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Markov Properties for Certain Random Fields

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Abstract

Lévy's Markov and sharp Markov properties for random fields are studied, first in a general setting, and then in the context of two-parameter processes. It is shown that if Lévy's Markov property holds relative to finite unions of sets in some neighborhood base, then it holds for all bounded open sets. Two-parameter Gaussian processes which satisfy the usual Markov property along certain one-parameter curves are shown to satisfy Lévy's Markov property; they are in fact transforms of the Brownian sheet. Finally, a new proof is given that the Poisson sheet satisfies Lévy's sharp Markov property relative to all bounded relatively convex open sets.

1 Introduction

Let E be a topological space and let $\{X_t,\ t \in E\}$ be a stochastic process with parameter set E. There are a number of legitimate generalizations of the Markov property for X. One of the most appealing was suggested by P. Lévy. It is roughly this. If $A \subset E$, then the processes $\{X_t,\ t \in A\}$ and $\{X_t,\ t \in A^c\}$ are conditionally independent given $\{X_t,\ t \in \partial A\}$. If one thinks of A as the "past", of A^c as the "future", and of ∂A as the "present", this becomes the statement that the past and future are conditionally independent given the present, which is just the ordinary Markov property. We call this the *sharp Markov property* below; there are others which are equally relevant in certain situations, and even the sharp Markov property must be modified slightly if it is to apply to a large class of processes.

[1]This paper is based on a portion of the first author's doctoral dissertation. Tragically, E. Carnal died while this manuscript was still in a preliminary state. The present version has been assembled by the second author from the preliminary version and the thesis, with an additional section suggested by the first author's unfinished work.

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Our aim is to look at several of these Markov-like properties and the relations between them. The basic question addressed in the first section is this: if the Markov property holds for a certain class of sets, when can it be extended to a larger class? The second and third sections are largely influenced by the Brownian sheet, a process whose parameter set is $\mathbb{R}_+^2$ so the Markov properties we look at are connected with the order properties of the plane. We concentrate on Gaussian processes in section two, and the Poisson sheet in section three.

2 Markov Properties and Conditional Independence

Let (E, d) be a separable metric space, and let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that for each subset $A \subset E$ there exists a sub-σ-field of $\mathcal{F}$, denoted $\mathcal{F}(A)$, and that for any sequence $A_1, A_2, \ldots$ of sets, $\mathcal{F}(\bigcup_n A_n) = \vee_n \mathcal{F}(A_n)$. Let X denote the collection of these σ-fields: $X = \{\mathcal{F}(A) : A \subset E\}$. If the $\mathcal{F}(A)$ are generated by a stochastic process, say $\mathcal{F}(A) = \sigma\{Y_t,\ t \in A\}$, we say that X is the natural filtration of the process. However, there are many situations in which no process enters, so we will concentrate on the σ-fields rather than on the processes.

For $A \subset E$, define

$$\mathcal{G}(A) = \bigcap_{\varepsilon > 0} \mathcal{F}(A_\varepsilon)\,.$$

where $A_\varepsilon = \{x : d(x, A) < \varepsilon\}$. We call $\mathcal{G}(A)$ the **germ field** of A.

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be σ-fields. We write $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ are conditionally independent given $\mathcal{C}$, and we say that $\mathcal{C}$ is a **splitting field** for $\mathcal{A}$ and $\mathcal{B}$. Let us recall several useful properties of conditional independence. The first lemma is immediate and the second is due to Hunt [5].

Lemma 2.1 *(i)* $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$ *if and only if for all* $A \in \mathcal{A}$,

$$P\{\mathcal{A} \mid \mathcal{B} \vee \mathcal{C}\} = P\{A \mid \mathcal{C}\}\;;$$

(ii) if $\mathcal{A}' \subset \mathcal{A}$ *and* $\mathcal{B}' \subset \mathcal{B}$*, then*

$$\mathcal{A} \perp \mathcal{B} \mid \mathcal{C} \Longrightarrow \mathcal{A}' \perp \mathcal{B}' \mid \mathcal{C}\;.$$

Lemma 2.2 *Suppose $\mathcal{D} \subset \mathcal{A} \cup \mathcal{B}$ is a collection of subsets of Ω. (Caution: $\mathcal{A} \cup \mathcal{B}$, not $\mathcal{A} \vee \mathcal{B}$.) If $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}$ then*
(i) $\mathcal{A} \vee \mathcal{C} \perp \mathcal{B} \vee \mathcal{C} \mid \mathcal{C}$;
(ii) $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C} \vee \sigma(\mathcal{D})$.

We will often use part (*ii*) in the following form: if $\mathcal{D}_1 \subset \mathcal{A}$ and $\mathcal{D}_2 \subset \mathcal{B}$ then $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C} \Rightarrow \mathcal{A} \perp \mathcal{B} \mid \mathcal{C} \vee \sigma(\mathcal{D}_1) \vee \sigma(\mathcal{D}_2)$.

Lemma 2.3 *Suppose $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, and $\mathcal{D}_2$ are σ-fields, with $\mathcal{C}_1 \cup \mathcal{C}_2 \subset \mathcal{B}$. Then*
(i) if $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}_i$, $i = 1, 2$, then $\mathcal{A} \perp \mathcal{B} \mid \mathcal{C}_1 \cap \mathcal{C}_2$.
(ii) If $\mathcal{A} \perp \mathcal{B} \mid \mathcal{D}_1 \vee \mathcal{D}_2$ and $\mathcal{A} \perp \mathcal{D}_2 \mid \mathcal{D}_1$, then $\mathcal{A} \perp \mathcal{B} \vee \mathcal{D}_2 \mid \mathcal{D}_1$.

PROOF. *(i)* is due to McKean [7], and *(ii)* follows from Lemma 2.1:

$$P\{A \mid \mathcal{B} \vee \mathcal{D}_1 \vee \mathcal{D}_2\} = P\{A \mid \mathcal{D}_1 \vee \mathcal{D}_2\} = P\{A \mid \mathcal{D}_2\}$$

for $A \in \mathcal{A}$. ♣

If the σ-fields X are generated by a Gaussian process, we have an additional property whose proof is elementary.

