The Structure of a Brownian Bubble

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Abstract

In this paper, we examine local geometric properties of level sets of the Brownian sheet, and in particular, we identify the asymptotic distribution of the area of sets which correspond to certain tall excursions of the sheet. It is equal to the area of certain individual connected components of the random set $\{(s,t) : B(t) > b(s)\}$, where B is a standard Brownian motion and b is (essentially) a Bessel process of dimension 3. This limit distribution is studied and, in particular, explicit formulas are given for the probability that a point belongs to a specific connected component, and for the expected area of a component given the height of the excursion of $B(t) - b(s)$ in this component. These formulas are evaluated numerically and compared with the results from direct simulations of B and b.

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1 Introduction

This paper is motivated by the authors' previous study [1] of level sets and excursions of a standard Brownian sheet $\{W(s,t),\ s \geq 0,\ t \geq 0\}$. The sample paths of the Brownian sheet are functions of two variables, so when we speak of an excursion of W above the level α, we mean the restriction of W to a single connected component of the set $\{(s,t) : W(s,t) > \alpha\}$. We call these components *bubbles.*

The focus of [1] was on the distribution and size of clusters of bubbles of $\{W > \alpha\}$, $\alpha > 0$, in a neighborhood of the point (S, t_0), where $t_0 > 0$ is fixed and $S = \inf\{s \geq 0 : W(s, t_0) = \alpha\}$. For simplicity, let us take $t_0 = \alpha = 1$. In particular, bounds on the height and width of bubbles that intersected certain curves were given [1, Theorems 3.9 and 3.11]. Here, we want to look at some properties of a single bubble. In particular, we will examine its area.

The key to the results of [1] was the following local decomposition of the Brownian sheet in the neighborhood of $(S, 1)$:

$$W(S-u, 1+v/S) = 1 + B(v) - b(u) - x(u, v/S), \tag{1}$$

where $B = \{B(v),\ v \geq 0\}$ is a standard Brownian motion started at the origin, $b = \{b(u),\ u \geq 0\}$ is a Bessel process of dimension 3 (or Bessel(3) process, for short) also started at the origin and independent of B, and, conditioned on $S = s$, $x = \{x(u,v),\ s \geq u \geq 0,\ v \geq 0\}$ is a standard Brownian sheet (see [1, (15) and Lemma 2.4]). In the neighborhood of $(S, 1)$, x is of smaller magnitude than B or b. Indeed, $B(v)$ and b(u) are on the order of $\sqrt{v}$ and $\sqrt{u}$ respectively, while $x(u,v)$ is on the order of $\sqrt{uv}$.

In view of (1), it is natural to expect that for small u and v, the set $\{(u,v) : W(S-u, 1+v) > 1\}$ would be well-approximated by the set

$$\{(u,v) : B(v) - b(u) > 0\}, \tag{2}$$

and that in particular, the bubbles themselves should be well-approximated by the connected components of the latter. Using the structure of the components of the set (2), we show in Section 5 that the area of bubbles corresponding to excursions of W high above level 1 does indeed converge in distribution as the bubbles approach $(S, 1)$ to the area of a component of the set (2).

It is then natural to study the limiting process

$$\{B(v) - b(u),\ u \geq 0,\ v \geq 0\} \tag{3}$$

and in particular the area of its excursions. This is analogous in spirit to studying an excursion of Brownian motion. While the excursion theory of the standard Brownian motion is well developed, we are not aware of any studies of these components for non-trivial planar processes, and processes of the form (3) are of particular interest, since such processes provide in fact local approximations to solutions of hyperbolic stochastic partial differential equations (s.p.d.e.'s).

Indeed, a typical hyperbolic partial differential equation in $I\!R^2$ can be transformed by a change of variables to the case where the second-order part of the differential operator is $\partial^2/\partial s \partial t$. Therefore the basic hyperbolic s.p.d.e. to consider is

$$\frac{\partial^2}{\partial s \partial t}\xi(s,t) = f(s,t,\xi(s,t))\,Y(ds,dt) + g(s,t,\xi(s,t)), \tag{4}$$

where ξ is given on the coordinate axes, Y is a non-atomic random measure, and f and g are smooth functions. Let $T = (U,V)$ be a (deterministic or random) point in $I\!R^2_+$. How does the solution ξ of (4) behave in the neighborhood of T, say in the region $D = \{(s,t) : s \geq U,\ t \geq V\}$. Of course, this depends on T, but in general ξ will satisfy an equation similar to (4). If T is random, the right-hand side of (4) may change slightly due to conditioning, so that in general, ξ will coincide in D with the solution ζ of the equation

$$\begin{gathered} \frac{\partial^2}{\partial s \partial t}\zeta(s,t) = X(ds,dt) \qquad \text{in } D, \\ \zeta(s,V) = \xi(s,V), \quad \zeta(U,t) = \xi(U,t), \qquad s \geq U,\ t \geq V, \end{gathered} \tag{5}$$

where X is some random measure. Now the solution to (5) can be decomposed into two parts, one of which accounts for the boundary condition and the other for the driving noise, that is, $\zeta = \zeta_1 + \zeta_2$, where ζ_1 satisfies

$$\begin{gathered} \frac{\partial^2}{\partial s \partial t}\zeta_1(s,t) = 0 \qquad \text{in } D, \\ \zeta_1(s,V) = \xi(s,V), \quad \zeta_1(U,t) = \xi(U,t), \qquad s \geq U,\ t \geq V, \end{gathered} \tag{6}$$

and ζ_2 satisfies

$$\begin{gathered} \frac{\partial^2}{\partial s \partial t}\zeta_2(s,t) = X(ds,dt) \qquad \text{in } D, \\ \zeta_2(s,V) = 0, \quad \zeta_2(U,t) = 0, \qquad s \geq U,\ t \geq V. \end{gathered} \tag{7}$$

The point of this is that typically, the dominant term in the neighborhood of T is ζ_1, which is the solution to a homogeneous wave equation with random boundary condition, and it is well-known that the solution to (6) is given by

$$\zeta_1(s,t) = \xi(s,V) + \xi(U,s),$$

which is of the same form as (3). For hyperbolic equations in which the differential operator does not have the reduced form of equation (4), the same type of local decomposition involving the sum of two single-parameter processes is possible.

In order to understand the structure of components of the set (3), and thereby describe the approximate local structure of excursions of solutions to hyperbolic s.p.d.e.'s, it is necessary to have a convenient way of recognizing whether two distinct points belong to the same component of this set (obviously, plotting the set is one way, but it is not helpful for calculations!). This is achieved by Algorithm A of Section 2, which can be applied to any pair of processes X_1 and X_2.

In Section 3, we then focus on the special case of importance for the Brownian sheet, namely when X_1 is a Brownian motion B and X_2 is a Bessel(3) process b, and we address the following question: what is the probability that a point in $\mathbb{R}^2_+$ belongs to a particular connected component of $\{B > b\} = \{(s,t) : B(t) > b(s)\}$? We establish an explicit formula for this probability. Then a simple integration over the point gives us the expected area of the component. We actually compute the conditional expected area, given (essentially) the height of the excursion. The formula is of the following type:

$$E\{ \text{ area of bubble } \mid \text{ height } \} = \sum_{n \in \mathbb{N}} p_n,$$

where p_n is an integral over a particular simplex in $7+6(n-1)$-dimensional Euclidean space, of functions f and g which are derived from probabilities concerning Bessel(3) processes. The computation of p_n is achieved by using the Markov property of a particular four-parameter diffusion. The same methods would apply if B and b were more general diffusions, but we have no need for this added generality.

In Section 4, we expand the functions f and g in series whose terms are explicitly derived from the standard Gaussian density. Since these series converge rapidly, they can be evaluated numerically, as can the p_n themselves. We have carried out this evaluation for p_1 and p_2. Since the p_n converge to 0 exponentially, this is in fact a reasonable approximation to the expected area of the whole bubble. These calculations have been checked by a direct simulation of the processes b and B. These results are reported in Remark 4.4.

As mentioned earlier, in Section 5 we use our results on the structure of components of the set (2) to prove that the area of bubbles which correspond to excursions of W high above level 1 near $(S, 1)$ has approximately the same distribution as the area of a component of the set (2).

2 Structure of clusters

In this section, we are going to study in detail the structure of components of the set $\{B > b\} = \{(s,t) : B(t) > b(s)\}$. The particular distribution of these processes will not play any role until the next section, and in fact, Proposition 2.2 applies to any pair of continuous single-parameter processes, and Proposition 2.4 applies to any pair of diffusions. We use the notation B and b here so that specific notation can be set up for later use.

The components of $\{B > b\}$ are not isolated; rather, they occur in clusters. In each region where such a cluster occurs, we can assume that b is making an excursion below some level $H > 0$. At the same time, B is necessarily positive, so it must be making an excursion above some level $L \geq 0$. We are going to assume that $L > 0$ and that $B = L$ at both extremities of its excursion, since the case of excursion intervals from the origin can in fact be derived from this (see Remark 3.2).

Fix $0 < L < H$. Suppose that $I_H \equiv]a_1, a_2[$ is an excursion interval of b below the level H, and $J_L \equiv]c_1, c_2[$ is an excursion interval of B above L. Let $R' = I_H \times J_L$, and let

$$\underline{M} = \inf_{a_1 \leq u \leq a_2} b(u) , \qquad \overline{M} = \sup_{c_1 \leq v \leq c_2} B(v) .$$

Let $\underline{S}$ and $\overline{T}$ be the (unique) times in $]a_1, a_2[$ and $]c_1, c_2[$ respectively such that

$$b(\underline{S}) = \underline{M}, \quad B(\overline{T}) = \overline{M}.$$

Fix two real numbers $\underline{m}$ and $\overline{m}$ such that $L < \underline{m} < \overline{m} < H$. We condition on the event $\{\underline{M} = \underline{m},\ \overline{M} = \overline{m}\}$, and set

$$\begin{array}{rclcrcl}
\sigma &=& \sup\{s \in (a_1, \underline{S}) : b(s) = \overline{m}\}, && \tau &=& \sup\{t \in (c_1, \overline{T}) : B(t) = \underline{m}\}, \\
\sigma' &=& \inf\{s \in (\underline{S}, a_2) : b(s) = \overline{m}\}, && \tau' &=& \inf\{t \in (\overline{T}, c_2) : B(t) = \underline{m}\}, \\
I_0 &=&]\sigma, \sigma'[\,, && J_0 &=&]\tau, \tau'[\,.
\end{array}$$

Let $R_0 = I_0 \times J_0$. There will be many components of $\{W > 1\}$—and of $\{B > b\}$—in R', and also in R_0. However, there is a *distinguished component* $\mathcal{C}_0$ of $\{B > b\} \cap R'$, namely the one which contains $(\underline{S}, \overline{T})$, the point of R' at which $B(t) - b(s)$ is maximal (see Figure 1). We call this distinguished component a *Brownian bubble* and will study it in some detail. In particular, we are interested in the expected area of $\mathcal{C}_0$: how much smaller is it than that of R_0, or than that of $\{B > b\} \cap R_0$? A standard application of Fubini's theorem shows that this area will be easy to compute once we have the probability of the event $F = \{(s,t) \in \mathcal{C}_0\}$. However, in order to compute $P(F)$, it is necessary to have a *convenient* way to recognize whether or not a given $\omega \in \Omega$ belongs to F. (Plotting the region $\mathcal{C}_0$ is one way, but it is not very convenient!). A more useful method is supplied by Algorithm A below.

Notice that the pair of intervals $I = I_0$ and $J = J_0$ have the following property.

Figure 1: A Brownian bubble surrounded by smaller components.

(P) *The interval I is an excursion interval of b below the maximum value $\overline{m}$ of B on J, J is an excursion interval of B above the minimum value $\underline{m}$ of b on I, and $\overline{m} - \underline{m} > 0$.*

Remark 2.1 Observe that whenever two intervals I and J satisfy (P), any component of $\{B > b\}$ that meets $I \times J$ is necessarily entirely contained in $I \times J$, for $B \leq b$ on the boundary. In particular, $\mathcal{C}_0 \subset R_0$.

Evidently, a point (s,t) belongs to $\mathcal{C}_0$ if and only if there exists a continuous curve from (s,t) to $(\underline{S}, \overline{T})$ contained in $\{B > b\}$. *A priori,* there is no restriction on the nature of this curve. Let us call $(I_0 \times \{\overline{T}\}) \cup (\{\underline{S}\} \times J_0)$ the *axes* of $\mathcal{C}_0$. They are contained in $\mathcal{C}_0$ and they divide the rectangle R, and hence $\mathcal{C}_0$, into four quadrants. Any curve from (s,t) which reaches the axes of $\mathcal{C}_0$ can reach $(\underline{S}, \overline{T})$ in one more step, so we can restrict our attention to paths which do not cross the axes, and therefore remain in one quadrant.

