THE EXACT $\frac{4}{3}$-VARIATION OF A PROCESS ARISING FROM BROWNIAN MOTION

L. C. G. ROGERS\textsuperscript{1} AND J. B. WALSH\textsuperscript{2}

1. Introduction.

Let $(B_t)_{t \geq 0}$ be a Brownian motion on $\mathbb{R}$, $B_0 = 0$, and define

$$A(t, x) \equiv \int_0^t I_{(B_u \leq x)} \, du = \int_{-\infty}^x L(t, y) \, dy,$$

where $\{L(t, x) : t \geq 0, x \in \mathbb{R}\}$ is the jointly continuous local time of $B$. The continuous adapted process $A(t, B_t)$ arose naturally in earlier work on the Brownian excursion filtration (see Rogers & Walsh [6], and particularly [7]), where it was important to know about the existence of a local time for $A(t, B_t)$. This would be a consequence of general results (Meyer [4], Yor [9]) if it were a semimartingale, but it is not; Rogers & Walsh [8] prove that the $p$-variation of

$$X_t \equiv X_t = A(t, B_t) - \int_0^t L(s, B_s) \, dB_s$$

is infinite if $p < \frac{4}{3}$, and is zero for $p > \frac{4}{3}$. Defining for $p, t > 0$

$$V_p^n(t) \equiv \sum_{j=1}^{[nt]} |X(j/n) - X((j - 1)/n)|^p,$$

the sole aim of this paper is to prove the following result.

**Theorem 1.1.** For each $t > 0$, $V_{4/3}^n(t)$ converges in probability and in $L^2$, the convergence is uniform for $t$ in bounded sets, and the limit is given by

$$\lim_{n \to \infty} V_{4/3}^n(t) = \gamma \int_0^t L(s, B_s)^{\frac{2}{3}} \, ds,$$

where

$$\gamma = \frac{2^\frac{2}{3}}{\sqrt{\pi}} \Gamma \left( \frac{7}{6} \right) E \left[ \left( \int_{-\infty}^\infty L(1, z)^2 \, dz \right)^{\frac{2}{3}} \right].$$

\textsuperscript{1}Research supported in part by SERC grant no. GR/E/35657.

\textsuperscript{2}Research supported in part by NSERC grant 5-88813.
The proof we present is long, though none of the steps is particularly difficult. We have to estimate the $p$th moment of $V_{4/3}(t) - \gamma \int_0^t L(s, B_s)^{2} ds$. If the random variables were Gaussian, this would be relatively easy, thanks to an elegant lemma of Kolmogorov (Lemma 1.3 below). However, they are not. In order to apply Kolmogorov’s Lemma we approximate the increments by stochastic integrals and then notice that if we condition on the proper $\sigma$-field, these approximations are Gaussian. This requires a rather careful decomposition of the increments. To guide the reader, here is an outline of it. For brevity, we write $h = \frac{1}{n}$ throughout, and frequently suppress the dependence on $n$ in the notation. We develop the increment

$$\Delta_h X_s \equiv X_{s+h} - X_s$$

$$= \int_{B_s}^{B_{s+h}} L(s, y) dy + \int_{s}^{s+h} I_{\{B_u \leq B_{s+h}\}} du - \int_{s}^{s+h} L(u, B_u) dB_u$$

$$= \int_{B_s}^{B_{s+h}} L(s, y) dy - \int_{s}^{s+h} L(s, B_u) dB_u + \left( \int_{0}^{h} I_{\{B_{s+u} - B_s \leq B_{s+h} - B_s\}} du \right)$$

$$- \int_{s}^{s+h} (L(u, B_u) - L(s, B_u)) dB_u$$

$$= \int_{B_s}^{B_{s+h}} L(s, y) dy - \int_{s}^{s+h} L(s, B_u) dB_u + \hat{X}_h,$$  \hspace{1cm} (1.3)

where $\hat{X}$ is defined exactly as $X$, but in terms of the process $\hat{B}_t \equiv B_{t+s} - B_s$ (which is independent of $(B_u)_{u \leq s}$). Now, as we shall see, the essential part of $\Delta_h X_s$ is the first two terms in (1.3); and if we write $f(x) \equiv L(s, x + B_s)$ we can rewrite these two terms as