Lemma 2.4 *Suppose X is the natural filtration of a Gaussian process. Let A, B_1, B_2, and C be subsets of E. If $\mathcal{F}(A) \perp \mathcal{F}(B_i) \mid \mathcal{F}(C)$, $i = 1, 2$, then $\mathcal{F}(A) \perp \mathcal{F}(B_1 \cup \mathcal{B}_2) \mid \mathcal{F}(C)$.*

Definition 2.1 *X has the* **sharp Markov property** *(SMP) relative to a set $A \subset E$ if $\mathcal{F}(A) \perp \mathcal{F}(A^c) \mid \mathcal{F}(\partial A)$. X has the* **Markov property** *(MP) relative to a set $A \subset E$ if $\mathcal{F}(A) \perp \mathcal{F}(A^c) \mid \mathcal{G}(\partial A)$.*

Here are some elementary facts about the SMP and the MP which follow easily from the above lemmas.

Proposition 2.5 *The following three statements are equivalent:*
(i) X has the SMP relative to A;
(ii) $\mathcal{F}(\bar{A}) \perp \mathcal{F}(\bar{A}^c) \mid \mathcal{F}(\partial A)$;
(iii) $\mathcal{F}(A \setminus \partial A) \perp \mathcal{F}(A^c \setminus \partial A) \mid \mathcal{F}(\partial A)$.

PROOF. $\mathcal{F}(\bar{A}) = \mathcal{F}(A) \vee \mathcal{F}(\partial A)$ and $\mathcal{F}(\bar{A}^c) = \mathcal{F}(A^c) \vee \mathcal{F}(\partial A)$, so $(i) \Rightarrow (ii)$ by Lemma 2.2 *(i)*. Note that $(iii) \Rightarrow (ii)$ by the same reasoning, while both implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$ follow from Lemma 2.1 *(ii)*. ♣

Proposition 2.6 *The following are equivalent:*

(i) X has the MP relative to A;

(ii) if G is a neighborhood of ∂A, then

$$\mathcal{F}(A) \perp \mathcal{F}(A^c) \mid \mathcal{F}(G)\,;$$

(iii) $\mathcal{G}(A) \perp \mathcal{G}(A^c) \mid \mathcal{G}(\partial A)\,.$

PROOF. Note that $\mathcal{F}(G) = \mathcal{F}(G \cap A) \vee \mathcal{F}(G \cap A^c)$, and let $\mathcal{D} = \mathcal{F}(G \cap A) \cup \mathcal{F}(G \cap A^c)$ in Lemma 2.2 *(ii)* to see that $(i) \Rightarrow (ii)$.

To see that $(ii) \Rightarrow (iii)$, let $\Lambda_1 \in \mathcal{G}(A)$, $\Lambda_2 \in \mathcal{G}(A^c)$, and let G_n be a $\frac{1}{n}$-neighborhood of ∂A. By *(ii)* and Lemma 2.2 *(i)*, $\mathcal{F}(\mathcal{A} \cup G_n) \perp \mathcal{F}(\mathcal{A}^c \cup G_n) \mid \mathcal{F}(G_n)$ so $P\{\Lambda_1 \cap \Lambda_2 \mid \mathcal{F}(G_n)\} = P\{\Lambda_1 \mid \mathcal{F}(G_n)\}P\{\Lambda_2 \mid \mathcal{F}(G_n)\}$. Note that $\bigcap_n \mathcal{F}(G_n) = \mathcal{G}(\partial A)$, so that the martingale convergence theorem can be applied to each term to see that $P\left\{\Lambda_1 \cap \Lambda_2 \mid \mathcal{G}(\partial A)\right\} = P\left\{\Lambda_1 \mid \mathcal{G}(\partial A)\right\} P\left\{\Lambda_2 \mid \mathcal{G}(\partial A)\right\}$.

Finally, $(iii) \Rightarrow (i)$ by Lemma 2.1 *(ii)*. ♣

Proposition 2.7 *If X has the SMP relative to A, it also has the MP relative to A.*

PROOF. By Lemma 2.2 *(ii)*, $\mathcal{F}(A) \perp \mathcal{F}(A^c) \mid \mathcal{F}(G)$ for any neighborhood G of ∂A, and the result follows from Proposition 2.6. ♣

Proposition 2.8 *Let A_1 and A_2 be subsets of E such that $\partial(A_1 \cup A_2) = \partial A_1 \cup \partial A_2$. If X has the SMP (respectively MP) relative to A_i, $i = 1, 2$, then X has the SMP (respectively MP) relative to $A_1 \cup A_2$.*

PROOF. Suppose X has the SMP relative to A_i, $i = 1, 2$. Let $D_i = \partial A_i$. By Proposition 2.5, $\mathcal{F}(\bar{A}_i) \perp \mathcal{F}(\bar{A}_i^c) \mid \mathcal{F}(D_i)$, $i = 1, 2$. Let $\mathcal{D} = \mathcal{F}(\bar{A}_2 \cap \bar{A}_1) \cup \mathcal{F}(\bar{A}_2 \cap \bar{A}_1^c)$. Then $\sigma(\mathcal{D}) = \mathcal{F}(\bar{A}_2)$, so by Lemma 2.2 *(ii)*,

$$\mathcal{F}(\bar{A}_1) \perp \mathcal{F}(\bar{A}_1^c) \mid \mathcal{F}(\mathcal{D}_1) \vee \mathcal{F}(\bar{A}_2)\,.$$

Similarly, $\mathcal{F}(D_1) = \mathcal{F}(D_1 \cap A_2) \vee \mathcal{F}(D_1 \cap A_2^c)$, so a second application of Lemma 2.2 *(ii)* shows that $\mathcal{F}(\bar{A}_2) \perp \mathcal{F}(\bar{A}_2^c) \mid \mathcal{F}(D_1 \cup D_2)$.