Given points $(s_1,t_1), \ldots, (s_n,t_n)$, let $\langle (s_1,t_1), \ldots, (s_n,t_n) \rangle$ be the polygonal curve which connects successive points. If the segments of this curve are alternately vertical and horizontal, and if it is non-self-intersecting, we will call it a *stepped path.* The *curvature number* of a stepped path is the number of right-angles in it; e.g. the curvature number of the stepped path $\langle (s_1,t_1), \ldots, (s_n,t_n) \rangle$ is $n-2$.

We are going to describe an algorithm which, given $(s,t) \in R$, determines whether or not (s,t) belongs to $\mathcal{C}_0$. When it does, the algorithm constructs a stepped path $\Gamma^* \subset \{B > b\}$ of (nearly) minimal curvature number which connects (s,t) and $(\underline{S}, \overline{T})$.

This will lead to a formula for the probability that these two points belong to the same connected component of $\{B > b\}$ (which in turn will give us an estimate for the probability that they are in the same component of $\{W > 1\}$).

The algorithm is most easily described in terms of the process $Y(s,t) = B(t) - b(s)$. This process takes on it maximum in $\mathcal{C}_0$ at $(\underline{S}, \overline{T})$. Fix (s,t) and suppose that $Y(s,t) > 0$. The algorithm is as follows: starting at (s,t), look at the horizontal segment through (s,t) which is contained in $\mathcal{C}_0$, and go to the point $(\underline{S}_1, t)$ on this segment where Y achieves its maximum on this segment. Then look at the vertical segment through $(\underline{S}_1, t)$ contained in $\mathcal{C}_0$ and move to the point $(\underline{S}_1, \overline{T}_1)$ where Y achieves its maximum on this vertical segment. Repeat these steps, looking alternatively at horizontal and vertical segments until either $(\underline{S}, \overline{T})$ is reached, or until you fail to find a new maximum value. In the first case, it is clear that (s,t) belongs to $\mathcal{C}_0$, and we are going to show that in the second case, it does not.

To do this, it is necessary to restate the procedure in terms of the processes b and B. This is done in the algorithm below, which outputs YES if the two points are in the same component of $\{B > b\}$ and NO otherwise.

Algorithm A. *Let* $(s,t) \in R_0$.

Stage 0. *If* $b(s) \geq B(t)$, *output NO and stop. Otherwise set* $\overline{M}_0 = B(t)$, $\underline{S}_0 = s$, $\overline{T}_0 = t$, $\Gamma_0 = \{(s,t)\}$ *and go to stage 1.*

We define the stages by induction, starting with $n = 1$.

Stage $2n-1$. *Let* I_n *be the (open) excursion interval of* b *below* $\overline{M}_{n-1}$ *which contains* $\underline{S}_{n-1}$. *Define* $\underline{M}_n$ *and* $\underline{S}_n$ *by*

$$\underline{M}_n = b(\underline{S}_n) = \min_{u \in I_n} b(u)\,,$$

and set $\Gamma_{2n-1} = \Gamma_{2n-2} \cup \langle (\underline{S}_{n-1}, \overline{T}_{n-1}), (\underline{S}_n, \overline{T}_{n-1}) \rangle$.

(a) *If* $\underline{S}_n = \underline{S}$, *then let* $\Gamma^* = \Gamma_{2n-1} \cup \langle (\underline{S}, \overline{T}_{n-1}), (\underline{S}, \overline{T}) \rangle$, *output YES, and stop.*
(b) *If* $\underline{S}_n = \underline{S}_{n-1}$, *output NO and stop.*
(c) *Otherwise, go to stage* $2n$.

Stage $2n$. *Let* J_n *be the open excursion interval of* B *above* $\underline{M}_n$ *which contains* $\overline{T}_{n-1}$. *Define* $\overline{M}_n$ *and* $\overline{T}_n$ *by*

$$\overline{M}_n = B(\overline{T}_n) = \max_{v \in J_n} B(v)\,,$$

and set $\Gamma_{2n} = \Gamma_{2n-1} \cup \langle (\underline{S}_n, \overline{T}_{n-1}), (\underline{S}_n, \overline{T}_n) \rangle$.

(a) *If* $\overline{T}_n = \overline{T}$, *then let* $\Gamma^* = \Gamma_{2n} \cup \langle (\underline{S}_n, \overline{T}), (\underline{S}, \overline{T}) \rangle$, *output YES, and stop.*
(b) *If* $\overline{T}_n = \overline{T}_{n-1}$, *then output NO and stop.*
(c) *Otherwise, go to stage* $2n+1$.

Proposition 2.2 *Consider* $0 < L < H$, R', R_0, $\underline{S}$ *and* $\overline{T}$ *as above. Assume* $\underline{M} < \overline{M}$ *and let* $(s,t) \in R_0$. *Then Algorithm A outputs YES if* (s,t) *and* $(\underline{S}, \overline{T})$ *belong to the same connected component of* $\{B > b\}$, *and outputs NO otherwise. In particular, with probability one, it terminates in a finite number of stages. When it outputs YES, the number of stages is equal to the curvature number of the stepped path* Γ^*. *This curvature number is at most one more than the minimal curvature number among all stepped paths in* $\{B > b\}$ *from* (s,t) *to* $(\underline{S}, \overline{T})$.

Proof. Let us first check that $\Gamma_n \subset \{B > b\}$ for all n. At stage zero, either the algorithm terminates immediately or $\Gamma_0 = \{(s,t)\} \subset \{B > b\}$.

Supppose by induction that we have constructed sequences $\underline{M}_k$, $\overline{M}_k$, $\underline{S}_k$, $\overline{T}_k$, I_k, and J_k, $k = 0, \dots, n-1$ and a stepped path Γ_{2n-2} with endpoints (s,t) and $(\underline{S}_{n-1}, \overline{T}_{n-1})$ such that

$$\begin{gathered} \underline{M} < \underline{M}_{n-1} < \underline{M}_{n-2} < \cdots < \underline{M}_1 < \underline{M}_0 < \overline{M}_0 < \cdots < \overline{M}_{n-2} < \overline{M}_{n-1} < \overline{M}\,, \\ I_1 \subset I_2 \subset \cdots \subset I_{n-1} \subset I_0 \quad \text{and} \quad J_1 \subset J_2 \subset \cdots \subset J_{n-1} \subset J_0, \\ \Gamma_{2n-2} \subset \{B > b\}\,. \end{gathered} \tag{8}$$

where

$$\begin{cases} I_k \text{ is an excursion interval of } b \text{ below } \overline{M}_{k-1}\,, \\ J_k \text{ is an excursion interval of } B \text{ above } \underline{M}_k\,, \\ \underline{M}_k = b(\underline{S}_k) = \min_{I_k} b \quad \text{and} \quad \overline{M}_k = B(\overline{T}_k) = \max_{J_k} B\,. \end{cases} \tag{9}$$

At stage $2n-1 \geq 1$, I_n is the excursion interval of b below $\overline{M}_{n-1}$, so $\overline{M}_{n-2} < \overline{M}_{n-1} < \overline{M}$ implies $I_{n-1} \subset I_n \subset I_0$. It follows that $\underline{M} \leq \underline{M}_n \leq \underline{M}_{n-1}$. If u is between $\underline{S}_{n-1}$ and $\underline{S}_n$, then $u \in I_n$ so $b(u) < \overline{M}_{n-1}$ and $B(\overline{T}_{n-1}) = \overline{M}_{n-1} > b(u)$. Thus $\langle(\underline{S}_{n-1}, \overline{T}_{n-1}), (\underline{S}_n, \overline{T}_{n-1})\rangle \subset \{B > b\}$, which implies that $\Gamma_{2n-1} \subset \{B > b\}$. A similar argument shows that $\Gamma_{2n} \subset \{B > b\}$.

Next we check that the algorithm terminates correctly. It returns NO in three cases. The first is trivial: if $b(s) \geq B(t)$ at stage zero, then $(s,t) \notin \{B > b\}$. The second case is at stage $2n-1$, if $\underline{S}_n = \underline{S}_{n-1}$ while $\underline{S}_n \neq \underline{S}$. Since $\underline{S}$ and $\underline{S}_n$ are the minimum points of their respective excursions, evidently $\underline{S} \notin I_n$. Moreover, $\underline{M}_n = \underline{M}_{n-1}$, so, if we were to go on to the next stage, we would find that $J_n = J_{n-1}$—they are not disjoint, and they are both excursion intervals of B above the same level $\underline{M}_{n-1}$—and hence that $\overline{T}_n = \overline{T}_{n-1}$. Evidently $\overline{T} \notin J_n$. But $(s,t) \in I_n \times J_n$ and if $\overline{T}_n = \overline{T}_{n-1}$ and $\underline{S}_n = \underline{S}_{n-1}$, then I_n and J_n satisfy (P). Thus $(s,t) \in \mathcal{C}_0$ implies $\mathcal{C}_0 \subset I_n \times J_n$, a contradiction. It follows that (s,t) and $(\underline{S}, \overline{T})$ are in different components. The third case occurs at stage $2n$, when $\overline{T}_n = \overline{T}_{n-1}$. This is the same as the previous case—just interchange s and t.

Observe that having reached stage $2n-1$, if the algorithm does go on to stage $2n$, then $\underline{S} \neq \underline{S}_n \neq \underline{S}_{n-1}$, so $\underline{M} < \underline{M}_n < \underline{M}_{n-1}$. Similarly, having reached stage $2n$, if the algorithm does go on to stage $2n+1$, then $\overline{T} \neq \overline{T}_n \neq \overline{T}_{n-1}$, so $\overline{M}_{n-1} < \overline{M}_n < \overline{M}$.

The algorithm returns YES in two cases: at stage $2n-1$ when $\underline{S}_n = \underline{S}$ and at stage $2n$ when $\overline{T}_n = \overline{T}$. We will check the first; the second is similar. At stage $2n-1$, $\Gamma_{2n-1} \in \{B > b\}$ and Γ_{2n-1} ends at $(\underline{S}, \overline{T}_{n-1})$, which is on the axes of $\mathcal{C}_0$. These are contained in $\mathcal{C}_0$, so we need only add on a segment of the axes to Γ_{n-1} to reach $(\underline{S}, \overline{T})$. Thus $(s,t) \in \mathcal{C}_0$.

We now check that the algorithm terminates with probability one. If it has not terminated by stage $2n-2$, we have constructed sequences $\underline{M}_k$, $\overline{M}_k$, I_k and J_k as in (8)-(9). The intervals

$$I_k \equiv]U_k, U'_k[\,, \qquad J_k \equiv]V_k, V'_k[\tag{10}$$

increase strictly with k. By construction, $b(U_n) = \overline{M}_{n-1} = b(U'_n)$ and $B(V_n) = \underline{M}_n = B(V'_n)$. Now if $\underline{M}_n < \underline{M}_{n-1}$, then either $U_n < \underline{S}_n < U_{n-1}$ or $U'_{n-1} < \underline{S}_n < U'_n$. Assume the former, since the latter case is similar. Then on $[U_n, U_{n-1}]$, b goes from $\overline{M}_{n-1}$ down to $\underline{M}_n$ and back up to $\overline{M}_{n-2}$. To do so, it has to cross the non-empty interval $]\underline{M}_1, \overline{M}_0[$. In particular, the number of upcrossings of $]\underline{M}_1, \overline{M}_0[$ by b during I_n is at least one larger than the number of upcrossings of this interval during I_{n-1}.

The same is true for B when $\overline{M}_n > \overline{M}_{n-1}$. At each stage, one of the two, $\underline{M}_k$ or $\overline{M}_k$, must change, so that either b or B has to have at least $n/2$ upcrossings of $]\underline{M}_1, \overline{M}_0[$ in the respective intervals I_0, J_0. Since both are continuous functions, the number of upcrossings of a non-empty interval must be finite, so the algorithm must terminate. This proves that the algorithm will decide correctly whether or not $(s,t) \in \mathcal{C}_0$.

It remains to show that the curvature number of Γ^* is (nearly) minimal. Assume that $(s,t) \in \mathcal{C}_0$—so Γ^* does in fact exist—and let $\Gamma \subset \{B > b\}$ be any stepped path from (s,t) to $(\underline{S}, \overline{T})$. Let $R_1, R_2, R_3, \dots$ be the sequence of rectangles $I_1 \times \{t\}, I_1 \times J_1, I_2 \times J_1, I_2 \times J_2, \dots, I_n \times J_{n-1}, I_n \times J_n, \dots$. Notice that $(s,t) \in R_1 \subset R_2 \subset \dots$. Let N be the largest n for which $(\underline{S}, \overline{T})$ is *not* in the closure $\bar{R}_n$ of R_n. Then Γ must cross the boundaries of $R_1, \dots, R_N$ (since R_1 is a segment, $R_1 = \partial R_1$; Γ will not exit R_1 through an extremity).

