculate the indefinite integral $\int e^{-2x} \sin x dx$.

Answer: $-\frac{1}{5}e^{-2x}\cos x - \frac{2}{5}e^{-2x}\sin x + C$

Solution: We integrate by parts once using $u = e^{-2x}$ and $dv/dx = \sin x$. Then, $du/dx = -2e^{-2x}$ and $v = -\cos x$, and we get

$$I = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx$$

We then integrate by parts one more time using $u = e^{-2x}$ and $dv/dx = \cos x$. Then, $du/dx = -2e^{-2x}$ and $v = \sin x$. We get

$$I = -e^{-2x}\cos x - 2\left[e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\right] = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4I\,.$$

Solving for I, we get $5I = -e^{-2x}\cos x - 2e^{-2x}\sin x$, which upon dividing by five gives the final result.

Definite Integrals

3. 8 marks Each part is worth 4 marks. Please write your answers in the boxes.

(a) Calculate $\int_0^{\pi/4} \sec^4(x)\tan^3(x)\,dx$.

Answer: $\frac{5}{12}$

Solution: This is a trigonometric integral that is calculated as:

$$\begin{aligned} I = \int_0^{\pi/4} \sec^4(x)\tan^3(x)\,dx &= \int_0^{\pi/4} \sec^3(x)\tan^2(x)\sec(x)\tan(x)\,dx \\ &= \int_0^{\pi/4} \sec^3(x)(\sec^2(x)-1)\sec(x)\tan(x)\,dx \\ &= \int_0^{\pi/4} (\sec^5(x)-\sec^3(x))\sec(x)\tan(x)\,dx \end{aligned}$$

which gives, upon substituting $u = \sec(x)$ and $du = \sec(x)\tan(x)\,dx$:

$$\begin{aligned} I = \left[\frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x)\right]_0^{\pi/4} &= \frac{1}{6}\left(\left(\frac{2}{\sqrt{2}}^6\right)-1\right) + \frac{1}{4}\left(\left(\frac{2}{\sqrt{2}}^4\right)-1\right) \\ &= \frac{7}{6} - \frac{3}{4} = \frac{5}{12} \end{aligned}$$

(b) Calculate $\displaystyle\int_0^1 \frac{7x^2}{5x^2+5}\,dx$.

Answer: $\frac{7}{5}\left(1-\frac{\pi}{4}\right)$

Solution: We first rewrite the definite integral as

$$\begin{aligned} I = \int_0^1 \frac{7x^2}{5x^2+5}\,dx = \frac{7}{5}\int_0^1 \frac{x^2}{x^2+1}\,dx &= \frac{7}{5}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx \\ &= \frac{7}{5}\int_0^1 \left(1-\frac{1}{x^2+1}\right)\,dx \end{aligned}$$

In this form, the integrand is very easy to anti-differentiate and we finally get:

$$I = \frac{7}{5}\left[x - \arctan(x)\right]_0^1 = \frac{7}{5}\left(1-\frac{\pi}{4}\right)$$

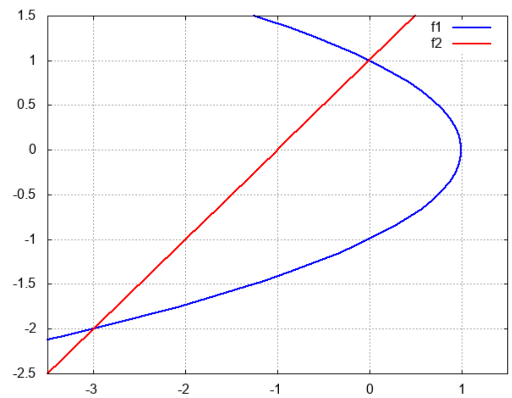
Areas, volumes and work

Please write your answers in the boxes. **Do not use absolute values in your expressions, always work out: (i) the outer function and the inner function for volumes or (ii) which function lies above the other function for areas.**

4. (a) [2 marks] Sketch by hand the finite area enclosed between the curves defined by the functions $y^2 + x = 1$ and $x = y - 1$

Answer:

Solution: The area is the region enclosed between the red and blue curves:



(b) [4 marks] Write the definite integral with specific limits of integration that determines this finite area.

Answer: $\int_{-2}^{1} (2 - y - y^2)\ dy$

Solution: We first find the intersection points of the two curves, given by the solution of:

$$1 - y^2 = y - 1 \Leftrightarrow (y-1)(y+2) = 0.$$

The intersection points are therefore $(-3, -2)$ and $(0, 1)$. We then label the curve $x_B = 1 - y^2$ and $x_R = y - 1$ and notice that $x_B \geq x_R$ for $-2 \leq y \leq 1$. The area is therefore given by the following definite integral:

$$A = \int_{-2}^{1} \left(1 - y^2 - y + 1\right)\ dy = \int_{-2}^{1} \left(2 - y - y^2\right)\ dy$$

(c) 2 marks Evaluate the integral.