$$\int_{0}^{h} f(y) dy - \int_{0}^{h} f(\hat{B}_u) d\hat{B}_u$$

which, by a formal application of Itô’s formula, is equal to

$$\frac{1}{2} \int_{0}^{h} f'(\hat{B}_u) du = \frac{1}{2} \int_{-\infty}^{\infty} f'(y) \hat{L}(h, y) dy,$$

where $\hat{L}(t, x) = L(s + t, x + B_s) - L(s, x + B_s)$ is the local time of $\hat{B}$. This is only formal, since $f$ is not differentiable. Nonetheless, $(L(s, y))_{y \in \mathbb{R}}$ is a semimartingale in the excursion filtration (see Perkins [5], Jeulin [2], p. 261) and can be used for stochastic integration. An integration by parts leads us to interpret this expression as the stochastic integral $\int_{-\infty}^{\infty} \frac{1}{2} \hat{L}(h, y - B_s) L(s, dy)$. We must be careful, though; this integral does not exist as an Itô integral. Even though the function $g(y) \equiv \hat{L}(h, y)$ is independent of the excursion filtration of...
(B_u)_{0 \leq u \leq s}, the integrand \( \hat{L}(h, y - B_s) \) will not be adapted for \( y < B_s \), even if we augment the excursion filtration with the independent process \( g \).

We shall see that we must interpret it as a “split integral”, and that the correct formulation of this argument leads to

\[
\int_0^{B_s} L(s, y + B_s) \, dy - \int_0^h L(s, \hat{B}_u + B_s) \, d\hat{B}_u = \frac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s) L(s, dy),
\]

where \( \int \) denotes the integral split at \( B_s \), running forward up from \( B_s \) to \( \infty \), and backward down from \( B_s \) to \( -\infty \); this will be made precise in \( \S 2 \).

For a subsequent stage of the argument, it turns out to be essential to approximate increments of \( X \) by (conditionally) Gaussian variables. The semimartingale decomposition of \( (L(s, x))_{x \in \mathbb{R}} \) shows us what to expect.

**Theorem 1.2** (Perkins [5]). Let \( (\mathcal{E}_x^s)_{x \in \mathbb{R}} \) be the excursion filtration of \( (B_u)_{0 \leq u \leq s} \). Then the process

\[
M_x = L(s, x) - \int_{-\infty}^x \alpha(y) \, dy
\]

is a continuous \( (\mathcal{E}_x^s) \)-martingale with quadratic-variation process

\[
4 \int_{-\infty}^x L(s, y) \, dy.
\]

Here,

\[
\alpha(y) = I_{\{y \geq \bar{B}_s\}} \left[ 2I_{\{y \leq 0\}} + 2I_{\{y \leq B_s\}} + I_{\{y \leq B_s\}} L(s, y) \left( \frac{4I_{\{y \geq B_s\}}}{L(s, y) + 2y^{-}} - \frac{L(s, y) + 2y^{-}}{s - A(s, y)} \right) \right]
\]

with \( \bar{B}_s \equiv \sup\{B_u : u \leq s\} \), \( B_s \equiv \inf\{B_u : u \leq s\} \).

In view of this, we may write

\[
\int_{B_s}^{\infty} \hat{L}(h, x - B_s) L(s, dx) = \int_{B_s}^{\infty} \hat{L}(h, x - B_s) \left( 2\sqrt{L(s, x)} W_+(dx) + \alpha(x) \, dx \right)
\]

and since the support of \( \hat{L}(h, \cdot) \) is small, the stochastic integral with respect to \( W_+ \) turns out to be approximately

\[
2\sqrt{L(s, B_s)} \int_{B_s}^{\infty} \hat{L}(h, x - B_s) W_+(dx),
\]

which is Gaussian, conditional on the \( \sigma \)-field

\[
\mathcal{A}_s \equiv \sigma(B_s, L(s, B_s), (B_{s+u} - B_s)_{u \geq 0}).
\]
The sense in which the increment $\Delta h X_s$ may be approximated by (1.6) and the analogous integral over $(-\infty, B_s)$ is explained and proved in §3.

