Problem 1

1. $$\boxed{\begin{array}{l} G_x''(z) + 4G_x(z) = \delta(z-x) \quad , \quad 0 < z < \frac{\pi}{4} \\ G_x(0) = G_x\left(\frac{\pi}{4}\right) = 0 \end{array}}$$

2. For $z \neq x$, we solve $G_x''(z) + 4G_x(z) = 0$

So the solution is of the form $A\cos(2z) + B\sin(2z)$

$$G_x(z) = \begin{cases} A\cos(2z) + B\sin(2z) & , \quad 0 \leq z < x \\ C\cos(2z) + D\sin(2z) & , \quad x < z \leq \frac{\pi}{4} \end{cases}$$

$G_x(0) = 0 = A$

$G_x\left(\frac{\pi}{4}\right) = C\cos\left(\frac{\pi}{2}\right) + D\sin\left(\frac{\pi}{2}\right) = 0$

$$G_x(z) = \begin{cases} B\sin(2z) & , \quad 0 \leq z < x \\ C\cos(2z) & , \quad x < z \leq \frac{\pi}{4} \end{cases}$$

Continuity: $B\sin(2x) = C\cos(2x) \Rightarrow C = B\cdot\tan(2x)$

Jump condition: $G_x'(x^+) - G_x'(x^-) = 1 = -2C\sin(2x) - 2B\cos(2x)$

$$\Rightarrow -2B\Big(\underbrace{\tan(2x)\sin(2x) + \cos(2x)}\Big) = 1$$

$$= \frac{\sin^2(2x)}{\cos(2x)} + \cos(2x) = \frac{1}{\cos(2x)}$$

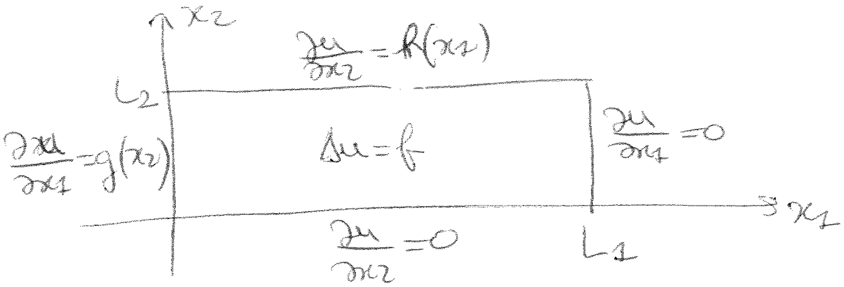
$$\Rightarrow \boxed{B = -\frac{1}{2}\cos(2x)} \quad \text{and} \quad \boxed{C = -\frac{1}{2}\sin(2x)}$$

So $$\boxed{G_x(z) = -\frac{1}{2}\begin{cases} \cos(2x)\sin(2z) & , \quad 0 \leq z \leq x \\ \sin(2x)\cos(2z) & , \quad x \leq z \leq \frac{\pi}{4} \end{cases}}$$

3. $$\boxed{\begin{aligned} u(x) = &-\frac{1}{2}\cos(2x)\int_0^x f(z)\sin(2z)\,dz \\ &-\frac{1}{2}\sin(2x)\int_x^{\frac{\pi}{4}} f(z)\cos(2z)\,dz \end{aligned}}$$

Problem 2

1.



2. $u^* = 1$ is a non-zero solution of the homogeneous problem

$$\int_D (\underbrace{\Delta u}_{f} \cdot \underbrace{u^*}_{1} - u \underbrace{\Delta u^*}_{=0}) dx = \int_{\partial D} (\underbrace{u^*}_{1} \frac{\partial u}{\partial n} - u \underbrace{\frac{\partial u^*}{\partial n}}_{=0}) dS(x)$$

$$\int_D f\, dx = \int_{\partial D} \frac{\partial u}{\partial n} dS(x) \Rightarrow \boxed{\int_0^{L_1}\int_0^{L_2} f(x_1,x_2)\, dx_1 dx_2 = \int_0^{L_1} h(x_1)\, dx_1 - \int_0^{L_2} g(x_2)\, dx_2}$$

Rem: the $-$ sign comes from $\left(\frac{\partial u}{\partial n}\right)_{x_1=0} = -\left(\frac{\partial u}{\partial x_1}\right)_{x_1=0} = -g(x_2)$

3. $$\boxed{\begin{cases} \Delta \tilde{G}_x(y) = \delta(y-x) - \frac{1}{L_1 L_2} \\ \frac{\partial \tilde{G}_x}{\partial n} = 0 \quad \text{on } \partial D \end{cases}}$$

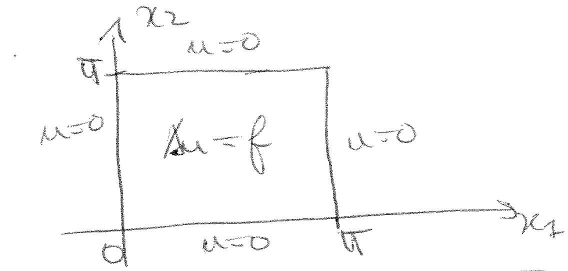
4. $$\int_D (\tilde{G} \underbrace{\Delta u}_{f} - u \underbrace{\Delta \tilde{G}}_{\delta - 1}) dy = \int_{\partial D} (\tilde{G} \frac{\partial u}{\partial n} - u \underbrace{\frac{\partial \tilde{G}}{\partial n}}_{=0}) dS(y)$$

$$\Rightarrow u = A + \int_D \tilde{G} f\, dy - \int_{\partial D} \tilde{G} \frac{\partial u}{\partial n} dS(y) \quad \text{where } A \text{ is a constant}$$

$$\boxed{u(x) = A + \int_0^{L_1}\int_0^{L_2} \tilde{G}_x(y_1,y_2) f(y_1,y_2)\, dy_1 dy_2 + \int_0^{L_2} \tilde{G}_x(0,y_2) g(y_2)\, dy_2 - \int_0^{L_1} \tilde{G}_x(y_1,L_2) h(y_1)\, dy_1}$$

Problem 3

1.