By construction, $\underline{M}_n \le b \le \overline{M}_{n-1}$ on I_n, and $b = \overline{M}_{n-1}$ at the endpoints. Similarly, $\underline{M}_n \le B \le \overline{M}_n$ on J_n and $B = \underline{M}_n$ at the endpoints. Consequently, if k is even, so that R_k is of the form $I_n \times J_n$, then $B = \underline{M}_n \le b$ on the horizontal portion $\bar{I}_n \times \partial J_n$ of the boundary; thus this is in $\{B > b\}^c$, and Γ cannot intersect it. Now Γ is a stepped path, made up of vertical and horizontal segments. If it must cross the boundary of R_k, k even, it must do so through one of the sides; this can only happen during a horizontal segment of Γ. Similarly, for odd k, where R_k is of the form $I_{n+1} \times J_n$, the vertical portion of the boundary is in $\{B > b\}^c$, since $B \le \overline{M}_n = b$ there. This time the path Γ cannot cross the sides, but must cross the top or bottom, which it must do during a vertical segment. Thus Γ must exit R_1 vertically, cross ∂R_2 horizontally, ∂R_3 vertically, ∂R_4 horizontally, and so on. The horizontal segments H_n and H_m crossing the boundary of R_{2n} and R_{2m} are distinct when $n < m$, since H_n cannot cross the vertical portions of the boundary of R_{2n+1}. Similarly, the vertical segments V_n and

V_m crossing the boundary of R_{2n+1} and R_{2m+1} are distinct when $n < m$. Since Γ is a stepped path, it must make at least one right angle between each of these crossings. There are N crossings in addition to the vertical segment which leaves R_1, so that the curvature number of Γ must be at least N.

In fact, let Γ^o be any path with minimal curvature number N^o. If the segment of Γ^o containing (s,t) is horizontal, then there is at least 1 right-angle to exit R_1, one additional right-angle to exit R_2, ..., so $N^o \geq N$. If the segment of Γ^o containing (s,t) is vertical, then no right angle is necessary to exit R_1, there is at least one right-angle to exit R_2, one additional right-angle to exit R_3, ..., so $N^o \geq N-1$.

When the algorithm terminates at stage K, the curvature number of Γ^* is also K, since each stage except the first adds one right-angle to Γ^*, and there is one additional righ-angle added just before the algorithm outputs YES. If $K = 2n-1$, then $\underline{S}_n = \underline{S}$ or, equivalently, $\underline{S} \in I_n$ and if $K = 2n$ then $\overline{T} \in J_n$. Thus, when $K = 2n-1$, $\underline{S} \in I_n$ but $\overline{T} \notin J_{n-1}$, hence $(s,t) \notin I_n \times J_{n-1} = R_{2n-1}$. It follows that $N = 2n-1 = K$ in this case. It is easy to see that if $K = 2n$, we also have $N = K$. In particular, if the segment of Γ^o containing (s,t) is horizontal, then $N^o \geq N = K$ by the above, so the algorithm has indeed constructed a minimal path. Otherwise, it might be the case that $N^o = N-1 = K-1$, so the curvature number of Γ^* could be one more than minimal. ♣

Remark 2.3 *(a)* There is one arbitrary aspect to Algorithm A, namely the way it starts: the first segment of Γ^* is always horizontal. It could just as easily have been vertical. If (s,t) belongs to $\mathcal{C}_0$ but Algorithm A does not construct a path with minimal curvature number, then the vertical counterpart to it will. This follows from the last lines of the proof above.

(b) All components of $\{B > b\}$ are Brownian bubbles, i.e. are distinguished components for some choice of rationals $H > L$ and excursion intervals I_H and J_L. To see this, assume $B(t) > b(s)$, and let $\mathcal{C}_0$ be the component of $\{B > b\}$ which contains (s,t). This component is bounded. Indeed, the excursion interval J' of B above 0 which contains t is finite, so the maximum H' of B on this interval is also finite. Now the first hit ρ of H' by b is finite, and clearly, $\mathcal{C}_0 \subset [0,\rho] \times J'$. Let $(\underline{S}, \overline{T})$ be the point in $\mathcal{C}_0$ where Y achieves its maximum, let I_0 be the excursion interval of b below $\overline{M} = B(\overline{T})$ which contains $\underline{S}$, and let J_0 be the excursion interval of B above $\underline{M} = b(\underline{S})$ which contains $\overline{T}$.

It is not difficult to see that the intervals I_0, J_0 and levels $\underline{M}$, $\overline{M}$ satisfy (P). Choosing rationals H slightly larger than $\overline{M}$ and L slightly smaller than $\underline{M}$, and considering the excursion intervals $I_H \supset I_0$ and $J_L \supset J_0$, it is easy to see that $\mathcal{C}_0$ is the distinguished component in $R' = I_H \times J_L$.

(c) All components of $\{B > b\}$ are obtained by enumerating rationals $0 < L < H$,

excursions I_L of B above L, and excursions I_H of b below H, such that

$$L < \underline{M} \equiv \min_{I_H} b < \overline{M} \equiv \max_{I_L} B < H.$$

If we move along a stepped path starting from $(\underline{S}, \overline{T})$, our successive directions of motion can be described by a sequence $(i, D_1, D_2, \ldots)$, where $i \in \{1, 2, 3, 4\}$ indicates the initial direction (1 for right, 2 for up, ...), and the $D_k \in \{\text{right, left}\}$ indicate the direction of each turn. The following theorem indicates just how complex the component $\mathcal{C}_0$ may be.

Theorem 2.4 *For any finite sequence of rights and lefts, there is a point (s, t) in $\mathcal{C}_0$ for which the stepped path from $(\underline{S}, \overline{T})$ to (s, t) with minimal curvature number has exactly that sequence of right and left turns.*

PROOF. Since this theorem is not used in the rest of the paper, we will not give a complete proof. Rather, we indicate how to construct a point (s, t) for which the minimal path Γ from $(\underline{S}, \overline{T})$ to this point is described by the sequence (1, left, left, left, right). It will be clear from the continuity and nowhere-differentiability of the paths of b and B that a similar construction is possible for any $n \in I\!\!N$ and finite sequence $(i, D_1, D_2, \ldots, D_n)$. For $s \geq \underline{S}$ and $t \geq \overline{T}$, set

$$b^*(s) = \max_{\underline{S} \leq u \leq t} b(u), \qquad B_*(t) = \min_{\overline{T} \leq v \leq t} B(v).$$

Let $I_2' \subset I_0$ be an open interval to the right of $\underline{S}$ where b accomplishes an excursion below b^*, and let $\underline{M}_2$, $\overline{K}_2$ and $\underline{S}_2$ be such that

$$\underline{M}_2 = b(\underline{S}_2) = \min_{I_2'} b < \max_{I_2'} b = \overline{K}_2,$$

and let J_2 be the excursion interval of B above $\underline{M}_2$ which contains $\overline{T}$. The first segment of Γ will be $[\underline{S}, \underline{S}_2] \times \{\overline{T}\}$. (We are starting the construction of Γ from $(\underline{S}, \overline{T})$, whereas the algorithm starts from the opposite endpoint (s, t). This is why we define $\underline{M}_2$ and $\underline{S}_2$ before $\underline{M}_1$ and $\underline{S}_1$.)

Since the first turn of Γ is to the left, we let t increase from $\overline{T}$ until the first time τ_1 that B hits level $(\overline{K}_2 + \underline{M}_2)/2$. This level is in $]\underline{M}, \overline{M}[$, so $\tau_1 \in J_0$. The paths of B are continuous and nowhere differentiable, so there is $\varepsilon_1 > 0$ such that $\underline{M}_2 < B(v) < \overline{K}_2$ for $v \in [\tau_1, \tau_1 + \varepsilon_1]$, and there is an open interval $J_2' \subset]\tau_1, \tau_1 + \varepsilon_1[$ on which B is accomplishing an excursion above B_*. Let $\underline{L}_2$, $\overline{M}_2$, $\overline{T}_2$ be such that

$$\overline{M}_2 = B(\overline{T}_2) = \max_{J_2'} B > \min_{J_2'} B = \underline{L}_2.$$

The second segment of Γ will be $\{\underline{S}_2\} \times [\overline{T}, \overline{T}_2]$. We now let $I_2 \subset I_0$ be the excursion interval of b below $\overline{M}_2$ which contains $\underline{S}_2$.

Since the second turn of Γ is to the left, we let s decrease from $\underline{S}_2$ until the first time σ_1 that b hits $(\overline{M}_2 + \underline{L}_2)/2$. Since this level is in $]\underline{M}_2, \overline{M}_2[$, σ_1 belongs to I_2. Now there is $\varepsilon_2 > 0$ such that $[\sigma_1 - \varepsilon_2, \sigma_1] \subset I_2$ and $\underline{L}_2 < b(u) < \overline{M}_2$ for $u \in [\sigma_1 - \varepsilon_2, \sigma_1]$, and there is an open interval $I_1' \subset]\sigma_1 - \varepsilon_2, \sigma_1[$ on which b is accomplishing an excursion below b^*. Let $\underline{M}_1$, $\overline{K}_1$ and $\underline{S}_1$ be such that

$$\underline{M}_1 = b(\underline{S}_1) = \min_{I_1'} b < \max_{I_1'} b = \overline{K}_1,$$

and let J_1 be the excursion interval of B above $\underline{M}_1$ which contains $\overline{T}_2$. The third segment of Γ will be $[\underline{S}_1, \underline{S}_2] \times \{\overline{T}_2\}$.

Since the third turn of Γ is to the left, we let t decrease from $\overline{T}_2$ until the first time τ_2 that B hits level $(\overline{K}_1 + \underline{M}_1)/2$. Since this level is in $]\underline{M}_1, \overline{M}_2[$, τ_2 belongs to J_2. Now there is $\varepsilon_3 > 0$ such that $[\tau_2 - \varepsilon_3, \tau_2] \subset J_2$ and $\underline{M}_1 < B(v) < \overline{K}_1$ for $v \in [\tau_2 - \varepsilon_3, \tau_2]$, and there is an open interval $J_1' \subset]\tau_2 - \varepsilon_3, \tau_2[$ on which B is accomplishing an excursion above B_*. Let $\underline{L}_1$, $\overline{M}_1$, $\overline{T}_1$ be such that

$$\overline{M}_1 = B(\overline{T}_1) = \max_{J_1'} B > \min_{J_1'} B = \underline{L}_1.$$

The fourth segment of Γ will be $\{\underline{S}_1\} \times [\overline{T}_1, \overline{T}_2]$. Let I_1 be the excursion interval of b below $\overline{M}_1$ which contains $\underline{S}_1$.

Since the fourth turn is to the right, we let s decrease from $\underline{S}_1$ until the first time σ_0 that b hits level $(\overline{M}_1 + \underline{L}_1)/2$. Since this level is in $]\underline{M}_1, \overline{M}_1[$, $\sigma_0 \in I_1$. Set $s = \sigma_0$, $t = \overline{T}_1$, and let $[s, \underline{S}_1] \times \{\overline{T}_1\}$ be the fifth and last segment of Γ.

Starting from $(\underline{S}, \overline{T})$, the successive turns of Γ are (1, left, left, left, right), and we claim that Algorithm A applied to (s, t) constructs the path Γ. Indeed, the points $\underline{S}_1$, $\underline{S}_2$, $\overline{T}_1$, $\overline{T}_2$, and the intervals I_1, I_2, J_1, and J_2 are exactly those constructed by the algorithm. Moreover, the path Γ is a minimal path from (s, t) to $(\underline{S}, \overline{T})$. Details are left to the reader. ♣

3 The expected area of a Brownian bubble

In this section and in the remainder of the paper, we make explicit use of the fact that B is a Brownian motion and b is a Bessel(3) process independent of B, though the reader will notice that the methods and proof of Proposition 3.6 apply to all pairs of independent time-homogeneous diffusions.

Let $\mathcal{C}_0$ be a connected component of $\{B > b\}$. Since this component is bounded (see Remark 2.3 (b)), there is a point $(\underline{S}, \overline{T}) \in \mathcal{C}_0$ at which $B(t) - b(s)$ is maximal in $\mathcal{C}_0$, equal to M, say. Given M, we are interested in probabilistic properties of $\mathcal{C}_0$; in particular, we are going to exhibit a formula for its expected area. Before doing this, let us examine the excursions of b and B which give rise to $\mathcal{C}_0$.

