MATH 401 : Mid-Term Exam, February 12th 2016

Duration : 50 minutes. No lecture notes, no textbooks, no calculators.

Content : 3 problems, Total : 30 points

Provide clear and justified answers. Write all your answers on that document.

Name: Student ID: Signature:

Problem 1 (11pts)

Consider the following ODE boundary value problem for $u(x)$ in $[0:1]$:

$$u'' = f(x) \quad , \quad 0 < x < 1 \tag{1}$$

$$u(0) = 0 \quad , \quad u'(1) - u(1) = 0 \tag{2}$$

1. (2pts) Show that the problem is self-adjoint.
2. (3pts) Find whether a solvability condition is needed on $f(x)$, and if yes write this solvability condition.
3. (6pts) Assuming this condition is satisfied, calculate the modified Green's function and give the solution formula for $u(x)$.

Tip: The modified Green's function solves the following problem

$$L\tilde{G}_x(z) = \delta(z-x) - \frac{u^*(x)}{(u^*, u^*)} u^*(z) \ , \ \textit{with homogeneous BCs}$$

Problem 2 (12pts)

Let D be a two-dimensional region ($D \subset \Re^2$) bounded by a smooth surface ∂D, and consider the following (Neumann) problem for $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$, in D:

$$\Delta u = 0 \quad in \quad D \tag{3}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = g(\boldsymbol{x}) \quad on \quad \partial D \tag{4}$$

1. (4pts) Find the solvability condition on g, write the problem a modified Green's function should solve (but do not solve it), and represent the family of solutions $u(\boldsymbol{x})$ in terms of it.

2. (6pts) If D is the rectangle $[0 : \pi] \times [0 : \pi]$, solve the modified Green's function problem by finding a modified Green's function in the form of an eigenfunction expansion.

3. (2pts) Prove (for a general D) that any two solutions of problem (3)-(4) must differ by a constant function (i.e. uniqueness up to the addition of a constant).

Tip: Green's second identity for two functions v_1 and v_2 is

$$\int_D (v_1 \Delta v_2 - v_2 \Delta v_1) d\boldsymbol{y} = \int_{\partial D} (v_1 \frac{\partial v_2}{\partial \boldsymbol{n}} - v_2 \frac{\partial v_1}{\partial \boldsymbol{n}}) dS(\boldsymbol{y})$$

Tip: The eigenvalues/eigenfunctions of the Δ operator in $[0 : \pi] \times [0 : \pi]$ with homogeneous Neumann BCs were given in Lecture (k). The eigenfunctions involve a product of cosine functions.

Remark: Question 3 is independent of question 2. You can solve it without solving 2.

Problem 3 (7pts)

Use the method of images to find the solution $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3)$, of the following (Dirichlet) Poisson problem in the positive x_3 half-space of $\Re^3$:

$$\Delta u = f(\boldsymbol{x}) \quad in -\infty < x_1 < +\infty\,,\ -\infty < x_2 < +\infty\,,\ x_3 > 0 \tag{5}$$

$$u(x_1, x_2, 0) = h(x_1, x_2) \tag{6}$$

1. (3pts) Write the problem the Green's function should solve, and solve it using the method of images. Graphically illustrate where to place the image charge(s) and show that the constructed Green's function indeed solves the Green's function problem (PDE + boundary condition).

2. (4pts) Give the solution formula for $u(\boldsymbol{x})$.

Tip: the free-space Green's function in $\Re^3$ *is* $G^f_{\boldsymbol{x}}(\boldsymbol{y}) = -\frac{1}{4\pi|\boldsymbol{y}-\boldsymbol{x}|}$