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TIME-REVERSAL IN HYPERBOLIC S.P.D.E.'S

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This paper studies questions of changes of variables in a class of hyperbolic stochastic partial differential equations in two variables driven by white noise. Two types of changes of variables are considered: naive changes of variables which do not involve a change of filtration, which affect the equation much as though it were deterministic, and changes of variables that do involve a change of filtration, such as time-reversals. In particular, if the process in reversed coordinates does satisfy an s.p.d.e., then we show how this s.p.d.e. is related to the original one. Time-reversals for the Brownian sheet and for equations with constant coefficients are considered in detail. A necessary and sufficient condition is provided under which the reversal of the solution to the simplest hyperbolic s.p.d.e. with certain random initial conditions again satisfies such an s.p.d.e. This yields a negative conclusion concerning the reversal in time of the solution to the stochastic wave equation (in one spatial dimension) driven by white noise.

1. Introduction. This paper was motivated by questions of changes of variables in stochastic partial differential equations (s.p.d.e.'s). To illustrate the issues, consider first the analogous question for a stochastic differential equation of the form

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \qquad X_0 = x_0. \tag{1.1}$$

Given a smooth increasing function φ: $\mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi'(u) > 0$, for all $u \geq 0$, set $Y_u = X_{\varphi(u)}$. Then (Y_u) is a (weak) solution of the following equation:

$$dY_u = b\big(\varphi(u), Y_u\big)\varphi'(u)\,du + \sigma\big(\varphi(u), Y_u\big)\sqrt{\varphi'(u)}\,d\tilde{B}(u),$$

for some Brownian motion $\tilde{B}$. That is, the change of variables $t = \varphi(u)$ affects equation (1.1) much as though it were an ordinary differential equation.

On the other hand, consider the change of variables $t = 1 - u$, namely, time-reversal. It is well known [8] that the process $(\hat{X}_u = X_{1-u},\ 0 \leq u \leq 1)$ is a solution of the stochastic differential equation

$$d\hat{X}_u = \hat{b}(u, \hat{X}_u)\,du + \hat{\sigma}(u, \hat{X}_u)\,d\hat{B}_u, \qquad \hat{X}_0 = X_1,$$

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where $\hat{B}$ is a Brownian motion independent of X_1, and $\hat{b}$ and $\hat{\sigma}$ are given by the formulas

$$\begin{aligned} &\hat{\sigma}(u,x) = \sigma(1-u,x), \\ &\hat{b}(u,x) = -b(1-u,x) + \frac{\frac{\partial}{\partial x}(\rho_{1-u}(x)\sigma(1-u,x))}{\rho_{1-u}(x)}, \end{aligned} \tag{1.2}$$

where $\rho_t(\cdot)$ is the density function of X_t. In the simplest case in which $b \equiv 0$, $\sigma \equiv 1$, X is a standard Brownian motion and these formulas give the following equation for $\hat{X}$:

$$d\hat{X}_u = -\frac{\hat{X}_u}{1-u}\,du + d\hat{B}_u.$$

As expected, the reversal of Brownian motion is a Brownian bridge.

These considerations have been considerably extended [5, 11], to include infinite systems of stochastic differential equations. Of course, the presence of the density of X_{1-u} and the derivative in x makes the extensions highly non-trivial, but under certains conditions, the formulas above (suitably reinterpreted) give an equation for the reversed process.

With s.p.d.e.'s, there is a much wider choice of changes of variables than with s.d.e.'s. However, the fundamental issue is similar to that of s.d.e.'s: if the change of variables respects the filtration, then the s.p.d.e. in the new variables is easily obtained from the s.p.d.e. in the old variables, almost as for deterministic p.d.e.'s (see Section 3). However, if the change of variables implies a change of filtration, then the situation is much more delicate. The aim of this paper is to examine this issue in the context of hyperbolic s.p.d.e.'s in two variables, driven by two-parameter white noise.

If one considers a change of variables such as time reversal in an s.p.d.e., one might be tempted to make use of the results of [5, 11]. Indeed, in an abstract sense, an s.p.d.e. can be interpreted as an infinite system of s.d.e's. However, the class of s.p.d.e.'s is only a small subset of the class of infinite systems of s.d.e.'s, and there is no reason to expect, given an infinite system of s.d.e.'s for the reversed process, that it will correspond to an s.p.d.e.

The outline of this paper is as follows. In Section 2, we recall the basic existence theory for hyperbolic s.p.d.e.'s in two variables, and establish some properties of the solution that will be needed in the sequel, including questions related to its planar quadratic variation. In Section 3, we consider changes of variables which do not involve a change of filtration, and we give in Theorem 3.2 the equation that is satisfied by the process in the new variables. In Section 4, we relate linear s.p.d.e.'s to random p.d.e.'s interpreted as equations in the space of (Schwarz) distributions. In Section 5, we consider specifically the issue of reversal in one or two coordinates, and show in Proposition 5.2 how the equation for the process in reversed coordinates, if there is one, must be related to the original equation. Section 6 particularizes to reversals of the Brownian sheet, while Section 7 extends this to certain hyperbolic s.p.d.e.'s with constant coefficients. Finally, in Section 8, we consider hyperbolic s.p.d.e.'s with certain random initial conditions, and establish in Theorem 8.1 a necessary and sufficient condition for the reversal in two coordinates to satisfy an s.p.d.e. with local coefficients. This theorem implies that the reversal in time of the solution to the wave equation driven by space-time white noise (with vanishing initial conditions) does not satisfy an s.p.d.e. with local coefficients (see Remarks 8.2 and 8.3).

2. Existence theory for hyperbolic s.p.d.e.'s in the plane. Consider the (reduced) hyperbolic s.p.d.e.

$$\frac{\partial^2 X}{\partial s\partial t} + a_1(s,t)\frac{\partial X}{\partial s} + a_2(s,t)\frac{\partial X}{\partial t} + a_3(s,t,X) = a_4(s,t)\dot{W}, \tag{2.1}$$

with initial data

$$X(s,0) = X_0 + M_s^1, \qquad X(0,t) = X_0 + M_t^2.$$

Here, $\dot{W}$ is a space-time white noise. The coefficients $a_1,\dots,a_4$ are deterministic functions: a_1, a_2 and a_3 are continuously differentiable and have bounded first partials, a_4 is bounded and continuous, and $a_3(s,t,X) = a_3(s,t,X(s,t))$. The boundary conditions X_0, M^1 and M^2 are (possibly) random, independent of the white noise $\dot{W}$, and M^1 and M^2 are continuous processes, with $M_0^1 = M_0^2 = 0$.

Equation (2.1) was studied in [4] using the theory of two-parameter processes. It was also studied in [14], [15], where it was formulated in mild form, using the Green's function, and it was shown that the two-parameter form, the mild form, and the weak form (see below) are equivalent, and have a unique solution.

To get the weak form of (2.1), multiply both sides by a test function $\phi \in C^{(2)}(\mathbb{R}^2)$, and integrate over the rectangle $R_{st} \stackrel{\text{def}}{=} [0,s]\times[0,t]$ to get

$$\begin{aligned}&\iint_{R_{st}} \phi(u,v)\left(\frac{\partial^2 X}{\partial u\partial v} + a_1(u,v)\frac{\partial X}{\partial u} + a_2(u,v)\frac{\partial X}{\partial v}\right) du\,dv\\ &\qquad = \iint_{R_{st}} \phi(u,v)\big[a_4(u,v)W(du\,dv) - a_3(u,v,X)\,du\,dv\big].\end{aligned} \tag{2.2}$$

Use the integration by parts formula

$$\begin{aligned}&\int_a^b dx\int_c^d dy\, f(x,y)\frac{\partial^2 g}{\partial x\partial y}\\ &\qquad = f(b,d)g(b,d) - f(a,d)g(a,d) - f(b,c)g(b,c) + f(a,c)g(a,c)\\ &\qquad\quad - \int_a^b \left[\frac{\partial f}{\partial x}(x,d)g(x,d) - \frac{\partial f}{\partial x}(x,c)g(x,c)\right] dx\end{aligned} \tag{2.3}$$

$$- \int_c^d \left[\frac{\partial f}{\partial y}(b,y) g(b,y) - \frac{\partial f}{\partial y}(a,y) g(a,y) \right] dy$$
$$+ \int_a^b dx \int_c^d dy \frac{\partial^2 f}{\partial x \partial y} g(x,y),$$

with $f = \phi$, $g = X$, to get all the derivatives onto ϕ:

$$\begin{aligned}
&X(s,t)\phi(s,t) - X(s,0)\phi(s,0) - X(0,t)\phi(0,t) + X(0,0)\phi(0,0) \\
&\quad - \int_0^s \Big(X(u,t) \Big[\frac{\partial \phi}{\partial u}(u,t) - a_2(u,t)\phi(u,t) \Big] \\
&\qquad\qquad - X(u,0) \Big[\frac{\partial \phi}{\partial u}(u,0) - a_2(u,0)\phi(u,0) \Big] \Big) du \\
&\quad - \int_0^t \Big(X(s,v) \Big[\frac{\partial \phi}{\partial v}(s,v) - a_1(s,v)\phi(s,v) \Big] \\
&\qquad\qquad - X(0,v) \Big[\frac{\partial \phi}{\partial v}(0,v) - a_1(0,v)\phi(0,v) \Big] \Big) dv \\
&\quad + \iint_{R_{st}} X(u,v) \Big[\frac{\partial^2 \phi}{\partial u \partial v}(u,v) - \frac{\partial}{\partial u}\big(a_1(u,v)\phi(u,v)\big) \\
&\qquad\qquad - \frac{\partial}{\partial v}\big(a_2(u,v)\phi(u,v)\big) \Big] du\, dv \\
&= \iint_{R_{st}} \phi(u,v) \big[a_4(u,v) W(du\, dv) - a_3(u,v,X)\, du\, dv \big].
\end{aligned} \tag{2.4}$$

We say that a jointly measurable and locally integrable process $(X(s,t),\ (s,t) \in \mathbb{R}_+^2)$ is a *weak solution of* (2.1) if (2.4) holds a.s. for each $(s,t) \in \mathbb{R}_+^2$ and each function $\phi \in C^{(2)}(\mathbb{R}_+^2)$. A slight extension of [15, Theorem 1] (which only considers more restrictive initial conditions) shows that if $E(X_0^2) < \infty$, $E(\sup_{u \le s}(M_u^1)^2) < \infty$ and $E(\sup_{v \le t}(M_v^2)^2) < \infty$, then there exists a unique weak solution of (2.1) which has continuous sample paths, and which has the property that $\sup_{(u,v) \in R_{st}} E(X(u,v)^2) < \infty$.

The weak solution of (2.1) has an integral representation using the Green's function for equation (2.1). The Green's function and its properties are studied in [15, Propositions 10 and 11]: it is a function $\gamma(s,t;u,v)$ defined for $(s,t) \in \mathbb{R}_+^2$ and $(u,v) \in R_{st}$, which has the following properties.

(a) For fixed (S,T), for all $s \le S$ and $t \le T$, $\gamma(s,t;\cdot,\cdot)$ has continuous and uniformly bounded first derivatives and a continuous and uniformly bounded second order mixed derivative in R_{st}. For $u \le S$ and $v \le T$, $\gamma(\cdot,\cdot;u,v)$ has continuous and uniformly bounded first derivatives and a continuous and uniformly bounded second order mixed derivative in $R_{ST} \setminus R_{uv}$. (*Note*: The continuity statements are not made in [14], [15] because in those papers, a_1 and

a_2 are not assumed to be C^1. However, under this assumption, they follow easily from the proof in [14], Proposition 3.2.)

(b) For $(u, v) \in R_{st}$,

$$\gamma(s, t; u, v) = 1 - \int_v^t a_1(u, w)\gamma(s, t; u, w)\, dw - \int_u^s a_2(r, v)\gamma(s, t; r, v)\, dr;$$

(c) For $(u, v) \in R_{st}$,

$$\frac{\partial^2 \gamma}{\partial u \partial v}(s, t; u, v) - \frac{\partial}{\partial u}\big(a_1(u, v)\gamma(s, t; u, v)\big) - \frac{\partial}{\partial v}\big(a_2(u, v)\gamma(s, t; u, v)\big) = 0;$$

(d) $\frac{\partial \gamma}{\partial u}(s, t; u, t) - a_2(u, t)\gamma(s, t; u, t) = 0, u \leq s;$

(e) $\frac{\partial \gamma}{\partial v}(s, t; s, v) - a_1(s, v)\gamma(s, t; s, v) = 0, v \leq t;$

(f) $\gamma(s, t; s, t) = 1.$

Moreover, there exists a universal constant $C > 0$ such that:

(g) $\sup_{(s,t)\in\mathbb{R}_+^2} \sup_{(u,v)\in R_{st}} |\gamma(s, t; u, v)| \leq C;$

(h) $\sup_{\substack{s \geq u \vee r,\\ t \geq v \vee w}} |\gamma(s, t; u, v) - \gamma(s, t; r, w)| \leq C(|u - r| + |v - w|);$

(i) $\sup_{\substack{s \wedge r \geq u,\\ t \wedge w \geq v}} |\gamma(s, t; u, v) - \gamma(r, w; u, v)| \leq C(|s - r| + |t - w|).$

If we replace $\phi(u, v)$ by $\gamma(s, t; u, v)$ in (2.4) and use (c), (d) and (e), we get

$$\begin{aligned} X(s, t) &= \gamma(s, t; s, 0)X(s, 0) + \gamma(s, t; 0, t)X(0, t) - \gamma(s, t; 0, 0)X(0, 0) \\ &\quad - \int_0^s X(u, 0)\left[\frac{\partial \gamma}{\partial u}(s, t; u, 0) - a_2(u, 0)\gamma(s, t; u, 0)\right] du \\ &\quad - \int_0^t X(0, v)\left[\frac{\partial \gamma}{\partial v}(s, t; 0, v) - a_1(0, v)\gamma(s, t; 0, v)\right] dv \\ &\quad + \iint_{R_{st}} \gamma(s, t; u, v)\big[a_4(u, v)W(du\, dv) - a_3(u, v, X)\, du\, dv\big]. \end{aligned} \tag{2.5}$$

DEFINITION 2.1. If $\Delta =]a, b] \times]c, d] \subset \mathbb{R}_+^2$ is a rectangle, the *planar increment of X over Δ* is

$$X(\Delta) \stackrel{\text{def}}{=} X(b, d) - X(a, d) - X(b, c) + X(a, c).$$

It is shown in [15], Propositions 2.1 and 2.20 that the solution of (2.4) also satisfies (2.5). One can extend (2.4) to certain non-smooth ϕ, and in particular to indicator functions, as follows.