By Remark 2.3 (b), there are rationals $0 < L < H$ and a pair of excursions of b and B from H and L, respectively, for which $\mathcal{C}_0$ is the distinguished component. In particular, $\underline{S}$ (resp. $\overline{T}$) is the minimum (resp. maximum) point of an excursion of b (resp. B) below $\overline{M} = B(\overline{T})$ (resp. above $\underline{M} = b(\underline{S})$). Clearly, $M = \overline{M} - \underline{M}$.

Since b and B are independent, b is independent of $\overline{M}$ and B is independent of $\underline{M}$. It follows that if we let $I_0 \equiv]\sigma, \sigma'[$ be the excursion interval of b below $\overline{M}$ which contains $\underline{S}$, and let $J_0 \equiv]\tau, \tau'[$ be the excursion interval of B above $\underline{M}$ which contains $\overline{T}$, then given $\overline{M}$ and $\underline{M}$, these two intervals behave like ordinary excursion intervals of the two processes.

Lemma 3.1 *Given $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\overline{M} = \overline{m}$ and $\underline{M} = \underline{m}$, the processes*

$$\begin{aligned} X_1 &= \{\overline{m} - b(\sigma' - u),\ 0 \le u < \sigma' - \underline{s}\}, \quad & X_3 &= \{B(\tau' - v) - \underline{m},\ 0 \le v < \tau' - \overline{t}\}, \\ X_2 &= \{b(\underline{s} + u) - \underline{m},\ \ 0 \le u \le \sigma' - \underline{s}\}, \quad & X_4 &= \{\overline{m} - B(\overline{t} + v),\ 0 \le v \le \tau' - \overline{t}\}, \end{aligned}$$

are all Bessel(3) processes killed at the first hit of $\overline{m} - \underline{m}$.

Proof. X_2 and X_4 are Bessel(3) processes by a result of D. Williams (see e.g. [8, Chap.XII, Theorem (4,5)], since they represent a Bessel(3) process and a Brownian motion started respectively from the bottom and the top of an excursion. The other two processes are obtained by reversing X_2 and X_4 from their lifetimes, so they are also Bessel(3) processes [8, Chap.VII, Proposition (4,8)]. ♣

Remark 3.2 *(a)* If we are observing the process $\{Y(s,t) \equiv B(t) - b(s),\ (s,t) \in \mathbb{R}_+^2\}$, rather than the processes b and B, we cannot determine $\overline{M}$ and $\underline{M}$ in general. However, σ, σ', τ and τ' can be determined, since for instance $\sigma' = \inf\{s \ge \underline{S} : Y(\overline{T}, s) = 0\}$. In addition, probabilistic properties of $\mathcal{C}_0$ only depend on M, not on the particular values of $\overline{M}$ and $\underline{M}$ (as long as $\overline{M} - \underline{M} = M$). Indeed, consider for instance $\mathcal{C}_0 \cap \{(s,t) : s \ge \underline{s},\ t \ge \overline{t}\}$, and observe that $b(\underline{s} + u) < B(\overline{t} + v)$ if and only if $X_2(u) + X_4(v) < M$, and the distribution of these processes does not depend on $\overline{M}$ or $\underline{M}$ by Lemma 3.1.

(b) If we wanted to consider excursion intervals of b below L of the form $[0, a_2[$, we would only have to set $\underline{m} = 0$ and $\underline{s} = 0$. The process X_1 would no longer be relevant and the entire component would be contained in $[0, \sigma'] \times [\tau, \tau']$.

Let $Q(0, x; s, y)$ be the probability measure on $\Omega_0 = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ (equipped with its usual topology of uniform convergence on compact sets and Borel sigma-field) under which the canonical coordinate process χ has the following distribution: on $[0, s]$, χ is a Bessel(3) process started at position x and conditioned to hit level y for the first time at time s; on $[s, \infty]$, χ is the constant process $\chi \equiv y$. Fix $0 < \underline{m} < \overline{m}$, $0 < \underline{s} < \sigma_0'$, $0 < \overline{t} < \tau_0'$, set $m = \overline{m} - \underline{m}$ and consider the following probability measures on Ω_0:

$$\begin{array}{rclrcl} R^1_{x,u} & = & Q(0, \overline{m} - x; u - \underline{s}, m), & R^2_{x,u} & = & Q(0, x - \underline{m}; \sigma_0' - u, m), \\ R^3_{y,v} & = & Q(0, y - \underline{m}; v - \overline{t}, m), & R^4_{y,v} & = & Q(0, \overline{m} - y; \tau_0' - v, m). \end{array}$$

Other measures on Ω_0 of interest to us are P_0^x, under which χ is a Brownian motion started at x and killed when it first hits 0, and Q^x, under which χ is a Bessel(3) process started at x. Let $T(a) = \inf\{u \geq 0 : \chi(u) = a\}$, and let $(\mathcal{G}_u)$ be the canonical (completed) filtration on Ω_0. The following lemma states the well-known relationships between these measures.

Lemma 3.3 *(a) Suppose $0 < x \leq a$ and let $\Lambda \in \mathcal{G}_{T(a)}$. Then $Q^x\{\Lambda\} = \frac{a}{x} P_0^x\{\Lambda\}$.*
(b) Suppose $0 < x < a < c$, $0 < s < t$, and let $\Lambda \in \mathcal{G}_{T(a)}$. Then

$$Q(0, x; t, c)\{\Lambda, T(a) \in ds\} = P_0^x\{\Lambda, T(a) \in ds\} \frac{P_0^a\{s + T(c) \in dt\}}{P_0^x\{T(c) \in dt\}} \, .$$

Proof. The first part comes from the fact that a Bessel(3) is an h-transform of killed Brownian motion with the function $h(x) = x$ on $[0, \infty)$; since $T(a) < \infty$ a.s. for a Bessel from $x < a$, and since $\chi(T(a)) = a$ on $\{T(a) < \infty\}$, (a) follows from Doob's h-transform formula [2, Chap. 2.X]. The second part follows from the first by the strong Markov property, since the numerator is just $\frac{x}{c} Q^x\{\Lambda, T(a) \in ds, T(c) \in dt\}$ and the denominator is $\frac{x}{c} Q^x\{T(c) \in dt\}$. ♣

Lemma 3.4 *Fix $\underline{s} < s < \sigma_0'$ and $\overline{t} < t < \tau_0'$, and define*

$$\begin{array}{ll} \chi_1(u) \equiv \overline{M} - b(s-u),\ 0 \leq u < s - \underline{s}\,, & \chi_3(v) \equiv B(t-v) - \underline{M},\ 0 \leq v < t - \overline{t}\,, \\ \chi_2(u) \equiv b(s+u) - \underline{M},\ 0 \leq u < \sigma_0' - s\,, & \chi_4(v) \equiv \overline{M} - B(\overline{t} + v),\ 0 \leq v < \tau_0' - t\,. \end{array}$$

(a) Given $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\overline{M} = \overline{m}$, $\underline{M} = \underline{m}$, $\sigma' = \sigma_0'$, $\tau' = \tau_0'$, $b(s) = x$ and $B(t) = y$, these processes are independent, and the distribution of χ_i is $R^i_{x,s}$ for $i = 1, 2$, and is $R^i_{y,t}$ for $i = 3, 4$.
(b) For $i = 1, \ldots, 4$, let $(\mathcal{G}_i(u))$ be the natural (completed) filtration of χ_i and let T be a stopping time relative to this filtration. Then the conditional distribution of $(\chi_i(T + \cdot))$ given $\mathcal{G}_i(T)$ is $R^i_{\chi_i(T),\zeta_i}$, where $\zeta_1 = s - T$, $\zeta_2 = s + T$, $\zeta_3 = t - T$ and $\zeta_4 = t + T$.

Proof. We only consider the case $i = 1$, since the other three cases are similar. Given $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\overline{M} = \overline{m}$, $\underline{M} = \underline{m}$, and $\sigma' = \sigma'_0$, the distribution of the process X_1 defined in Lemma 3.1 is $R^1_{\overline{m},\sigma'_0}$ by definition. Now both (a) and (b) are consequences of the fact that for $\Lambda \in \mathcal{G}_1(\sigma'_0)$,

$$R^1_{\overline{m},\sigma'_0}(\Lambda) = \frac{Q^0\{\Lambda, \sigma' \in d\sigma'_0\}}{Q^0\{\sigma' \in d\sigma'_0\}}.$$

Indeed, χ_1 has the strong Markov property under Q^0, and a standard calculation shows that $R^1_{\overline{m},\sigma'_0}$ inherits this property from Q^0, and that given $b(s) = x$, the conditional distribution of χ_1 is $R^1_{x,s}$, proving (a). A similar calculation establishes (b). ♣

Remark 3.5 We are effectively translating the origin to the point (s, t). The processes χ_i represent b and B going in the four directions from this point, modified to make them into Bessel processes.

Let Σ be the sigma-field generated by $\underline{S}$, $\overline{T}$, $\overline{M}$, $\underline{m}$, σ, σ', τ, and τ'. We are going to determine a formula for the probability p that $(s,t) \in \mathcal{C}_0$ given that $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\overline{M} = \overline{m}$, $\underline{M} = \underline{m}$, $\sigma = \sigma_0$, $\sigma' = \sigma'_0$, $\tau = \tau_0$, $\tau' = \tau'_0$, $b(s) = x$ and $B(t) = y$. By symmetry, we can confine ourselves to those (s,t) for which $s \geq \underline{s}$ and $t \geq \overline{t}$. Let

$$p_n \equiv p_n(s, x; t, y; \underline{s}, \overline{t}, \underline{m}, \overline{m}, \sigma'_0, \tau'_0)$$

be the conditional probability that Algorithm A terminates and outputs YES either at stage $2n-1$ or at stage $2n$, given $\Sigma \vee \sigma\{b(s), B(t)\}$. We will give an explicit formula for p_n involving only the distribution of Brownian motion. Clearly,

$$p = \sum_{n \geq 1} p_n.$$

Let χ_i , $i = 1, \ldots, 4$ be the processes defined in Lemma 3.4, set

$$\chi^i_*(v) = \inf_{0 \leq u \leq v} \chi_i(u), \qquad \chi^*_i(v) = \sup_{0 \leq u \leq v} \chi_i(u),$$

and let $\chi^i_* = \chi^i_*(\infty)$ and $\chi^*_i = \chi^*_i(\infty)$ be the inf and sup over all time. Let $Q^x_i \otimes Q^y_j$ be the joint distribution of χ_i and χ_j, given that $\chi_i(0) = x$, $\chi_j(0) = y$. Define the hitting times S_i for the χ_i by

$$S_i(a) = \inf\{u \geq 0 : \chi_i(u) = a\},$$

with the usual convention that the sup of the empty set is ∞.

Consider the sequence of random variables $(\underline{M}_k)$ and $(\overline{M}_k)$ defined in (8)–(9), and the U_k, U'_k, V_k, V'_k defined in (10). Let

$$T_i(a) = \begin{cases} S_i(\overline{M} - a) & \text{if } i = 1, 4, \\ S_i(a - \underline{M}) & \text{if } i = 2, 3, \end{cases}$$

and

$$\begin{aligned} \underline{Z}(a) &= \min\{\overline{M} - \chi_1^*(T_1(a)),\ \underline{M} + \chi_*^2(T_2(a))\}, \\ \overline{Z}(a) &= \max\{\underline{M} + \chi_3^*(T_3(a)),\ \overline{M} - \chi_*^4(T_4(a))\}\,. \end{aligned} \tag{11}$$

Observe that on the set where $T_i(\overline{M}_{k-1})$, $i = 1, 2$ and $T_i(\underline{M}_k)$, $i = 3, 4$ are finite,

$$\begin{aligned} T_1^k &\equiv T_1(\overline{M}_{k-1}) = s - U_k\,, \quad & T_2^k &\equiv T_2(\overline{M}_{k-1}) = U'_k - s\,, \\ T_3^k &\equiv T_3(\underline{M}_k) = t - V_k\,, & T_4^k &\equiv T_4(\underline{M}_k) = V'_k - t\,, \end{aligned} \tag{12}$$

and

$$\underline{M}_n = \underline{Z}(\overline{M}_{n-1}), \qquad \overline{M}_n = \overline{Z}(\underline{M}_n). \tag{13}$$

Finally, consider the densities

$$\begin{aligned} &f_{\overline{m},\underline{s},\underline{m},\sigma'_0}(u, u', x, y; u_1, u_2, \underline{m}_1)\ du_1\ du_2\ d\underline{m}_1 \\ &\qquad = R^1_{x,u} \otimes R^2_{x,u'}\{u - T_1(y) \in du_1,\ u' + T_2(y) \in du_2,\ \underline{Z}(y) \in d\underline{m}_1\} \end{aligned}$$

and

$$\begin{aligned} &g_{\underline{m},\overline{t},\overline{m},\tau'_0}(v, v', y, x; v_1, v_2, \overline{m}_1)\ dv_1\ dv_2\ d\overline{m}_1 \\ &\qquad = R^3_{y,v} \otimes R^4_{y,v'}\{v - T_3(x) \in dv_1,\ v' + T_4(x) \in dv_2,\ \overline{Z}(x) \in d\overline{m}_1\}. \end{aligned}$$

We will generally omit the subscripts on f and g in what follows.