By Lemma 2.1 *(ii)*,

$$\mathcal{F}(\bar{A}_1) \perp \mathcal{F}(A_1^c \cap A_2^c) \mid \mathcal{F}(D_1) \vee \mathcal{F}(\bar{A}_2)$$

and

$$\mathcal{F}(\bar{A}_2) \perp \mathcal{F}(A_1^c \cap A_2^c) \mid \mathcal{F}(D_1 \cup D_2)\,.$$

By Lemma 2.3 *(ii)* with $\mathcal{D}_1 = \mathcal{F}(D_1 \cup D_2)$, and $D_2 = \mathcal{F}(\bar{A}_2)$,

$$\mathcal{F}(\bar{A}_1 \cup \bar{A}_2) \perp \mathcal{F}(A_1^c \cap A_2^c) \mid \mathcal{F}(D_1 \cup D_2)\,.$$

But $D_1 \cup D_2 = \partial(A_1 \cup A_2)$, so this implies the SMP relative to $A_1 \cup A_2$.

If X has the MP relative to A, repeat the argument with the D_i replaced by neighborhoods G_i of ∂A_i, and apply Proposition 2.6. ♣

Proposition 2.9 *Let (G_n) be a sequence of disjoint open sets. If X satisfies the SMP (respectively MP) relative to each G_n, it satisfies the SMP (respectively MP) relative to $\bigcup_n G_n$.*

Proof. Suppose X satisfies the SMP relative to each G_n. Since $\partial(\bigcup_1^N G_n) = \bigcup_1^N \partial G_n$ for each N, Proposition 2.8 and an induction argument imply that X has the SMP relative to $\bigcup_1^N G_n$. Fix N and let $\Lambda \in \mathcal{F}(\bigcup_1^N G_n)$. For $m > N$ the SMP implies that

$$P\{\Lambda \mid \mathcal{F}(\textstyle\bigcap_1^m G_n^c)\} = P\{\Lambda \mid \mathcal{F}(\bigcup_1^m \partial G_n)\}\,.$$

Let $m \to \infty$. By the martingale convergence theorem,

$$P\{\Lambda \mid \textstyle\bigcap_{m=1}^\infty \mathcal{F}(\bigcap_{n=1}^m G_n^c)\} = P\{\Lambda \mid \mathcal{F}(\bigcup_1^\infty \partial G_n)\}\,.$$

Let $G = \bigcup_1^\infty G_n$. Then $G^c = \bigcap_1^\infty G_n^c \supset \partial G \supset \bigcup_1^\infty \partial G_n$, so that

$$\textstyle\bigcap_{m=1}^\infty \mathcal{F}\left(\bigcap_1^m G_n^c\right) \supset \mathcal{F}\left(\bigcap_1^\infty G_n^c\right) = \mathcal{F}\left(G^c\right) \supset \mathcal{F}(\partial G) \supset \mathcal{F}\left(\bigcup_1^\infty \partial G_n\right).$$

Thus it follows that $P\{\Lambda \mid \mathcal{F}(G^c)\} = P\{\Lambda \mid \mathcal{F}(\partial G)\}$, since both sides equal $P\{\Lambda \mid \mathcal{F}(\bigcup_1^\infty \partial G_n)\}$. This is true for $\Lambda \in \mathcal{F}(\bigcup_1^N G_n)$ for any N, hence for $\Lambda \in \bigvee_1^\infty \mathcal{F}(G_n) = \mathcal{F}(G)$. By Lemma 2.1, we are done. The argument for the MP is similar. ♣

Corollary 2.10 *Suppose the space E is connected and locally connected and that the SMP (respectively the MP) holds for every open connected set whose complement is also connected. Then it holds for every open set.*

PROOF. If G is open, write $G = \bigcup_n G_n$, where the G_n are the connected components of G. Let $O_n = (\bar{G}_n)^c$ and let (O_{nj}) denote the connected components of O_n. If A is an open set containing $\bar{G}_n$, it must intersect each O_{nj}. Indeed, if it missed O_{nj}, say, then O_{nj} and $A \cup \left(\bigcup_{k\neq j} O_{nk}\right)$ would disconnect $\bar{G}_n \cup (\bigcup_k O_{nk}) = E$. It follows that $O_{nj}^c = \bar{G}_n \cup \left(\bigcup_{k\neq j} O_{nk}\right)$ is connected. Thus the hypothesis applies to the O_{nj}: the SMP (respectively MP) holds for each O_{nj}, hence for $O_n = \bigcup_n O_{nj}$ by Proposition 2.9. As $O_n^c = \bar{G}_n$, we have

$$\mathcal{F}(O_n) \perp \mathcal{F}(\bar{G}_n) \mid \mathcal{F}(\partial O_n)\,,$$

(respectively, $\mathcal{F}(O_n) \perp \mathcal{F}(\bar{G}_n) \mid \mathcal{G}(\partial O_n)$), hence by Proposition 2.5 (respectively Proposition 2.6 and Lemma 2.1 *(ii)*) the SMP (respectively MP) holds for G_n. A final application of Proposition 2.9 shows that the SMP (respectively MP) holds for G. ♣

Let $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference of A and B. For $x \in E$, let $d(x, B) = \inf\{d(s, y) : y \in B\}$. For any set B, let $N_\varepsilon(B) = \{x : d(x, B) < \varepsilon\}$ be the ε-neighborhood of B. The next results concern the MP rather than the SMP. We first show that if the MP holds for a class of sets which approximate A well enough, it also holds for A.