LEMMA 2.1. *Let* $0 < u_i < v_i$, $i = 1, 2$, *and set* $\Delta =]u_1, v_1] \times]u_2, v_2]$. *Suppose that* $(X(s,t))$ *is a weak solution of* (2.1). *Then*

$$
\begin{aligned}
X(\Delta) &- \iint_\Delta X(u,v)\left[\frac{\partial a_1}{\partial u}(u,v) + \frac{\partial a_2}{\partial v}(u,v)\right] du\,dv \\
&+ \int_{u_1}^{v_1} \big[X(u,v_2)a_2(u,v_2) - X(u,u_2)a_2(u,u_2)\big]\,du \\
&+ \int_{u_2}^{v_2} \big[X(v_1,v)a_1(v_1,v) - X(u_1,v)a_1(u_1,v)\big]\,dv \\
&= \iint_\Delta \big[a_4(u,v)W(du\,dv) - a_3(u,v,X)\,du\,dv\big].
\end{aligned}
\tag{2.6}
$$

Further, if $(X(s,t))$ *is a continuous process such that* (2.6) *holds for all rectangles* $\Delta \subset \mathbb{R}_+^2$, *then* (2.4) *holds for all* $\phi \in C^{(2)}(\mathbb{R}_+^2)$ *and therefore* X *is a weak solution of* (2.1).

PROOF. Fix $(s,t) \in \mathbb{R}_+^2$. We only consider the case where $s > v_1$ and $t > v_2$, as the other cases are similar. Let $\psi(x)$ be a non-negative smooth function with compact support, such that $\psi(0) > 0$ and $\int \psi(x)\,dx = 1$. Define

$$
\phi_{i\varepsilon}(x) = \frac{1}{\varepsilon}\int_0^x \left(\psi\left(\frac{y-u_i}{\varepsilon}\right) - \psi\left(\frac{y-v_i}{\varepsilon}\right)\right) dy,
$$

and let $\phi_\varepsilon(u,v) = \phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)$. If we put ϕ_ε into (2.4), the first three lines vanish if ε is small, and we get

$$
\begin{aligned}
\iint_{R_{st}} X(u,v)\Big[&\phi'_{1\varepsilon}(u)\phi'_{2\varepsilon}(v) - a_1(u,v)\phi'_{1\varepsilon}(u)\phi_{2\varepsilon}(v) \\
&-a_2(u,v)\phi_{1\varepsilon}(u)\phi'_{2\varepsilon}(v) \\
&-\phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)\left(\frac{\partial a_1}{\partial u}(u,v) + \frac{\partial a_2}{\partial v}(u,v)\right)\Big]\,du\,dv \\
= \iint_{R_{st}} &\phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)\big[a_4(u,v)W(du\,dv) - a_3(u,v,X)\,du\,dv\big].
\end{aligned}
\tag{2.7}
$$

Notice that as $\varepsilon \downarrow 0$, $\phi_{i\varepsilon}$ converges boundedly and pointwise to $1_{[u_i,v_i]}$ while $\phi'_{i\varepsilon}$ converges weakly to $\delta_{u_i} - \delta_{v_i}$. Since X, the a_i, and their first partials are continuous, the left-hand side of (2.7) converges to the left-hand side of (2.6). At the same time, the a_i are bounded and ϕ_ε converges pointwise and boundedly to the indicator function of Δ, so the right-hand side of (2.7) converges in L^2 to the right-hand side of (2.6), proving this equality.

Assume now that (2.6) holds for all $\Delta \in \mathbb{R}_+^2$. Fix $(s,t) \in \mathbb{R}_+^2$ and $\phi \in C^{(2)}(\mathbb{R}_+^2)$. We shall show that (2.4) holds. For $n \in \mathbb{N}$ and $i, j \in \{0,\dots,n\}$, set $s_i^n = is/n$,

$t_j^n = jt/n$, $\Delta_{i,j}^n =]s_i^n, s_{i+1}^n] \times]t_j^n, t_{j+1}^n]$, $\phi_{i,j} = \phi(s_i^n, t_j^n)$, and

$$\phi^n = \sum_{i=0}^{n} \sum_{j=0}^{n} \phi_{i,j} 1_{\Delta_{i,j}^n}.$$

Finally, set $X_{i,j} = X(s_i^n, t_j^n)$. Apply (2.6) to each $\Delta_{i,j}^n$, multiply by $\phi_{i,j}$, then sum over i and j, to see that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \phi_{i,j} (X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} + X_{i,j}) \\
&\qquad - \iint_{R_{st}} \phi^n(u,v) X(u,v) \Big[\frac{\partial a_1}{\partial u}(u,v) + \frac{\partial a_2}{\partial v}(u,v) \Big] \, du \, dv \\
&\qquad + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \phi_{i,j} \int_{s_i^n}^{s_{i+1}^n} \big[X(u, t_{j+1}^n) a_2(u, t_{j+1}^n) - X(u, t_j^n) a_2(u, t_j^n) \big] \, du \\
&\qquad + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \phi_{i,j} \int_{t_j^n}^{t_{j+1}^n} \big[X(s_{i+1}^n, v) a_1(s_{i+1}^n, v) - X(s_i^n, v) a_1(s_i^n, v) \big] \, dv \\
&\quad = \iint_{R_{st}} \phi^n(u,v) \big[a_4(u,v) W(du \, dv) - a_3(u,v,X) \, du \, dv \big].
\end{aligned} \tag{2.8}$$

By a double summation by parts (see also [7] or [12], page 24), the first double sum above can be written

$$\begin{aligned}
&\phi_{n-1,n-1} X_{n,n} - \phi_{n-1,0} X_{n,0} - \phi_{0,n-1} X_{0,n} + \phi_{0,0} X_{0,0} \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\phi_{i-1,j-1} - \phi_{i,j-1} - \phi_{i-1,j} + \phi_{i,j}) X_{i,j} \\
&\quad - \sum_{j=1}^{n-1} (\phi_{n-1,j} - \phi_{n-1,j-1}) X_{n,j} - \sum_{i=1}^{n-1} (\phi_{i,n-1} - \phi_{i-1,n-1}) X_{i,n} \\
&\quad + \sum_{j=1}^{n} (\phi_{0,j} - \phi_{0,j-1}) X_{0,j} + \sum_{i=1}^{n-1} (\phi_{i,0} - \phi_{i-1,0}) X_{i,0},
\end{aligned}$$

the second double sum in (2.8) can be written

$$\begin{aligned}
&- \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} (\phi_{i,j} - \phi_{i,j-1}) \int_{s_i^n}^{s_{i+1}^n} X(u, t_j^n) a_2(u, t_j^n) \, du \\
&\quad + \sum_{i=0}^{n-1} \int_{s_i^n}^{s_{i+1}^n} \big(\phi_{i,n-1} X(u, t_n^n) a_2(u, t_n^n) - \phi_{i,0} X(u, t_0^n) a_2(u, t_0^n) \big) \, du,
\end{aligned}$$

and the third double sum in (2.8) can be written

$$- \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} (\phi_{i,j} - \phi_{i-1,j}) \int_{t_j^n}^{t_{j+1}^n} X(s_i^n, v) a_1(s_i^n, v)\, dv$$

$$+ \sum_{j=0}^{n-1} \int_{t_j^n}^{t_{j+1}^n} \big(\phi_{n-1,j} X(s_n^n, v) a_1(s_n^n, v) - \phi_{0,j} X(s_0^n, v) a_1(s_0^n, v)\big)\, dv.$$

With these three expressions, we can let $n \to \infty$ in (2.8), to see that the left-hand side converges a.s. By comparing terms, this limit is easily identified with the left-hand side of (2.4). As for the right-hand side of (2.8), it clearly converges as $n \to \infty$ in L^2 to the right-hand side of (2.4). This proves the lemma. $\square$

2.1. *Semimartingale initial data.* We want to consider solutions with fairly regular initial values. In this context, "initial values" refers to the values of X on the boundary of $\mathbb{R}_+^2$, and "fairly regular" means that the boundary values should be well-behaved semimartingales.

Let $Y = (Y_t,\ t \geq 0)$ be a semimartingale with the decomposition $Y_t = M_t + V_t$, where M_t is a martingale (in some given filtration), and V_t is a process of locally finite variation. Let $\langle Y \rangle_t = \langle M \rangle_t$ be the predictable increasing process associated to Y.

DEFINITION 2.2. We say that a semimartingale Y is *smooth* if:

(i) M and V are continuous;
(ii) $t \mapsto \langle Y \rangle_t$ and $t \mapsto V_t$ are continuously differentiable;
(iii) $\frac{d\langle Y \rangle}{dt}$ is L^1-bounded in compact t-sets, and $\frac{dV}{dt}$ is L^2-bounded in compact t-sets.

Notice that a smooth semimartingale need not have smooth sample paths (quite the opposite, it will only have smooth sample paths if its martingale part is constant). It is the characteristics of the semimartingales, not the semimartingales themselves, which are smooth. One can think of a smooth semimartingale as the solution of a stochastic differential equation $dY = \sigma\, dW_t + \mu\, dt$, where $\sigma(x,t)$ and $\mu(x,t)$ are Lipschitz continuous.

REMARK 2.2. It is straightforward to show that if f is a bounded, continuous, adapted process and Y is a smooth semimartingale, then $Z_t \overset{\text{def}}{=} \int_0^t f(s)\, dY_s$ is also a smooth semimartingale.

ASSUMPTION A. *Let $Y_u^1 = X(u,0)$ and $Y_v^2 = X(0,v)$. $(Y_u^1,\ u \geq 0)$ and $(Y_v^2,\ v \geq 0)$ are smooth semimartingales (in their respective natural filtrations) which are independent of $\dot{W}$, with semimartingale decomposition $Y_u^i = M_u^i + V_u^i$, $i = 1, 2$.*

Under this assumption, denote

$$\sigma_i^2(u) \stackrel{\text{def}}{=} \frac{d\langle Y^i\rangle_u}{du}, \qquad i = 1, 2,$$

$$\mu_i(u) \stackrel{\text{def}}{=} \frac{dV_u^i}{du}, \qquad i = 1, 2,$$

and for $(s, t) \in \mathbb{R}_+^2$, set

$$\mathcal{F}_{s,t} = \sigma(Y_u^1,\ Y_v^2,\ \dot{W}_{u,v},\ u \le s,\ v \le t).$$

LEMMA 2.3. *Under Assumption* A, *for any* $(s, t) \in \mathbb{R}_+^2$, *the processes* $(X(u, t),\ 0 \le u \le s)$ *and* $(X(s, v),\ 0 \le v \le t)$ *are smooth semimartingales* (*in the respective filtrations* $(\mathcal{F}_{u,t},\ 0 \le u \le s)$ *and* $(\mathcal{F}_{s,v},\ 0 \le v \le t)$), *and the* L^p*-bounds on their characteristics are uniform for* (s, t) *in compact sets*.

Moreover, *if* $\Delta =]u_1, v_1] \times]u_2, v_2]$, (2.6) *can be written*

$$\begin{aligned} &X(\Delta) + \int_{u_2}^{v_2} dv \int_{u_1}^{v_1} a_1(u, v) X(du, v) + \int_{u_1}^{v_1} du \int_{u_2}^{v_2} a_2(u, v) X(u, dv) \\ &\quad = \iint_{\Delta} \bigl[a_4(u, v) W(du\, dv) - a_3(u, v, X)\, du\, dv\bigr]. \end{aligned} \tag{2.9}$$

PROOF. Since $X(u, 0) = Y_u^1$ and $X(0, v) = Y_v^2$ are semimartingales, we can integrate by parts in the first two integrals on the right-hand side of (2.5) to get

$$X(s, t) = \gamma(s, t; 0, 0) X(0, 0) + \int_0^s \gamma(s, t; u, 0)\, dY_u^1 + \int_0^t \gamma(s, t; 0, v)\, dY_v^2 \tag{2.10}$$

$$+ \int_0^s Y_u^1 a_2(u, 0) \gamma(s, t; u, 0)\, du + \int_0^t Y_v^2 a_1(0, v) \gamma(s, t; 0, v)\, dv$$

$$+ \iint_{R_{st}} \gamma(s, t; u, v) a_4(u, v) W(du\, dv)$$

$$- \iint_{R_{st}} \gamma(s, t; u, v) a_3(u, v, X)\, du\, dv \tag{2.11}$$

$$\stackrel{\text{def}}{=} I_1(s, t) + \cdots + I_7(s, t).$$

The integrals with respect to dY_u^1 and dY_v^2 are stochastic integrals relative to semimartingales. One can show that each of them has a version which is continuous in (s, t), and we will always take that version.

We will show that if we fix s or t, $I_1, \dots, I_7$ are smooth semimartingales in the remaining variable. By symmetry, it is enough to fix t. Let us decompose $I_1, \dots, I_7$ into their martingale and bounded variation parts in s.

Note that I_1, I_3, I_4, I_5 and I_7 are each $C^{(1)}$ and have no martingale part, so $\langle I_1\rangle = \langle I_3\rangle = \langle I_4\rangle = \langle I_5\rangle = \langle I_7\rangle \equiv 0$. Indeed, this is clear for I_1, I_5 and I_7 thanks to the differentiability of $s \mapsto \gamma(s,t;u,v)$ (property (a) above). In I_3, one can differentiate (with care!) inside the stochastic integral to see that

$$\frac{\partial}{\partial s} I_3(s,t) = \int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\,dY_v^2, \tag{2.12}$$

which is continuous in s by (a). For I_4, write

$$\gamma(s,t;u,v) = \gamma(u,t;u,v) + \int_u^s \frac{\partial}{\partial r}\gamma(r,t;u,v)\,dr$$

and use Fubini's theorem:

$$I_4(s,t) = \int_0^s Y_u^1 a_2(u,0)\gamma(u,t;u,0)\,du + \int_0^s dr \int_0^r Y_u^1 a_2(u,0)\frac{\partial}{\partial r}\gamma(r,t;u,0)\,du,$$

which is clearly differentiable in s. The same idea can be used in I_2 and I_6, although one has to use Fubini's theorem for mixed stochastic/Riemann integrals [16]:

$$\begin{aligned} I_2(s,t) &= \int_0^s \gamma(u,t;u,0)\,dY_u^1 + \int_0^s dr \int_0^r \frac{\partial}{\partial r}\gamma(r,t;u,0)\,dY_u^1 \\ &= \int_0^s \gamma(u,t;u,0)\,dM_u^1 + \int_0^s \gamma(u,t;u,0)\mu_1(u)\,du \\ &\quad + \int_0^s dr \int_0^r \frac{\partial}{\partial r}\gamma(r,t;u,0)\,dY_u^1 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} I_6(s,t) &= \iint_{R_{st}} \gamma(u,t;u,v)a_4(u,v)W(du\,dv) \\ &\quad + \int_0^s dr \iint_{R_{rt}} \frac{\partial}{\partial r}\gamma(r,t;u,v)a_4(u,v)W(du\,dv). \end{aligned}$$

This gives us the semimartingale decomposition of I_2 and I_6—so that we have the decomposition of all the I_j—and we see that

$$\begin{aligned} \frac{\partial}{\partial s}\langle I_2\rangle_{st} &= \gamma^2(s,t;s,0)\sigma_1^2(s), \\ \frac{\partial}{\partial s}\langle I_6\rangle_{st} &= \int_0^t \gamma^2(s,t;s,v)a_4^2(s,v)\,dv. \end{aligned}$$

Now, a_4 is bounded by hypothesis and γ is bounded by property (g), and both are continuous and deterministic. Further, $\sigma_1^2(s)$ is continuous and locally L^1-bounded by the smoothness of Y^1. So we conclude that the derivatives of the

$\langle I_j \rangle$ are all continuous and L_1-bounded, and the bound is uniform for (s,t) in bounded sets.