Proposition 3.6 *Set $\overline{m}_{-1} = b(s) = x$, $\underline{m}_0 = \overline{m}_0 = B(t) = y$, $u_0 = s = u'_0$, $v_0 = t = v'_0$ and, for $n \geq 1$, let A_n denote the set of all $((u_k, u'_k, v_k, v'_k, \underline{m}_k, \overline{m}_k),\ 1 \leq k \leq n-1) \in I\!R^{6(n-1)}$, such that*

$$\begin{aligned} \underline{m} < \underline{m}_{n-1} < \cdots < \underline{m}_1 < {}& x < \overline{m}_0 < \cdots < \overline{m}_{n-1} < \overline{m}\,, \\ \underline{s} < u_{n-1} < \cdots < u_1 < {}& s < u'_1 < \cdots < u'_{n-1} < \sigma'_0\,, \\ \overline{t} < v_{n-1} < \cdots < v_1 < {}& t < v'_1 < \cdots < v'_{n-1} < \tau'_0\,. \end{aligned} \tag{14}$$

Then p_1 is equal to

$$\begin{aligned} R^1_{x,s}\{\chi_*^1 > \overline{m} - y\} + \int_{\underline{s}}^{s} du_1 \int_s^{\sigma'_0} du_2 \int_{\underline{m}}^{x} d\underline{m}_1\ f(s, s, x, y; u_1, u_2, \underline{m}_1) \\ \times R^3_{y,t}\{\chi_*^3 > \overline{m}_1 - \underline{m}\}, \end{aligned} \tag{15}$$

and, for $n \geq 2$, p_n is equal to the following integral:

$$\begin{aligned}
&\int_{A_n} \Bigg(\prod_{k=1}^{n-1} du_k \, du'_k \, d\underline{m}_k \, dv_k \, dv'_k \, d\overline{m}_k \;\; f(u_{k-1}, u'_{k-1}, \overline{m}_{k-2}, \overline{m}_{k-1}; u_k, u'_k, \underline{m}_k) \\
&\quad \times g(v_{k-1}, v'_{k-1}, \underline{m}_{k-1}, \underline{m}_k; v_k, v'_k, \overline{m}_k) \Bigg) \Bigg(R^1_{\overline{m}_{n-2}, u_{n-1}} \{ \chi^1_* > \overline{m} - \overline{m}_{n-1} \} \\
&\quad + \int_{\underline{s}}^{u_{n-1}} du_n \int_{u'_{n-1}}^{\sigma'_0} du'_n \int_{\underline{m}}^{\underline{m}_{n-1}} d\underline{m}_n \;\; f(u_{n-1}, u'_{n-1}, \overline{m}_{n-2}, \overline{m}_{n-1}; u_n, u'_n, \underline{m}_n) \\
&\qquad\qquad\qquad\qquad \times R^3_{\underline{m}_{n-1}, v_{n-1}} \{ \chi^3_* > \underline{m}_n - \underline{m} \} \Bigg)
\end{aligned} \tag{16}$$

Remark 3.7 *(a)* The fact that formula (16) is valid comes from the Markov property of the processes χ_i, $i = 1, \ldots, 4$. Since the path Γ constructed by Algorithm A will wind around itself in general, it is not obvious how a Markov property can be brought into play. The key idea is that the sequence $R_1 \subset R_2 \subset \ldots$ of rectangles constructed in the proof of Proposition 2.2 forms the "past" for the 4-parameter diffusion

$$\bar{\chi} = (\chi_1(u_1), \ldots, \chi_4(u_4)),$$

and that once Γ leaves R_k, it never returns. So we will be using the Markov property for $\bar{\chi}$.

(b) If B and b were arbitrary time-homogeneous diffusions, the same formula would apply provided f and g were redefined using appropriate other measures rather than the measures $R^i_{x,u}$.

PROOF OF PROPOSITION 3.6. We are given $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\sigma' = \sigma'_0$, $\tau' = \tau'_0$, $b(\underline{s}) = \underline{m}$, $B(\overline{t}) = \overline{m}$, $b(s) = x$ and $B(t) = y$, such that $\underline{m} < x < y < \overline{m}$. For $n \geq 1$, we shall calculate the conditional probability that Algorithm A terminates successfully—i.e. it stops and outputs YES—at stage $2n-1$ or $2n$, given these values. By symmetry we may assume that $\underline{s} < s$ and $\overline{t} < t$.

Stages 1 and 2 are somewhat special since the algorithm cannot output NO at these times. It succeeds at stage 1 if $b(u) < y$ for all $u \in]\underline{s}, s[$, or equivalently, if $\chi^1_* > \overline{m} - y$ for in that case $\underline{s}_1 = \underline{s}$. If it does not succeed at stage 1, χ_1 must hit $\overline{m} - y$ before $s - \underline{s}$, which means that $T_1(y) < s - \underline{s}$. The algorithm then terminates successfully at stage 2 if $B(v) > \underline{M}_1$ for all $v \in]\overline{t}, t[$, or equivalently, if $\chi^3_* > \underline{Z}(y) - \underline{m}$. So by Lemma 3.4 (a),

$$p_1 = R^1_{x,s} \otimes R^2_{x,s} \otimes R^3_{y,t}(\{\chi^1_* > \overline{m} - y\} \cup \{T_1(y) < s - \underline{s}, \ \chi^3_* > \underline{Z}(y) - \underline{m}\}).$$

But this can be written

$$\begin{aligned}
R^1_{x,s}\{\chi^1_* > \overline{m} - y\} + \int_0^{s-\underline{s}} \int_{\underline{m}}^{x} R^1_{x,s} \otimes R^2_{x,s}\{T_1(y) \in du_1, \ \underline{Z}(y) \in d\underline{m}_1\} \\
\times R^3_{y,t}\{\chi^3_* > \underline{m}_1 - \underline{m}\}.
\end{aligned}$$

Formula (15) follows by the change of variables in $u_1 \mapsto s - u_1$ and the definition of f.

The above derivation for $n = 1$ is somewhat informal. We now fix $n \geq 2$ and give a formal proof in that case. We begin by a precise definition of the σ-fields relative to which we shall use the multiparameter Markov property. We refer the reader to [7] for definitions relating to multiparameter processes that we use below. Define $T_3^0 = T_4^0 = 0$, and for $n \geq 1$,

$$\hat{S}_n = (T_1^n, T_2^n, T_3^{n-1}, T_4^{n-1}), \qquad \hat{T}_n = (T_1^n, T_2^n, T_3^n, T_4^n).$$

These two random variables are stopping points relative to the four-parameter filtration

$$\mathcal{F}(\underline{u}) = \mathcal{G}_1(u_1) \vee \mathcal{G}_2(u_2) \vee \mathcal{G}_3(u_3) \vee \mathcal{G}_4(u_4), \qquad \underline{u} = (u_1, u_2, u_3, u_4),$$

where $\mathcal{G}_i(u)$ is defined in Lemma 3.4. Associated to these stopping points are the σ-fields $\hat{\mathcal{G}}_n = \mathcal{F}(\hat{S}_n)$ and $\hat{\mathcal{H}}_n = \mathcal{F}(\hat{T}_n)$ (these represent the information about the χ_i available at $\hat{S}_n$ and $\hat{T}_n$). Since $T_i^n \leq T_i^{n+1}$, $i = 1, \ldots, 4$, $\hat{\mathcal{G}}_n \subset \hat{\mathcal{H}}_n \subset \hat{\mathcal{G}}_{n+1}$, for all n.

Applying Lemma 3.4 (b) and (12), it is not difficult to see that the conditional distribution of $(\bar{\chi}(\hat{S}_n + \underline{u}))$ given $\hat{\mathcal{G}}_n$ is

$$R^1_{\overline{M}_{n-1}, U_n} \otimes R^2_{\overline{M}_{n-1}, U'_n} \otimes R^3_{\underline{M}_{n-1}, V_{n-1}} \otimes R^4_{\underline{M}_{n-1}, V'_{n-1}}. \tag{17}$$

Indeed, since the χ_i are independent, $\bar{\chi}$ has the strong Markov property [3, Theorem 3.3], and

$$\begin{aligned} \chi_1(T_1^n) &= \overline{m} - \overline{M}_{n-1}, & \chi_2(T_2^n) &= \overline{M}_{n-1} - \underline{m}, \\ \chi_3(T_3^{n-1}) &= \underline{M}_{n-1} - \underline{m}, & \chi_4(T_4^{n-1}) &= \overline{m} - \underline{M}_{n-1}. \end{aligned}$$

Similarly, the conditional distribution of $(\bar{\chi}(\hat{T}_n + \bar{u}))$ given $\hat{\mathcal{H}}_n$ is

$$R^1_{\overline{M}_{n-1}, U_n} \otimes R^2_{\overline{M}_{n-1}, U'_n} \otimes R^3_{\underline{M}_n, V_n} \otimes R^4_{\underline{M}_n, V'_n}. \tag{18}$$

Now fix $n \geq 2$ and assume that the algorithm has continued through stage $2n - 2$. It has then constructed a sequence $((U_k, U'_k, V_k, V'_k, \underline{M}_k, \overline{M}_k),\ 1 \leq k \leq n-1)$ as in (8), (9) and (10). In particular, this sequence belongs to A_n a.s. By (12), the inequalities on U_k, U'_k, V_k and V'_k in (14) are equivalent to $0 < T_i^1 < \cdots < T_i^{n-1} < \infty$, $i = 1, \ldots, 4$.

Algorithm A will terminate successfully at stage $2n - 1$ if $\underline{s} \in I_n$, which means that $b(u) < \overline{M}_{n-1}$ for all $u \in]\underline{s}, U_{n-1}[$, or equivalently that $\chi_*^1 > \overline{m} - \overline{M}_{n-1}$. The algorithm will end unsuccessfully if $\underline{M}_n = \underline{M}_{n-1}$, for then $\underline{s}_n = \underline{s}_{n-1}$. On the other hand, if $\underline{M}_n < \underline{M}_{n-1}$, then the new minimum $\underline{M}_n$ must be reached in the set $I_n - I_{n-1}$, or equivalently, either there is u satisfying $T_1^{n-1} < u < T_1^n$ and $\chi_1(u) = \underline{Z}(\overline{M}_{n-1})$, or there is u satisfying $T_2^{n-1} < u < T_2^n$ and $\chi_2(u) = \underline{Z}(\overline{M}_{n-1})$. In particular, we do not have to look back at previous portions of the paths of χ_1 or χ_2 to determine whether the algorithm continues. Algorithm A continues on to stage 2n when

$$T_1^n < s - \underline{s} \quad \text{and} \quad \underline{Z}(\overline{M}_{n-1}) < \underline{M}_{n-1} = \underline{Z}(\overline{M}_{n-2})\,. \tag{19}$$

Similarly, having attained stage $2n$, Algorithm A stops successfully if $\chi^3_* > \underline{M}_n - \underline{m}$, and it continues on if

$$T_3^n < t - \bar{t} \quad \text{and} \quad \overline{Z}(\underline{M}_n) > \overline{M}_{n-1} = \overline{Z}(\underline{M}_{n-1}). \tag{20}$$

Again, we do not have to look back at previous portions of the paths of χ_3 or χ_4 to verify the second inequality.

Putting (19) and (20) together, we see that the condition for continuing through stage $2n-2$ is that there exist $((u_k, u'_k, v_k, v'_k, \underline{m}_k, \overline{m}_k),\ 1 \le k \le n-1)$ in A_n such that

$$\begin{array}{rlrlrlrl} s - T_1^1 = & u_1, & \underline{Z}(y) = & \underline{m}_1, & t - T_3^1 = & v_1, & \overline{Z}(\underline{m}_1) = & \overline{m}_1, \\ s - T_1^2 = & u_2, & \underline{Z}(\overline{m}_1) = & \underline{m}_2, & t - T_3^2 = & v_1, & \overline{Z}(\underline{m}_2) = & \overline{m}_2, \\ \vdots & & \vdots & & \vdots & & \vdots & \\ s - T_1^{n-1} = & u_{n-1}, & \underline{Z}(\overline{m}_{n-2}) = & \underline{m}_{n-1}, & t - T_3^{n-1} = & v_{n-1}, & \overline{Z}(\underline{m}_{n-1}) = & \overline{m}_{n-1}. \end{array} \tag{21}$$

We also request that $T_2^k = s + u'_k$ and $T_4^k = t + v'_k$, $1 \le k \le n-1$.

