Ordinary Differential Equations

I) First Order Equations

- Homogeneous 1^{st} Order ODE with constants coefficients

$$\frac{du}{dt} + \alpha u = 0 , \quad u(0) = \beta$$

$$\frac{du}{u} = -\alpha dt \Rightarrow \int \frac{du}{u} = \int^t -\alpha ds , \quad \log|u| = -\alpha t + c$$

$$u = \pm e^{-\alpha t + c} = \underbrace{\pm e^{c}}_{=A} e^{-\alpha t} = A e^{-\alpha t}$$

$$u(t=0) = \beta = A \Rightarrow \boxed{u(t) = u(t=0) \cdot e^{-\alpha t}}$$

- Homogeneous 1^{st} Order ODE with variable coefficients

$$\frac{du}{dt} + p(t) u = 0 , \quad u(0) = \beta$$

$$\frac{du}{u} = -p(t) \Rightarrow \int \frac{du}{u} = \int^t -p(s) ds , \quad \log|u| = -\int^t p(s) ds + c$$

$$u = \underbrace{\pm e^{c}}_{=A} e^{-\int^t p(s) ds}$$

$$u(t=0) = \beta = A , \quad \boxed{u(t) = u(t=0) e^{-\int^t p(s) ds}}$$

- Non-homogeneous 1^{st} Order ODE with variable coefficients

$$\frac{du}{dt} + p(t) u = g(t) , \quad u(0) = \beta$$

Look for a function $\mu(t)$ / $\frac{d(\mu u)}{dt} = \mu \frac{du}{dt} + u \frac{d\mu}{dt} = \mu g(t)$ (1)

Dividing by $\mu(t) \neq 0$: $\frac{du}{dt} + u \cdot \frac{1}{\mu} \frac{d\mu}{dt} = g(t)$

Hence $\mu(t)$ satisfies $\frac{1}{\mu} \frac{d\mu}{dt} = p(t)$ (2)

From (1), the solution reads: $\mu u = \int^t \mu(s) g(s) ds + c$

$$\boxed{u(t) = \frac{\int^t \mu(s) g(s) ds + c}{\mu(t)}} \quad (A)$$

Solving (2) gives: $\boxed{\mu = \alpha e^{\int^{t} p(s)ds}}$ (B)

Plugging (B) into (A) yields:

$$u(t) = \frac{\int^{t} \alpha e^{\int^{s} p(t')dt'} \cdot g(s)ds + C}{\alpha e^{\int^{t} p(s)ds}}$$

Dividing by α and $D = \frac{C}{\alpha}$ gives the final expression:

$$\boxed{u(t) = \frac{\int^{t} e^{\int^{s} p(t')dt'} \cdot g(s)ds + D}{e^{\int^{t} p(s)ds}}}$$

D is determined by $u(t=0) = \beta$

$u(t)$ can be written as

$$u(t) = \underbrace{D e^{-\int^{t} p(s)ds}}_{\substack{\text{complementary} \\ \text{solution} \\ \text{(Homogeneous} \\ \text{equation)}}} + \underbrace{\frac{\int^{t} e^{\int^{s} p(t')dt'} \cdot g(s)ds}{e^{\int^{t} p(s)ds}}}_{\substack{\text{Particular} \\ \text{solution} \\ \text{(Non-homogeneous} \\ \text{equation)}}}$$

VI) 2nd order linear equations

$$p(t)\frac{d^2y}{dt^2} + q(t)\frac{dy}{dt} + r(t)y = g(t)$$

$\Rightarrow$ Constant coefficients, $p(t)=a$, $q(t)=b$, $r(t)=c$

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

- Homogeneous equation with constant coefficients

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

Look for a solution of the form e^{rx}

$$y = e^{rx} \Rightarrow ar^2\cancel{e^{rx}} + br\cancel{e^{rx}} + c\cancel{e^{rx}} = 0$$

$$\boxed{ar^2 + br + c = 0}$$

$$r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac$$

a) Real distinct roots : $\Delta > 0$

$$r_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

$$\boxed{y(t) = Ae^{r_1 t} + Be^{r_2 t}}$$

b) Double real root : $\Delta = 0$

$$r_{1,2} = -\frac{b}{2a} = r, \quad y_1(t) = e^{rt}$$

Look for $y_2(t)$ of the form $v(t)e^{rt}$

$$y' = v'e^{rt} + rve^{rt}$$

$$y'' = v''e^{rt} + rv'e^{rt} + rv'e^{rt} + r^2ve^{rt}$$

$$\Rightarrow a(v''\cancel{e^{rt}} + 2rv'\cancel{e^{rt}} + r^2v\cancel{e^{rt}}) + b(v'\cancel{e^{rt}} + rv\cancel{e^{rt}}) + cv\cancel{e^{rt}} = 0$$

$$av'' + (\underbrace{2ar + b}_{=0})v' + (\underbrace{ar^2 + br + c}_{=0})v = 0$$