We must also check that the $\frac{\partial}{\partial s}(V_s^j)$ are continuous and L^2-bounded, and that the bound is uniform for (s,t) in bounded sets. Since we have explicit formulas for $V^1, \dots, V^6$, this is straightforward. We will just check I_3, which contains a stochastic integral, and leave the rest to the reader. Fix $R_{s_0 t_0}$.

Since I_3 is $C^{(1)}$, $V_3 = I_3$ and from (2.12) we must bound

$$A(s,t) \stackrel{\text{def}}{=} E\left\{\left(\frac{\partial}{\partial s} I_3\right)^2\right\} = E\left\{\left(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\,dY_v^2\right)^2\right\}, \qquad (s,t) \in R_{s_0 t_0}.$$

Now $dY_v^2 = dM_v^2 + \mu_2(v)\,dv$, so

$$\begin{aligned} A(s,t) \le 2E\left\{\left(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\,dM_v^2\right)^2\right\} \\ + 2E\left\{\left(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\mu_2(v)\,dv\right)^2\right\}. \end{aligned}$$

The first expectation equals $2E\{\int_0^t (\frac{\partial}{\partial s}\gamma(s,t;0,v))^2\sigma_2^2(v)\,dv\}$. If $(u,v) \prec (s,t) \in R_{s_0 t_0}$, there is a constant $K = K_{s_0 t_0}$ such that $|\frac{\partial}{\partial s}\gamma(s,t;0,v)| \le K$ by (a). The second expectation is bounded by $2K^2 E\{(\int_0^{t_0} |\mu_2(v)|\,dv)^2\}$. Thus, by the Schwarz inequality, if $(s,t) \in R_{s_0 t_0}$,

$$A(s,t) \le 2K^2 t_0 \left(\sup_{v \le t_0} E\{\sigma_2^2(v)\} + \sup_{v \le t_0} E\{\mu_2^2(v)\}\right).$$

Now Y is a smooth semimartingale, so this is bounded independently of (s,t), hence $A(s,t)$ is uniformly bounded for $(s,t) \in R_{s_0 t_0}$, as claimed.

To get (2.9), integrate by parts in the first double integral on the left-hand side of (2.6). $\square$

The following is a direct consequence of Lemma 2.3.

COROLLARY 2.4. *Let* $\Delta =]s-h, s] \times]t-k, t]$. *Under Assumption* A, *for any* $(s_0, t_0) \in \mathbb{R}_+^2$ *there exists a constant* $C = C_{s_0 t_0}$ *such that if* $(s,t) \in R_{s_0 t_0}$,

$$E\{(X(\Delta))^2\} \le Chk,$$

$$E\{(X(s,t) - X(s-h, t-k))^2\} \le C(h+k).$$

2.2. *Iterated quadratic variation.* The aim of this section is to show that sample paths of the solution X of (2.1) determine the coefficient $a_4(s,t)$ in (2.1). This will be needed in Section 5. To this aim, we shall show that $a_4(\cdot,\cdot)$ can be determined by computing a quantity analogous to the planar quadratic variation (see [9]) of X.

Set $t_i^n = i\,2^{-n}$ and $\Delta_{ij}^{nm} =]t_{i-1}^n, t_i^n] \times]t_{j-1}^m, t_j^m]$. Define

$$Q_{n,m}(s,t) = \sum_{i=1}^{[2^n s]} \sum_{j=1}^{[2^m t]} |X(\Delta_{ij}^{nm})|^2$$

and

$$[X](s,t) = \lim_{n\to\infty} \lim_{m\to\infty} Q_{n,m}(s,t),$$

if this iterated limit exists a.s. We call $[X](s,t)$ the *iterated quadratic variation* of X. In general, even if $[X](s,t)$ exists, the limit in inverse order may not exist, but for solutions of (2.1), it turns out that the limit can be taken in either order.

LEMMA 2.5. *If X has finite iterated quadratic variation and if Y has zero iterated quadratic variation, then the iterated quadratic variation of $X+Y$ is equal to that of X.*

PROOF. Clearly,

$$|X(\Delta_{ij}^{nm}) + Y(\Delta_{ij}^{nm})|^2 = |X(\Delta_{ij}^{nm})|^2 + 2X(\Delta_{ij}^{nm})Y(\Delta_{ij}^{nm}) + |Y(\Delta_{ij}^{nm})|^2.$$

Sum each term over $i = 1, \dots, [2^n s]$ and $j = 1, \dots, [2^m t]$, let $m \to \infty$, then $n \to \infty$. The third term has iterated limit 0 by hypothesis, and, using the Cauchy–Schwarz inequality, one sees that the second does too. Therefore, $[X+Y](s,t) = [X](s,t)$. □

LEMMA 2.6. *If*

$$Y(s,t) = y_0 + f_1(s) + f_2(t) + \iint_{R_{st}} f(u,v)\,du\,dv$$

for some integrable function f, continuous functions f_1 and f_2, and $y_0 \in \mathbb{R}$, then $[Y](s,t) = 0$ for all s and t.

PROOF. Notice that the planar increments of the first three terms vanish, so

$$[Y](s,t) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{[2^n s]} \sum_{j=1}^{[2^m t]} \left(\iint_{\Delta_{ij}^{nm}} f(u,v)\,du\,dv \right)^2.$$

For fixed n and i, $y \mapsto \iint_{]t_{i-1}^n, t_i^n] \times [0,y]} f(u,v)\,du\,dv$ is a function with bounded variation, so

$$\lim_{m\to\infty} \sum_{j=1}^{[2^m t]} \left(\iint_{\Delta_{ij}^{nm}} f(u,v)\,du\,dv \right)^2 = 0,$$

and therefore $[Y](s,t) = 0$. □

PROPOSITION 2.7. *Under Assumption* A, *the iterated quadratic variation of the weak solution X of* (2.1) *is*

$$[X](s,t) = \iint_{R_{st}} a_4^2(u,v)\,du\,dv.$$

PROOF. Consider the decomposition (2.10) of $X(s,t)$ into the sum of $I_1(s,t)$, $\dots, I_7(s,t)$. Each of these terms can be expressed in the form needed to apply Lemma 2.6: indeed, for I_7, for instance, the differentiability properties of $\gamma(\cdot,\cdot;u,v)$ listed in Section 2.1 imply that $(s,t)\mapsto I_7(s,t)$ is absolutely continuous with respect to Lebesgue measure, so

$$I_7(s,t) = \iint_{R_{st}} \frac{\partial^2 I_7}{\partial u\,\partial v}(u,v)\,du\,dv,$$

and the terms I_1, I_4 and I_5 can be expressed in an analogous way. It follows therefore from Lemma 2.6 that $[I_1]\equiv[I_4]\equiv[I_5]\equiv[I_7]\equiv 0$. Notice that as in (2.13),

$$\begin{aligned} I_2(s,t) &= \int_0^s \gamma(u,t;u,0)\,dY_u^1 + \int_0^s dr\int_0^r \frac{\partial}{\partial r}\gamma(r,t;u,0)\,dY_u^1 \\ &\stackrel{\text{def}}{=} I_2^1(s,t) + I_2^2(s,t), \end{aligned}$$

and similarly,

$$\begin{aligned} I_3(s,t) &= \int_0^t \gamma(s,v;0,v)\,dY_v^2 + \int_0^t dy\int_0^y \frac{\partial}{\partial y}\gamma(s,y;0,v)\,dY_v^2 \\ &\stackrel{\text{def}}{=} I_3^1(s,t) + I_3^2(s,t) \end{aligned}$$

and

$$\begin{aligned} I_6(s,t) = {} & \iint_{R_{st}} a_4(u,v)W(du\,dv) \\ & + \int_0^t dy \iint_{R_{sy}} \frac{\partial}{\partial y}\gamma(u,y;u,v)a_4(u,v)W(du\,dv) \\ & + \int_0^s dr \iint_{R_{rt}} \frac{\partial}{\partial r}\gamma(r,t;u,v)a_4(u,v)W(du\,dv) \\ \stackrel{\text{def}}{=} {} & I_6^1(s,t) + I_6^2(s,t) + I_6^3(s,t). \end{aligned}$$

As above (for $I_7(s,t)$), it clearly follows from Lemma 2.6 that

$$[I_2^2]\equiv[I_3^2]\equiv[I_6^2]\equiv[I_6^3]\equiv 0.$$

Observe that

$$[I_2^1](s,t) = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{[2^n s]}\sum_{j=1}^{[2^m t]}\left(\int_{t_{i-1}^n}^{t_i^n}\left(\gamma(u,t_j^m;u,0)-\gamma(u,t_{j-1}^m;u,0)\right)dY_u^1\right)^2.$$

Because the function $t \mapsto \int_{t_{i-1}^n}^{t_i^n} \gamma(u, t; u, 0)\, dY_u^1$ has bounded variation, its quadratic variation vanishes, so for fixed n, the limit as $m \to \infty$ above vanishes, and therefore $[I_2^1](s,t) = 0$.

Turning to I_3^1, notice that

$$[I_3^1](s,t) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{[2^n s]} \sum_{j=1}^{[2^m t]} \left(\int_{t_{j-1}^m}^{t_j^m} \big(\gamma(t_i^n, v; 0, v) - \gamma(t_{i-1}^n, v; 0, v)\big)\, dY_v^2 \right)^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{[2^n s]} \int_0^t \big(\gamma(t_i^n, v; 0, v) - \gamma(t_{i-1}^n, v; 0, v)\big)^2\, d\langle Y^2\rangle_v.$$

But the map $u \mapsto \int_0^t \gamma(u, v; 0, v)\, d\langle Y^2\rangle_v$ has bounded variation, so $[I_3^1](s,t) = 0$.

Finally,

$$[I_6^1](s,t) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{[2^n s]} \sum_{j=1}^{[2^m t]} \left(\iint_{\Delta_{ij}^{nm}} a_4(u,v) W(du\, dv) \right)^2.$$

Now

$$\tilde{Z}_i^n(y) \stackrel{\text{def}}{=} \iint_{]t_{i-1}^n, t_i^n] \times [0,y]} a_4(u,v) W(du\, dv)$$

is a continuous martingale with quadratic variation

$$\langle \tilde{Z}_i^n \rangle_y = \iint_{]t_{i-1}^n, t_i^n] \times [0,y]} a_4^2(u,v)\, du\, dv,$$

so

$$[I_6^1](s,t) = \lim_{n\to\infty} \sum_{i=1}^{[2^n s]} \iint_{]t_{i-1}^n, t_i^n] \times [0,t]} a_4^2(u,v)\, du\, dv$$

$$= \iint_{[0,s]\times[0,t]} a_4^2(u,v)\, du\, dv.$$

Together with Lemma 2.5, this proves the proposition. □

REMARK 2.8. Proposition 2.7 implies that $|a_4(s,t)|$ can be determined from the sample paths of $(X(s,t))$. In fact, as long as a_4 is never zero, one can determine the sign of a_4 as well, since $\dot{W}$ is then X-measurable and the iterated covariation (defined by polarization) of X and W is $[X, W]_{st} = \iint_{R_{st}} a_4(u,v)\, du\, dv$.

3. Naive changes of variables. Changes of variables in deterministic ordinary and partial differential equations are well-understood and are usually handled by a judicious use of the chain rule. This is no longer true with stochastic equations, however. These are more delicate because of the complicating factor of the filtration. A change of variables may involve an implicit change of filtration, and this can affect the equation in more than one way.

First, it can change the stochastic calculus. Itô integrals depend in a fundamental way on the underlying filtration. A change of variables may involve a change in coordinates, which in turn may call for a new filtration. For instance, the usual filtration for the Brownian sheet is a two-parameter filtration which depends strongly on the coordinates: the "past" at point (s,t) is generally taken to be $\mathcal{P}_{s,t} = R_{s,t}$, and one sets $\mathcal{F}_{s,t} = \sigma\{\dot{W}_{u,v},\ (u,v) \in \mathcal{P}_{s,t}\}$. However, a rotation by 45° changes the Brownian sheet into a solution of the stochastic wave equation [16], and the most natural filtration for such an evolution equation may be a one-parameter filtration $(\hat{\mathcal{F}}_t)$ ordered by time: the "past" at time t and position x is $\mathcal{P}_{t,x} = \{(s,y): s \le t,\ y \in \mathbb{R}\}$ (which does not depend on x), and the sigma-field $\hat{\mathcal{F}}_t$ is generated by the white noise in $\mathcal{P}_{t,x}$. So a change of variables which includes a change of filtration may involve a delicate transformation of stochastic integrals.

Second, a change in filtration may change the nature of some underlying processes. If the equation involves a given martingale measure or white noise, for example, there is a chance that it may no longer be either a martingale measure or a white noise *relative to the new filtration*.

This can occur even with the simplest linear stochastic differential equations when they are reversed in time. Consider, for instance, a stationary Ornstein–Uhlenbeck process $(X_t,\ 0 \le t \le 1)$. This satisfies the Itô stochastic differential equation

$$dX_t = dW_t - X_t\,dt, \qquad X_0 \text{ given}, \tag{3.1}$$

for some Brownian motion (W_t). In the notations of this paper, we re-write this equation as

$$\frac{dX}{dt} = \dot{W} - X,$$

where $\dot{W}$ is a white noise on the line. Let us make the change of variables $s = 1 - t$ and set $\hat{X}_s = X_{1-s}$. If we could use the chain rule as we would with an ordinary differential equation, we would have

$$\frac{d\hat{X}}{ds} = -\dot{W} + \hat{X}. \tag{3.2}$$

White noise is symmetric, so if $\dot{W}$ is a white noise, so is $-\dot{W}$, but the drift term, $\hat{X}$, now apparently makes the process drift away from zero. On the other hand, the change of variables $s = 1 - t$ is just a time-reversal and a stationary Ornstein–Uhlenbeck process is reversible, so $\hat{X}$ has the same distribution as X and must satisfy

$$\frac{d\hat{X}}{ds} = \dot{\hat{W}} - \hat{X}, \tag{3.3}$$

for some white noise $\dot{\hat{W}}$. The drift term is now plainly toward zero.