In order to compute p_n, we condition successively on $\hat{\mathcal{G}}_n$, $\hat{\mathcal{H}}_{n-1}$, $\hat{\mathcal{G}}_{n-1}$, $\ldots$, $\hat{\mathcal{H}}_1$, $\hat{\mathcal{G}}_1$. If we use the Markov property on (21) and the conditional distributions (17) and (18), we see that p_n is equal to

$$\begin{aligned} \int_{A_n} \Bigg(\prod_{k=1}^{n-1} & R^1_{\overline{m}_{k-2},u_{k-1}} \otimes R^2_{\overline{m}_{k-2},u'_{k-1}} \{u_{k-1} - T_1^k \in du_k,\ u'_{k-1} + T_2^k \in du'_k,\ \underline{Z}(\overline{m}_{k-1}) \in d\underline{m}_k\} \\ & \times R^3_{\underline{m}_{k-1},v_{k-1}} \otimes R^4_{\underline{m}_{k-1},v'_{k-1}} \{v_{k-1} - T_3^k \in dv_k,\ v'_{k-1} + T_4^k \in dv'_k,\ \overline{Z}(\underline{m}_k) \in d\overline{m}_k\} \Bigg) \\ \times \Bigg(& R^1_{\overline{m}_{n-2},u_{n-1}} \{\chi^1_* > \overline{m} - \overline{m}_{n-1}\} \\ & + \int_{\underline{s}}^{u_{n-1}} \int_{\underline{m}}^{\underline{m}_{n-1}} R^1_{\overline{m}_{n-2},u_{n-1}} \otimes R^2_{\overline{m}_{n-1},u'_{n-1}} \{u_{n-1} - T_1^n \in du_n, \underline{Z}(\overline{m}_{n-1}) \in d\underline{m}_n\} \\ & \qquad R^3_{\underline{m}_{n-1},v_{n-1}} \{\chi^3_* > \underline{m}_n - \underline{m}\} \Bigg). \end{aligned}$$

Now (16) follows from the definitions of f and g. ♣

Remark 3.8 A good approximation to p can be obtained from the sum of the first few terms of the series $\sum_{n \in \mathbb{N}} p_n$. Indeed, there is $0 < c < 1$ such that

$$\sum_{k=n}^{\infty} p_k \le c^n. \tag{22}$$

This comes from the following. Let F_n be the event "Algorithm A stops at stage $2n-1$ or later". Then

$$\sum_{k=n}^{\infty} p_k = P(F_n) = P(F_{n-1} \cap F_n) = \int_{F_{n-1}} P(F_n \mid \hat{\mathcal{G}}_{2n-1})\ dP \le \lambda_{1,n}\lambda_{2,n} P(F_{n-1}),$$

where $\lambda_{1,n}$ is the $R^1_{\overline{M}_{n-2},U_{n-1}} \otimes R^2_{\overline{M}_{n-2},U'_{n-1}}$-probability that χ_1 has at least one downcrossing of $[\overline{m}-\overline{M}_{n-1}, \overline{m}-\overline{M}_{n-2}]$ or that χ_2 has at least one upcrossing of the interval $[\underline{M}_{n-2}, \overline{M}_{n-1}]$. $\lambda_{2,n}$ is defined similarly relative to χ_3 and χ_4. Clearly, $\lambda_{1,n}$ is not greater than the $R^1_{y,s} \otimes R^2_{y,s}$-probability that χ_1 has at least on downcrossing of $[\overline{m}-y, \overline{m}-x]$ or that χ_2 has at least one upcrossing of $[x, y]$. A similar inequality is valid for $\lambda_{2,n}$, hence (22).

4 Expressions for the densities and numerical results

In this section, we complete the calculation of the expected area of $\mathcal{C}_0$. We will provide an exact explicit formula and will discuss the results of a numerical evaluation and of simulations.

The densities f and g and the probabilities appearing in (16) can be written in terms of one function and its derivatives. Let $I \subset \mathbb{R}_+$ be an interval with endpoints a and c, and let $x \in I$. Let P^x be the probability measure on Ω_0 under which χ is a Brownian motion started at x. Let

$$K(x, a, c, u) \equiv P^x\{T(c) \leq u \wedge T(a)\}$$

and observe that when $a < x < c$,

$$K(x, a, c, u) = \begin{cases} P_0^x\{T(c) \leq u,\ \chi_*(T(c)) > a\} & \text{if } a < x < c, \\ P_0^x\{T(c) \leq u,\ \chi^*(T(c)) < a\} & \text{if } c < x < a. \end{cases}$$

since $I \subset \mathbb{R}_+$. Define

$$\phi(x, a, c, u) \equiv \frac{\partial}{\partial u} K(x; a, b, u), \quad \hat{\phi}(x, a, c, u) \equiv \frac{\partial}{\partial a} \phi(x, a, c, u),$$

and observe that if $a < x < c$, then

$$\phi(x, a, c, u)\ du = P_0^x\{T(c) \in du,\ \chi_*(T(c)) > a\} \tag{23}$$

and $-\hat{\phi}(x, a, c, u)$ gives the joint density of $(T(c), \chi_*(T(c)))$, whereas if $c < x < a$, then

$$\phi(x, a, c, u)\ du = P_0^x\{T(c) \in du,\ \chi^*(T(c)) < a\} \tag{24}$$

and $\hat{\phi}(x, a, c, u)$ gives the joint density of $(T(c), \chi^*(T(c)))$. A closed form expression is available for ϕ: let

$$p(u, x, y) \equiv (2\pi u)^{-1/2} e^{-(y-x)^2/(2u)},$$

and

$$\tilde{p}(u, x, y) \equiv \frac{\partial}{\partial x} p(u, x, y), \qquad \hat{p}(u, x, y) \equiv \frac{\partial}{\partial y} \tilde{p}(u, x, y) = -\frac{\partial}{\partial x} \tilde{p}(u, x, y)\,.$$

Then

$$\phi(x, a, c, u) = -\sum_{n=-\infty}^{+\infty} \tilde{p}(u, |c - x|, 2n|c - a|) \tag{25}$$

(see [6, Chap. 2, (8.25) and (8.26)]), and it follows that

$$\hat{\phi}(x,a,c,u) = \operatorname{sgn}(c-a) \sum_{n=-\infty}^{+\infty} 2n\ \hat{p}(u, |c-x|, 2n|c-a|).$$

From these expressions, we can write the functions appearing in Proposition 3.6 in closed form.

Lemma 4.1 *For $\underline{m} \le \underline{m}_1 \le x \le y \le \overline{m}$, $0 \le \underline{s} \le u_1 \le u$ and $0 \le \overline{t} \le u' \le u_2$, the function $f(u,u',x,y;u_1,u_2,\underline{m}_1)$ is equal to*

$$\begin{aligned}
(26)\qquad & (-\phi(\overline{m}-x,\overline{m}-\underline{m}_1,\overline{m}-y,u-u_1)\hat{\phi}(x-\underline{m},\underline{m}_1-\underline{m},y-\underline{m},u_2-u') \\
& \quad + \hat{\phi}(\overline{m}-x,\overline{m}-\underline{m}_1,\overline{m}-y,u-u_1)\phi(x-\underline{m},\underline{m}_1-\underline{m},y-\underline{m},u_2-u')) \\
& \quad \times \frac{\phi(\overline{m}-y,0,m,u_1-\underline{s})}{\phi(\overline{m}-x,0,m,u-\underline{s})}\ \frac{\phi(y-\underline{m},0,m,\sigma_0'-u_2)}{\phi(x-\underline{m},0,m,\sigma_0'-u')}
\end{aligned}$$

and

$$(27)\qquad R^1_{x,u}\{\chi^1_* > \overline{m}-y\} = \frac{\phi(\overline{m}-x,\overline{m}-y,m,u-\underline{s})}{\phi(\overline{m}-x,0,m,u-\underline{s})},$$

$$(28)\qquad R^3_{y,v}\{\chi^3_* > x-\underline{m}\} = \frac{\phi(y-\underline{m},x-\underline{m},m,v-\overline{t})}{\phi(y-\underline{m},0,m,v-\overline{t})}.$$

Proof. For $\Lambda \in \mathcal{G}_1(T_1(y))$, $R^1_{x,u}\{\Lambda,\ T_1(y)\in d(u-u_1)\}$ is equal to

$$Q^{\overline{m}-x}\{\Lambda,\ T(\overline{m}-y)\in d(u-u_1)\}\ \frac{Q^{\overline{m}-y}\{T(m)\in d(u_1-\underline{s})\}}{Q^{\overline{m}-x}\{T(m)\in d(u-\underline{s})\}}$$

by Lemma 3.3 (b), and the ratio in this expression is equal to

$$\frac{\overline{m}-x}{\overline{m}-y}\ \frac{\phi(\overline{m}-y,0,m,u_1-\underline{s})}{\phi(\overline{m}-x,0,m,u-\underline{s})}$$

by Lemma 3.3 (a) and (23), and similarly, for $\Lambda \in \mathcal{G}_2(T_2(y))$, $R^2_{x,u'}\{\Lambda,\ T_2(y)\in d(u_2-u')\}$ is equal to

$$Q^{x-\underline{m}}\{\Lambda,\ T(y-\underline{m})\in d(u_2-u')\}\ \frac{x-\underline{m}}{y-\underline{m}}\ \frac{\phi(y-\underline{m},0,m,\sigma_0'-u_2)}{\phi(x-\underline{m},0,m,\sigma_0'-u')}.$$

Since $\underline{Z}(y)$ is $\mathcal{G}_1(T_1(y))\otimes\mathcal{G}_2(T_2(y))$-measurable, (26) will follow once we compute the density

$$Q^{\overline{m}-x}\otimes Q^{x-\underline{m}}\{u-T_1(y)\in du_1,\ u'+T_2(y)\in du_2,\ \underline{Z}(y)\in d\underline{m}_1\}.$$

Using (11), we have

$$\underline{Z}(y) = \min\{\overline{m} - \chi_1^*(T_1(y)),\ \underline{m} + \chi_*^2(T_2(y))\},$$

and there are two ways for the minimum to be in $d\underline{m}_1$: either $\chi_1^*(T_1(y)) < \overline{m} - \underline{m}_1$ while $\chi_*^2(T_2(y)) \in d(\underline{m}_1 - \underline{m})$, or else $\chi_1^*(T_1(y)) \in d(\overline{m} - \underline{m}_1)$ while $\chi_*^2(T_2(y)) > \underline{m}_1 - \underline{m}$. Since χ_1 and χ_2 are independent, the above density is equal to

$$\begin{aligned}
&Q^{\overline{m}-x}\{T(\overline{m} - y) \in d(u - u_1),\ \chi^*(T(\overline{m} - y)) < \overline{m} - \underline{m}_1\}\\
&\qquad \times Q^{x-\underline{m}}\{T(y - \underline{m}) \in d(u_2 - u'),\ \chi_*(T(y - \underline{m})) \in d(\underline{m}_1 - \underline{m})\}\\
&+Q^{\overline{m}-x}\{T(\overline{m} - y) \in d(u - u_1),\ \chi^*(T(\overline{m} - y)) \in d(\overline{m} - \underline{m}_1)\}\\
&\qquad \times Q^{x-\underline{m}}\{T(y - \underline{m}) \in d(u_2 - u'),\ \chi_*(T(y - \underline{m})) > \underline{m}_1 - \underline{m}\}.
\end{aligned}$$

Applying Lemma 3.3 (a) to transform the Q^x-measures into P_0^x-measures and taking (23) and (24) into account, (26) follows.

Formula (27) follows from (23) by Lemma 3.3 (a), since

$$R^1_{x,u}\{\chi^1_* > \overline{m} - y\} = \frac{Q^{\overline{m}-x}\{T(m) \in d(u - \underline{s}),\ \chi_*(T(m)) > \overline{m} - y\}}{Q^{\overline{m}-x}\{T(m) \in d(u - \underline{s})\}},$$

and the proof of (28) is similar. Details are left to the reader. ♣

Remark 4.2 There is a similar expression for the function g, but it is simpler to notice the following relationship between f and g:

$$\begin{aligned}
&g_{\underline{m},\overline{t},\overline{m},\tau_0'}(v, v', y, x; v_1, v_2, \overline{m}_1) =\\
&\qquad f_{\overline{m},\overline{t},\underline{m},\tau_0'}(v, v', \overline{m} + \underline{m} - y, \overline{m} + \underline{m} - x; v_1, v_2, \overline{m} + \underline{m} - \overline{m}_1)
\end{aligned}$$

by definition of f and g, since

$$R^1_{x,u} = R^3_{\overline{m}+\underline{m}-x,u} \quad \text{and} \quad R^2_{x,u} = R^4_{\overline{m}+\underline{m}-x,u}$$

provided $\overline{t}$ and τ_0' are replaced in the definition of these measures by $\underline{s}$ and σ_0', respectively, and since the symmetry with respect to $(\overline{m} + \underline{m})/2$ transforms $\overline{Z}(x)$ into $\underline{Z}(\overline{m} + \underline{m} - x)$.