The strategy for proving the theorem is to show that for any $\varepsilon > 0$, and $T > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that if $0 \leq s \leq t \leq T$, and $n \geq N$, and $t - s < \delta$, then

$$\left\| V^n \left( \frac{t}{3} \right) - V^n \left( \frac{s}{3} \right) - \gamma \int_s^t L(u, B_u) \frac{2}{3} \, du \right\|_2 \leq \varepsilon |t - s|. \quad (1.7)$$

From this, $\| V^n \left( \frac{t}{3} \right) - \gamma \int_0^t L(u, B_u) \frac{2}{3} \, du \|_2 \leq \varepsilon t$, and the theorem will follow. To estimate the $L^2$ norms appearing in (1.7), we have to know about the covariance of $|\Delta h X_s|^{\frac{3}{2}}$ and $|\Delta h X_t|^{\frac{3}{2}}$. It is hard to get hold of this for general random variables, but here we use the following result of Kolmogorov, which we learned from J. Bretagnolle. Let $\rho(X, Y) \equiv \text{cov}(X, Y)/(\text{var}(X)\text{var}(Y))^{\frac{1}{2}}$ be the correlation coefficient of $X, Y \in L^2$.

**Lemma 1.3** (Kolmogorov). Let $(X, Y)$ have a bivariate Gaussian distribution, and suppose that $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are such that $\varphi(X), \psi(Y) \in L^2$. Then

$$|\rho(\varphi(X), \psi(Y))| \leq |\rho(X, Y)|. \quad (1.8)$$

Moreover, if $E\{\varphi(X)(X - E\{X\})\} = 0$, then

$$|\rho(\varphi(X), \psi(Y))| \leq \rho(X, Y)^2. \quad (1.9)$$

A proof of this pretty result appears in an appendix. It will allow us to estimate the correlation of (the conditionally Gaussian approximations to) $\Delta h X_s$ and $\Delta h X_t$. This final part of the proof occupies §4.

To conclude the introduction, we record a few simple results which we shall use repeatedly. Firstly, for $p \geq 1$, and positive $a$ and $b$,

$$|b^p - a^p| \leq p |b - a| (b^{p-1} \vee a^{p-1}), \quad (1.10)$$

$$|\sqrt{b} - \sqrt{a}| \leq \sqrt{|b - a|},$$

and secondly for any $p > 0$, there exists $c_p$ such that for all $x \in \mathbb{R}$, and $t > 0$,

$$\| L(t, x) - L(t, 0) \|_p \leq c_p (|x| \wedge t^{rac{1}{p}} + (x^2 t)^{rac{1}{2}}). \quad (1.11)$$
The inequalities in (1.10) are elementary, and (1.11) is proved in Rogers and Walsh [6]. From (1.11), it follows easily that for \( x, y \in \mathbb{R}, t \geq 0 \)
\[
\|L(t, x) - L(t, y)\|_p \leq c_p (|x - y| + t^{\frac{1}{2}} + ((x - y)^2 t)\frac{1}{4}).
\]

\[
(1.12)
\]

Finally, let us note that the processes \( \{B_t, t \geq 0\} \) and \( \{cB_t/c^2, t \geq 0\} \) have the same distribution, since they are both standard Brownian motions. Since \( A \) and \( L \) are both functions of the Brownian motion, we can compute them for both of these Brownian motions and compare to get the following scaling lemma, which we shall use repeatedly below.

**Lemma 1.4.** The vector-valued processes \( \{(B_t, A(t, x), L(t, x)), t \geq 0, x \in \mathbb{R}\} \) and \( \{(cB_t/c^2, c^2 A(t/c^2, x/c), cL(t/c^2, x/c)), t \geq 0, x \in \mathbb{R}\} \) have the same distribution.

### 2. Integral representation of increments of \( X \)

Recall the decomposition (1.3) of the increment \( \Delta_h X_s \equiv X_{s+h} - X_s \):
\[
\Delta_h X_s = \int_{B_s}^{B_{s+h}} L(s, y) \, dy - \int_s^{s+h} L(s, B_u) \, dB_u + \hat{X}_h,
\]
so that
\[
\Delta_h X_s - \hat{X}_h = \lim_{\varepsilon \downarrow 0} \left[ \int_{B_s}^{B_{s+h}} (\varphi_\varepsilon * L(s, \cdot))(y) \, dy - \int_s^{s+h} (\varphi_\varepsilon * L(s, \cdot))(B_u) \, dB_u \right],
\]