Theorem 2.11 *Let $A \in E$ and let $\{A_\alpha,\ \alpha \in T\}$ be a family of sets with the property that for each $\varepsilon > 0$ there is an $\alpha \in T$ such that $A \triangle A_\alpha \subset N_\varepsilon(\partial A)$. Then if X has the MP relative to A_α for each $\alpha \in T$, it also has the MP relative to A.*

PROOF. Let $\varepsilon > 0$ and let $G = N_\varepsilon(\partial A)$. There exists α such that $A_\alpha \triangle A \subset G$. Thus $A \setminus G = A_\alpha \setminus G$ and $A^c \setminus G = A_\alpha^c \setminus G$. It follows that $\partial A_\alpha \subset G$. By hypothesis and Proposition 2.6, $\mathcal{F}(A_\alpha) \perp \mathcal{F}(A_\alpha^c) \mid \mathcal{F}(G)$, hence by Lemma 2.2 *(i)*,

$$\mathcal{F}(A_\alpha \cup G) \perp \mathcal{F}(\mathcal{A}_\alpha^c \cup G) \mid \mathcal{F}(G)\,.$$

Since $A \subset A_\alpha \cup G$ and $A^c \subset A_\alpha^c \cup G$, Lemma 2.1 *(ii)* implies that $\mathcal{F}(A) \perp \mathcal{F}(A^c) \mid \mathcal{F}(G)$, and the result follows from Proposition 2.6. ♣

Notice that we gain nothing by assuming that X has the SMP—not just the MP—relative to each A_α, for the conclusion would still only be that X satisfied the MP relative to A. One can use this theorem when the A_α are sets which either increase or decrease to A. Here is a typical application.

Corollary 2.12 *Let $\mathcal{A}$ be a class of open subsets of E which contains a neighborhood basis of connected subsets of E. Let $\mathcal{A}_f$ be the class of all finite unions of sets in $\mathcal{A}$. Then if X has the MP relative to each set in $\mathcal{A}_f$, it has the MP relative to each relatively compact open subset of E.*

PROOF. Let G be open and relatively compact in E, and let $\varepsilon > 0$. Then $\bar{G}$ is compact, and can be covered by a finite number, say $A_1, \ldots, A_n$ of connected elements of $\mathcal{A}$ which have diameter less than $\varepsilon/2$. By throwing away some of the A_i if necessary, we may assume that each A_i intersects $\bar{G}$. Let $A = \bigcup_{i=1}^n A_i$. Now $G \subset \bar{G} \subset A$, so $\partial G \subset A \setminus G = A \triangle G$. If $z \in A \triangle G$, then $z \in A_j$ for some j; since $A_j \cap \bar{G} \neq \phi$, A_j must contain points of $\bar{G}$ and of G^c. Since A_i is connected by hypothesis, it must contain points of ∂G (otherwise, $A_i \cap G$ and $A_i \cap (\bar{G})^c$ would disconnect A_i). Thus $d(z, \partial G) < \varepsilon$, hence $A \triangle G \subset N_\varepsilon(\partial G)$, and the result follows from Theorem 2.11. ♣

3 Markov Properties in the Plane

Let I and J be intervals, not necessarily bounded or open, and let $E = I \times J$. By *rectangle*, we mean a bounded sub-rectangle of E of the form $I' \times J'$, where $I' \subset I$ and $J' \subset J$. We define the partial order $\prec$ and its complementary order $\overset{c}{\prec}$ by

$$\begin{aligned} (s,t) &\prec (s',t') \text{ iff } s \leq s' \text{ and } t' \leq t\,; \\ (s,t) &\overset{c}{\prec} (s',t') \text{ iff } s \leq s' \text{ and } t \geq t'\,. \end{aligned}$$

Generalizations of the results of this section from $\mathbb{R}^2$ to $\mathbb{R}^n$ are straightforward for the most part, though they may be messy. For instance, one needs 2^{n-1} partial orders in $\mathbb{R}^n$ in place of the two above. We will confine our remarks to $\mathbb{R}^2$ for simplicity.

We will define several versions of the Markov property. Our main concern is with Gaussian processes, so we will consider the case where X is the natural filtration of a Gaussian process $Y = \{Y_z,\, z \in E\}$. If $z = (s,t)$, we will often write $Y(s,t)$ in place of Y_z.

Definition 3.1 *X satisfies the* **Markov property** *(resp.* **sharp Markov property***) if it has the MP (resp. SMP) relative to all open relatively compact subsets of E. X has the* **elementary Markov property** *if it has the MP relative to all finite unions of open rectangles. X has the* **order Markov property** *if, whenever $\gamma = \{\gamma(u),\, u \in [0,1]\}$ is a parameterized curve in E with the property that γ is increasing relative to either of the two partial orders $\prec$ or $\overset{c}{\prec}$, that $\{Y_{\gamma(u)},\, u \in [0,1]\}$ is a Markov process.*

There are many other Markov-type properties which have been suggested for two-parameter processes. Wong and Zakai [14] have studied a property which is related to both the MP and the order MP, and Nualart and Sanz [10] and Korezlioglu et al [6] have considered a closely related property.

Proposition 3.1 *The MP and the elementary MP are equivalent.*

PROOF. The MP clearly implies the elementary MP. The converse follows from Corollary 2.12 since the rectangles form a base for the topology of E.♣

We will now specialize to centered Gaussian processes, for which Markov properties are more tractable. For a mean-zero Gaussian process $\{Y_z,\ z \in E\}$, the order MP reduces to the following:

if $z_1,\, z_2,$ and z_3 are points in E for which either $z_1 \prec z_2 \prec z_3$ or $z_1 \overset{c}{\prec} z_2 \overset{c}{\prec} z_3$, then $Y_{z_1} \perp Y_{z_2} \mid Y_{z_3}$.

If $\Gamma(z,z')$ is the covariance function of Y, then it is easily seen that $Y_{z_1} \perp Y_{z_2} \mid Y_{z_3}$ if and only if

$$\Gamma(z_1,z_3) = \Gamma(z_1,z_2)\Gamma(z_2,z_3)(\Gamma(z_2,z_2))^{-1} \tag{1}$$

with the convention that $\frac{0}{0} = 0$.