Hence $v'' = 0 \Rightarrow v(t) = Ct + B$

and $y_2(t) = (Ct+B)e^{rt}$

Finally: $y(t) = Ae^{rt} + D(Ct+B)e^{rt}$

$$= \underbrace{(A+DB)}_{\tilde{A}}e^{rt} + \underbrace{DC}_{\tilde{B}}te^{rt}$$

$$\boxed{y(t) = \tilde{A}e^{rt} + \tilde{B}te^{rt}}$$

c) Complex distinct roots, $\Delta < 0$

$$r_{1,2} = \frac{-b \pm i\sqrt{\Delta}}{2a} = \alpha \pm i\beta \quad \text{with } \alpha = -\frac{b}{2a} \text{ and } \beta = \frac{\sqrt{-\Delta}}{2a}$$

$$y_1(t) = e^{(\alpha+i\beta)t}, \quad y_2(t) = e^{(\alpha-i\beta)t}$$

Using sine / cosine functions:

$$y_1(t) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$y_2(t) = e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t)) = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

Since any combination of y_1 and y_2 is also a solution, we can write

$$\tilde{y}_1(t) = \frac{1}{2}(y_1(t) + y_2(t)) = e^{\alpha t}\cos(\beta t)$$

$$y_2(t) = \frac{1}{2i}(y_1(t) - y_2(t)) = e^{\alpha t}\sin(\beta t)$$

Now the general solution reads: $y(t) = A\tilde{y}_1(t) + B\tilde{y}_2(t)$

i.e. $y(t) = e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$

- Homogeneous equation with non-constant coefficients by reduction of order ③

Let's assume $y_1(t)$ is a solution, the second solution is sought in the form : $\boxed{y_2(t) = v(t) y_1(t)}$ (Generalization of constant coef and $\Delta = 0$)

Ex 1: $\boxed{2t^2 y'' + ty' - 3y = 0}$

$y_1(t) = t^{-1}$ is a solution

Let's seek $y_2(t) = v(t) \cdot t^{-1}$

$y' = v' t^{-1} - t^{-2} v$

$y'' = v'' t^{-1} - 2t^{-2} v' + 2t^{-3} v$

$\Rightarrow 2t^2 (v'' t^{-1} - 2t^{-2} v' + 2t^{-3} v) + t(v' t^{-1} - t^{-2} v) - 3v t^{-1} = 0$

$2t v'' + (-4+1) v' + \underbrace{(4t^{-1} - t^{-1} - 3t^{-1})}_{=0} v = 0$

$2t v'' - 3v' = 0$

Set $f = v'$ $\Rightarrow$ $2t f' - 3f = 0$ $\Rightarrow$ $f' - \frac{3}{2t} f = 0$

$f = \alpha e^{\int^t \frac{3}{2s} ds} = \alpha e^{\ln t^{3/2}} = \alpha t^{3/2}$

$v' = \alpha t^{3/2}$ $\Rightarrow$ $v = \frac{2}{5} \alpha t^{5/2}$

and $y_2(t) = \frac{2}{5} \alpha t^{5/2} \cdot t^{-1} = \frac{2}{5} \alpha t^{3/2}$

And the general solution is $\boxed{y(t) = A \cdot t^{-1} + B \cdot t^{3/2}}$

Ex 1 is an "Euler equation"

- Case of Euler equations

$\boxed{a x^2 y'' + b x y' + c y = 0}$

We seek solutions of the form: x^r

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$

Let's plug this solution into the ODE

$a x^2 r(r-1) x^{r-2} + b x r x^{r-1} + c x^r = 0$

$a r(r-1) \cancel{x^r} + b r \cancel{x^r} + c \cancel{x^r} = 0$

$\therefore$ $\boxed{a r^2 + (b-a) r + c = 0}$

$\Delta = (b-a)^2 - 4ac$ and $r_{1,2} = \dfrac{a - b \pm \sqrt{\Delta}}{2a}$

a) Real distinct roots, $\Delta > 0$

$\boxed{y(x) = A x^{r_1} + B x^{r_2}}$

b) Double real root, $\Delta = 0$

$r_{1,2} = r = \dfrac{a-b}{2a}$, $y_1(x) = x^r$

Look for $y_2(x)$ of the form $v(x) x^r$

$y' = v' x^r + v r x^{r-1}$

$y'' = v'' x^r + 2 v' r x^{r-1} + v r(r-1) x^{r-2}$

$\Rightarrow a x^2 (v'' x^r + 2 v' r x^{r-1} + v r(r-1) x^{r-2}) + b x (v' x^r + v r x^{r-1}) + c v x^r = 0$