2. $$\boxed{\begin{array}{ll} \Delta G_x = \delta(y-x) \quad , & y \in D \\ G_x(y) = 0 \quad , & y \text{ on } \partial D \end{array}}$$

We seek a solution of the form $G = \sum_{n_1,n_2=1}^{\infty} g_{n_1,n_2} \sin(n_1 y_1)\sin(n_2 y_2)$

$$\Delta G = \sum_{n_1,n_2=1}^{\infty} -(n_1^2+n_2^2) g_{n_1,n_2} \sin(n_1 y_1)\sin(n_2 y_2) = \delta(y-x)$$

Multiplying by $\sin(iy_1)\sin(jy_2)$ and integrating over $[0,\pi]^2$

$$\sum_{n_1,n_2=1}^{\infty} -(n_1^2+n_2^2) g_{n_1,n_2} \int_0^{\pi}\int_0^{\pi} \sin(n_1 y_1)\sin(n_2 y_2)\sin(iy_1)\sin(jy_2)\,dy_1 dy_2$$
$$= \int_0^{\pi}\int_0^{\pi} \delta(y-x)\sin(iy_1)\sin(jy_2)\,dy_1 dy_2$$

$$\int_0^{\pi}\int_0^{\pi} \sin(n_1 y_1)\sin(n_2 y_2)\sin(iy_1)\sin(jy_2)\,dy_1 dy_2 = \begin{cases} \frac{\pi^2}{4}, & i=n_1, j=n_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad -(i^2+j^2)\frac{\pi^2}{4} g_{i,j} = \sin(ix_1)\sin(jx_2)$$

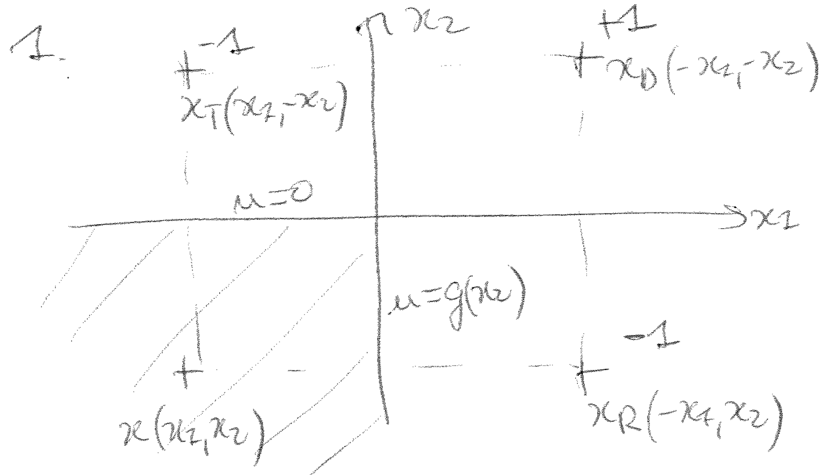
$$\boxed{g_{i,j} = -\frac{4}{\pi^2}\cdot\frac{1}{i^2+j^2}\sin(ix_1)\sin(jx_2)}$$

and $$\boxed{G_x(y) = -\frac{4}{\pi^2}\sum_{n_1,n_2=1}^{\infty}\frac{1}{(n_1^2+n_2^2)}\sin(n_1x_1)\sin(n_2x_2)\sin(n_1y_1)\sin(n_2y_2)}$$

3. $$\boxed{\begin{aligned} u(x) &= \int_D G_x(y) f(y)\,dy \\ &= -\frac{4}{\pi^2}\sum_{n_1,n_2=1}^{\infty}\frac{1}{(n_1^2+n_2^2)}\sin(n_1x_1)\sin(n_2x_2)\int_0^{\pi}\int_0^{\pi} f(y_1,y_2)\sin(n_1y_1)\sin(n_2y_2)\,dy_1 dy_2 \end{aligned}}$$

Prob 4

1.



$$\boxed{\begin{aligned} &\Delta G = \delta(y-x),\ y_1<0,\ y_2<0 \\ &G(y_1,0)=0 \\ &G(0,y_2)=0 \end{aligned}}$$

2. $$\boxed{G_x(y) = \frac{1}{2\pi}\left(\log|y-x| - \log|y-x_R| - \log|y-x_T| + \log|y-x_D|\right)}$$

3. $$\int_D (u \underbrace{\Delta G}_{\delta} - G \underbrace{\Delta u}_{=f})\, dy = \int_{\partial D} \left(u \frac{\partial G}{\partial n} - \underbrace{G}_{0} \frac{\partial u}{\partial n}\right) dS(y)$$

$$u(x) = \int_D G f\, dy + \int_{\partial D} u \frac{\partial G}{\partial n}\, dS(y)$$

$$\boxed{u(x) = \int_{-\infty}^{0}\int_{-\infty}^{0} G_x(y_1,y_2)\, f(y_1,y_2)\, dy_1\, dy_2 + \int_{-\infty}^{0} g(y_2) \left(\frac{\partial G}{\partial n}\right)_{y_1=0} dy_2}$$