MID-TERM EXAM, Feb 12th 2016, SOLUTIONS

Problem 1

1) $(Lu, v) = \int_0^1 u'' v \, dy = [u'v]_0^1 - \int_0^1 u'v' \, dy$

$$= [u'v - uv']_0^1 + \int_0^1 u v'' \, dy$$

$$= (u, Lv) + \underbrace{u'(1)}_{=u(1)} v(1) - u(1) v'(1) - u'(0) v(0) + \underbrace{u(0)}_{=0} v'(0)$$

$$= (u, Lv) + \underbrace{u(1)\left(v(1) - v'(1)\right) - v(0) u'(0)}_{= BT}$$

If $\begin{cases} v(0) = 0 \\ v'(1) - v(1) = 0 \end{cases}$, then $BT = 0$

Operator L is self-adjoint
BCs on v are the same as the ones on u $\Big\} \Rightarrow$ Problem is self-adjoint

2) Homogeneous problem for $u(x)$ is : $\begin{cases} u'' = 0, \quad 0 < x < 1 \\ u(0) = 0, \quad u'(1) - u(1) = 0 \end{cases}$ (HP)

If there is a non-trivial solution to (HP), then a solvability condition on $f(x)$ is needed.

$u'' = 0 \Rightarrow u = ax + b$

$u(0) = 0 \Rightarrow b = 0$

$u'(1) - u(1) = a - a = 0$, satisfied for any a

So $u^*(x) = ax$, $a \in \mathbb{R}$ satisfies (HP), we can take $\boxed{u^*(x) = x}$

Solvability condition: $\boxed{\int_0^1 x \cdot f(x) \, dx = 0}$

3) Modified Green's function $\tilde{G}_x(z)$ solves:

$$\begin{cases} \tilde{G}_x'' = \delta(z-x) - \dfrac{u^*(x)}{(u^*|u^*)}u^*(z), & 0<z<1 \\ \tilde{G}_x(0)=0, \quad \tilde{G}_x'(1)-\tilde{G}_x(1)=0 \end{cases}$$

$$\frac{u^*(x)}{(u^*|u^*)}u^*(z) = \frac{xz}{\int_0^1 x^2\,dx} = \frac{xz}{\left[\frac{1}{3}x^3\right]_0^1} = 3xz$$

For $z \neq x$: $\tilde{G}_x'' = -3xz$

$\Rightarrow \tilde{G}_x' = -\frac{3}{2}xz^2 + \alpha$ and $\tilde{G}_x = -\frac{1}{2}xz^3 + \alpha z + \beta$, $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$

Hence we have:

$$\boxed{\tilde{G}_x(z) = \begin{cases} -\frac{1}{2}xz^3 + Az + B, & 0<z<x \\ -\frac{1}{2}xz^3 + Cz + D, & x<z<1 \end{cases}}$$

$\tilde{G}_x(0)=0 \Rightarrow \boxed{B=0}$

$\tilde{G}_x'(1)-\tilde{G}_x(1)=0 \Rightarrow -\frac{3}{2}x + \cancel{C} = -\frac{1}{2}x + \cancel{C} + D \Rightarrow \boxed{D=-x}$

which give:

$$\tilde{G}_x(z) = \begin{cases} -\frac{1}{2}xz^3 + Az, & 0<z<x \\ -\frac{1}{2}xz^3 + Cz - x, & x<z<1 \end{cases}$$

Matching conditions at $z=x$:

- Continuity: $\cancel{-\frac{1}{2}x^4} + Ax = \cancel{-\frac{1}{2}x^4} + Cx - x$

  $Ax = Cx - x \Rightarrow \boxed{C = 1+A}$

- Jump condition: $\tilde{G}_x'(x^+) - \tilde{G}_x'(x^-) = 1$

  $\Rightarrow \cancel{-\frac{3}{2}x^3} + C + \cancel{\frac{3}{2}x^3} - A = 1 \Rightarrow \boxed{C = 1+A}$

Finally, $\tilde{G}_x(z)$ reads:

$$\boxed{\tilde{G}_x(z) = -\frac{1}{2}xz^3 + \begin{cases} Az, & 0 < z \leq x \\ Az + z - x, & x \leq z < 1 \end{cases}}$$