The MP for a Gaussian process evidently reduces to a study of the covariance function. We say a covariance function $\Gamma(z,z')$ is of **Markov tensor product type**, or more simply, of **product type**, if there exist covariance functions $\Gamma_1(s,s')$, $s,s'\in I$ and $\Gamma_2(t,t')$, $t,t'\in J$, each of which is the covariance function of a one-parameter Gaussian Markov process, and a function f on E such that, if $z=(s,t)$ and $z'=(s',t')$, then

$$\Gamma(z,z') = f(z)f(z')\Gamma_1(s,s')\Gamma_2(t,t')\,. \tag{2}$$

For example, the Brownian sheet $\{W(s,t),\ s\geq 0, t\geq 0\}$ is a Gaussian process which has a covariance function of product type:

$$\Gamma((s,t'),(s',t')) = (s\wedge s')(t\wedge t')\,. \tag{3}$$

Theorem 3.2 *Suppose Y is a mean zero Gaussian process with a continuous covariance function which does not vanish on the diagonal. Then the following are equivalent.*

(i) Y has the order MP;

(ii) Γ is of product type;

(iii) there exist continuous increasing functions f on I and g on J, a continuous function h on E, and a Brownian sheet W, such that

$$Y(s,t) \equiv h(s,t)W(f(s),g(t))\,,$$

where "$\equiv$" means equivalence in distribution.

Proof. $(i)\Rightarrow(ii)$. By taking $Z_z=\Gamma(z,z)^{-\frac{1}{2}}Y_z$ if necessary, we can assume that $\Gamma(z,z)=1$. Let z_1, z_2, and z_3 be in E. The condition (2) that $Y_{z_1}\perp Y_{z_2}\mid Y_{z_3}$ becomes

$$\Gamma(z_1,z_3) = \Gamma(z_1,z_2)\Gamma(z_2,z_3)\,. \tag{4}$$

Let $z=(s,t)\prec(s',t')=z'$ and put $\bar{z}=(s,t')$ and $\underline{z}=(s',t)$. Both $z\prec\bar{z}\prec z'$ and $z\prec\underline{z}\prec z'$, so

$$\Gamma(z,z') = \Gamma(z,\bar{z})\Gamma(\bar{z},z') = \Gamma(z,\underline{z})\Gamma(\underline{z},z')\,. \tag{5}$$

Furthermore, both $\bar{z}\overset{c}{\prec}z'\overset{c}{\prec}\underline{z}'$ and $\bar{z}\overset{c}{\prec}z\overset{c}{\prec}\underline{z}$ so

$$\Gamma(\bar{z},\underline{z}) = \Gamma(\bar{z},z')\Gamma(z',\underline{z}) = \Gamma(\bar{z},z)\Gamma(z,\underline{z}) \tag{6}$$

Divide the second two terms of (5) by the corresponding terms of (6). Since Γ is symmetric, we see that $\Gamma(z,\bar{z})^2=\Gamma(\underline{z},z')^2$, so that $\Gamma(z,\bar{z})=\pm\Gamma(\underline{z},z')$

and hence that $\Gamma(z,\bar{z}) = \Gamma(\underline{z}, z')$. Indeed, Γ is continuous and non-vanishing by (4) and the plus sign holds if $z = \bar{z} = \underline{z}$, so it holds everywhere.

In terms of s and t,

$$\Gamma((s,t),(s,t')) = \Gamma((s',t),(s',t'))$$

so that the left-hand side depends only on (t,t'), not on s. By symmetry,

$$\Gamma((s,t),(s',t)) = \Gamma((s,t'),(s',t'))\,,$$

which depends only on (s,s'). Putting these together, we define $\Gamma_1(s,s') = \Gamma((s,t),(s',t))$ and $\Gamma_2(t,t') = \Gamma((s,t)(s,t'))$; then by (5)

$$\Gamma((s,t),(s',t')) = \Gamma_1(s,s')\Gamma_2(t,t')\,.$$

Now Γ_1 is the covariance function of the process $\{Y(s,t),\ s,t \in I\}$ for fixed t, which is a Gaussian Markov process since Y is Markovian along horizontal lines by hypothesis. By symmetry, Γ_2 is also a Markovian covariance function, so Γ is of product type.

(ii) $\Rightarrow$ *(iii)*: By [7, §3.1] there exist continuous functions φ_i and ψ_i such that φ_i/ψ_i is increasing, and $\Gamma_i(u,v) = \varphi_i(u \wedge v)\psi_i(u \vee v)$. It follows that if $W(s,t)$ is a Brownian sheet, the process

$$Z(s,t) = \psi_1(s)\psi_2(t)\, W\left(\frac{\varphi_1(s)}{\psi_1(s)}, \frac{\varphi_2(t)}{\psi_2(t)}\right)$$

has covariance function $\Gamma_1\Gamma_2$. Thus we take $f(s) = \varphi_1(s)\psi_1(s)^{-1}$, $g(s) = \varphi_2(s)\psi_2(s)^{-1}$, and $h(s,t) = \psi_1(s)\psi_2(t)$.

(iii) $\Rightarrow$ *(i)*: The map $(s,t) \mapsto (f(s),\, g(t))$ preserves both partial orders "$\prec$" and "$\stackrel{c}{\prec}$", so we need only show that the Brownian sheet satisfies the order MP. This follows by direct calculation using (3). ♣

Corollary 3.3 *Suppose that Y is a centered Gaussian process whose covariance is continuous and non-zero on the diagonal. If Y has the order MP, it also has the MP, and it has the SMP for finite unions of rectangles.*

PROOF. The mapping $(s,t) \mapsto (f(s), g(t))$ of E into $\mathbb{R}^2_+$ preserves order and set operations, and maps rectangles onto rectangles. By Theorem 3.2, it is enough to prove the Corollary for the Brownian sheet W. By [11, 3, 2][3] W satisfies the SMP for finite unions of rectangles. Thus it satisfies the MP by Proposition 3.1. ♣

Remark 3.1 The condition that the covariance function be continuous in Theorem 3.2 and Corollary 3.3 is not necessary, but it simplifies the proof enormously. The results are proved in [2] without the continuity condition.