where \( \varphi_\varepsilon(x) = \varepsilon^{-1} \varphi(x/\varepsilon) \) for some \( C^\infty \) function \( \varphi \) which is non-negative, supported in \([-1, 1]\), and satisfies \( \int \varphi(x) \, dx = 1 \). The convergence of the first integral is almost sure. The second converges in \( L^2 \), as we see using (1.12). Hence by Itô’s formula
\[
\Delta_h X_s - \hat{X}_h = \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_s^{s+h} (\varphi_\varepsilon * L(s, \cdot))(B_u) \, dB_u
\]
\[
= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s)(\varphi_\varepsilon * L(s, \cdot))(y) \, dy
\]
\[
= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s) \left( \int_{-\infty}^{\infty} \varphi_\varepsilon'(y - x) L(s, x) \, dx \right) \, dy.
\]

By splitting the \( x \)-integral at \( B_s \), we get two terms, one of which is
\[
\frac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s) \left( \int_{B_s}^{\infty} \varphi_\varepsilon'(y - x) L(s, x) \, dx \right) \, dy.
\]

(2.2)

Now abbreviate
\[
g(y) \equiv \hat{L}(h, y)
\]
and notice that \( g \) is independent of \( \{B_u : u < s\} \), so we may condition on \( \{\hat{B}_u : u \geq 0\} \equiv \{B_{u+s} - B_s : u \geq 0\} \) and treat \( g \) as a deterministic...
function. As we recalled in Theorem 1.2, \( L(s, x) \) is a semimartingale in the filtration \((\mathcal{E}_x^\varepsilon)_{x \in \mathbb{R}}\) if \( \tau^s(t, x) \equiv \inf\{u : A(u, x) > t\} \wedge s \), then \( \mathcal{E}_x^\varepsilon \) is the \( \sigma \)-field generated by \( (B(\tau^s(t, x)))_{t \geq 0} \) — see Perkins [5], Jeulin [2]). Thus we may integrate by parts in (2.2) to give

\[
\frac{1}{2} \int_{-\infty}^{\infty} g(y - B_s) \left( \varphi\varepsilon(y - B_s)L(s, B_s) + \int_{B_s}^{\infty} \varphi\varepsilon(y - x)L(s, dx) \right) dy
= \frac{1}{2} L(s, B_s) \int_{-\infty}^{\infty} \varphi\varepsilon(y - B_s)g(y - B_s) dy
+ \frac{1}{2} \int_{B_s}^{\infty} \left( \int_{-\infty}^{\infty} g(y - B_s)\varphi\varepsilon(y - x) dy \right) L(s, dx).
\]

The interchange of orders of integration is achieved by taking Riemann-sum approximations in (2.2) — which will be finite sums since \( g \) and \( \varphi\varepsilon \) are of compact support — then interchanging sum and integral and finally letting the mesh go to zero, so that the integrands converge almost surely uniformly to \( \int g(y - B_s)\varphi\varepsilon(y - x) dy \), again by compact support. The representation (1.4)–(1.5) ensures the conclusion. Finally we let \( \varepsilon \downarrow 0 \) in (2.3) to get

\[
\frac{1}{2} L(s, B_s) g(0) + \frac{1}{2} \int_{B_s}^{\infty} g(x - B_s) L(s, dx).
\]

The term coming from the \( x \)-integral over \((-\infty, B_s)\) is handled similarly, except that now we work in the filtration \((\mathcal{E}_x^\varepsilon)_{x \in \mathbb{R}}\), where we define \( \mathcal{E}_x^\varepsilon \) to be the \( \sigma \)-field generated by \( (B(\bar{\tau}^s(t, -x)))_{t \geq 0} \), with

\[
\bar{\tau}^s(t, x) \equiv \inf\{u : u - A(u, x) > t\} \wedge s.
\]