Solution formula for $u(x)$

$$u(x) = (\delta_x(z), u(z)) = \left(\tilde{G}_x^{u}(z) + \frac{u^*(x)}{(u^*, u^*)} u^*(z), u(z) \right)$$

$$= \left(\tilde{G}_x^{u}(z), u(z) \right) + \underbrace{\frac{(u^*, u)}{(u^*, u^*)}}_{\substack{= \text{constant that can be chosen} \\ \text{arbitrarily as we can always} \\ \text{add any multiple of } u^*(x)}} u^*(x)$$

$$= \left(\tilde{G}_x(z), u''(z) \right) + E \cdot u^*(x)$$

$$= \left(\tilde{G}_x(z), f(z) \right) + E \cdot u^*(x)$$

$$= \int_0^x \left(-\frac{1}{2}xz^3 + Az \right) f(z)\,dz + \int_x^1 \left(-\frac{1}{2}xz^3 + Az + z - x \right) f(z)\,dz + E \cdot x$$

$$= x \underbrace{\int_0^1 \left(-\frac{1}{2}z^3 \right) f(z)\,dz}_{\substack{= \text{this is a number,} \\ \text{let's call it } F}} + A \underbrace{\int_0^1 z f(z)\,dz}_{\substack{=0 \\ \text{solvability} \\ \text{condition}}} + \int_x^1 (z - x) f(z)\,dz + E \cdot x$$

$$= \underbrace{(F + E)}_{= H} \cdot x + \int_x^1 (z - x) f(z)\,dz$$

$$= H \cdot x + \underbrace{\int_x^1 z f(z)\,dz}_{= -\int_0^x z f(z)\,dz \text{ from solvability condition}} - x \int_x^1 f(z)\,dz$$

And finally:

$$\boxed{u(x) = H\cdot x - \int_0^x z f(z)dz - x\int_x^1 f(z)dz} \quad , H\in\mathbb{R}$$

Problem 2

1) Homogeneous problem is $\begin{cases} \Delta u = 0 \text{ in } D \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \end{cases}$ (HP)

$u^*(x) = 1$ is a non-trivial solution of (HP) (actually any constant)

Green's second identity with u and u^* writes

$$\int_D (u^* \underbrace{\Delta u}_{=0} - u \underbrace{\Delta u^*}_{=0}) dx = \int_{\partial D} (\underbrace{u^*}_{=1} \underbrace{\frac{\partial u}{\partial n}}_{g} - u \underbrace{\frac{\partial u^*}{\partial n}}_{=0}) dS(x)$$

$$\Rightarrow \boxed{\int_{\partial D} g(x) dS(x) = 0} \quad \text{solvability condition on } g$$

Modified Green's function problem:

$$\begin{cases} \Delta \tilde{G}_x(y) = \delta(y-x) + C \text{ in } D, \quad C\in\mathbb{R} \\ \frac{\partial \tilde{G}_x}{\partial n} = 0 \text{ in } \partial D \end{cases}$$

To determine C: $1 = \int_D \delta(y-x)dy = \int_D (\Delta\tilde{G}_x(y) - C)dy$

$$= \int_{\partial D} \underbrace{n\cdot\nabla\tilde{G}_x(y)}_{=\frac{\partial \tilde{G}_x}{\partial n}=0} dS(y) - C\underbrace{\int_D dy}_{|D|}$$

so $\boxed{C = -\frac{1}{|D|}}$

Green's second identity for $\tilde{G}_x$ and u

$$\int_D (\Delta\tilde{G}_x(y)\cdot u(y) - \tilde{G}_x(y)\Delta u(y)) dy = \int_{\partial D} \left(\frac{\partial \tilde{G}_x(y)}{\partial n(y)} u(y) - \tilde{G}_x(y)\frac{\partial u(y)}{\partial n(y)} \right) dS(y)$$

Now $\Delta \tilde{G}_x(y) = \delta(y-x) - \frac{1}{|D|}$, $\Delta u(y) = 0$, $\frac{\partial \tilde{G}_x(y)}{\partial n(y)} = 0$ and (3)