As it turns out, both (3.2) and (3.3) are correct equations, and there is no contradiction. Both equations involve white noises, to be sure, but the filtrations relative to which $\dot{W}$ and $\dot{\hat{W}}$ are white noises are not apparent in (3.2) or (3.3). Writing the integral forms of (3.2) and (3.3) clarifies the situation: (3.2) should be interpreted as

$$\hat{X}_1 - \hat{X}_s = -W[0, 1-s] + \int_s^1 \hat{X}_u \, du,$$

which is equivalent to

$$X_{1-s} - X_0 = W[0, 1-s] - \int_0^{1-s} X_v \, dv, \tag{3.4}$$

and this is precisely (3.1), whereas (3.3) should be interpreted as

$$\hat{X}_s - \hat{X}_0 = \hat{W}[0, s] - \int_0^s \hat{X}_u \, du,$$

which is equivalent to

$$X_{1-s} - X_1 = \hat{W}[0, s] - \int_{1-s}^1 X_v \, dv. \tag{3.5}$$

Implicit in (3.3) is the fact that $\hat{W}$ is a white noise in the natural filtration of $\hat{X}$ and is independent of $\hat{X}_0 = X_1$. Equating (3.4) and (3.5), we find the relationship between W and $\hat{W}$:

$$\hat{W}[0, s] = X_0 - X_1 + W[0, 1-s] - \int_0^{1-s} X_v \, dv + \int_{1-s}^1 X_v \, dv.$$

Differentiating rather informally, this translates into $\dot{\hat{W}} = -\dot{W} + 2\hat{X}$, which is just enough to reconcile (3.2) and (3.3).

In short, when we change variables in an s.p.d.e., we must be careful to specify how the filtrations transform.

When we speak of a *naive* change of variables ζ in an s.p.d.e., we mean that the new filtration is the image of the old one under ζ: if $\mathcal{F}_t = \sigma(\dot{W}_{u,v},\ (u, v) \in \mathcal{P}_t)$, then $\hat{\mathcal{F}}_t = \sigma(\dot{\hat{W}}_{u,v},\ (u, v) \in \zeta^{-1}(\mathcal{P}_t))$. We will see that naive changes of variables in s.p.d.e.'s work as expected. It is only when the filtrations change that we find new phenomena.

3.1. *Changing variables in stochastic integrals.* Let $O \subset \mathbb{R}^2$ be an open set and let ζ be a one-to-one $C^{(\infty)}$ map of O onto an open set $D \subset \mathbb{R}^2$. Suppose the Jacobian J of ζ never vanishes. Then for a Borel subset $A \subset D$ and an integrable function f on A,

$$\int_A f(z)\,dz = \int_{\zeta^{-1}(A)} f(\zeta(\xi))J(\xi)\,d\xi. \tag{3.6}$$

If W is a white noise on D, define a set function $\hat{W}$ on O by

$$\hat{W}(B) = \int_{\zeta(B)} \frac{1}{\sqrt{J(\zeta^{-1}(z))}} W(dz). \tag{3.7}$$

LEMMA 3.1. *$\hat{W}(B)$ is a standard white noise on O, and if A is a Borel subset of D and if f is a deterministic square-integrable function on A,*

$$\int_A f(z)W(dz) = \int_{\zeta^{-1}(A)} f(\zeta(\xi))\sqrt{J(\xi)}\hat{W}(d\xi). \tag{3.8}$$

PROOF. $\hat{W}(B)$ is clearly a mean zero Gaussian random variable (if finite) and from (3.7) and (3.6),

$$\begin{aligned} E\{\hat{W}(B)^2\} &= \int_{\zeta(B)} J(\zeta^{-1}(z))^{-1}\,dz \\ &= \int_B J(\xi)^{-1}J(\xi)\,d\xi \\ &= |B|, \end{aligned}$$

which shows that $\hat{W}$ is defined and has the correct variance on sets of finite Lebesgue measure. Moreover, if A and B are disjoint subsets of O, $\zeta(A)$ and $\zeta(B)$ are disjoint in D, so $\hat{W}(A)$ and $\hat{W}(B)$ are independent, being stochastic integrals of W over disjoint sets.

Equation (3.8) holds by (3.7) if f is of the form $f(z) = 1_B(z)$, hence it holds for simple f by linearity, and for square-integrable f by the usual functional completion argument. $\square$

3.2. *Changing variables in s.p.d.e.'s.* Let X be a weak solution of (2.1) and let $a_1, \dots, a_4$ satisfy the assumptions stated at the beginning of Section 2. Let $\partial_1 = \partial/\partial s$, $\partial_2 = \partial/\partial t$, and set

$$L = \partial_1\partial_2 + a_1\partial_1 + a_2\partial_2,$$

so that the formal adjoint of L is

$$L^*\phi = \partial_1\partial_2\phi - \partial_1(a_1\phi) - \partial_2(a_2\phi).$$

Then for $\phi \in C^{(2)}(\mathbb{R}_+^2)$, X will satisfy (2.4), which we write in the form

$$
\begin{aligned}
&(X\phi)(R_{st}) + \oint_{\partial R_{st}} X(z)\big[\nabla\phi(z) - \phi(z)(a_2(z)\hat{\mathbf{i}} - a_1(z)\hat{\mathbf{j}})\big]\cdot \mathbf{T}\,ds \\
&\qquad + \int_{R_{st}} \big(X(z)L^*\phi(z) + a_3(z,X)\phi(z)\big)\,dz = \int_{R_{st}} \phi(z)a_4(z)W(dz),
\end{aligned} \tag{3.9}
$$

where $\mathbf{T}$ is the unit tangent vector, $\hat{\mathbf{i}} = (1,0)$, $\hat{\mathbf{j}} = (0,1)$, ds is the element of arc length, and $X = X_0$ on the boundary of $\mathbb{R}_+^2$, where $X_0(s,0) \overset{\text{def}}{=} X_0 + M_s^1$, $X_0(0,t) = X_0 + M_t^2$, as in (2.1).

Let ζ be a $C^{(\infty)}$ homeomorphism of an open set O onto an open set $D \subset \mathbb{R}^2$ such that $D \supset \mathbb{R}_+^2$, and let $\hat{D} = \zeta^{-1}(\mathbb{R}_+^2)$. Let $\hat{W}$ be the white noise on O which is related to W by (3.7). If $\hat{\phi}(\xi) \overset{\text{def}}{=} \phi(\zeta(\xi))$, then a straightforward calculation gives us a differential operator $\hat{L}^*$ on $\hat{D}$ for which

$$(L^*\phi)\big(\zeta(\xi)\big) = \hat{L}^*\hat{\phi}(\xi).$$

We let $\hat{L}$ be the formal adjoint of $\hat{L}^*$, define

$$\hat{X}(\xi) \overset{\text{def}}{=} X\big(\zeta(\xi)\big),$$

and for $i = 3,4$, we set $\hat{a}_i(\xi,x) = a_i(\zeta(\xi),x)$.

THEOREM 3.2. *The process $\hat{X}$ is a weak solution of the stochastic partial differential equation*

$$\hat{L}(J\hat{X}) + \hat{a}_3 J = \hat{a}_4\sqrt{J}\,\dot{\hat{W}}, \tag{3.10}$$

with boundary values $\hat{X}(\xi) = X_0(\zeta(\xi))$ on $\zeta^{-1}(\partial\mathbb{R}_+^2)$. [*Note*: *Formally*, *equation* (3.10) *is interpreted as equation* (3.12) *below*.]

PROOF. The map ζ is a smooth homeomorphism on a neighborhood of $\mathbb{R}_+^2$, so its restriction is a smooth homeomorphism of $\hat{D}$ onto the closed set $\mathbb{R}_+^2$, which takes the boundary of $\mathbb{R}_+^2$ onto a (possibly proper) subset of the boundary of $\hat{D}$. Clearly $\hat{X}$ has the correct boundary values, so we need only check that (3.10) holds in the interior. For this, we check the weak form of the equation for $\phi \in C_K^{(\infty)}(\mathbb{R}_+^2)$ whose support is in the interior of $\mathbb{R}_+^2$. If we choose (s,t) large enough so that the support of ϕ is in the interior of R_{st}, the boundary terms of (3.9) drop out and we are left with

$$\int_{R_{st}} \big(X(z)L^*\phi(z) + a_3(z,X)\phi(z)\big)\,dz = \iint_{R_{st}} \phi(z)a_4(z)W(dz). \tag{3.11}$$

The left-hand side is a Riemann integral and transforms under the mapping ζ in the usual way, while the right-hand side is a stochastic integral which transforms according to Lemma 3.1. Since the homeomorphism induces a one-to-one map of X to $\hat{X}$, there is a function $\hat{a}_3$ such that $\hat{a}_3(\xi, \hat{X}) = a_3(\zeta(\xi), X)$. So, setting $\xi = \zeta^{-1}(z)$, we have $X(z) = \hat{X}(\xi)$, $L^*\phi(z) = \hat{L}^*\hat{\phi}(\xi)$, and (3.11) becomes

$$\int_{\hat{D}} \big(\hat{X}(\xi)\hat{L}^*\hat{\phi}(\xi) + \hat{a}_3(\xi, \hat{X})\hat{\phi}(z)\big) J(\xi)\, d\xi = \int_{\hat{D}} \hat{\phi}(\xi)\hat{a}_4(\xi)\sqrt{J(\xi)}\,\dot{\hat{W}}(d\xi), \tag{3.12}$$

which is the weak form of (3.10). $\square$

EXAMPLE 3.1. This example will be used in the proof of Theorem 7.1. Assume that $a_i = a_i(s,t)$, $i = 1, \dots, 4$, and $O = D = \mathbb{R}^2_+$. Let $\zeta(x, y) = (s(x), t(y))$, where

$$s(x) = \frac{e^{2a} - e^{2a(1-x)}}{2a}, \qquad t(y) = \frac{e^{2b} - e^{2b(1-y)}}{2b}.$$

Suppose that $X(s, t)$ satisfies

$$\frac{\partial^2 X}{\partial s \partial t} + a_1 \frac{\partial X}{\partial s} + a_2 \frac{\partial X}{\partial t} + a_3 X = a_4 \dot{W}, \tag{3.13}$$

with initial conditions $X(s, 0) \equiv X(0, t) \equiv 0$. Set $\hat{a}_i(x, y) = a_i(s(x), t(y))$. With the notations above, $J(x, y) = s'(x)t'(y)$,

$$(L^*\phi)(s(x), t(y)) = \frac{\partial^2 \phi}{\partial s \partial t}(s(x), t(y)) - \frac{\partial(a_1\phi)}{\partial s}(s(x), t(y)) - \frac{\partial(a_2\phi)}{\partial t}(s(x), t(y))$$

and

$$\hat{L}^*\hat{\phi}(x, y) = \frac{1}{s'(x)t'(y)} \frac{\partial^2 \hat{\phi}}{\partial x \partial y}(x, y) - \frac{1}{s'(x)} \frac{\partial(\hat{a}_1\hat{\phi})}{\partial x}(x, y) - \frac{1}{t'(y)} \frac{\partial(\hat{a}_2\hat{\phi})}{\partial y}(x, y).$$

Therefore,

$$\hat{L}\hat{\phi} = \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{s'(x)t'(y)}\hat{\phi}\right) + \hat{a}_1 \frac{\partial}{\partial x}\left(\frac{1}{s'(x)}\hat{\phi}\right) + \hat{a}_2 \frac{\partial}{\partial y}\left(\frac{1}{t'(y)}\hat{\phi}\right).$$

Let $\hat{X}(x, y) = X(s(x), t(y))$. Then by (3.10), $\hat{X}$ satisfies

$$\begin{aligned} &\frac{\partial^2 \hat{X}}{\partial x \partial y} + \hat{a}_1 e^{2b(1-y)} \frac{\partial \hat{X}}{\partial x} + \hat{a}_2 e^{2a(1-x)} \frac{\partial \hat{X}}{\partial y} + \hat{a}_3 \hat{X} e^{2a(1-x)+2b(1-y)} \\ &\qquad = \hat{a}_4 e^{a(1-x)+b(1-y)} \dot{\hat{W}}, \end{aligned}$$

with initial conditions $\hat{X}(s, 0) \equiv \hat{X}(0, t) \equiv 0$.

EXAMPLE 3.2. Assume a_1 and a_2 are constants, $a_3 = a_3(s, t, x)$ and $a_4 = a_4(s, t)$. Suppose $O = (-\infty, 1]^2$ and $\zeta(u, v) = (1-u, 1-v)$. Suppose that $X(s, t)$ satisfies

$$\frac{\partial^2 X}{\partial s \partial t} + a_1 \frac{\partial X}{\partial s} + a_2 \frac{\partial X}{\partial t} + a_3 X = a_4 \dot{W},$$

with initial conditions $X(s,0) \equiv X(0,t) \equiv 0$. With the notations above, $J(u,v) \equiv 1$, and

$$L^*\phi(1-u,1-v) = \frac{\partial^2 \phi}{\partial s \partial t}(1-u,1-v) - a_1 \frac{\partial \phi}{\partial s}(1-u,1-v) - a_2 \frac{\partial \phi}{\partial t}(1-u,1-v),$$

so

$$\hat{L}\hat{\phi} = \frac{\partial^2 \hat{\phi}}{\partial u \partial v} - a_1 \frac{\partial \hat{\phi}}{\partial u} - a_2 \frac{\partial \hat{\phi}}{\partial v}.$$

Therefore, $\hat{X}(u,v) = X(1-u,1-v)$ satisfies

$$\hat{L}\hat{X}(u,v) + \hat{a}_3(u,v,\hat{X}) = \hat{a}_4(u,v)\dot{\hat{W}},$$

with boundary conditions $\hat{X}(u,1) \equiv \hat{X}(1,v) \equiv 0$. This statement should be compared with the very different conclusion of Theorem 6.3, in which the change of variables is the same but the boundary conditions and the underlying filtration are different.

4. Linear s.p.d.e.'s as distributional p.d.e.'s. Let us specialize to the linear case, where $a_3(s,t,X) = a_3(s,t)X(s,t)$. There is a meta-theorem which states that linear s.p.d.e.'s are simply random p.d.e.'s with distribution values. We will illustrate this.

Let $L = \sum_{i,j} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i + c$ be a partial differential operator on a domain $D \subset \mathbb{R}^2_+$, whose coefficients a_{ij}, b_i, and c are deterministic Lipschitz functions, with $a_{ij} \in C^{(2)}(D)$, $b_i \in C^{(1)}(D)$ and $c \in C(D)$. Let $F \in L^1(D)$, $G \in L^2(D)$ be deterministic functions, and consider the s.p.d.e. in D,

$$LX = F + G\dot{W}. \tag{4.1}$$

We say that X is a *weak solution* of (4.1) in D if for each $\phi \in C_K^{(\infty)}(D)$,

$$\int_D X(z)(L^*\phi)(z)\,dz = \int_D \phi(z)\big[G(z)W(dz) + F(z)\,dz\big] \tag{4.2}$$

with probability one.