Thus \((\mathcal{E}_x^\varepsilon)\) is simply the filtration \((\mathcal{E}_x^\varepsilon)\) for the Brownian motion \(-B. If we write \( \hat{L}(s, x) \equiv L(s, -x) \), then \( \hat{L}(s, x) \) is a semimartingale in the filtration \((\mathcal{E}_x^\varepsilon)\), and

\[
\frac{1}{2} \int_{-\infty}^{\infty} g(y - B_s) \left( \int_{-\infty}^{B_s} dx \varphi\varepsilon'(y - x)L(s, x) \right) dy
= \frac{1}{2} \int_{-\infty}^{\infty} g(y - B_s) \left\{ -\varphi\varepsilon(y - B_s)L(s, B_s) - \int_{-B_s}^{\infty} \varphi\varepsilon(y + x)\hat{L}(s, dx) \right\} dy
\to -\frac{1}{2} L(s, B_s) g(0) - \frac{1}{2} \int_{-B_s}^{\infty} g(-x - B_s) \hat{L}(s, dx)
\]

as \( \varepsilon \downarrow 0 \). Combining with (2.4), we have then that

\[
\Delta_hX_s - \hat{X}_h = \frac{1}{2} \int_{B_s}^{\infty} g(x - B_s)L(s, dx) - \frac{1}{2} \int_{-B_s}^{\infty} g(-x - B_s)\hat{L}(s, dx)
= \frac{1}{2} \int_{-\infty}^{\infty} g(x - B_s)L(s, dx)
\]

as a piece of shorthand notation.
The aim now is to express (2.5) in terms of the semimartingale decomposition of \( L \). From Theorem 1.1, we have that
\[
M_x \equiv L(s, x) - \int_{-\infty}^{x} \alpha(y) \, dy
\]
and
\[
\tilde{M}_x \equiv \tilde{L}(s, x) - \int_{-\infty}^{x} \tilde{\alpha}(y) \, dy
\]
are martingales in the filtrations \((\mathcal{E}_x^s)\), \((\tilde{\mathcal{E}}_x^s)\) respectively, where \( \tilde{\alpha} \) is defined exactly as \( \alpha \), but in terms of \(-B\):
\[
\tilde{\alpha}(y) = I_{\{y \leq -B_s\}} \left[ 2 I_{\{y \leq -B_s\}} + 2 I_{\{y \leq -B_s\}} \frac{4 I_{\{y \geq -B_s\}}}{L(s, y) + 2y} - \frac{L(s, -y) + 2y}{s - \int_{-\infty}^{y} L(s, v) \, dv} \right].
\]

We shall suppose that the probability space is enlarged if need be, and the filtrations \((\mathcal{E}_x^s)_{x \in \mathbb{R}}\), respectively \((\tilde{\mathcal{E}}_x^s)_{x \in \mathbb{R}}\), are enlarged so that there exist an \((\mathcal{E}_x^s)\)-Brownian motion \( \beta^+_x \) and an \((\tilde{\mathcal{E}}_x^s)\)-Brownian motion \( \beta^-_x \) which are independent of \( B \) and of each other. This is not essential, but simplifies the statements of results.

**Proposition 2.1.** There is an \((\mathcal{E}_x^s)\)-Brownian motion \( W^+ \) independent of \( \mathcal{A}_s \equiv \sigma \{ B_t, L(s, B_t), (B_{s+u} - B_s)_{u \geq 0} \} \) such that
\[
\int_{B_s}^{y \vee B_s} \sqrt{L(s, x)} \, dW_x^+ + \int_{B_s}^{y \vee B_s} \alpha(x) \, dx.
\]

Similarly, there is an \((\tilde{\mathcal{E}}_x^s)\)-Brownian motion \( W^- \) independent of \( \mathcal{A}_s \) such that
\[
\int_{-B_s}^{(-B_s) \vee y} \sqrt{L(s, x)} \, dW_x^- + \int_{-B_s}^{(-B_s) \vee y} \tilde{\alpha}(x) \, dx.
\]

Moreover, \( W^+ \) and \( W^- \) are independent.