$\frac{\partial u(y)}{\partial n(y)} = g(y)$ give

$$\underbrace{u(x) - \frac{1}{|D|}\int_D u(y)\,dy}_{} = -\int_{\partial D} \tilde{G}_x(y) \cdot g(y)\,dS(y)$$

= constant that can be chosen arbitrarily as we can always add any multiple of $u^*(x) = 1$

Finally: $$\boxed{u(x) = A - \int \tilde{G}_x(y)\, g(y)\, dS(y)}\ , \quad A \in \mathbb{R}$$

2] We look for $\tilde{G}_x(y)$ written as:

$$\tilde{G}_x(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}(x_1,x_2)\, \underbrace{\cos(ny_1)\cos(my_2)}_{}$$

eigenfunctions of Δ in $[0,\pi]\times[0,\pi]$ with homogeneous Neumann BCs

Plugging into the equation for $\tilde{G}_x(y)$, we get

$$\Delta \tilde{G}_x(y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda_{n,m}\, c_{n,m}(x_1,x_2)\cos(ny_1)\cos(my_2) = \delta(y-x) - \frac{1}{\pi^2}$$

where $\lambda_{n,m} = -(n^2+m^2)$ are the corresponding eigenvalues.

Multiplying by $\cos(ky_1)\cos(ly_2)$ and integrating over $\int_0^\pi\int_0^\pi$, we get:

$$\lambda_{k,l}\, c_{k,l}(x_1,x_2) = \frac{1}{\int_0^\pi\int_0^\pi \cos^2(ky_1)\cos^2(ly_2)\,dy_1dy_2}\left[\int_0^\pi\int_0^\pi \delta(y-x)\cos(ky_1)\cos(ly_2)\,dy_1dy_2 - \frac{1}{\pi^2}\underbrace{\int_0^\pi\int_0^\pi \cos(ky_1)\cos(ly_2)\,dy_1dy_2}_{=\begin{cases}0, & k\neq 0 \text{ or } l\neq 0\\ \pi^2, & k=l=0\end{cases}}\right]$$

as $n=k$ and $m=l$ is the only non-zero contribution.

$$\Rightarrow \lambda_{k,l}\, c_{k,l}(x_1,x_2) = \begin{cases} 0 & , (k,l)=(0,0) \\ \frac{2}{\pi^2}\cos(kx_1)\cos(lx_2) & , k=0 \text{ or } l=0 \\ \frac{4}{\pi^2}\cos(kx_1)\cos(lx_2) & , k\geq 1 \text{ and } l\geq 1 \end{cases}$$

Using again the (n,m) notation, we have:

$$C_{n,m}(x_1,x_2) = \begin{cases} \text{free} & , (n,m)=(0,0) \\ -\dfrac{2}{\pi^2(n^2+m^2)} \cos(nx_1)\cos(mx_2), & n=0 \text{ or } m=0 \\ -\dfrac{4}{\pi^2(n^2+m^2)} \cos(nx_1)\cos(mx_2), & n\geq 1 \text{ and } m\geq 1 \end{cases}$$

And $\tilde{G}_x(y)$ finally writes:

$$\boxed{\begin{aligned} \tilde{G}_x(y) = \tilde{A} &- \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx_1)\cos(ny_1) \\ &- \frac{2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos(mx_2)\cos(my_2) \\ &- \frac{4}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^2+m^2}\cos(nx_1)\cos(mx_2)\cos(ny_1)\cos(ny_2) \end{aligned}}$$

3) If u_1 and u_2 are 2 solutions, we have:

$$\begin{cases} \Delta u_1 = 0 & \text{in } D \\ \dfrac{\partial u_1}{\partial n} = g & \text{on } \partial D \end{cases}, \text{ and } \begin{cases} \Delta u_2 = 0 & \text{in } D \\ \dfrac{\partial u_2}{\partial n} = g & \text{on } \partial D \end{cases}$$

So the difference $w = u_1 - u_2$ solves $\begin{cases} \Delta w = 0 & \text{in } D \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial D \end{cases}$