Answer: $\frac{9}{2} = 4.5$

Solution:

$$A = \int_{-2}^{1} \left(2 - y - y^2\right)\, dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1} = 6 + \frac{3}{2} - 3 = \frac{9}{2}$$

5. [4 marks] Write a definite integral, with specified limits of integration, for the volume obtained by revolving the bounded region between $x = \frac{(y+1)^2}{25}$ and $x = y - 3$ about the horizontal line $y = -2$. **Do not evaluate the integral.**

Answer: $\pi \int_1^{16} (5\sqrt{x}+1)^2 - (x+5)^2 \, dx$

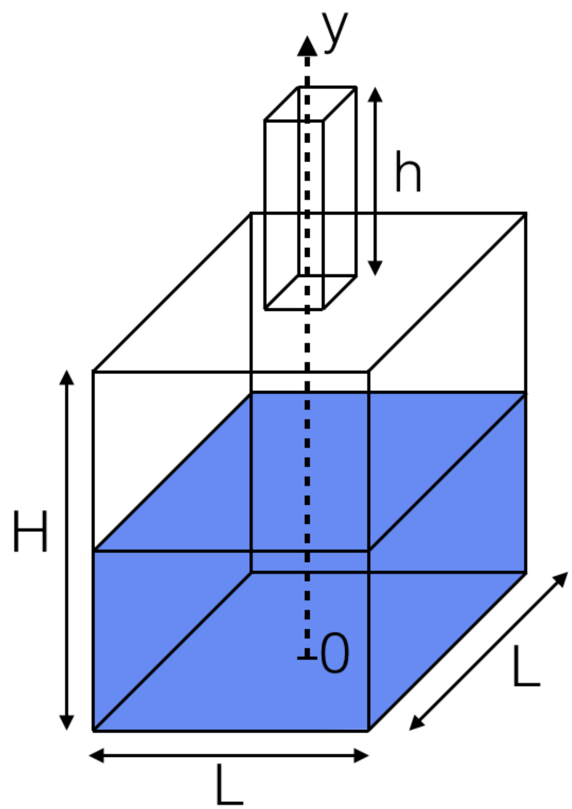
Solution: Intersection points are given by $\frac{(y+1)^2}{25} = y - 3$.
Solving for y, we determine the 2 intersection points

$$I_1 = (1, 4) \quad , \quad I_2 = (16, 19).$$

We integrate in x, hence we write y as a function of x for the 2 curves and apply a shift of +2, we finally establish:

$$\pi \int_1^{16} (5\sqrt{x}+1)^2 - (x+5)^2 \, dx.$$

6. A tank of height H and of square cross section of edge length L is half full with water of density $\rho = 1000kg/m^3$. The top of the tank features a spout of height h. We take the vertical axis y upwards oriented with its origin at the bottom of the tank. We assume gravity acceleration is $g = 10m/s^2$.
We take $H = 8m$, $L = 2m$ and $h = 3m$.



(a) [2 marks] Formulate the total work to pump the water out of the tank by the top of the spout as a definite integral.

Answer: $4 \cdot 10^4 \int_0^4 (11 - y)\, dy$

Solution: The cross section of the tank as a function of y is constant and equal to L^2. So the elementary volume, mass and force of a slice of height Δy read:

$$\Delta V = L^2 \Delta y$$
$$\Delta M = \rho L^2 \Delta y$$
$$\Delta F = g\rho L^2 \Delta y$$

The displacement of a slice of height Δy at position y is $H + h - y$, and the elementary work of that slice is:

$$\Delta W = g\rho L^2 (H + h - y)\Delta y = g\rho L^2 (11 - y)\Delta y$$

Now we integrate from bottom $y = 0$ to half height $H/2 = 8/2 = 4$ as

$$W = \int_0^4 g\rho L^2(11 - y)\, dy = 4 \cdot 10^4 \int_0^4 (11 - y)\, dy$$

(b) 2 marks Evaluate the definite integral.

Answer: $1.44 \cdot 10^6 J$

Solution:

$$W = 4 \cdot 10^4 \int_0^4 (11 - y)\, dy = 4 \cdot 10^4 \left[11y - \frac{y^2}{2}\right]_0^4 = 4 \cdot 10^4 \left(11 \cdot 4 - \frac{4^2}{2}\right)$$
$$= 4 \cdot 10^4 \cdot 36 = 1.44 \cdot 10^6 J$$