**Proof.** We begin by defining
\[
dW_x^+ = I_{\{x \leq B_s\}} \, d\beta^+_x + (2 \sqrt{L(s, x)})^{-1} I_{\{B_s \leq x < B_s\}} \, dM_x + I_{\{x \geq B_s\}} \, d\beta^+_x,
\]
\[
dW_x^- = I_{\{x \leq -B_s\}} \, d\beta^-_x + (2 \sqrt{L(s, -x)})^{-1} I_{\{-B_s \leq x \leq -B_s\}} \, d\tilde{M}_x + I_{\{x \geq -B_s\}} \, d\beta^-_x,
\]
\[
W_0^+ = W_0^- = 0.
\]
It is clear that \( W^+ \) (respectively, \( W^- \)) is a Brownian motion in \((\mathcal{E}_x^s)\) (respectively, \((\tilde{\mathcal{E}}_x^s)\)), and that (2.6), (2.7) hold by construction. To
prove the independence properties, take \( f_+, f_- \in \mathcal{C}_K^{\infty}, \xi \in b\mathcal{A}_s \), and compute

\[
E \left\{ \exp \left(i\xi \int f_+(x) \, dW_x^+ + i\xi \int f_-(x) \, dW_x^- \right) \right\} \\
= E \left\{ \exp \left(i\xi \int_{-\infty}^{\infty} f_+(x) \, dW_x^+ + i\xi \int_{-B_s}^{\infty} f_-(x) \, dW_x^- - \frac{\xi^2}{2} \int_{-\infty}^{-B_s} f_-(x)^2 \, dx \right) \right\},
\]

since \( \beta^- \) is independent of \( \beta^+ \) and \( B \);

\[
= E \left\{ \exp \left(-\frac{\xi^2}{2} \int_{B_s}^{\infty} f_+(x)^2 \, dx + i\xi \int_{-\infty}^{B_s} f_+(x) \, dW_x^+ \\
+ i\xi \int_{-B_s}^{\infty} f_-(x) \, dW_x^- - \frac{\xi^2}{2} \int_{-\infty}^{-B_s} f_-(x)^2 \, dx \right) \right\},
\]

by conditioning on \( \mathcal{E}_s \cup \sigma(\beta^-) \), since \( \int_{-B_s}^{\infty} f_-(x) \, dW_x^- \) is measurable on this \( \sigma \)-field and \( W^+ \) is a Brownian motion in the filtration \( \mathcal{E}_s \cup \sigma(\beta^-) \);

\[
= E \left\{ \exp \left(-\frac{\xi^2}{2} \int f_+(x)^2 \, dx + i\xi \int_{-\infty}^{\infty} f_-(x) \, dW_x^- - \frac{\xi^2}{2} \int_{-\infty}^{-B_s} f_-(x)^2 \, dx \right) \right\},
\]

since \( \beta^+ \) is independent of \( B, \beta^- \);

\[
= E \left\{ \exp \left(-\frac{\xi^2}{2} \int f_+(x)^2 \, dx - \frac{\xi^2}{2} \int f_-(x)^2 \, dx \right) \right\},
\]

by conditioning on \( \mathcal{E}_{-B_s} \supseteq \mathcal{A}_s \). This proves that \( W^\pm \) are independent Brownian motions, independent of \( \mathcal{A}_s \).

In view of this, then, we can re-express (2.5) as

\[
\Delta_h X_s - \hat{X}_h = \int_{-\infty}^{\infty} g(x-B_s) \sqrt{L(s,x)} \, dW_x + \frac{\alpha(x)}{2} \int_{-\infty}^{\infty} g(x-B_s) \, dx,
\]

where the two split integrals are defined by

\[
\int_{-\infty}^{\infty} g(x-B_s) \sqrt{L(s,x)} \, dW_x \\
= \int_{B_s}^{\infty} g(x-B_s) \sqrt{L(s,x)} \, dW_x^+ - \int_{-B_s}^{\infty} g(-x-B_s) \sqrt{L(s,-x)} \, dW_x^-,
\]

and

\[
\int_{-\infty}^{\infty} g(x-B_s) \alpha(x) \, dx = \int_{B_s}^{\infty} g(x-B_s) \alpha(x) \, dx - \int_{-B_s}^{\infty} g(-x-B_s) \hat{\alpha}(x) \, dx.
\]
Note that the splitting point of the split integral depends on the value of $B_s$. We suppress this from the notation since in most cases the splitting point will be clear from the context.