From question 1), for a general D, it means $w =$ constant

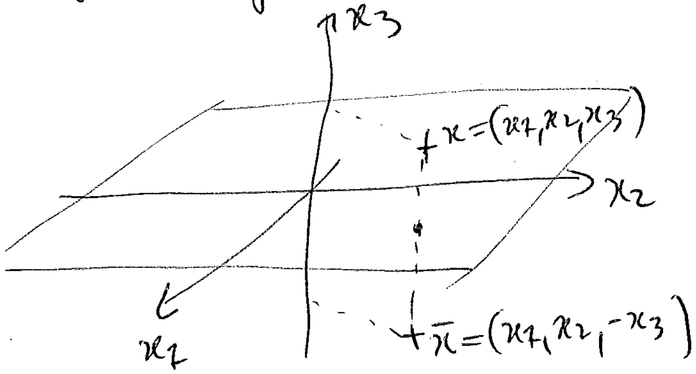
So $u_1 - u_2 =$ constant $\in \mathbb{R}$, which means that any two solutions differ by a constant.

Problem 3

1) The Green's function problem is:

$$\boxed{\begin{cases} \Delta G_x(y) = \delta(y-x), & -\infty < y_1 < +\infty, \ -\infty < y_2 < +\infty, \ y_3 > 0 \\ G_x(y_1,y_2,0) = 0 \end{cases}}$$

Using the method of images, we place an image charge at $\bar{x} = (x_1, x_2, -x_3)$ of intensity -1 and hence construct the following Green's function: $G_x(y) = G^f_x(y) - G^f_{\bar{x}}(y)$



$$\boxed{G_x(y) = \frac{1}{4\pi}\left(-\frac{1}{|y-x|} + \frac{1}{|y-\bar{x}|}\right)}$$

Since $|y-x| = |y-\bar{x}|$ for $y_3 = 0$ we have $G_x(y) = 0$ for $y_3 = 0$, so the BC is satisfied

$$\Delta G_x(y) = \Delta G^f_x(y) - \Delta G^f_{\bar{x}}(y) = \delta(y-x) - \underline{\delta(y-\bar{x})} = \delta(y-x)$$

$$= \delta(y_1 - x_1)\delta(y_2 - x_2)\underline{\delta(y_3 - x_3)} = 0$$

$$= 0 \text{ for } y_3 > 0, x_3 > 0$$

and the PDE is satisfied too.

2) Solution formula: $u(x) = \int_D G f \, dy + \int_{\partial D} u \frac{\partial G}{\partial n} dS(y)$

We need to calculate $\left(\frac{\partial G}{\partial n}\right)_{y_3 = 0}$

$$\left(\frac{\partial G}{\partial n}\right)_{y_3=0} = n_{y_3=0} \cdot \nabla G|_{y_3=0}$$

$$\nabla G = \frac{1}{4\pi|y-x|^2} \cdot \frac{y-x}{|y-x|} - \frac{1}{4\pi|y-\bar{x}|^2} \cdot \frac{y-\bar{x}}{|y-\bar{x}|} = \frac{y-x}{4\pi|y-x|^3} - \frac{y-\bar{x}}{4\pi|y-\bar{x}|^3}$$

$$\Rightarrow \nabla G|_{y_3=0} = \frac{\bar{x} - x}{4\pi|y-x|^3_{y_3=0}} \quad \text{as } |y-x| = |y-\bar{x}| \text{ for } y_3 = 0$$

Since $n_{y_3=0} = (0, 0, -1)$, we find

$$\boxed{\left(\frac{\partial G}{\partial n}\right)_{y_3=0} = \frac{2x_3}{4\pi|y-x|^3_{y_3=0}} = \frac{x_3}{2\pi\left((y_1-x_1)^2 + (y_2-x_2)^2 + x_3^2\right)^{3/2}}}$$

Finally, we write $u(x)$ as :

$$u(x) = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{0}^{\infty}\left(\frac{1}{|y-x|}-\frac{1}{|y-\bar{x}|}\right)f(y)\,dy_1\,dy_2\,dy_3$$

$$+\frac{x_3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{h(y_1,y_2)}{\left((y_1-x_1)^2+(y_2-x_2)^2+x_3^2\right)^{3/2}}\cdot dy_1\,dy_2$$