4 Markov Properties of the Poisson Sheet

Let Π be a homogeneous Poisson random measure on the Borel sets $\mathcal{B}$ of $\mathbb{R}^2_+$, that is

(i) $\Pi(A)$ is a Poisson r.v. with parameter $|A|$, $A \in \mathcal{B}$;

(ii) If $A_1, \ldots, A_n$ are disjoint, $\Pi(A_i)$, $i = 1, \ldots, n$ are independent and $\Pi(\bigcup_i A_i) = \sum_i \Pi(A_i)$,

where $|A|$ is the Lebesgue measure of A. The measure Π is a sum of point masses. The points form a Poisson point process on $\mathbb{R}^2_+$. We will use the symbol Π for both the random measure and the point process. "Points" below will refer to the points of this process.

If $z = (s,t) \in \mathbb{R}^2_+$, let R_z (or R_{st}) denote the rectangle $[0,s] \times [0,t]$. The **Poisson sheet** $\{X_z,\ z \in \mathbb{R}^2_+\}$ is defined by $X_z = \Pi(R_z)$.

It is not hard to see from (i) and (ii) that X satisfies the SMP for rectangles, but it is not obvious that X will satisfy the SMP relative to any much larger class of sets. The Brownian sheet, for instance, satisfies the SMP for finite unions of rectangles, but not for many other sets [11, 12, 13, 3]. However, it turns out that the Poisson sheet satisfies the SMP for all bounded open sets.

The reason for the difference between the Poisson sheet and the Brownian sheet is in the global behavior: the Brownian sheet is continuous, while the Poisson sheet has discontinuities which propagate on lines. These discontinuities are the key to the SMP. In fact, all two parameter processes of

[3] This result is difficult. A short and elegant proof has recently been found by Z-M Yang [15].

independent increments with no Gaussian part satisfy the SMP with respect to bounded open sets, though this is a rather delicate fact in the case where the processes can have negative as well as positive jumps [4].

We will not prove this general theorem here, even for the Poisson sheet; we will limit ourselves to the special case of relatively convex open sets. (In fact, because of a delay in publishing this article—due entirely to the second author—it has been necessary to delete a conjecture in the original version, to the effect that the result should be true without the restriction of relative convexity. This conjecture has been proved in part by Merzbach and Nualart [8], who have shown that the SMP holds for a large class of point processes and for domains with piecewise-monotone boundaries; and by Dalang and the second author [4].) Both of the above proofs depend on earlier non-trivial results. By limiting ourselves to relatively convex sets, we can give a proof which is elementary and self-contained, and which avoids most of the cumbersome technicalities which come with increased generality.

The proper notion of convexity for the Poisson sheet is relative convexity. A set $A \subset \mathbb{R}_+^2$ is **relatively convex** if for each horizontal or vertical line L, $L \cap A$ is connected. Thus "relatively convex" means convex relative to horizontal and vertical lines. A convex set is relatively convex, but the converse is not true. The cross in the Swiss flag, for instance, is relatively convex but not convex. A relatively convex set need not be connected.

Theorem 4.1 *The Poisson sheet satisfies the sharp Markov property relative to all bounded open relatively convex sets, but not with respect to all unbounded open relatively convex sets. If A is a bounded open relatively convex set, then $\mathcal{F}(\partial A)$ is the minimal splitting field of $\mathcal{F}(A)$ and $\mathcal{F}(A^c)$.*

We will deal with the unbounded sets first. Let T be the triangular region $\{(s,t) : s > t\} \subset \mathbb{R}_+^2$. Then ∂T is the diagonal of the first quadrant, and $\mathcal{F}(\partial T) = \sigma\{X(t,t),\ t \geq 0\}$. Let

$$\begin{aligned} \Lambda_1 &= \{\exists \text{ exactly one point in } T \cap R_{11}\} \\ \Lambda_2 &= \{\exists \text{ exactly one point in } T^c \cap R_{11}\} \\ \Lambda_3 &= \{\exists \text{ exactly one point in } R_{11}\}\,. \end{aligned}$$

Now $\Lambda_3 = \{X(1,1) = 1\} \in \mathcal{F}(\partial T)$ and $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3 = \phi$, so

$$P\left\{\bigcap_1^3 \Lambda_i \mid \mathcal{F}(\partial T)\right\} \neq P\left\{\Lambda_1 \cap \Lambda_3 \mid \mathcal{F}(\partial T)\right\} P\left\{\Lambda_2 \cap \Lambda_3 \mid \mathcal{F}(\partial T)\right\}$$

since the first probability is zero and each of the last two is strictly positive. Thus $\mathcal{F}(T)$ is not conditionally independent of $\mathcal{F}(T^c)$ given $\mathcal{F}(\partial T)$.

Before dealing with the rest of the theorem we need some facts about relatively convex sets. The reader who is willing to limit himself to convex sets can skip points 1°–6°. In what follows, A is an open, bounded, connected, relatively convex subset of $\mathbb{R}^2_+$. Let $I = \{s : \exists t \ni (s,t) \in A\}$. A is connected, so I is an interval, say $I = (\alpha, \beta)$. For $s \in I$, define $M(s) = \sup\{t : (s,t) \in A\}$ and $m(s) = \inf\{t : (s,t) \in A\}$.

1° If $M(u) < \lambda < M(v)$, there exists an s between u and v such that $(s, \lambda) \in A$.

PROOF. If not, the half planes $\{t < \lambda$ and $\{t > \lambda\}$ disconnect A.

2° There exist s_0 and s_1, not necessarily unique, such that M increases on (α, s_1) and decreases on (s_1, β) ; and m decreases on (α, s_0) and increases on (s_0, β) .

PROOF. If the statement for M is false there exist λ and $u < v < w$ such that $M(v) < \lambda < M(u) \wedge M(w)$. By 1°, there exist $s' < v < s''$ such that the points (s', λ) and (s'', λ) are in A. By relative convexity, $(v, \lambda) \in A$ contradicting the definition of $M(v)$.

Extend M and m to α and β by continuity and define $\bar{M}(s) = \sup\{t : (s,t) \in \bar{A}\}$ and $\bar{m}(s) = \inf\{t : (s,t) \in \bar{A}\}$ for $s \in [\alpha, \beta]$. The following facts are now elementry, so we omit the proofs. We state them for M; analogous statements hold for m.