$a x^2 v'' + (\underbrace{2ar + b}_{= a}) x v' + (\underbrace{a r(r-1) + b r + c}_{= 0}) v = 0$

$\Rightarrow x v'' + v' = 0$

Set $f = v'$ $\Rightarrow x f' + f = 0$ $\Rightarrow f' + \dfrac{1}{x} f = 0$

$f = \alpha e^{-\int^x \frac{1}{s} ds} = \alpha e^{-\ln x} = \dfrac{\alpha}{x}$

$v' = \dfrac{\alpha}{x}$ $\Rightarrow v = \alpha \ln x$ $\Rightarrow y_2(x) = x^r . \ln x$

And $\boxed{y(x) = A x^r + B x^r \ln x = x^r (A + B \ln x)}$

c) Complex distinct roots, $\Delta < 0$

$y(x) = A\, x^{\alpha + i\beta} + B e^{\alpha - i\beta}$ with $\alpha = \frac{a-b}{2a}$, $\beta = \frac{\sqrt{-\Delta}}{2a}$

$x^{\alpha+i\beta} = e^{\ln x^{(\alpha+i\beta)}} = e^{(\alpha+i\beta)\ln x} = e^{\alpha \ln x} e^{i\beta \ln x}$

$$= x^{\alpha}\left(\cos(\beta \ln x) + i \sin(\beta \ln x)\right)$$

Similarly, $x^{\alpha - i\beta} = x^{\alpha}\left(\cos(-\beta \ln x) + i \sin(-\beta \ln x)\right)$

$$= x^{\alpha}\left(\cos(\beta \ln x) - i \sin(\beta \ln x)\right)$$

By linear combination, we get

$\tilde{y}_1(t) = \frac{1}{2}\left(y_1(t) + y_2(t)\right) = x^{\alpha}\cos(\beta \ln x)$

$\tilde{y}_2(t) = \frac{1}{2i}\left(y_1(t) - y_2(t)\right) = x^{\alpha}\sin(\beta \ln x)$

and $\boxed{y(t) = x^{\alpha}\left(A\cos(\beta \ln x) + B \sin(\beta \ln x)\right)}$

Remark: all these solutions hold for $x > 0$

If $x < 0$, by a simple change of variable $z = -x$, $z > 0$, we can show that the solutions can be re-written with $-x$

Hence, for $x > 0$, solutions with x

for $x < 0$, ——— $-x$

means for any x, solutions for $|x|$, i.e.

$$\boxed{\begin{aligned} \Delta > 0 &: \ y = A|x|^{r_1} + B|x|^{r_2} \\ \Delta = 0 &: \ y = |x|^{r}\left(A + B \ln|x|\right) \\ \Delta < 0 &: \ y = |x|^{\alpha}\left(A\cos(\beta \ln|x|) + B\sin(\beta \ln|x|)\right) \end{aligned}}$$

• Fundamental set of solutions

Let's $y_1(t)$ and $y_2(t)$ be the solutions of

$$p(t)y''(t)+q(t)y'(t)+r(t)y(t)=0 \quad \text{with } y(t_0)=y_0,\ y'(t_0)=y'_0$$

The general solution is $y(t)=Ay_1(t)+By_2(t)$

Applying the IC : $y_0 = Ay_1(t_0)+By_2(t_0)$

$y'_0 = Ay'_1(t_0)+By'_2(t_0)$

$$A = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0)\end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0)\end{vmatrix}} \quad \text{and} \quad B = -\frac{\begin{vmatrix} y_0 & y_1(t_0) \\ y'_0 & y'_1(t_0)\end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0)\end{vmatrix}}$$

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0)\end{vmatrix} = W(y_1,y_2)(t_0) \neq 0$$

If 2 solutions satisfying $\tilde{y}_1(t_0)=1,\ \tilde{y}'_1(t_0)=0$ and $\tilde{y}_2(t_0)=0,\ \tilde{y}'_2(t_0)=1$ (with $\tilde{y}(t)=Ay_1(t)+By_2(t)$) have a $W(\tilde{y}_1,\tilde{y}_2)(t_0)\neq 0$, there are a fundamental set of solutions

Given two functions $h(x)$ and $k(x)$ defined on an interval $[\alpha,\beta]$

(A) If $\exists x_0 \,/\, W(h,k)(x_0)\neq 0 \Rightarrow h$ and k are linearly independant

(B) If h and k are linearly dependant $\Rightarrow W(h,k)(x)=0,\ \forall x\in[\alpha,\beta]$

