MATH 401 : Mid-Term Exam, February 17th 2017

Duration : 50 minutes.

No lecture notes, no textbooks, no calculators or electronic devices of any kind.

Content : 4 problems, Total : 30 points.

- Provide clear and justified answers.
- Write all your answers on that document.
- If you run out of space, continue on a separate sheet of paper and write your name and student number on the top. If you have many additional sheets of paper, number them. Make clear which problem they refer to. At the end of the exam, make sure you slide them in this booklet.
- Take a couple of minutes to read the whole exam first and pay attention to the tips.

First Name: Last Name:

Student-No: Section:

Signature:

Problem	1	2	3	4	Total
Points	8	8	8	6	30
Score					

Problem 1 (8pts)

Consider the following ODE boundary value problem (BVP) for $u(x)$ in $[0:\pi/4]$:

$$u'' + 4u = f(x) \quad , \quad 0 < x < \pi/4 \tag{1}$$
$$u(0) = 0 \quad , \quad u(\pi/4) = 0 \tag{2}$$

Tip: As you can easily notice, the problem is self-adjoint and the BVP is of the Sturm-Liouville type. Also, the only solution to the homogeneous problem is $u(x) = 0$, so we will solve this problem with a standard Green's function.

1. (2pts) Write the Greens's function problem.
2. (4pts) Calculate the Green's function as a solution of the Green's function problem formulated in question 1.
3. (2pts) Give the solution formula for $u(x)$.

Problem 2 (8pts)

Let D be a two-dimensional rectangular region ($D \subset \Re^2$) defined as $D = [0 : L_1] \times [0 : L_2]$ and consider the following (Neumann) problem for $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$, in D:

$$\Delta u = f(x_1, x_2) \quad in \quad D \tag{3}$$

$$\frac{\partial u}{\partial x_1} = g(x_2) \quad on\; x_1 = 0\;,\; x_2 \in [0 : L_2] \tag{4}$$

$$\frac{\partial u}{\partial x_1} = 0 \quad on\; x_1 = L_1\;,\; x_2 \in [0 : L_2] \tag{5}$$

$$\frac{\partial u}{\partial x_2} = 0 \quad on\; x_2 = 0\;,\; x_1 \in [0 : L_1] \tag{6}$$

$$\frac{\partial u}{\partial x_2} = h(x_1) \quad on\; x_2 = L_2\;,\; x_1 \in [0 : L_1] \tag{7}$$

1. (2pts) Sketch the problem.
2. (2pts) Find the solvability condition involving f, g and h. Write it in cartesian coordinates.
3. (2pts) Write the problem a modified Green's function should solve (but do not solve it).
4. (2pts) Represent the family of solutions $u(\boldsymbol{x})$ in terms of the modified Green's function.

Tips: 1. The homogeneous problem has an obvious non-zero solution $u^*(\boldsymbol{x})$, we have discussed this at length over the Lectures.
2. Green's second identity for two functions v_1 and v_2 is

$$\int_D (v_1 \Delta v_2 - v_2 \Delta v_1) d\boldsymbol{y} = \int_{\partial D} (v_1 \frac{\partial v_2}{\partial \boldsymbol{n}} - v_2 \frac{\partial v_1}{\partial \boldsymbol{n}}) dS(\boldsymbol{y})$$

Problem 3 (8pts)

Let D be a two-dimensional rectangular region ($D \subset \Re^2$) defined as $D = [0:\pi] \times [0:\pi]$ and consider the following (Dirichlet) problem for $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$, in D:

$$\Delta u = f(x_1, x_2) \quad in \quad D \tag{8}$$

$$u = 0 \quad on\ S = \partial D\ (= boundary\ of\ D) \tag{9}$$

1. (2pt) Sketch the problem.

2. (4pts) Write the problem a standard Green's function should solve and solve it using eigenfunction expansion.

3. (2pts) Give the solution formula for $u(\boldsymbol{x})$.

Tip: The eigenvalues/eigenfunctions of the Δ operator in $[0:\pi]\times[0:\pi]$ with homogeneous Dirichlet BCs were given in Lectures & Assignments. The eigenfunctions involve a product of sine functions.

Problem 4 (6pts)

Use the method of images to find the solution $u(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2)$, of the following (Dirichlet) Poisson problem in the negative quadrant $x_1 < 0$, $x_2 < 0$ of $\Re^2$:

$$\Delta u = f(\boldsymbol{x}) \quad in - \infty < x_1 < 0 \, , \, -\infty < x_2 < 0 \tag{10}$$

$$u(x_1, 0) = 0 \, , \, x_1 < 0 \tag{11}$$

$$u(0, x_2) = g(x_2) \, , \, x_2 < 0 \text{ (with } g(0) = 0) \tag{12}$$

1. (1pt) Sketch the problem.

2. (3pts) Write the problem the Green's function should solve, and solve it using the method of images. On your sketch illustrate where to place the image charge(s) and explain briefly why (i.e. <u>without calculations</u>).

3. (2pts) Give the solution formula for $u(\boldsymbol{x})$ <u>without calculating</u> $\left(\frac{\partial G_{\boldsymbol{x}}}{\partial n}\right)_{y_1=0}$.

Tip: the free-space Green's function in $\Re^2$ *is* $G^f_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{1}{2\pi} log|\boldsymbol{y} - \boldsymbol{x}|$