3° $\bar{M} \geq M$ and $\bar{M}(s) = M(s)$ at points of continuity of M. If M has a jump at s, the vertical line segment with endpoints $(s, M(s))$ and $(s, \bar{M}(s))$ forms part of ∂A. The set $N \equiv \{$ jumps of $\bar{M}$ $\}$ is countable.

4° ∂A is composed of points $\{(s, M(s)) :$ s a continuity point of $M\}$, $\{(s, m(s)) :$ s a continuity point of $m\}$, and vertical segments.

For a more graphical description of A, let $a = \bar{m}(s_0)$ and $b = \bar{M}(s_1)$. The points (s_0, a) and (s_1, b) are the "lowest" and "highest" points of ∂A. Similarly, there exist t_0 and t_1 such that (α, t_0) and (β, t_1) are in ∂A. We call these four points *relative extreme points* of A. In terms of these, we have the following picture of ∂A.

5° ∂A consists of four curves, (some of which may be degenerate):

the *upper left segment* UL with end points (α, t_0) and (s_1, b);
the *upper right segment* UR with end points (s_1, b) and (β, t_1);
the *lower right segment* LR with end points (s_0, a) and (β, t_1);
the *lower left segment* LL with end points (α, t_0) and (s_0, a).

For example, if A is a disc, the relative extreme points are at three, six, nine and twelve o'clock, while the curves UL, UR, LR, and LL are respectively from nine to twelve o'clock, from twelve to three o'clock, from three to six o'clock, and from six to nine o'clock.

6° Let $t = c$ be a horizontal line. Then there is a countable set N_1 such that if $c \notin N_1$ and $a < c < b$, then $t = c$ meets ∂A in exactly two points. Furthermore, if it meets, say, UL at the point (s, c), either
(i) (s, c) is in the interior of a vertical segment of UL, or
(ii) M is continuous at s and $M(s - \varepsilon) < M(s) < M(s + \varepsilon)$ for small $\varepsilon > 0$.

The Poisson sheet X has integer values. Each point of the point process gives rise to a jump discontinuity of X or, rather, since X has a two-dimensional parameter set, it gives rise to a pair of discontinuities which originate at the point. One runs horizontally, the other vertically (see figure 1).

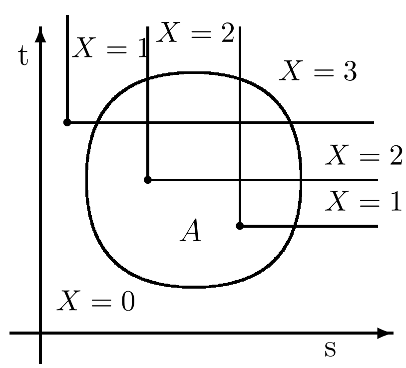

Figure 1: The discontinuities of X

When we speak of a discontinuity of X we mean one of these semi-infinite line segments, either horizontal or vertical, along which S has a jump discontinuity. If either discontinuity crosses ∂A, it causes a discontinuity of $X|_{\partial A}$. These jumps of $X|_{\partial A}$ give us information about the Poisson measure Π. To be more explicit, let us define

$$\Delta Y_U(s) = \lim_{r\downarrow\downarrow s,\, r\in\mathbf{Q}} X(r, M(r)) - \lim_{r\uparrow\uparrow s,\, r\in\mathbf{Q}} X(r, M(r))$$

$$\Delta Y_L(s) = \lim_{r\downarrow\downarrow s,\, r\in\mathbf{Q}} X(r, m(r)) - \lim_{r\uparrow\uparrow s,\, r\in\mathbf{Q}} X(r, m(r))\,.$$

If we interchange the roles of s and t, we get analogous processes $\Delta\tilde{Y}_U(t)$ and $\Delta\tilde{Y}_L(t)$. All four are clearly $\mathcal{F}(\partial A)$-measurable.

We will use several facts about the Poisson point process without special mention, such as the fact that with probability one, no two points fall on any single horizontal or vertical line, and if K is any given set of measure zero, with probability one no point falls in K, and no two discontinuities of X can cross in K.

Lemma 4.2 *For a.e. ω, the following hold.*
(i) If $\Pi(\{(s,t)\}) = 1$ then
(a) $t < m(s)$ and $s \in (\alpha,\beta) \setminus N \Rightarrow \Delta Y_L(s) = 1$;
(b) $t < M(s)$ and $s \in (\alpha,\beta) \setminus N \Rightarrow \Delta Y_U(s) = 1$.
(ii) Conversely, suppose $s \in (s_1,\beta) \setminus N$ and $\Delta Y_U(s) = 1$, where s_1 is defined in $2°$. Then $\exists\, t < M(s) \ni \Pi(\{(s,t)\}) = 1$.
(iii) If $s \in (\alpha, s_1) \setminus N$ and $\Delta Y_U(s) = 1$. Then either
(a) $\exists\, t < M(s) \ni \Pi(\{(s,t)\}) = 1$ or
(b) $\exists\, s' < s \ni \Pi(\{(s', M(s))\}) = 1$, and $\Delta\tilde{Y}_U(M(s)) \neq 0$.

Proof. *(i)* See figure 1.
(ii) If $\Delta Y_U(s) \neq 0$ and $s \notin N$, there is a discontinuity of S passing through ∂A at $(s, M(s))$. Since $s > s_1$, the point is in UR, and $s \mapsto M(s)$ is strictly decreasing by $6°$. If the discontinuity is horizontal, $\Delta Y_U(s) = -1$ (see figure 1), a contradiction. Thus it is vertical, and comes from a point located at (s,t) for some $t < M(s)$.
(iii) If *(a)* doesn't hold, the discontinuity through $(s, M(s))$ is horizontal. By $6°$ M is strictly increasing there, so the discontinuity must enter A at $(s, M(s))$, and it must then exit to the right at some point $(s'', M(s))$, causing a discontinuity of $X|_{\partial A}$, hence $\Delta\tilde{Y}_U(M(s)) \neq 0$. ♣

Let $\mathcal{D}_0$ be the collection of sets of the forms

$$V_U(c,d) \equiv \{(s,t) : c < s < d,\ 0 < t < M(s)\}\,;$$

$$V_L(c,d) \equiv \{(s,t) : c < s < d,\ 0 < t < m(s)\}\,;$$

along with their counterparts $H_U(c,d)$ and $H_L(c,d)$, gotten by interchanging s and t, and sets of the form R_z, for $z \in \partial A$. (V_U and V_L are vertical strips bounded above by portions of ∂A, and H_U and H_L are horizontal strips bounded to the right by portions of ∂A.) Let $\mathcal{D}'$ be the class of sets of the form $B \setminus A$, for sets $A \subset B$ such that $A, B \in \mathcal{D}_0$. Then let $\mathcal{D}$ be the class of finite disjoint unions of sets in $\mathcal{D}'$. The following lemma is the key to the SMP.