Proof of (A) : Find $(c_1,c_2) \,/\, c_1 h + c_2 k = 0$

Then $c_1 h' + c_2 k' = 0 \Rightarrow c_1 = -\frac{k}{h}c_2,\ c_2(hk'-kh')=0$

If $W(h,k)\neq 0 \Rightarrow c_2=0 \Rightarrow c_1=0$ hence h and k are linearly independant

(B) is straightforward

Abel's Theorem

Let y_1 and y_2 be solutions of $py'' + qy' + ry = 0$
with p, q, r continuous on $[a, b]$
The Wronskian is given by $W(y_1, y_2)(t) = ce^{-\int^t \frac{q(s)}{p(s)} ds}$
$W(y_1, y_2)(t)$ is then either 0 for all t in $[a, b]$ and never 0 in $[a, b]$

Easy to prove: $W = y_1 y_2' - y_1' y_2$

$$W' = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_1' y_2'} - y_1'' y_2 = y_1 y_2'' - y_2 y_1''$$

Since y_1 and y_2 are solutions of the ODE:

$$p y_1'' + q y_1' + r y_1 = 0 \quad \times (-y_2)$$
$$p y_2'' + q y_2' + r y_2 = 0 \quad \times (y_1)$$

$$p\Big(\underbrace{y_1 y_2'' - y_1'' y_2}_{W'}\Big) + q\Big(\underbrace{y_1 y_2' - y_2 y_1'}_{=W}\Big) + r\big(\cancel{y_1 y_2 - y_1 y_2}\big) = 0$$

$$\Rightarrow W' + \frac{q}{p} W = 0 \quad \Rightarrow W = ce^{-\int^t \frac{q(s)}{p(s)} ds}$$

- Non-homogeneous equation with non-constant coefficients

The solution of $py'' + qy' + ry = g$ is the sum of

* $\tilde{y}(t) = c_1 y_1(t) + c_2 y_2(t)$, solution of the homogeneous eq.

* and a particular solution $y_p(t)$ of the non-homogeneous eq.

$$\boxed{y(t) = \tilde{y}(t) + y_p(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)}$$

How to determine $y_p(t)$:

1) Undetermined coefficients: guess a form of $y_p(t)$ like
$y_p(t) = \sum_{n=0}^{N} A_n t^n$ or $A\cos(\beta t) + B\sin(\beta t)$ ...
and determine A_n, A, B, β, etc based on algebraic eq.

2) Method of variation of parameters

• Method of variation of parameters

Assume $y_1(t)$ and $y_2(t)$ are linearly independant solutions, a particular solution $y_p(t)$ can be written as :

$$\boxed{y_p(t) = -y_1(t)\int^t \frac{y_2(s)}{W(y_1,y_2)(s)}\cdot\frac{g(s)}{p(s)}\,ds + y_2(t)\int^t \frac{y_1(s)}{W(y_1,y_2)(s)}\cdot\frac{g(s)}{p(s)}\,ds}$$

Proof : We are looking for a solution like $y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$

so $y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

and we assume that $\underset{(A)}{u_1'y_1 + u_2'y_2 = 0} \Rightarrow y_p' = u_1y_1' + u_2y_2'$

$y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$

Plug this into the ODE :

$$p(t)(\cancel{u_1y_1''} + u_1'y_1' + \cancel{u_2y_2''} + u_2'y_2') + q(t)(\cancel{u_1y_1'} + \cancel{u_2y_2'})$$
$$+ r(t)(\cancel{u_1y_1} + \cancel{u_2y_2}) = g(t)$$

$$u_1(\overbrace{py_1'' + qy_1' + ry_1}^{=0}) + u_2(\overbrace{py_2'' + qy_2' + ry_2}^{=0})$$
$$+ p(t)(u_1'y_1' + u_2'y_2') = g(t)$$

$\Rightarrow u_1'y_1' + u_2'y_2' = g(t)/p(t)$ (B)

Using (A) : $u_1' = -u_2'\cdot\frac{y_2}{y_1}$ into (B) ; $-u_2'\frac{y_2}{y_1}y_1' + u_2'y_2' = \frac{g(t)}{p(t)}$

hence $u_2'(y_1y_2' - y_2y_1') = \frac{g(t)}{p(t)}\cdot y_1$

$$\boxed{u_2' = \frac{y_1}{W(y_1,y_2)}\cdot\frac{g}{p}} \quad \text{and } u_1' = -\frac{y_2}{W(y_1,y_2)}\cdot\frac{g}{p}$$

$$\Rightarrow u_1 = -\int^t \frac{y_2}{W(y_1,y_2)}\,\frac{g}{p}\,ds \quad \text{and } u_2 = \int^t \frac{y_1}{W(y_1,y_2)}\,\frac{g}{p}\,ds$$