Let $f(x) = g(x - B_s)\sqrt{L(s, x)}$ and apply Burkholder’s inequality to each stochastic integral in (2.9) to see that the $L^p$ norm of the split integral in (2.8) satisfies

$$\left\| \int_{-\infty}^{\infty} f(y) dW_y \right\|_p \leq C_p \left\| \int_{-\infty}^{\infty} f(y + B_s)^2 dy \right\|^{1/2}_{p/2}. \quad (2.11)$$

3. The main estimates

We now take $0 < s < t \leq 1$ and estimate $\Delta_hX_s$ and $\Delta_hX_t$ in turn. We decompose $\Delta_hX_s$ (using (2.8), (2.9)) as

$$\Delta_hX_s = \int_{-\infty}^{\infty} g(x - B_s)\sqrt{L(s, B_s)} dW_x$$

$$+ \int_{-\infty}^{\infty} g(x - B_s)\left(\sqrt{L(s, x)} - \sqrt{L(s, B_s)}\right) dW_x$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} g(x - B_s)\alpha(x) dx + \hat{X}_h$$

$$\equiv I_1 + I_2 + I_3 + \hat{X}_h,$$ say. Firstly we take $I_2$. By (2.11)

$$\| I_2 \|_p \leq C_p \left\| \int_0^\infty g(y)^2 |L(s, B_s + y) - L(s, B_s)| dy \right\|_{p/2}^{1/2}. \quad (3.2)$$

Write the norm in terms of expectations and use Hölder’s inequality with $p/2$ and $p/(p - 2)$; this is

$$\leq C_p \left( E \left\{ \int_0^\infty g(y)^2 |L(s, B_s + y) - L(s, B_s)| dy \right\}^{p/2} \right)^{1/p};$$

change order and use the fact that $g$ is independent of $L(s, \cdot)$:

$$\leq C_p \left( \int_0^\infty E \left\{ |L(s, y) - L(s, 0)|^{p/2} \right\} E \left\{ g(y)^2 \left( \int_0^\infty g(x)^2 dx \right)^{p/2 - 1} \right\} dy \right)^{1/p};$$

now use (1.11) on the first expectation and the Schwartz inequality on the second:

$$\leq C_p \left( \int_0^\infty \left( y \wedge \sqrt{s} + y^{\frac{3}{2}} s^{\frac{1}{2}} \frac{p}{2} E \left\{ g(y)^4 \right\} \right)^{\frac{1}{2}} \left( E \left\{ \left( \int_0^\infty g(x)^2 dx \right)^{p/2 - 1} \right\} \right)^{\frac{1}{2}} dy \right)^{1/p};$$
using (1.11) on the first term. Now recall that \( g(y) = \hat{L}(h, y) \overset{D}{=} \sqrt{h}L(1, y/\sqrt{h}) \) so that
\[
E \left\{ g(y)^4 \right\} = h^2 E \left\{ \hat{L}(1, y/\sqrt{h})^4 \right\} 
\leq h^2 P \left\{ H_{y/\sqrt{h}} < 1 \right\} E \left\{ \hat{L}(1, 0)^4 \right\},
\]
where \( H_x = \inf \{ u : B_u = x \} \)
\[
\leq ch^2 \Phi(y/\sqrt{h})
\]
where \( \Phi \) is the tail of the standard normal distribution. Moreover,
\[
E \left\{ \left( \int_0^\infty g(x)^2 dx \right)^{p-2} \right\} = h^{p-2} E \left\{ \left( \int_0^\infty \hat{L}(1, y/\sqrt{h})^2 dy \right)^{p-2} \right\} = h^{3(p-2)/2} E \left\{ \left( \int_0^\infty \hat{L}(1, y)^2 dy \right)^{p-2} \right\} \leq Ch^{3(p-2)/2},
\]
since by the results of Barlow and Yor [1], \( L^*(1) \in \cap_p L^p, \) where \( L^*(s) \equiv \sup_y L(s, y). \) Thus
\[
\| I_2 \|_p \leq C_p h^{3/2} \left( \int_0^\infty (y \wedge \sqrt{s} + y^{1/2} s^{1/4})^{p/2} \Phi(y/\sqrt{h})^{1/2} h^{-1/2} dy \right)^{1/p} \leq C_p h^{5/2}, \tag{3.3}
\]
since \( s \) is bounded.