PROPOSITION 4.1. *If $X = (X(z),\ z \in D)$ is a weak solution of* (4.2) *with continuous sample paths, and if $\hat{D}$ is an open, relatively compact subdomain of D, then X defines a random distribution on $\hat{D}$. With probability one, it is a distributional solution of* (4.1) *on $\hat{D}$.*

PROOF. To say X is a distribution is to say it is a continuous linear functional on a nuclear space. Let us choose the nuclear space to be the completion of $C_K^{(\infty)}(\hat{D})$ in the vector space topology generated by the seminorms

$$F_n(\phi) = \|\phi\|_2^2 + \sum_{k=1}^{n}\sum_{\ell=1}^{n} \|\partial_1^k \partial_2^\ell \phi\|_2^2,$$

where $\|\phi\|$ is the norm of ϕ in $L^2(\hat{D})$.

Let $L^*\phi = \sum_{ij} \partial_i \partial_j (a_{ij}\phi) - \sum_i \partial_i (b_i \phi) + c\phi$ be the formal adjoint of L. If ω is such that $z \mapsto X(z,\omega)$ is continuous on D, $X(\cdot,\omega)$ defines a distribution on $\hat{D}$ by

$$X(\phi,\omega) = \int_{\hat{D}} \phi(z) X(z,\omega)\,dz,$$

and LX is also a distribution: $LX(\phi) = X(L^*\phi)$.

On the right hand side of (4.2), $F + G\dot{W}$ also defines a distribution: $(F + G\dot{W})(\phi) = \int_{\hat{D}} \phi(z)[F(z)\,dz + G(z)W(dz)]$ a.s. for each ϕ (see [16], Chapter 4). Then (4.2) says that for a fixed $\phi \in C_K^{(\infty)}$,

$$(LX)(\phi,\omega) = (F + G\dot{W})(\phi,\omega) \tag{4.3}$$

for a.e. ω. This is true simultaneously for a countable dense set of ϕ, hence for all ϕ by continuity, since both sides are distributions. $\square$

We chose a particularly simple space of distributions to avoid having to discuss the boundary behavior of X. It should be clear that one can extend this to include boundaries.

In other words, equation (4.2), and even equation (2.1), is an equation in distribution space which holds for a.e. ω. Consequently, all operations which are legal on such equations are legal on this one—*as long as they do not change the definition of the stochastic integral in* (2.4). It is interesting to consider the previous section from this point of view. In particular, if we multiply X by a deterministic $C^{(\infty)}$ function, we can just use the usual calculus to see what s.p.d.e. it satisfies.

COROLLARY 4.2. *Suppose that $A = (a_{ij})$ and $f \in C^{(2)}(D)$, $f > 0$. Let $L_1 = L - c$. If X satisfies* (4.1) *and if $f\tilde{X} = X$, then $\tilde{X}$ satisfies*

$$L_1\tilde{X} + \frac{1}{f}\nabla f \cdot (A + A^T)\nabla^T \tilde{X} + \frac{Lf}{f}\tilde{X} = \frac{1}{f}(F + f^{-1}G\dot{W}). \tag{4.4}$$

EXAMPLE 4.1. Suppose X satisfies (3.13). Let $Y(s,t) = e^{as+bt}X(s,t)$. Then $Y(s,t)$ satisfies

$$\frac{\partial^2 Y}{\partial s \partial t} + (a_1 - b)\frac{\partial Y}{\partial s} + (a_2 - a)\frac{\partial Y}{\partial t} + (a_3 + ab - a_1 a - a_2 b)Y = e^{as+bt} a_4\,\dot{W}. \tag{4.5}$$

5. Changing filtrations: final values as initial conditions. We now want to consider some changes of variables which involve changes of filtration, namely time-reversal.

Consider the linear form of (2.1):

$$\frac{\partial^2 X}{\partial s \partial t} + a_1(s,t)\frac{\partial X}{\partial s} + a_2(s,t)\frac{\partial X}{\partial t} + a_3(s,t)X(s,t) = a_4(s,t)\dot{W}, \tag{5.1}$$

where the initial values $X(s,0)$ and $X(0,t)$ are given, and satisfy Assumption A.

Thus, let X be a weak solution of (5.1). We will consider two fundamental types of time reversal.

- Reversal in one coordinate: $(s,t) \mapsto (1-s,t)$.

Let $\hat{X}(s,t) = X(1-s,t)$, $0 \le s \le 1$, and let $\hat{\mathcal{F}}_s$ be the one-parameter filtration $\hat{\mathcal{F}}_s = \sigma\{\hat{X}(u,v): u \le s\} = \sigma\{X(u,v): u \ge 1-s\}$.

By symmetry, results for this type of reversal will translate directly to the reversal $(s,t) \mapsto (s,1-t)$.

- Reversal in two coordinates: $(s,t) \mapsto (1-s,1-t)$.

Let $\hat{X}(s,t) = X(1-s,1-t)$, $0 \le s \le 1$, $0 \le t \le 1$, and let $(\hat{\mathcal{F}}_{st})$ be the two-parameter filtration defined by $\hat{\mathcal{F}}_{st} = \sigma\{\hat{X}(u,v): u \le s,\ v \le t\} = \sigma\{X(u,v): u \ge 1-s,\ v \ge 1-t\}$.

Let us suppose that $Y = \hat{X}$ is the weak solution of an s.p.d.e. of the form

$$\frac{\partial^2 Y}{\partial s \partial t} + \hat{a}_1(s,t)\frac{\partial Y}{\partial s} + \hat{a}_2(s,t)\frac{\partial Y}{\partial t} + \hat{a}_3(s,t)Y = \hat{a}_4(s,t)\dot{\hat{W}}, \qquad s \ge 0,\ t \ge 0, \tag{5.2}$$

where the initial values for Y are specified on the axes of $\mathbb{R}^2_+$: in the case of one parameter reversal, $Y(0,t) = X(1,t)$, $Y(s,0) = X(1-s,0)$, and in the case of two-parameter reversal, $Y(0,t) = X(1,1-t)$, $Y(s,0) = X(1-s,1)$; and $\hat{a}_1, \dots, \hat{a}_4$ satisfy the smoothness conditions of Section 2 and $\dot{\hat{W}}$ is a white noise relative to the new filtration $(\hat{F}_t)$, independent of the boundary values of Y.

The first question we shall ask is this: "If the reversed process actually is the solution of (5.2), what can we say about the coefficients $\hat{a}_1, \dots, \hat{a}_4$?"

Let us first establish a property of the original solution, which clarifies the independence of the solution and the white noise. Let

$$\mathcal{G}_{st} = \sigma\{X(u,v): u \le s \ \text{ or } \ v \le t\},$$
$$\mathcal{H}_{st} = \sigma\{W(A): \text{Borel } A \subset]s,\infty[\times]t,\infty[\}.$$

Note that $\mathcal{H}_{st}$ represents information in the strict future of (s,t), while $\mathcal{G}_{st}$ represents information in the wide-sense past, which is roughly everything not in the strict future.

PROPOSITION 5.1. *Let X be a weak solution of* (5.1). *Then for each $s \ge 0$, $t \ge 0$, $\mathcal{G}_{st}$ and $\mathcal{H}_{st}$ are independent.*

PROOF. From (2.5), $X(s,t)$ is measurable with respect to $\mathcal{F}^0_{st} \stackrel{\text{def}}{=} \sigma\{Y^1_u,\ u \le s\} \vee \sigma\{Y^2_v,\ v \le t\} \vee \sigma\{W([0,u]\times[0,v]),\ u \le s \text{ and } v \le t\}$.

If $A \subset]s,\infty[\times]t,\infty[$, and either $u \le s$ or $v \le t$, then $W(A)$ is independent of $W([0,u]\times[0,v])$. White noise is a Gaussian process, so it follows that $\mathcal{H}_{st}$ is independent of $\sigma\{W([0,u]\times[0,v]),\ u\le s \text{ or } v \le t\}$. Since the Y^i are independent of the white noise, it follows that $\mathcal{H}_{st}$ is independent of $\vee_{u\le s \text{ or } v\le t}\mathcal{F}^0_{uv} \supset \mathcal{G}_{st}$. $\square$

Set $\Delta = [s-h,s]\times[t-h,t]$ and $\hat\Delta = [1-s,1-s+h]\times[1-t,1-t+h]$, and consider a two-parameter reversal. If $\hat X$ is a weak solution of (5.2), Proposition 5.1 implies that $\dot{\hat W}|_{\hat\Delta}$ is independent of

$$\begin{aligned}\hat{\mathcal{G}}_{1-s,1-t} &\stackrel{\text{def}}{=} \sigma\{Y(u,v),\ u\le 1-s \text{ or } v \le 1-t\}\\ &= \sigma\{X(u,v),\ u \ge s \text{ or } v \ge t\}.\end{aligned} \tag{5.3}$$

PROPOSITION 5.2. *Consider reversal in two coordinates, and set $\hat s = 1-s$, $\hat t = 1-t$. Suppose that the reversed process $Y = \hat X$ is a solution of* (5.2) *in the above sense. Then the a_i and $\hat a_i$ are related as follows*:

$$\hat a_4(\hat s,\hat t) = a_4(s,t); \tag{5.4}$$

$$\begin{aligned}&E\left\{\int_\Delta a_4(s,t)W(ds\,dt) \mid \hat{\mathcal{G}}_{\hat s\hat t}\right\}\\ &\quad= \big(a_1(s,t)+\hat a_1(\hat s,\hat t)\big)\big(X(s,t)-X(s-h,t)\big)h\\ &\qquad+\big(a_2(s,t)+\hat a_2(\hat s,\hat t)\big)\big(X(s,t)-X(s,t-h)\big)h\\ &\qquad+\big(a_3(s,t)-\hat a_3(\hat s,\hat t)\big)X(s,t)h^2\\ &\qquad+\mathcal{E}(s,t;h),\end{aligned} \tag{5.5}$$

where

$$E\{\mathcal{E}(s,t;h)^2\} \le Ch^4. \tag{5.6}$$

REMARK 5.3. If we consider reversal in one coordinate, then we would set $Y(s,t) = \hat X(s,t) = X(1-s,t)$, $\hat s = 1-s$, $\hat t = t$ and

$$\hat{\mathcal{G}}_{1-s,t} \stackrel{\text{def}}{=} \sigma\{Y(u,v),\ u \le 1-s \text{ or } v \le t\} = \sigma\{X(u,v),\ u \ge s \text{ or } v \le t\}.$$

Then $\hat\Delta = [1-s,1-s+h]\times[t-h,t]$ and Proposition 5.1 implies that $\dot{\hat W}|_{\hat\Delta}$ is independent of $\hat{\mathcal{G}}_{\hat s\hat t}$. So with these definitions, formula (5.5) remains valid.

PROOF OF PROPOSITION 5.2. Equality (5.4) follows from Proposition 2.7. From (2.9),

$$\begin{aligned}&\hat X(\hat\Delta) + \iint_{\hat\Delta}\hat a_1(u,v)\hat X(du,v)\,dv + \iint_{\hat\Delta}\hat a_2(u,v)\,du\,\hat X(u,dv)\\ &\quad+\iint_{\hat\Delta}\hat a_3(u,v)\hat X(u,v)\,du\,dv = \iint_{\hat\Delta}\hat a_4(u,v)\hat W(du\,dv).\end{aligned} \tag{5.7}$$

On the other hand,

$$
\begin{aligned}
(5.8)\qquad & X(\Delta) + \iint_\Delta a_1(u,v)X(du,v)\,dv + \iint_\Delta a_2(u,v)\,du\,X(u,dv) \\
& \quad + \iint_\Delta a_3(u,v)X(u,v)\,du\,dv = \iint_\Delta a_4(u,v)W(du\,dv).
\end{aligned}
$$

By definition, $\hat{X}(\hat{\Delta}) = X(\Delta)$, and $\hat{X}(\hat{s},\hat{t}) = X(s,t)$. Subtract these equations to see that

$$
\begin{aligned}
& \iint_\Delta a_4(u,v)W(du\,dv) - \iint_{\hat{\Delta}} \hat{a}_4(u,v)\hat{W}(du\,dv) \\
(5.9)\qquad & \quad = \iint_\Delta a_1(u,v)X(du,v)\,dv - \iint_{\hat{\Delta}} \hat{a}_1(u,v)\hat{X}(du,v)\,dv \\
& \qquad + \iint_\Delta a_2(u,v)\,du\,X(u,dv) - \iint_{\hat{\Delta}} \hat{a}_2(u,v)\,du\,\hat{X}(u,dv) \\
& \qquad + \iint_\Delta a_3(u,v)X(u,v)\,du\,dv - \iint_{\hat{\Delta}} \hat{a}_3(u,v)\hat{X}(u,v)\,du\,dv.
\end{aligned}
$$

Approximate $a_i(u,v)$ and $X(u,v)$ by $a_i(s,t)$ and $X(s,t)$ to see that

$$
\begin{aligned}
& \iint_\Delta a_4(u,v)W(du\,dv) - \iint_{\hat{\Delta}} \hat{a}_4(u,v)\hat{W}(du\,dv) \\
& \quad = a_1(s,t)\big(X(s,t) - X(s-h,t)\big)h \\
(5.10)\qquad & \qquad - \hat{a}_1(\hat{s},\hat{t})\big(\hat{X}(\widehat{s-h},\hat{t}) - \hat{X}(\hat{s},\hat{t})\big)h + \mathcal{E}_1 - \hat{\mathcal{E}}_1 \\
& \qquad + a_2(s,t)\big(X(s,t) - X(s,t-h)\big)h \\
& \qquad - \hat{a}_2(\hat{s},\hat{t})\big(\hat{X}(\hat{s},\widehat{t-h}) - \hat{X}(\hat{s},\hat{t})\big)h + \mathcal{E}_2 - \hat{\mathcal{E}}_2 \\
& \qquad + \big(a_3(s,t) - \hat{a}_3(\hat{s},\hat{t})\big)X(s,t)h^2 + \mathcal{E}_3 - \hat{\mathcal{E}}_3,
\end{aligned}
$$

where the $\mathcal{E}_i$ and $\hat{\mathcal{E}}_i$ are the errors in the respective approximations. Now condition on $\hat{\mathcal{G}}_{\hat{s}\hat{t}}$. Note that $\hat{W}$ is a white noise with respect to the reversed filtration and that $Y = \hat{X}$ is a solution of (5.2), so Proposition 5.1 implies that the white noise on $\hat{\Delta}$ is independent of $\hat{\mathcal{G}}_{\hat{s},\hat{t}}$, and therefore

$$
E\left\{\iint_{\hat{\Delta}} \hat{a}_4(u,v)\hat{W}(du\,dv) \mid \hat{\mathcal{G}}_{\hat{s},\hat{t}}\right\} = 0.
$$

On the other hand, all the terms on the right-hand side except the errors are $\hat{\mathcal{G}}_{\hat{s},\hat{t}}$-measurable, so that we get (5.5) with $\mathcal{E}(s,t;h) = \sum_{i=1}^3 E\{\mathcal{E}_i - \hat{\mathcal{E}}_i \mid \hat{\mathcal{G}}_{\hat{s},\hat{t}}\}$.

