

# MATH 401 : Mid-Term Exam, February 17th 2017

Duration : 50 minutes.

No lecture notes, no textbooks, no calculators or electronic devices of any kind.

Content : 4 problems, Total : 30 points.

- Provide clear and justified answers.
- Write all your answers on that document.
- If you run out of space, continue on a separate sheet of paper and write your name and student number on the top. If you have many additional sheets of paper, number them. Make clear which problem they refer to. At the end of the exam, make sure you slide them in this booklet.
- Take a couple of minutes to read the whole exam first and pay attention to the tips.

First Name:

Last Name:

Student-No:

Section:

Signature:

| Problem | 1 | 2 | 3 | 4 | Total |
|---------|---|---|---|---|-------|
| Points  | 8 | 8 | 8 | 6 | 30    |
| Score   |   |   |   |   |       |

## Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).



## Problem 1 (8pts)

Consider the following ODE boundary value problem (BVP) for  $u(x)$  in  $[0 : \pi/4]$ :

$$u'' + 4u = f(x) \quad , \quad 0 < x < \pi/4 \quad (1)$$

$$u(0) = 0 \quad , \quad u(\pi/4) = 0 \quad (2)$$

*Tip: As you can easily notice, the problem is self-adjoint and the BVP is of the Sturm-Liouville type. Also, the only solution to the homogeneous problem is  $u(x) = 0$ , so we will solve this problem with a standard Green's function.*

1. (2pts) Write the Green's function problem.
2. (4pts) Calculate the Green's function as a solution of the Green's function problem formulated in question 1.
3. (2pts) Give the solution formula for  $u(x)$ .



## Problem 2 (8pts)

Let  $D$  be a two-dimensional rectangular region ( $D \subset \mathfrak{R}^2$ ) defined as  $D = [0 : L_1] \times [0 : L_2]$  and consider the following (Neumann) problem for  $u(\mathbf{x})$ ,  $\mathbf{x} = (x_1, x_2)$ , in  $D$ :

$$\Delta u = f(x_1, x_2) \quad \text{in } D \quad (3)$$

$$\frac{\partial u}{\partial x_1} = g(x_2) \quad \text{on } x_1 = 0, x_2 \in [0 : L_2] \quad (4)$$

$$\frac{\partial u}{\partial x_1} = 0 \quad \text{on } x_1 = L_1, x_2 \in [0 : L_2] \quad (5)$$

$$\frac{\partial u}{\partial x_2} = 0 \quad \text{on } x_2 = 0, x_1 \in [0 : L_1] \quad (6)$$

$$\frac{\partial u}{\partial x_2} = h(x_1) \quad \text{on } x_2 = L_2, x_1 \in [0 : L_1] \quad (7)$$

1. (2pts) Sketch the problem.
2. (2pts) Find the solvability condition involving  $f$ ,  $g$  and  $h$ . Write it in cartesian coordinates.
3. (2pts) Write the problem a modified Green's function should solve (but do not solve it).
4. (2pts) Represent the family of solutions  $u(\mathbf{x})$  in terms of the modified Green's function.

*Tips: 1. The homogeneous problem has an obvious non-zero solution  $u^*(\mathbf{x})$ , we have discussed this at length over the Lectures.*

*2. Green's second identity for two functions  $v_1$  and  $v_2$  is*

$$\int_D (v_1 \Delta v_2 - v_2 \Delta v_1) d\mathbf{y} = \int_{\partial D} (v_1 \frac{\partial v_2}{\partial \mathbf{n}} - v_2 \frac{\partial v_1}{\partial \mathbf{n}}) dS(\mathbf{y})$$



### Problem 3 (8pts)

Let  $D$  be a two-dimensional rectangular region ( $D \subset \mathbb{R}^2$ ) defined as  $D = [0 : \pi] \times [0 : \pi]$  and consider the following (Dirichlet) problem for  $u(\mathbf{x})$ ,  $\mathbf{x} = (x_1, x_2)$ , in  $D$ :

$$\Delta u = f(x_1, x_2) \quad \text{in } D \tag{8}$$

$$u = 0 \quad \text{on } S = \partial D \quad (= \text{boundary of } D) \tag{9}$$

1. (2pt) Sketch the problem.
2. (4pts) Write the problem a standard Green's function should solve and solve it using eigenfunction expansion.
3. (2pts) Give the solution formula for  $u(\mathbf{x})$ .

*Tip: The eigenvalues/eigenfunctions of the  $\Delta$  operator in  $[0 : \pi] \times [0 : \pi]$  with homogeneous Dirichlet BCs were given in Lectures & Assignments. The eigenfunctions involve a product of sine functions.*





### Problem 4 (6pts)

Use the method of images to find the solution  $u(\mathbf{x})$ ,  $\mathbf{x} = (x_1, x_2)$ , of the following (Dirichlet) Poisson problem in the negative quadrant  $x_1 < 0$ ,  $x_2 < 0$  of  $\mathbb{R}^2$ :

$$\Delta u = f(\mathbf{x}) \quad \text{in } -\infty < x_1 < 0, -\infty < x_2 < 0 \quad (10)$$

$$u(x_1, 0) = 0, \quad x_1 < 0 \quad (11)$$

$$u(0, x_2) = g(x_2), \quad x_2 < 0 \quad (\text{with } g(0) = 0) \quad (12)$$

1. (1pt) Sketch the problem.
2. (3pts) Write the problem the Green's function should solve, and solve it using the method of images. On your sketch illustrate where to place the image charge(s) and explain briefly why (i.e. without calculations).
3. (2pts) Give the solution formula for  $u(\mathbf{x})$  without calculating  $(\frac{\partial G_{\mathbf{x}}}{\partial n})_{y_1=0}$ .

*Tip: the free-space Green's function in  $\mathbb{R}^2$  is  $G_{\mathbf{x}}^f(\mathbf{y}) = \frac{1}{2\pi} \log|\mathbf{y} - \mathbf{x}|$*