Lemma 4.3 *If $D \in \mathcal{D}$, then $\Pi(D)$ is $\mathcal{F}(\partial A)$-measurable.*

Proof. It is enough to show measurability for the sets in $\mathcal{D}_0$. By Lemma 4.2 *(i)*,

$$\Pi(V_L(c,d)) = \#\{s \in (c,d) \setminus N : \Delta Y_L(s)\Delta Y_U(s) \neq 0\}\,,$$

where N is the countable set of 3°. This is $\mathcal{F}(\partial A)$-measurable. If $c > s_1$,

$$\Pi(V_U(c,d)) = \#\{s \in (c,d) \setminus N : \Delta Y_U(s) = 1\}$$

by Lemma 4.2 *(i)* and *(ii)*, while if $d \le s_1$, Lemma 4.2 *(i)* and *(iii)* imply that

$$\Pi(V_U(c,d)) = \#\{s \in (c,d) \setminus N : \Delta Y_U(s) = 1 \text{ and } \Delta \tilde{Y}_U(M(s)) = 0\}\,.$$

Thus these are $\mathcal{F}(\partial A)$-measurable.

The same is true of $H_U(c,d)$ and $H_L(c,d)$ by symmetry, and $\Pi(R_z) = X_z$ which is $\mathcal{F}(\partial A)$-measurable if $z \in A$. ♣

Let

$$\begin{aligned} X_z^1 &= \Pi(R_z \cap A) \\ X_z^2 &= \Pi(R_z \cap A^c)\,. \end{aligned}$$

Then X^1 and X^2 are independent processes, and $X_z = X_z^1 + X_z^2$.

Lemma 4.4 *(i)* $z \in \bar{A} \Rightarrow X^2_z \in \mathcal{F}(\partial A)$;
(ii) $z \in A^c \Rightarrow X^1_z \in \mathcal{F}(\partial A)$.

Proof. Let $z = (s,t) \in \bar{A}$. We claim $R_z \cap A^c \in \mathcal{D}$, which will prove *(i)* by Lemma 4.3. We can choose $s' \leq s$ and $t' \leq t$ such that $(s',t') \in \mathrm{LL}$. Then

$$R_{st} \cap A^c = R_{s't'} \cup V_L(s',s) \cup H_L(t',t) \in \mathcal{D} .$$

Similarly, $R_z \cap A \in \mathcal{D}$ if $z \in A^c$. We leave the verification to the reader. ♣

We can now prove the theorem.

Proof of Theorem 4.1. Let A be a bounded open relatively convex set. Each connected component is necessarily relatively convex and bounded, so by Proposition 2.9 we may as well assume that A itself is open, connected, bounded and relatively convex. Notice that

$$\mathcal{F}(\bar{A}) = \sigma(X^1) \vee \mathcal{F}(\partial A) , \qquad \mathcal{F}(A_c) = \sigma(X^2) \vee \mathcal{F}(\partial A) . \tag{7}$$

Indeed, $X = X^1 + X^2$ and $\sigma(X^1) \in \mathcal{F}(\bar{A})$, so $\sigma(X^1) \vee \mathcal{F}(\partial A) \subset \mathcal{F}(\bar{A})$. On the other hand, $X^2|_{\partial A} \in \mathcal{F}(\partial A)$ by Lemma 4.4 *(i)* so if $z \in \bar{A}$, $X_z \in \sigma(X^1) \vee \mathcal{F}(\partial A)$. Thus $\mathcal{F}(\bar{A}) \subset \sigma(X_1) \vee \mathcal{F}(\partial A)$, proving the first half of (7). The second half is similar, using Lemma 4.4 *(ii)*.

Now both $X^1|_{\partial A}$ and $X^2|_{\partial A}$ are $\mathcal{F}(\partial A)$-measurable, so

$$\mathcal{F}(\partial A) = \sigma\left(X^1|_{\partial A}\right) \vee \sigma\left(X^2|_{\partial A}\right) . \tag{8}$$

Let $\mathcal{C}$ be the trivial σ-field. X^1 and X^2 are independent, so certainly $\sigma(X^1) \perp \sigma(X^2) \mid \mathcal{C}$. By (8) and Lemma 2.2 *(i)*, we see that $\sigma(X^1) \perp \sigma(X^2) \mid \mathcal{F}(\partial A)$, hence $\sigma(X^1) \vee \mathcal{F}(\partial A) \perp \sigma(X^2) \vee \mathcal{F}(\partial A) \mid \mathcal{F}(\partial A)$, by Lemma 2.2 *(i)*, and we are done by (7).

Finally, to see that $\mathcal{F}(\partial A)$ is the minimal splitting field, let $\mathcal{S}$ be a splitting field and let $\Lambda \in \mathcal{F}(\partial A) \subset \mathcal{F}(A^c)$. Since $\mathcal{S}$ is a splitting field and Λ is in both $\mathcal{F}(\bar{A})$ and $\mathcal{F}(A^c)$

$$P\left\{\Lambda \mid \mathcal{S}\right\} = P\left\{\Lambda \mid \mathcal{F}(\bar{A})\right\} = I_\Lambda \Rightarrow \Lambda \in \mathcal{S} .$$

♣

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