In order to complete the proof of the proposition, we need only show that there exists $C > 0$ such that $E\{\mathcal{E}_i^2\} \le Ch^4$ and $E\{\hat{\mathcal{E}}_i^2\} \le Ch^4$ for $i = 1, \dots, 3$. Consider

$$\begin{aligned}\mathcal{E}_1 &= \iint_\Delta \bigl(a_1(u, v) - a_1(s, t)\bigr) X(du, v)\, dv \\ &\quad + a_1(s, t) \int_{t-h}^t \bigl(X(s, v) - X(s - h, v) - \bigl(X(s, t) - X(s - h, t)\bigr)\bigr)\, dv \\ &\stackrel{\text{def}}{=} I_1 + I_2.\end{aligned} \tag{5.11}$$

Let $X(du, v) = dM_u^v + dV_u^v$ be the semimartingale decomposition of $X(\cdot, v)$ and write

$$I_1 = \int_{t-h}^t \int_{s-h}^s \bigl(a_1(u, v) - a_1(s, t)\bigr)(dM_u^v + dV_u^v)\, dv$$

and use the Schwarz inequality:

$$\begin{aligned}E\{I_1^2\} &\le 2hE\left\{\int_{t-h}^t \int_{s-h}^s \bigl(a_1(u, v) - a_1(s, t)\bigr)^2 d\langle M^v\rangle_s\, dv\right\} \\ &\quad + 2hE\left\{\int_{t-h}^t \left(\int_{s-h}^s \bigl(a_1(u, v) - a_1(s, t)\bigr)\, dV_u^v\right)^2 dv\right\}.\end{aligned} \tag{5.12}$$

By Lemma 2.3, $d\langle M^v\rangle_u = \sigma^2(u, v)\, du$ and $dV_u^v = \mu(u, v)\, du$. Moreover, a_1 has uniformly bounded derivatives, so $|a_1(u, v) - a_1(s, t)| \le C(|s - u| + |t - v|) \le 2Ch$. Thus, this is less than or equal to

$$\begin{aligned}&8Ch^3 \int_{t-h}^t \int_{s-h}^s E\{\sigma^2(u, v)\}\, du\, dv \\ &\quad + 8Ch^3 \int_{t-h}^t E\left\{\left(\int_{s-h}^s |\mu(u, v)|\, du\right)^2\right\} dv.\end{aligned} \tag{5.13}$$

By Lemma 2.3, $E\{\mu^2(u, v)\}$ and $E\{\sigma^2(u, v)\}$ are bounded for (u, v) in compact sets, so there is a constant C' for which $E\{I_1^2\}$ is bounded by $C'h^5$.

Let $Z(v) = X(]s - h, s] \times]v, t])$ and note that

$$I_2 = -a_1(s, t) \int_{t-h}^t Z(v)\, dv,$$

so that

$$\begin{aligned}E\{I_2^2\} &= a_1^2(s, t) \int_{t-h}^t \int_{t-h}^t E\bigl\{Z(u)Z(v)\bigr\}\, du\, dv \\ &\le a_1^2(s, t) \int_{t-h}^t \int_{t-h}^t E\bigl\{Z^2(u)\bigr\}^{1/2} E\bigl\{Z^2(v)\bigr\}^{1/2}\, du\, dv.\end{aligned}$$

From Corollary 2.4, $E\{Z^2(u)\} \le Ch(t-v) \le Ch^2$, so this is bounded by, say, $C''a_1^2(s,t)h^4$. Thus, for small h,

$$E\{\mathcal{E}_1^2\} \le 2E\{I_1^2\} + 2E\{I_2^2\} \le C'h^5 + C''h^4 \le Ch^4$$

for a suitable constant C which depends only on $s+t$, the coefficients a_i, and the smoothness of the initial semimartingales Y^i. The errors $\hat{\mathcal{E}}_1$, $\mathcal{E}_2$, and $\hat{\mathcal{E}}_2$ are similar.

Moving to $\mathcal{E}_3$, we have

$$\begin{aligned}\mathcal{E}_3 &= \iint_\Delta \big(a_3(u,v)X(u,v) - a_3(s,t)X(s,t)\big)\,du\,dv \\ &= \iint_\Delta \big(a_3(u,v) - a_3(s,t)\big)X(u,v)\,du\,dv \\ &\quad + a_3(s,t)\iint_\Delta \big(X(u,v) - X(s,t)\big)\,du\,dv \\ &\stackrel{\text{def}}{=} J_1 + J_2.\end{aligned}$$

Using the same reasoning as above, we see that

$$\begin{aligned}E\{J_1^2\} &\le E\left\{\left(\iint_\Delta ChX(u,v)\,du\,dv\right)^2\right\} \\ &= C^2h^2\int_{\Delta\times\Delta} E\{X(u,v)X(u',v')\}\,du\,dv\,du'\,dv'.\end{aligned}$$

But $E\{X(u,v)X(u',v')\} \le \sup_{(u,v)\in\Delta} E\{X^2(u,v)\} \le C'$, so $E\{J_1^2\} \le Ch^6$. Further,

$$E\{J_2^2\} = a_3^2(s,t)\int_{\Delta\times\Delta} E\big\{\big(X(u,v)-X(s,t)\big)\big(X(u',v')-X(s,t)\big)\big\}\,du\,dv\,du'\,dv',$$

while $\sup_{(u,v)\in\Delta} E\{(X(u,v)-X(s,t))^2\} \le 2Ch$ by Corollary 2.4. Thus, this is bounded by $2Ca_3^2(s,t)h^5$. The same bound holds for $\hat{\mathcal{E}}_3$ by symmetry.

Adding the errors together, we see that $E\{\mathcal{E}^2(s,t;h)\} \le Ch^4$ for small h. $\square$

REMARK 5.4. The only error term above which has order as large as $O(h^4)$ is I_2. The others are all $O(h^5)$ or smaller.

6. Reversals of the Brownian sheet.

6.1. *Reversal in one coordinate.*

THEOREM 6.1. *Let $(W(s,t))$ be a standard Brownian sheet. Set $Y(s,t) = W(1-s,t)$. Then there is a standard Brownian sheet $(B(s,t))$ independent of $(W(1,t),\ t\ge 0)$ such that $(Y(s,t))$ is the weak solution on $[0,1[\,\times\mathbb{R}_+$ of*

$$\frac{\partial^2 Y}{\partial s\partial t} + \frac{1}{1-s}\frac{\partial Y}{\partial t} = \frac{\partial^2 B}{\partial s\partial t}, \tag{6.1}$$

with initial conditions $Y(0,t) = W(1,t)$, $Y(s,0) = 0$.

REMARK 6.2. Set $\Delta = [s-h, s] \times [t-h, t]$. One easily checks (with $\hat{\mathscr{G}}_{\hat{s},\hat{t}}$ as in Remark 5.3), that

$$E\big(W(\Delta) \mid \hat{\mathscr{G}}_{\hat{s},\hat{t}}\big) = \frac{h}{s}\big(W(s,t) - W(s,t-h)\big),$$

so from (5.5), we guess that $\hat{a}_1(\hat{s},\hat{t}) = 1/s$, that is, $\hat{a}_1(s,t) = 1/(1-s)$, and $\hat{a}_2 \equiv \hat{a}_3 \equiv 0$. Therefore, Proposition 5.2 suggests that (6.1) should hold. Of course, we still must prove that (6.1) does indeed hold.

PROOF OF THEOREM 6.1. According to Lemma 2.1, with (2.6) written as in (2.9), it suffices to show that the following expression is a Brownian sheet:

$$Y(s,t) - Y(s,0) - Y(0,t) + Y(0,0) + \int_0^s \frac{du}{1-u}\big(Y(u,t) - Y(u,0)\big).$$

Replace $Y(s,t)$ by $W(1-s,t)$ and do the change of variables $u \mapsto 1-u$ to see that this expression equals

$$W(1-s,t) - W(1,t) + \int_{1-s}^1 \frac{du}{u} W(u,t).$$

Do the change of variables $x = 1/u$ to get

$$W(1-s,t) - W(1,t) + \int_1^{\frac{1}{1-s}} \frac{dx}{x} W(1/x,t). \tag{6.2}$$

Let $\xi(s,t) = sW(1/s,t)$. Then $(\xi(s,t))$ is a standard Brownian sheet [16] and the expression above can be written

$$(1-s)\xi\left(\frac{1}{1-s}, t\right) - \xi(1,t) + \int_1^{\frac{1}{1-s}} \frac{dx}{x^2}\xi(x,t).$$

Integrate by parts to see that this expression is equal to

$$\int_1^{\frac{1}{1-s}} \xi(dx,t)\frac{1}{x} = \int_1^{\frac{1}{1-s}}\int_0^t \xi(dx,dy)\frac{1}{x} \stackrel{\text{def}}{=} B(s,t).$$

It is not difficult to check that $(B(s,t))$ is a Brownian sheet. For instance, if $s < s'$ and $t < t'$, then

$$E\big(B(s,t)B(s',t')\big) = \int_1^{\frac{1}{1-s}} dx \int_0^t dy \frac{1}{x^2} = t\frac{1}{x}\bigg|_{\frac{1}{1-s}}^1 = t\big(1-(1-s)\big) = st,$$

while if $s < s'$ and $t' < t$, this covariance is st'.

We now check that $(B(s,t))$ is independent of $(W(1,t),\ t \geq 0)$. More generally, fix $a \geq 1$, $b \geq 0$, and show that $(B(s,t))$ is independent of $W(a,b)$. If $0 \leq s \leq 1$

and $t \leq b$, then we use the fact that $B(s,t)$ is equal to the expression in (6.2) to write

$$\begin{aligned} E\big(B(s,t)W(a,b)\big) &= E\left(\left[W(1-s,t) - W(1,t) + \int_{1-s}^{1} \frac{du}{u} W(u,t)\right] W(a,b)\right) \\ &= -st + \int_{1-s}^{1} \frac{du}{u}\, ut \\ &= -st + st \\ &= 0. \end{aligned}$$

If $t \geq b$, then

$$E\big(B(s,t)W(a,b)\big) = -sb + \int_{1-s}^{1} \frac{du}{u} ub = -sb + sb = 0.$$

This proves the desired independence and completes the proof. □

6.2. *Reversal in two coordinates.*

THEOREM 6.3. *Let $(W(s,t))$ be a standard Brownian sheet. Set*

$$Y(s,t) = W(1-s, 1-t).$$

Then there is a standard Brownian sheet $(B(s,t))$ independent of $(W(x,1), W(1,x), 0 \leq x \leq 1)$ such that $(Y(s,t))$ is a weak solution on $[0,1[^2$ of

$$\frac{\partial^2 Y}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial Y}{\partial s} + \frac{1}{1-s}\frac{\partial Y}{\partial t} + \frac{1}{(1-s)(1-t)} Y(s,t) = \frac{\partial^2 B}{\partial s \partial t}, \tag{6.3}$$

with initial conditions $Y(0,x) = W(1, 1-x)$, $Y(x,0) = W(1-x, 1)$, $0 \leq x \leq 1$.

REMARK 6.4. With Δ and $\hat{\mathscr{G}}_{\hat{s},\hat{t}}$ defined as in (5.3), it is not difficult to check that

$$\begin{aligned} E\big(W(\Delta) \mid \hat{\mathscr{G}}_{\hat{s},\hat{t}}\big) &= \frac{h}{t}\big(W(s,t) - W(s-h,t)\big) + \frac{h}{s}\big(W(s,t) - W(s,t-h)\big) \\ &\quad + \frac{h^2}{st} W(s,t). \end{aligned}$$

This formula also can be obtained from [3], Theorem 4.2. From (5.5), we guess that

$$\hat{a}_1(\hat{s},\hat{t}) = \frac{1}{t}, \qquad \hat{a}_2(\hat{s},\hat{t}) = \frac{1}{s}, \qquad \hat{a}_3(\hat{s},\hat{t}) = \frac{1}{st}.$$

Proposition 5.2 suggests, therefore, that equation (6.3) should hold.

PROOF OF THEOREM 6.3. Again according to Lemma 2.1, with (2.6) written as in (2.9), it suffices to show that the following expression is a Brownian sheet:

$$Y(s,t)-Y(s,0)-Y(0,t)+Y(0,0)+\int_0^t \frac{dv}{1-v}\big(Y(s,v)-Y(0,v)\big) + \int_0^s \frac{du}{1-u}\big(Y(u,t)-Y(u,0)\big)+\int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v} Y(u,v).$$

Replace $Y(s,t)$ by $W(1-s,1-t)$ and do the change of variables $(u,v)\mapsto(1-u, 1-v)$ to get

$$\begin{aligned}
&W(1-s,1-t)-W(1-s,1)-W(1,1-t)+W(1,1)\\
&\quad+\int_{1-t}^1 \frac{dv}{v}\big(W(1-s,v)-W(1,v)\big)+\int_{1-s}^1 \frac{du}{u}\big(W(u,1-t)-W(u,1)\big)\\
&\quad+\int_{1-s}^1 \frac{du}{u}\int_{1-t}^1 \frac{dv}{v} W(u,v).
\end{aligned}\tag{6.4}$$

Now do the change of variables $x=1/u$, $y=1/v$, to see that this equals

$$\begin{aligned}
&W\big(]1-s,1]\times]1-t,1]\big)+\int_1^{\frac{1}{1-t}} \frac{dy}{y}\big(W(1-s,1/y)-W(1,1/y)\big)\\
&\quad+\int_1^{\frac{1}{1-s}} \frac{dx}{x}\big(W(1/x,1-t)-W(1/x,1)\big)\\
&\quad+\int_1^{\frac{1}{1-s}} \frac{dx}{x}\int_1^{\frac{1}{1-t}} \frac{dy}{y} W(1/x,1/y).
\end{aligned}$$

Let $\xi(s,t)=stW(1/s,1/t)$. Then $(\xi(s,t))$ is a standard Brownian sheet, and the expression above can be written

$$\begin{aligned}
&(1-s)(1-t)\xi\left(\frac{1}{1-s},\frac{1}{1-t}\right)-(1-s)\xi\left(\frac{1}{1-s},1\right)\\
&\quad-(1-t)\xi\left(1,\frac{1}{1-t}\right)+\xi(1,1)\\
&\quad-\int_1^{\frac{1}{1-t}}\left[(1-s)\frac{-1}{y^2}\xi\left(\frac{1}{1-s},y\right)-\frac{-1}{y^2}\xi(1,y)\right]dy\\
&\quad-\int_1^{\frac{1}{1-s}}\left[(1-t)\frac{-1}{x^2}\xi\left(x,\frac{1}{1-t}\right)-\frac{-1}{x^2}\xi(x,1)\right]dx\\
&\quad+\int_1^{\frac{1}{1-s}} dx\int_1^{\frac{1}{1-t}} dy\,\frac{1}{x^2y^2}\xi(x,y).
\end{aligned}\tag{6.5}$$

Using formally the formula for integration by parts (2.3) (whose use is justified in Remark 6.5 below), with $f(x,y)=1/(xy)$ and $g(x,y)=\xi(x,y)$, we see that this equals

$$\iint_{[1,\frac{1}{1-s}\times[1,\frac{1}{1-t}]} \frac{1}{xy}\xi(dx,dy) \stackrel{\text{def}}{=} B(s,t). \tag{6.6}$$

It is now straightforward to check that $(B(s,t))$ so defined is a standard Brownian sheet. For instance, if $s < s'$ and $t' < t$, then

$$E\big(B(s,t)B(s',t')\big) = \int_1^{\frac{1}{1-s}} dx \int_1^{\frac{1}{1-t'}} dy \frac{1}{x^2y^2} = \frac{-1}{x}\bigg|_1^{\frac{1}{1-s}} \cdot \frac{-1}{y}\bigg|_1^{\frac{1}{1-t'}} = st'.$$

This proves that (6.3) holds.