Define a function $f_1(u, x, y)$ by

$$\begin{aligned}
f_1(u, x, y) &= \sum_{n \in Z} (\tilde{p}(u, 0, x + 2ny) - \tilde{p}(u, 0, -x - 2ny))\\
&= \sum_{n \in Z} 2(x + 2ny) \exp(-(x + 2ny)^2/(2u))/(2\pi u^3)^{\frac{1}{2}}.
\end{aligned}$$

Theorem 4.3 *The expected area of* $\mathcal{C}_0 \cap \{(s,t) : s \geq \underline{s}, t \geq \overline{t}\}$ *given* $\underline{S} = \underline{s}$, $\overline{T} = \overline{t}$, $\overline{M} = \overline{m}$, $\underline{M} = \underline{m}$, $\sigma' = \sigma'_0$ *and* $\tau' = \tau'_0$ *is expressed (explicitly) by the formula*

$$\int_{\underline{s}}^{\sigma'_0} ds \int_{\overline{t}}^{\tau'_0} dt \int_{\underline{m}}^{\overline{m}} dx \int_{x}^{\overline{m}} dy \; p_s(x) q_t(y) \sum_{n \geq 1} p_n(s, x; t, y; \underline{s}, \overline{t}, \underline{m}, \overline{m}, \sigma'_0, \tau'_0),$$

where p_n *is given in Proposition 3.6 (together with Lemma 4.1) and*

$$p_s(x) \;=\; -f_1(s - \underline{s}, x - \underline{m}, m) \, \frac{\phi(x - \underline{m}, 0, m, \sigma'_0 - s)}{\hat{\phi}(0, 0, m, \sigma'_0 - \underline{s})}\,, \tag{29}$$

$$q_t(y) \;=\; -f_1(t - \overline{t}, \overline{m} - y, m) \, \frac{\phi(\overline{m} - y, 0, m, \tau'_0 - t)}{\hat{\phi}(0, 0, m, \tau'_0 - \overline{t})}\,. \tag{30}$$

PROOF. By Fubini's Theorem, the expected area of $\mathcal{C}_0 \cap \{(s,t) : s > \underline{s}, \; t > \overline{t}\}$ is the integral over the rectangle $\tilde{R} = [\underline{s}, \sigma'_0] \times [\overline{t}, \tau'_0]$ of $P\{(s,t) \in \mathcal{C}_0\}$ (P denotes the conditional probability given the six variables in the statement of the theorem). Conditioning in addition with respect to the values of $b(s)$ and $B(t)$ and letting $p_s(x)$ and $q_t(y)$ be the respective densities of $b(s)$ and $B(t)$, this is equal to

$$\int_{\tilde{R}} \int_{\{\underline{m} \leq x \leq y \leq \overline{m}\}} p_s(x) q_t(y) P\{(s,t) \in \mathcal{C}_0 \mid b(s) = x, B(t) = y\}.$$

By Lemma 3.1, the conditional probability is exactly $\sum_{n \geq 1} p_n$ by definition of p_n, and so we only need to check that (29) and (30) are indeed expressions for the densities of b and B, more precisely for $\underline{m} + X_2(s - \underline{s})$ and $\overline{m} - X_4(t - \overline{t})$ under $R^2_{\underline{m},\underline{s}}$ and $R^4_{\overline{m},\overline{t}}$, respectively. This will become clear once we compute the densities of $X_2(s - \underline{s})$ and $X_4(t - \overline{t})$ under $R^2_{\underline{m},\underline{s}}$ and $R^4_{\overline{m},\overline{t}}$, respectively.

By definition of $R^2_{\underline{m},\underline{s}}$ and the strong Markov property of Q^0, $R^2_{\underline{m},\underline{s}}\{X_2(s - \underline{s}) \in dx\}$ is equal to

$$Q^0\{X_2(s - \underline{s}) \in dx, \, T(m) > s - \underline{s}\} \, \frac{Q^x\{T(m) \in d(\sigma'_0 - s)\}}{Q^0\{T(m) \in d(\sigma'_0 - \underline{s})\}}\,. \tag{31}$$

By Lemma 3.3 and (23), the denominator is equal to

$$\begin{aligned} \lim_{a \downarrow 0} Q^a\{T(m) \in d(\sigma'_0 - \underline{s})\} \;&=\; m \lim_{a \downarrow 0} \frac{1}{a} \phi(a, 0, m, \sigma'_0 - \underline{s}) \\ &=\; m \frac{\partial}{\partial x} \phi(0, 0, m, \sigma'_0 - \underline{s}). \end{aligned}$$

Using (25), it is not difficult to see that this is equal to $-m\hat{\phi}(0, 0, m, \sigma'_0 - \underline{s})$, hence the denominator in (29). By (23), the second factor in the numerator of (31) is

$$\frac{m}{x} P^x_0\{T_m \in d(\sigma'_0 - \underline{s})\} = \frac{m}{x} \phi(x, 0, m, \sigma'_0 - \underline{s}),$$

and by [5, (3.1)] the first factor is equal to

$$\lim_{a\downarrow 0}\frac{x}{a}P_0^a\{X_2(s-\underline{s})\in dx,\ T(m)>s-\underline{s}\},$$

which, by [6, Chap.2, (8.12)] is equal to $xf_1(s-\underline{s},x,m)$. Formula (29) follows. The proof of (30) is similar and is left to the reader. ♣

Remark 4.4 The main interest of Theorem 4.3 is that it provides a formula for the expected area of $\mathcal{C}_0\cap\{(s,t): s\geq\underline{s}, t\geq\overline{t}\}$ which is both *exact* and *explicit.* In addition, this formula is amenable to numerical integration, since the various series which enter into it converge very rapidly. We have carried out the computation of

$$\int_{\underline{s}}^{\sigma_0'}ds\int_{\overline{t}}^{\tau_0'}dt\int_{\underline{m}}^{\overline{m}}dx\int_x^{\overline{m}}dy\ p_s(x)q_t(y)p_n(s,x;t,y;\underline{s},\overline{t},\underline{m},\overline{m},\sigma_0',\tau_0'),\tag{32}$$

for $n=1,2$ using the Monte-Carlo method as follows (by Remark 3.8, p_1+p_2 is a good approximation of p!). First, consider the case $n=1$. Using Proposition 3.6, (32) can be rewritten as

$$\begin{aligned}&\int_{\underline{s}}^{\sigma_0'}ds\int_{\overline{t}}^{\tau_0'}dt\int_{\underline{m}}^{\overline{m}}dx\int_x^{\overline{m}}dy\int_{\underline{s}}^{s}du_1\int_s^{\sigma_0'}du_2\int_{\underline{m}}^{x}d\underline{m}_1\ \Big(p_s(x)q_t(y)\\ &(\phi(\overline{m}-x,\overline{m}-y,m,s-\underline{s})/((s-\underline{s})(\sigma_0'-s)(x-m)\phi(\overline{m}-x,0,m,s-\underline{s}))\\ &-f(s,s,x,y;u_1,u_2,\underline{m}_1)\phi(y-\underline{m},\underline{m}_1-m,m,t-\overline{t})/\phi(y-\underline{m},0,m,t-\overline{t})\Big)\end{aligned}$$

The integral is then evaluated by picking a point $(u_1,s,u_2;t;m_1,x,y)$ at random according to the uniform distribution on the product of three simplices

$$\{\underline{s}\leq u_1\leq s\leq u_1'\leq\sigma_0'\}\times\{\overline{t}\leq t\leq\tau_0'\}\times\{\underline{m}\leq m_1\leq x\leq y\leq\overline{m}\},$$

taking the average value of the function in parenthesis at these points, and multiplying by the product of the volumes of the simplices. The case $n=2$ is similar, but the integral is then 13-dimensional. The computations were carried out on a Sun Sparcstation using a program written in C. The random number generator used was the standard *random()* function supplied with C-compilers. A point was chosen at random on a simplex by first picking the point at random uniformly in the corresponding hyper-rectangle, then ordering the components. The computation was done for $m=1$, $\sigma_0'-\underline{s}=1$ and $\tau_0'-\overline{t}=1$ using k random points, with various values of k (between 100 and 10000) and initial seed for the random number generator. Since the denominator in formulas (26)–(28) and (29)–(30) can be arbitrarily close to 0, we in fact fixed $\varepsilon>0$ and picked a point at random in the product of three simplices

$$\{\underline{s}+\varepsilon\leq u_1\leq u_1'\leq\sigma_0'-\varepsilon\}\times\{\overline{t}+\varepsilon\leq t\leq\tau_0'-\varepsilon\}\times\{\underline{m}+\varepsilon\leq m_1\leq x\leq y\leq\overline{m}-\varepsilon\},$$

Figure 2: Distribution of the percentage of bubble area.

yielding an underestimate of the actual integral. Averaging out these results gave the following values for the average area a_n of points with curvature number n for $n = 1, 2$:

$$a_1 = 0.318, \qquad a_2 = 0.060.$$

Hence, a reasonable lower bound for the expected area of points with curvature number ≤ 2 would be 0.378. For comparison, the expected are of $\{B > b\} \cap ([\underline{s}, \sigma_0'] \times [\overline{t}, \tau_o'])$ was also computed, and found to be near 0.512.

These numerical results have been checked by doing 1000 direct simulations on a NeXT station of the two processes B and b started at the bottom of an excursion (so each is a Bessel(3) process), constructing the bubble from these and determining the area of the set of points with a given curvature number. The approximate Bessel(3) processes are constructed by taking the modulus of a three-dimensional random walk. The average area a_n of points with curvature number n is given below for $n = 1, \ldots, 5$.

$$a_1 = 0.328,\ a_2 = 0.099,\ a_3 = 0.018,\ a_4 = 0.0026,\ a_5 = 0.0003\,.$$

An empirical approximation of the conditional density function of the random variable

$$Z = 100\ \times\ \text{area of } \mathcal{C}_0 / \text{ area of } R_0,$$

given $m = 1$, was obtained by counting the proportion q_j of times the plotted area fell into the interval $]j/10, (j+1)/10]$. The results are given in Figure 2.

5 The asymptotic area of high local excursions of the Brownian sheet

In this section, we use the results of the previous sections to determine the asymptotic distribution of the area of components of $\{W > 1\}$ which correspond to high excursions that we encounter as we approach $(S, 1)$ along certain curves, where $\{W(s,t),\ s \geq 0,\ t \geq 0\}$ is the standard Brownian sheet and $(S, 1)$ is the point defined in the introduction. Consider the processes B, b and x that enter into the decomposition (1) of W near $(S, 1)$.

We recall briefly the result of [1, Section 3]. For $\beta > 0$, let

$$\psi_\beta(s) = s(\log(1/s))^{-2}(\log\log(1/s))^{-\beta},$$

fix $\kappa > 0$, and for $n \in \mathbb{N}$, let $I_n = [e^{-n}, e^{1-n}]$. In the proof of [1, Theorem 3.1] (see also [1, Remark 3.2], it was shown that if $\beta \leq 2$, then there are infinitely many s for which $1 - W(S - s, 1) = b(s)$ is unusually small while $W(S - s, 1 + \psi_\beta(s))$ is comparatively large. More precisely, let ν_n be the unique time in I_n for which $b(\nu_n) = \min_{I_n} b$. Then there are infinitely many n such that

$$(33) \quad W(S - \nu_n, 1 + \psi_\beta(e^{-n})/S) > 1 + \kappa\, \psi_\beta(e^{-n})^{\frac{1}{2}} \qquad \text{and} \qquad b(\nu_n) < \psi_\beta(e^{-n})^{\frac{1}{2}}.$$

In view of (1), the components of $\{W > 1\}$ which correspond to the highest excursions of W above 1 are those which intersect the vertical segment $\{S - \nu_n\} \times [1, \infty)$. Notice that the typical magnitude of $b(\nu_n)$ is $\sqrt{\nu_n}$, so (33) describes a somewhat unusual event.