It remains to prove that $(B(s,t))$ is independent of $(W(a,1),\ W(1,a),\ 0 \le a \le 1)$. For this, it suffices to compute the covariance between the expression in (6.4) and $W(a,1)$, then $W(1,a)$. We omit the second computation and do the first.

From the covariance of the Brownian sheet and elementary geometric considerations, using the fact that $B(s,t)$ is equal to the expression in (6.4), we see that for $a \le 1-s$,

$$E\big(B(s,t)W(a,1)\big) = \int_{1-s}^1 \frac{du}{u}(-at) + \int_{1-s}^1 \frac{du}{u}\int_{1-t}^1 \frac{dv}{v}(av) = 0,$$

and for $1-s \le a \le 1$,

$$\begin{aligned} E\big(B(s,t)W(a,1)\big) &= (a-1+s)t + \int_{1-t}^1 \frac{dv}{v}(-v(a-1+s)) \\ &\quad + \int_{1-s}^a \frac{du}{u}(-ut) + \int_a^1 \frac{du}{u}(-at) \\ &\quad + \int_{1-s}^a \frac{du}{u}\int_{1-t}^1 \frac{dv}{v}(uv) + \int_a^1 \frac{du}{u}\int_{1-t}^1 \frac{dv}{v}(av) \\ &= 0. \end{aligned}$$

This completes the proof. $\square$

REMARK 6.5. We justify here the two-parameter integration by parts used in (6.5) and (6.6) above. Set $a = c = 1$, $b = 1/(1-s)$, $d = 1/(1-t)$, $f_1(x) = 1/x$, $f_2(y) = 1/y$. The expression in (6.6) is equal to

$$I = \iint_{[a,b]\times[c,d]} f_1(x)f_2(y)\xi(dx,dy) = \int_a^b f_1(x)Z_1(dx),$$

where $(Z_1(x),\ a \le x \le b)$ is the martingale defined by

$$Z_1(x) = \iint_{[a,x]\times[c,d]} f_2(y)\xi(dx,dy).$$

Using the standard integration by parts formula for semimartingales [13], Chapter IV, Proposition (3.11), in which the mutual variation term vanishes because f_1 has bounded variation, we see that

$$\begin{aligned} I &= f_1(b)Z_1(b) - f_1(a)Z_1(a) - \int_a^b Z_1(x)\frac{\partial f_1}{\partial x}(x)\,dx \\ &= f_1(b)\iint_{[a,b]\times[c,d]} f_2(y)\xi(dx,dy) \\ &\quad - \int_a^b \left(\iint_{[a,x]\times[c,d]} f_2(y)\xi(du,dy)\right)\frac{\partial f_1}{\partial x}(x)\,dx. \end{aligned}$$

Now, for fixed x, let

$$Z_2^x(y) = \iint_{[a,x]\times[c,y]} \xi(du,dv) = \xi(x,y) - \xi(x,c) - \xi(a,y) + \xi(a,c).$$

Then

$$I = f_1(b)\int_c^d f_2(y)Z_2^b(dy) - \int_a^b dx\,\frac{\partial f_1}{\partial x}(x)\int_c^d f_2(y)Z_2^x(dy).$$

In both stochastic integrals, use again the standard integration by parts formula, to see that this equals

$$\begin{aligned} &f_1(b)\left[f_2(d)Z_2^b(d) - f_2(c)Z_2^b(c) - \int_c^d Z_2^b(y)\frac{\partial f_2}{\partial y}(y)\,dy\right] \\ &\quad - \int_a^b dx\,\frac{\partial f_1}{\partial x}(x)\left(f_2(d)Z_2^x(d) - f_2(c)Z_2^x(c) - \int_c^d Z_2^x(y)\frac{\partial f_2}{\partial y}\,dy\right). \end{aligned}$$

Now replace a, b, c, d, $f_1(x)$ and $f_2(y)$ by their values, to see, after simplification, that this is the expression in (6.5).

REMARK 6.6. Equation (6.3) is reminiscent of the equation for a Brownian bridge $(X_s,\ 0 \le s \le 1)$:

$$dX_s + \frac{X_s}{1-s} = dB_s,$$

where (B_s) is a standard Brownian motion. The law of the reversed process $(B(1-s),\ 0 \le s \le 1)$, is the same as the law of (Y_t), where

$$Y_t = (1-t)Z + X_t,$$

and Z is a standard normal random variable independent of the Brownian bridge (X_t). A similar identity in law occurs for the Brownian sheet, as is shown in the following theorem. This identity is related to some results of [1], [10].

THEOREM 6.7. *Let* $(W(s,t))$ *and* $(X(s,t))$ *be standard Brownian sheets. Set*

(6.7) $U(s,t) = X(s,t) - sX(1,t) - tX(s,1) + stX(1,1)$,

(6.8) $Z(s,t) = (1-s)X(1,1-t) + (1-t)X(1-s,1) - (1-s)(1-t)X(1,1)$.

Then U *and* Z *are independent, and* $Y = U + Z$ *has the same law as* $(W(1-s, 1-t),\ (s,t) \in [0,1]^2)$. *In particular,* Y *is a weak solution of* (6.3) *with initial conditions* $Y(0,x) = W(1,1-x)$, $Y(x,0) = W(1-x,1)$, $0 \le x \le 1$.

The proof of this theorem relies on two lemmas.

LEMMA 6.8. *Z is a weak solution of the equation*

$$\frac{\partial^2 Z}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial Z}{\partial s} + \frac{1}{1-s}\frac{\partial Z}{\partial t} + \frac{1}{(1-s)(1-t)} Z(s,t) = 0. \tag{6.9}$$

PROOF. Again according to Lemma 2.1, with (2.6) written as in (2.9), it suffices to show that the following integral vanishes:

$$Z(\Delta) + \int_0^t \frac{dv}{1-v}\big(Z(s,v) - Z(0,v)\big) + \int_0^s \frac{du}{1-u}\big(Z(u,t) - Z(u,0)\big) + \int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v} Z(u,v).$$

Use formula (6.8) to see, after simplification, that this expression is indeed equal to 0. □

LEMMA 6.9.

$$E\big(Z(s,t)Z(s',t')\big) = \big(1 - (s \wedge s')(t \wedge t')\big)(1 - s \vee s')(1 - t \vee t') \tag{6.10}$$

and

$$E\big(U(s,t)U(s',t')\big) = (s \wedge s')(t \wedge t')(1 - s \vee s')(1 - t \vee t'). \tag{6.11}$$

PROOF. Using elementary algebra, one checks that

$$\begin{aligned}
Z(s,t) &= \big(1 - s + 1 - t - (1-s)(1-t)\big)X(1-s,1-t) \\
&\quad + \big(1 - s - (1-s)(1-t)\big)\big(X(1,1-t) - X(1-s,1-t)\big) \\
&\quad + \big(1 - t - (1-s)(1-t)\big)\big(X(1-s,1) - X(1-s,1-t)\big) \\
&\quad - (1-s)(1-t)X\big([1-s,1] \times [1-t,1]\big) \\
&= (1 - st)X(1-s,1-t) \\
&\quad + t(1-s)\big(X(1,1-t) - X(1-s,1-t)\big) \\
&\quad + s(1-t)\big(X(1-s,1) - X(1-s,1-t)\big) \\
&\quad - (1-s)(1-t)X\big([1-s,1] \times [1-t,1]\big).
\end{aligned}$$

The four terms in the last expression are independent. It is now a tedious but elementary calculation, using the covariance of the Brownian sheet, to check that $E(Z(s,t)Z(s',t'))$ is given by formula (6.10). This is left to the reader, as is the similar calculation that establishes formula (6.11). $\square$

PROOF OF THEOREM 6.7. From Lemma 6.9, we see that $Y = U + Z$ has the same covariance, hence the same law, as $(W(1-s, 1-t))$. This of course implies that there is a white noise $\dot{B}$ such that Y is the solution of equation (6.3), but we prefer to give a direct derivation. By Lemma 6.8, it suffices to check that there is a Brownian sheet $(\xi(s,t))$ such that $(U(s,t))$ is the solution of

$$\frac{\partial^2 U}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial U}{\partial s} + \frac{1}{1-s}\frac{\partial U}{\partial t} + \frac{1}{(1-s)(1-t)}U(s,t) = \frac{\partial^2 \xi}{\partial s \partial t}. \tag{6.12}$$

Let $(\hat{W}(s,t),\ (s,t) \in [0,1]^2)$ be the standard Brownian sheet defined by

$$\hat{W}(s,t) = X\big([1-s, 1] \times [1-t, 1]\big),$$

so that

$$X(s,t) = \hat{W}\big([1-s, 1] \times [1-t, 1]\big). \tag{6.13}$$

Because U vanishes on the axes, the double integral of the left-hand side of (6.12) over $\Delta = [0,s] \times [0,t]$ is equal to

$$U(s,t) + \int_0^t \frac{dv}{1-v}U(s,v) + \int_0^s \frac{du}{1-u}U(u,t) + \int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v}U(u,v).$$

Replace $U(\cdot,\cdot)$ by its expression in terms of X given in (6.7) to get

$$\begin{aligned} &X(s,t) - sX(1,t) - tX(s,1) + stX(1,1) \\ &\quad + \int_0^t \frac{dv}{1-v}\big(X(s,v) - sX(1,v) - vX(s,1) + svX(1,1)\big) \\ &\quad + \int_0^s \frac{du}{1-u}\big(X(u,t) - uX(1,t) - tX(u,1) + utX(1,1)\big) \\ &\quad + \int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v}\big(X(u,v) - uX(1,v) - vX(u,1) + uvX(1,1)\big). \end{aligned}$$

Rearrange the terms and simplify to get

$$\begin{aligned} &X(s,t) + \int_0^t \frac{dv}{1-v}\big(X(s,v) - X(s,1)\big) \\ &\qquad + \int_0^s \frac{du}{1-u}\big(X(u,t) - X(1,t)\big) \end{aligned} \tag{6.14}$$

$$
\begin{aligned}
+\int_0^s du \int_0^t dv \biggl(X(1,1) - \frac{X(1,v)}{1-v} + \frac{vX(1,1)}{1-v} \\
- \frac{X(u,1)}{1-u} + \frac{uX(1,1)}{1-u} \\
+ \frac{X(u,v) - uX(1,v) - vX(u,1) + uvX(1,1)}{(1-u)(1-v)} \biggr).
\end{aligned}
$$

The integrand in the double integral simplifies to

$$
\frac{X(u,v) - X(1,v) - X(u,1) + X(1,1)}{(1-u)(1-v)}.
$$

Now replace $X(\cdot,\cdot)$ by its expression in terms of $\hat{W}$ given in (6.13) and do the changes of variables $u \mapsto 1-u$, $v \mapsto 1-v$, to see that (6.14) is equal to

$$
\begin{aligned}
&\hat{W}(1-s,1-t) - \hat{W}(1-s,1) - \hat{W}(1,1-t) + \hat{W}(1,1) \\
&\quad + \int_{1-t}^1 \frac{dv}{v} \bigl(\hat{W}(1-s,v) - \hat{W}(1,v)\bigr) + \int_{1-s}^1 \frac{du}{u} \bigl(\hat{W}(u,1-t) - \hat{W}(u,1)\bigr) \\
&\quad + \int_{1-s}^1 \frac{du}{u} \int_{1-t}^1 \frac{dv}{v} \hat{W}(u,v).
\end{aligned}
$$

This is exactly the expression in (6.4), with W replaced by $\hat{W}$, and we have shown in the lines that follow (6.4) that this expression is a standard Brownian sheet that is independent of $(\hat{W}(1-x,1),\ \hat{W}(1,1-x),\ 0 \le x \le 1)$. $\square$

7. Reversal in hyperbolic s.p.d.e.'s. We shall consider the reversal in two coordinates of the weak solution of the hyperbolic equation with constant coefficients

$$
\frac{\partial^2 X}{\partial s \partial t} + a_1 \frac{\partial X}{\partial s} + a_2 \frac{\partial X}{\partial t} + a_3 X(s,t) = \dot{W}, \tag{7.1}
$$

with vanishing initial conditions $X(s,0) = 0$, $X(0,t) = 0$. The reversal in one coordinate could be done similarly, and in fact, more simply. In this equation, the case $a_3 \neq a_1 a_2$ corresponds to the stochastic telegraph equation [6], Chapter IV, Section 43, whereas in the special case where $a_3 = a_1 a_2$, equation (7.1) can be transformed into the wave equation by a change of variables and parameters. We shall restrict ourselves to this special case.

THEOREM 7.1. *Fix $a_1, a_2, a_3 \in \mathbb{R}$ and suppose $a_3 = a_1 a_2 \neq 0$. Let $(X(s,t))$ be the weak solution of* (7.1) *with vanishing initial conditions, and set $\hat{X}(s,t) = X(1-s,1-t)$. Then there is a Brownian sheet $(B(s,t))$ independent of $(X(u,1), X(1,u),\ 0 \le u \le 1)$ such that $(\hat{X}(s,t))$ is the solution on $[0,1[^2$ of*

$$
\frac{\partial^2 \hat{X}}{\partial s \partial t} + \hat{a}_1(s,t) \frac{\partial \hat{X}}{\partial s} + \hat{a}_2(s,t) \frac{\partial \hat{X}}{\partial t} + \hat{a}_3(s,t) \hat{X}(s,t) = \frac{\partial^2 B}{\partial s \partial t}, \tag{7.2}
$$

with initial conditions $\hat{X}(s,0) = X(1-s,1)$, $\hat{X}(0,t) = X(1,1-t)$, *where*

$$\hat{a}_1(s,t) = \frac{2a_1 e^{2a_1(1-s)}}{e^{2a_1(1-s)} - 1} - a_1, \qquad \hat{a}_2(s,t) = \frac{2a_2 e^{2a_2(1-s)}}{e^{2a_2(1-s)} - 1} - a_2, \tag{7.3}$$

and $\hat{a}_3(s,t) = \hat{a}_1(s,t)\hat{a}_2(s,t)$.

REMARK 7.2. The case $a_1 = a_2 = 0$ has been discussed in Theorem 6.3. In order to recover this case from the theorem above, it is not possible to set $a_i = 0$ in (7.3), but there is no problem in taking the limit as $a_i \to 0$. Doing this for $i = 1, 2$ leads to equation (6.3).

PROOF OF THEOREM 7.1. Define $\tilde{X}(s,t) = e^{a_2 s + a_1 t} X(s,t)$. From Example 4.1, we see that $\tilde{X}$ satisfies the equation

$$\frac{\partial^2 \tilde{X}}{\partial s \partial t} = e^{a_2 s + a_1 t} \dot{W}.$$

Therefore, there is a Brownian sheet $\tilde{W}$ such that

$$\tilde{X}(s,t) = \tilde{W}\left(\frac{e^{2a_2 s} - 1}{2a_2}, \frac{e^{2a_1 t} - 1}{2a_1}\right),$$

and therefore,

$$X(s,t) = e^{-a_2 s - a_1 t} \tilde{W}\left(\frac{e^{2a_2 s} - 1}{2a_2}, \frac{e^{2a_1 t} - 1}{2a_1}\right)$$

and

$$\hat{X}(s,t) = e^{-a_2(1-s) - a_1(1-t)} \tilde{W}\left(\frac{e^{2a_2(1-s)} - 1}{2a_2}, \frac{e^{2a_1(1-t)} - 1}{2a_1}\right).$$

Set

$$Z(s,t) = \tilde{W}\left(\frac{e^{2a_2} - 1}{2a_2} - s, \frac{e^{2a_1} - 1}{2a_1} - t\right).$$

Then

$$\hat{X}(s,t) = e^{-a_2(1-s) - a_1(1-t)} Z\left(\frac{e^{2a_2} - e^{2a_2(1-s)}}{2a_2}, \frac{e^{2a_1} - e^{2a_1(1-t)}}{2a_1}\right).$$

By Theorem 6.3, $(Z(s,t))$ is a solution of the equation

$$\frac{\partial^2 Z}{\partial s \partial t} + f(a_1,t)\frac{\partial Z}{\partial s} + f(a_2,s)\frac{\partial Z}{\partial t} + f(a_1,s) f(a_2,t) Z = \dot{B},$$

where B is a Brownian sheet independent of

$$\left(Z\left(\frac{e^{2a_2} - 1}{2a_2}, x\right),\ Z\left(x, \frac{e^{2a_1} - 1}{2a_1}\right),\ 0 \le x \le 1\right)$$

and

$$f(a,x)=\left(\frac{e^{2a}-1}{2a}-x\right)^{-1}.$$

Let

$$Y(s,t)=Z\left(\frac{e^{2a_2}-e^{2a_2(1-s)}}{2a_2},\frac{e^{2a_1}-e^{2a_1(1-t)}}{2a_1}\right).$$

From Example 3.1, we conclude that $(Y(s,t))$ is a solution of the equation

$$\frac{\partial^2 Y}{\partial s\partial t}+g(a_1,s)\frac{\partial Y}{\partial s}+g(a_2,t)\frac{\partial Y}{\partial t}+g(a_1,s)g(a_2,t)Y=e^{a_2(1-s)+a_1(1-t)}\dot{\hat{W}},$$

where $\dot{\hat{W}}$ is a white noise and

$$g(a,x)=\frac{2ae^{2a(1-x)}}{e^{2a(1-x)}-1}.$$

Again by Example 4.1, we conclude that $(\hat{X}(s,t))$ solves equation (7.2). This proves the theorem. $\square$

8. Reversal with initial conditions.

THEOREM 8.1. *Let X_0 be a $N(0,\sigma^2)$ random variable, (M_s^1) and (M_t^2) be Gaussian martingales such that $M_0^1=M_0^2=0$ and $E((M_u^i)^2)=f_i(u)$, $i=1,2$. We assume that X_0, (M_s^1) and (M_t^2) are independent.*

Let X be the weak solution of the s.p.d.e.

$$\frac{\partial^2 X}{\partial s\partial t}=\dot{W}, \tag{8.1}$$

with the initial conditions $X(s,0)=X_0+M_s^1$ and $X(0,t)=X_0+M_t^2$, $s,t\geq 0$. Then there exists a Brownian sheet $(B(s,t))$ such that $\hat{X}(s,t)=X(1-s,1-t)$ satisfies an s.p.d.e. of the form

$$\frac{\partial^2 \hat{X}}{\partial s\partial t}+a_1(s,t)\frac{\partial \hat{X}}{\partial s}+a_2(s,t)\frac{\partial \hat{X}}{\partial t}+a_3(s,t)\hat{X}=a_4(s,t)\dot{B} \tag{8.2}$$

if and only if $a_4\equiv 1$ and there are real numbers $T_1>0$ and $T_2>0$ such that $f_i(u)=T_{3-i}u$ and $T_1T_2=\sigma^2$. In other words, X can be embedded into a Brownian sheet $\tilde{W}$ as follows:

$$X(s,t)=\tilde{W}(T_1+s,T_2+t),\qquad (s,t)\in\mathbb{R}^2. \tag{8.3}$$

PROOF. We know from Theorem 6.3 that the reversal of $\tilde{W}$ in both coordinates does satisfy an s.p.d.e. of the form (8.2). So we assume that $\hat{X}$ satisfies such an s.p.d.e. and show that X can be embedded into a Brownian sheet (the fact that a_4 must be identically equal to 1 follows immediately from Proposition 2.7).

Fix s, t such that $s + t = 2 - r$, and let $\Delta = [s - h, s] \times [t - h, t]$. According to (5.5),

$$E\big(W(\Delta) \mid \hat{\mathcal{F}}(r)\big) = a_1(s,t)h\big(X(s,t) - X(s-h,t)\big) \\ + a_2(s,t)h\big(X(s,t) - X(s,t-h)\big) + a_3(s,t)h^2X(s,t) \\ + \varepsilon(s,t;h),$$

or, equivalently, for $u + v \geq s + t$,

$$\begin{aligned} &E\big(\big[W(\Delta) - a_1h\big(X(s,t) - X(s-h,t)\big) \\ &\quad - a_2h\big(X(s,t) - X(s,t-h)\big) - a_3h^2X(s,t) - \varepsilon\big]X(u,v)\big) = 0. \end{aligned} \tag{8.4}$$

Because X solves (8.1), Lemma 2.3 implies that

$$X(s,t) = X_0 + M_s^1 + M_t^2 + W(s,t),$$

and therefore,

$$X(s,t) - X(s-h,t) = M_s^1 - M_{s-h}^1 + W(s,t) - W(s-h,t),$$

$$X(s,t) - X(s,t-h) = M_t^2 - M_{t-h}^2 + W(s,t) - W(s,t-h).$$

Write (8.4) for $u \leq s - h$ to get

$$\begin{aligned} &- a_2h\big(hf_2'(t) + o(h) + uh\big) - a_3h^2\big(\sigma^2 + f_1(u) + f_2(t) + ut\big) \\ &\quad - E\big(\varepsilon X(u,v)\big) = 0, \end{aligned} \tag{8.5}$$

for $u \geq s$ and $v \geq t$ to get

$$\begin{aligned} &h^2 - a_1h\big(hf_1'(s) + o(h) + ht\big) - a_2h\big(hf_2'(t) + o(h) + hs\big) \\ &\quad - a_3h^2\big(\sigma^2 + f_1(s) + f_2(t) + st\big) - E\big(\varepsilon X(u,v)\big) = 0, \end{aligned} \tag{8.6}$$

and for $v \leq t - h$ to get

$$\begin{aligned} &- a_1h\big(hf_1'(s) + o(h) + hv\big) - a_3h^2\big(\sigma^2 + f_1(s) + f_2(v) + sv\big) \\ &\quad + E\big(\varepsilon X(u,v)\big) = 0. \end{aligned} \tag{8.7}$$

Divide the three equations by h^2, let $h \downarrow 0$ and use the fact that $\operatorname{Var} \varepsilon(s,t;h) = o(h^4)$ to get the three equations

$$-a_2\big(f_2'(t) + u\big) - a_3\big(\sigma^2 + f_1(u) + f_2(t) + ut\big) = 0, \tag{8.8}$$

$$1 - a_1\big(f_2'(s) + t\big) - a_2\big(f_2'(t) + s\big) - a_3\big(\sigma^2 + f_1(s) + f_2(t) + st\big) = 0, \tag{8.9}$$

$$-a_1\big(f_1'(s) + v\big) - a_3\big(\sigma^2 + f_1(s) + f_2(v) + sv\big) = 0 \tag{8.10}$$

(the first equation is valid for $u \le s$, the third for $v \le t$). From (8.8), we get

$$f_1(u) = -f_2(t) - \frac{a_2}{a_3} f_2'(t) - \sigma^2 - u\left(t + \frac{a_2}{a_3}\right), \tag{8.11}$$

and from (8.10) we get

$$f_2(v) = -f_1(s) - \frac{a_1}{a_3} f_1'(s) - \sigma^2 - v\left(s + \frac{a_1}{a_3}\right). \tag{8.12}$$

Therefore, f_1 and f_2 are affine functions of u and v, respectively. Because $f_1(0) = f_2(0) = 0$, there are numbers $T_1 > 0$ and $T_2 > 0$ such that

$$f_1(u) = T_2 u, \qquad f_2(v) = T_1 v. \tag{8.13}$$

Identifying coefficients in (8.11) and (8.12) with those in (8.13), we see that

$$T_1 = -s - a_1/a_3, \qquad T_2 = -t - a_2/a_3,$$

and these expressions cannot depend on s and/or t. In addition,

$$-f_2(t) - \frac{a_2}{a_3} f_2'(t) - \sigma^2 = 0,$$

and from (8.13), the left-hand side is equal to

$$-T_1 t - \frac{a_2}{a_3} T_1 - \sigma^2 = 0.$$

Because $-t - a_2/a_3 = T_2$, we conclude that

$$T_1 T_2 = \sigma^2.$$

This completes the proof. □

REMARK 8.2. Theorem 8.1 implies the following fact regarding the reversal in time of the weak solution of the stochastic wave equation

$$\frac{\partial u}{\partial \tau^2}(\tau, x) - \frac{\partial u}{\partial x^2}(\tau, x) = \dot{W}(\tau, x), \qquad \tau > 0,\ x \in \mathbb{R},$$

with vanishing initial conditions. Indeed, one would like to know if the reversed process $\hat{u}(\tau, x) = u(1 - \tau, x)$ also satisfies an s.p.d.e., and if so, which one.

Set

$$X(s,t) = u\left(\frac{s+t}{\sqrt{2}}, \frac{t-s}{\sqrt{2}}\right).$$

It is not difficult to see, as in Theorem 3.2, for instance, that X is a solution of the s.p.d.e.

$$\frac{\partial^2 X}{\partial s \partial t} = \dot{\tilde{W}}, \qquad s + t > 0,$$

where $\dot{\tilde{W}}$ is again a white noise, and the initial conditions are zero along the line $s+t=0$. In the region where $(s,t)\in\mathbb{R}_+^2$, X therefore also solves the initial value problem

$$\frac{\partial^2 X}{\partial s\,\partial t}=\dot{\tilde{W}},\qquad X(s,0)=M_s^1,\ X(0,t)=M_t^2,$$

where $M_s^1 \stackrel{\text{def}}{=} X(s,0)$ [resp. $M_t^2 \stackrel{\text{def}}{=} X(0,t)$] is the Gaussian martingale such that $M_0^i\equiv 0$ and $E((M_x^i)^2)=x^2/2$.

By Theorem 8.1, the process $\hat{X}(s,t)=X(1-s,1-t)$ does not satisfy an s.p.d.e. of the form (8.2), and therefore, again by Theorem 3.2, $\hat{u}$ does not satisfy a second order hyperbolic s.p.d.e. of the form

$$\frac{\partial u}{\partial\tau^2}-\frac{\partial u}{\partial x^2}+a(\tau,x)\frac{\partial\hat{u}}{\partial\tau}+b(\tau,x)\frac{\partial\hat{u}}{\partial x}+c(\tau,x)\hat{u}(\tau,x)=f(\tau,x)\dot{B}(\tau,x)\tag{8.14}$$

were $\dot{B}$ is a white noise independent of $u(1,\cdot)$.

Of course, the coefficients in (8.14) are *local* [i.e., only depend on τ and x], so it still may be possible that $\hat{u}$ satisfies an s.p.d.e. in which the term $c(\tau,x)\hat{u}(\tau,x)$ in (8.14) is replaced by $C(\tau,x,\hat{u}(\tau,\cdot))$, where $C(\tau,x,v(\cdot))$ is a linear functional of $v(\cdot)$.

REMARK 8.3. Requesting that the reversal $\hat{X}$ of the weak solution to (8.1) satisfy a linear equation is natural, since $\hat{X}$ is Gaussian. On the other hand, it is the fact that the terms in (8.2) are local [i.e. only depend on $X(s,t)$ and its derivatives at (s,t)] that prevents $\hat{X}$ from satisfying such an equation unless X is a Brownian sheet. It is interesting to point out that even in the setting of d-dimensional diffusions, with d *decoupled* equations, most kinds of initial conditions will lead to *coupled* equations for the reversed process. The simplest example, suggested to the first author by E. Mayer-Wolf and O. Zeitouni, is the following. Let $B=(B^1,\dots,B^d)$ be a d-dimensional Brownian motion

$$dX_t^i=dB_t^i,\quad X_0^i=Y^i,\qquad i=1,\dots,d,$$

where $(Y^1,\dots,Y^d)$ is an $\mathbb{R}^d$-valued and centered Gaussian random variable, independent of B, with covariance matrix Ξ. Then the law of X_t is $N(0,\Xi+uI)$, where I is the $d\times d$ identity matrix. According to the d-dimensional version of (1.2), the system of diffusion equations for $\hat{X}_u=(X_{1-u}^1,\dots,X_{1-u}^d)$ is

$$d\hat{X}_u^i=d\hat{B}_u^i-\sum_{j=1}^d a_{i,j}(u)\hat{X}_u^j\,du,\tag{8.15}$$

where $(a_{i,j}(u))=(\Xi+(1-u)I)^{-1}$. Unless Ξ is diagonal (that is, $Y^1,\dots,Y^d$ are independent), the drift in (8.15) is "non-local," in that it depends on all components of $\hat{X}_u^j$.

This example, Theorem 8.1 and Remark 8.2 suggest that the only type of equation that the reversal of (8.1) may satisfy is an equation with *non-local* coefficients. This should motivate the development of an existence theory for such equations.

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