Let Q_n be the component of $\{(s,t) : W(S - s, 1 + t/S) > 1\}$ which contains $(\nu_n, \psi_\beta(e^{-n}))$. This component may be empty if for instance, $W < 1$ at this point, but according to the above, it will be non-empty for infinitely many n.

In [1, Theorems 3.9 and 3.11], we gave bounds on the asymptotic height and width of Q_n. Here, we want to go deeper into the study of functionals of Q_n which are authentically two-dimensional. The most natural such functional is the area A_n of Q_n, and we are going to identify the asymptotic conditional distribution of A_n given conditions (33).

Let $\{c(u),\ u \in \mathbb{R}\}$ be a process such that $c(0)$ is uniform on $[0, 1]$ and $(c(u) - c(0),\ u \geq 0)$ and $(c(-u) - c(0),\ u \geq 0)$ are independent Bessel(3) processes independent of $c(0)$. Let $\{\tilde{B}(v),\ v \geq 0\}$ be a Brownian motion independent of c, and let A be the area of the component Q of $\{(u,v) : \tilde{B}(v) - c(u) > 0\}$ which contains $(0, 1)$.

Theorem 5.1 *As $n \to \infty$, the conditional distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of $\psi_\beta(e^{-n})^{-2} A_n$ given conditions (33) converges to the conditional distribution of A given $\tilde{B}(1) - c(0) > \kappa$.*

Remark 5.2 *(a)* Theorem 5.1 describes the following situation. An observer approaches the point $(S, 1)$ along the curve $t = 1 + \psi_\beta(S - s)$. He observes a sequence of excursions of W high above level 1, and chooses a subsequence by looking at both their height and the values of W along the line $t = 1$ used to define S. The theorem says that the conditional distribution of the area of these bubbles, suitably normalized, converges to that of the random variable A.

(b) The main idea in the proof of the theorem is that if the height of an excursion of W above 1 is large compared to the error term in the approximation (1), then this component is not very different from the components of $\{B > b\}$ that we studied in the previous sections. Our particular method for selecting the component is not essential for the proof of this result or for the distribution of the limiting processes $\tilde{B}$ and c, but different methods would lead to a different conditioning of the limit process and possibly to a different normalization factor.

Lemma 5.3 *Suppose (B_n) and (c_n) are sequences of continuous processes that converge weakly to $\tilde{B}$ and c respectively. Let A'_n be the area of the component Q'_n of*

$$\{(u, v) : B_n(v) - c_n(u) > 0\}$$

which contains $(0, 1)$. Then A'_n converges weakly to A.

Proof. By a theorem of Skorohod [4, Chapter 1, Theorem 2.7], we can assume that B_n, c_n, $\tilde{B}$ and c are all defined on the same probability space, and that $B_n(\cdot) \to \tilde{B}(\cdot)$ and $c_n(\cdot) \to c(\cdot)$ uniformly on compact sets with probability one. We are going to prove that in this case, $A'_n \to A$ a.s., which will establish the lemma.

Notice that if $(u, v) \in Q$, then there is $\varepsilon > 0$ and a path $\Gamma \subset Q$, with extremites $(0, 1)$ and (u, v), along which $\tilde{B} - c > \varepsilon$. Therefore, for sufficiently large n, $\tilde{B}_n - c_n > 0$ along this path, and so $(u, v) \in Q_n$. This implies that $Q \subset \liminf Q_n$, so $A \leq \liminf A'_n$ a.s.

We now show that

$$\limsup Q_n \subset (Q \cup Q_0), \tag{34}$$

where $Q_0 \subset \{(u, v) : \tilde{B}(v) - b(u) = 0\}$. Since the u-sections of this set are level sets of $\tilde{B}$, hence have Lebesgue measure 0, this last set a.s. has zero two-dimensional Lebesgue measure, and so (34) will imply $\limsup A'_n \leq A$ a.s., and the proof will be complete.

To prove (34), we need the following fact. If b_1 and b_2 are two independent non-degenerate diffusions, and E_i is the set of local extremum values of b_i, $i = 1, 2$, then $P\{E_1 \cap E_2 \neq \phi\} = 0$ (Since the local minimum and maximum of b_1 on each dyadic interval has a continuous distribution and b_1 and b_2 are independent, each local extremum value of b_1 will belong with probability 0 to the countable set of local extremum values of b_2). In particular, the sets of local extremum values of $\tilde{B}$ and c are a.s. disjoint.

Assume that $(u_0, v_0) \in Q_n$ for infinitely many n. In order to establish (34), it is sufficient to prove that if $\tilde{B}(v_0) - b(u_0) > 0$, then $(u_0, v_0) \in Q$.

In view of Remark 2.3 (b) and by property (P) of Section 2, Q is inscribed in a rectangle $R = [\sigma, \sigma'] \times [\tau, \tau']$ with the following properties:

$$c(\sigma) = c(\sigma') = \max_{v \in [\tau, \tau']} B(v) \qquad \text{and} \qquad \tilde{B}(\tau) = \tilde{B}(\tau') = \min_{u \in [\sigma, \sigma']} c(u),$$

on $[\sigma, \sigma']$, c is accomplishing an excursion below level $c(\sigma)$ and on $[\tau, \tau']$, $\tilde{B}$ is accomplishing an excursion above level $\tilde{B}(\tau)$. Assume to begin with that $(u_0, v_0) \notin R$. By the fact mentioned above, c does not have a local maximum value at level $c(\sigma)$, nor does $\tilde{B}$ have a local minimum value at level $\tilde{B}(\tau)$. Due to the continuity of $\tilde{B}$ and c, there are $\varepsilon, \varepsilon', \eta, \eta' > 0$ and $\alpha > 0$ such that

$$c(\sigma - \varepsilon) = c(\sigma' + \varepsilon') > c(\sigma) + \alpha, \qquad \tilde{B}(\tau - \eta) = \tilde{B}(\tau' + \eta') < \tilde{B}(\tau) - \alpha,$$

and

$$\min_{u \in [\sigma - \varepsilon, \sigma' + \varepsilon']} c(u) = \min_{u \in [\sigma, \sigma']} c(u), \qquad \max_{v \in [\tau - \eta, \tau' + \eta']} \tilde{B}(v) = \max_{v \in [\tau, \tau']} \tilde{B}(v).$$

Therefore, on the boundary of the rectangle $[\sigma - \varepsilon, \sigma' + \varepsilon'] \times [\tau - \eta, \tau' + \eta']$, we have $\tilde{B} - c < -\alpha$, and therefore (u_0, v_0) does not belong to Q_n for sufficiently large n, a contradiction.

It follows that (u_0, v_0) is in the interior of R. Assume that $(u_0, v_0) \notin Q$. In this case, we apply Algorithm A with B replaced by $\tilde{B}$ and b replaced by c. The algorithm will stop after k steps, say, and output NO. When this occurs, the algorithm has in fact constructed the rectangle $\tilde{R}$ which circumscribes the component of $\{(u, v) : \tilde{B}(v) - c(u) > 0\}$ which contains (u_0, v_0), and which satisfies property (P) of Section 2. On the boundary of $\tilde{R}$, we have $\tilde{B} - c \leq 0$, but if we extend $\tilde{R}$ slightly as we did above and use the fact that the local extremum values of $\tilde{B}$ and c are disjoint, then we obtain a larger rectangle containing (u_0, v_0) on the boundary of which $\tilde{B} - c < -\alpha$ for some $\alpha > 0$. Therefore, $(u_0, v_0) \notin Q_n$ for all sufficiently large n, a contradiction. It follows that $(u_0, v_0) \in Q$, and the lemma is proved. ♣

PROOF OF THEOREM 5.1. Define

$$B'_n(t) = \psi_\beta(e^{-n})^{-\frac{1}{2}} B(\psi_\beta(e^{-n})\, t), \qquad b'_n(s) = \psi_\beta(e^{-n})^{-\frac{1}{2}} b(e^{-n} + \psi_\beta(e^{-n})\, s),$$

and $a_n = e^{-n}(e-1)/\psi_\beta(e^{-n})$. Set $\varepsilon_n = e^{3(1-n)/4}$. By [1, (17)–(18)], the component of

$$\{(s, t) : B(t) - b(s) \pm \varepsilon_n > 0, \quad s \geq e^{-n},\ t \geq 0\} \tag{35}$$

which contains $(\nu_n, \psi_\beta(\nu_n))$ is a superset/subset of Q_n. Notice that components C of the set (35) are in one-to-one correspondence with components C' of the set

$$\{(s, t) : B'_n(t) - b'_n(s) \pm \varepsilon_n/\psi_\beta(e^{-n}) > 0, \quad 0 \leq s \leq a_n,\ 0 \leq t\}, \tag{36}$$

and the ratio of the area of C to that of C' is $\psi_\beta(e^{-n})^2$. The component Q_n corresponds to the component Q'_n of the set (36) which contains $(\nu'_n, 1)$, where $\nu'_n = (\nu_n - e^{-n})/\psi_\beta(e^{-n})$.

Notice that B'_n is a Brownian motion independent of b'_n, for all n, so distributional properties are unchanged if we replace B'_n by $\tilde{B}$. By the Markov property and scaling, b' is a Bessel(3) process. The distribution of $b'_n(0)$ is that of $e^{-n/2}\psi_\beta(e^{-n})^{-\frac{1}{2}}b(1)$. In particular, $b'_n(0) \to \infty$.

For $-\nu'_n \le u \le a_n$, let $c_n(u) = b'_n(\nu'_n + u)$. In order to complete the proof, we only need to check that the conditional distribution of the process $(c_n(u))$ given

$$\tilde{B}(1) - c_n(0) \pm \varepsilon_n/\psi_\beta(e^{-n}) > \kappa \qquad \text{and} \qquad c_n(0) < 1 \tag{37}$$

converges to the conditional distribution of c given $\tilde{B}(1) - c(0) > \kappa$, since the conclusion of the theorem will then follow from Lemma 5.3 and the fact that $\varepsilon_n/\psi_\beta(e^{-n})^{\frac{1}{2}} \to 0$.

For this, let α be a random variable with the distribution of $b(1)$ under Q^0, set $d_n = e^{-n/2}/\psi_\beta(e^{-n})^{\frac{1}{2}}$, $\alpha_n = d_n\,\alpha$ and $T_1^n = \inf\{s \ge 0 : b'_n(s) = 1\}$. By the strong Markov property of b'_n and [8, Chap. XII, Cor. (4.4)], the $C(\mathbb{R}_+, \mathbb{R}) \times (C(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R})$-valued random variable

$$((b'_n(T_n^1 - u),\ 0 \le u \le T_n^1),\ ((b'_n(T_n^1 + u),\ u \ge 0),\ \min_{[0,a_n]} b'_n))$$

has the same conditional distribution given (37) as Z_n given $\tilde{B}(1) - \min_{[0,a_n - \sigma_{\alpha_n - 1}]} \hat{b} \pm \varepsilon_n/\psi_\beta(e^{-n}) > \kappa$, where

$$Z_n = ((1 + \tilde{b}(u),\ 0 \le u \le \sigma_{\alpha_n - 1}),\ ((\hat{b}(u),\ u \ge 0),\ \min_{[0,a_n - \sigma_{\alpha_n - 1}]} \hat{b})),$$

$\tilde{b}$ and $\hat{b}$ are independent Bessel(3) processes started respectively at 0 and 1, independent of α, and for $a \in \mathbb{R}_+$, $\sigma_a = \inf\{u \ge 0 : \tilde{b}(u) = a\}$. Notice that $\alpha_n \to \infty$, so $\sigma_{\alpha_n - 1} \to \infty$, and $a_n - \sigma_{\alpha_n - 1} \sim d_n^2(e - 1 - \sigma_\alpha) \to \infty$ in distribution, so

$$Z_n I_{\{\tilde{B}(1) - \min_{[0,a_n - \sigma_{\alpha_n - 1}]} \hat{b} \pm \varepsilon_n/\psi_\beta(e^{-n})^{\frac{1}{2}} > \kappa\}} \to Z I_{\{\tilde{B}(1) - \min_{[0,\infty)} \hat{b} > \kappa\}}$$

in distribution, where $Z = ((1 + \tilde{b}(u),\ u \ge 0),\ ((\hat{b}(u),\ u \ge 0),\ \min_{[0,\infty)} \hat{b}))$. Let $T_1 = \sup\{u \le 0 : c(u) = 1\}$. Then since $\min_{[0,\infty)} \hat{b}$ is uniform on $[0, 1]$ by [8, Chap.VII, Cor. (3.4)], the strong Markov property, [8, Chap.VII, Cor. (4.6)] and [8, Chap.VI, Prop. (3.10)] imply that Z has the same distribution as $((c(T_1 - u),\ u \ge 0),\ (c(T_1 + u),\ u \ge 0), c(0)))$, so the proof is complete. ♣

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