Next we estimate \( I_3, \) a more delicate task. Write it as the sum of two integrals, \( \int_0^\infty + \int_{-\infty}^0 = I_3^+ + I_3^- \), say. By symmetry we only have to treat the first one. By (1.5) we have
\[
2I_3^+ = \int_{B_s} g(y - B_s) \left( 2I_{\{y \leq 0\}} + I_{\{y \leq B_s\}} \frac{L(s, y)}{s - A(s, y)} \right) dy.
\]
The integral \( \int g(y - B_s)I_{\{y \leq 0\}} dy \) is at most \( h \), so that will cause no trouble. What remains is the sum of two terms,
\[
\int_{B_s} g(y - B_s) \frac{L(s, y)^2}{s - A(s, y)} dy + 2I_{\{B_s < 0\}} \int_{B_s} g(y - B_s) \frac{|y|L(s, y)}{s - A(s, y)} dy.
\]
If we make the replacements \( B_u \mapsto -B_{s-u} + B_s \) in the first term and \( B_u \mapsto -B_u \) in the second, we see that they are equal in distribution to
\[
J_1 = \int_{B_s} g(y) L(s, y)^2 \frac{dy}{A(s, y)},
\]
and

\[ J_2 \equiv I_{\{B_s > 0\}} \int_0^{B_s} g(y - B_s) \frac{2yL(s, y)}{A(s, y)} \, dy \]

respectively. Now by Jensen’s inequality for \( p \geq 1 \)

\[ |J_1|^p \leq h^{p-1} \int_{B_s}^0 g(y) \left| \frac{L(s, y)^2}{A(s, y)} \right|^p \, dy. \quad (3.4) \]

For \( y \in (B_s, 0) \), \( L(s, y)^2/A(s, y) \overset{D}{=} L(s - H_y, 0)^2/A(s - H_y, 0) \), and, by scaling, \( L(t, 0)^2/A(t, 0) \overset{D}{=} L(1, 0)^2/A(1, 0) \). Thus we have the estimate

\[ E \{ |J_1|^p \} \leq h^p E \left\{ \frac{L(1, 0)^{2p}}{A(1, 0)^p} \right\} \equiv h^p \gamma_p, \quad (3.5) \]

say, and according to Karatzas and Shreve [3] who compute the joint distribution of \((B_t, L(t, 0), A(t, 0))\),

\[ P \{ L(1, 0) \in db, A(1, 0) \in ds \} = \frac{b}{\pi(s(1-s))^{\frac{p}{2}}} \exp(-b^2/2s(1-s)) \, db \, ds, \]

hence

\[ \gamma_p = \int_0^\infty db \int_0^1 ds \frac{b^{2p}}{s^{sp}} \frac{b}{\pi(s(1-s))^{\frac{p}{2}}} \exp(-b^2/2s(1-s)) \]

\[ = \int_0^\infty dv \int_0^1 ds (2v(1-s))^p \frac{e^{-v}}{\pi \sqrt{s(1-s)}} \]

\[ = \Gamma(p+1)2^p \int_0^1 ds (1-s)^{p}/\sqrt{s(1-s)} \]

\[ < \infty. \]

To summarize, then,

\[ \| J_1 \|^p \leq C_p h. \quad (3.6) \]

The estimation of \( J_2 \) proceeds along similar lines: for \( p \geq 1 \)

\[ |J_2|^p \leq h^{p-1} \int_0^\infty g(y - B_s) \left| \frac{2yL(s, y)}{A(s, y)} \right|^p I_{\{y < B_s\}} \, dy; \quad (3.7) \]

\[ E \left\{ \left| \frac{L(s, y)}{A(s, y)} \right|^p ; y < B_s \right\} = \int_0^s \frac{ye^{-y^2/2(s-u)}}{\sqrt{2\pi(s-u)^3}} E \left\{ \left| \frac{L(u, 0)}{s-u + A(u, 0)} \right|^p ; B_u > 0 \right\} \, du. \]
Using once again the result of Karatzas and Shreve,

\[
E \left\{ \left| \frac{L(u,0)}{s - u + A(u,0)} \right|^p : B_u > 0 \right\}
\]

\[
= \int_0^\infty db \int_0^u dv \frac{b}{\pi(v^3(u-v))^{1/2}} \exp \left( -\frac{b^2}{2} \frac{u}{v(u-v)} \right) \left| \frac{b}{s - u + v} \right|^p
\]

\[
= C_p \int_0^u dv v^{(p-1)/2} \left( u - v \right)^{(p+1)/2} u^{-\frac{p+2}{2}(s - u + v)^{-p}}
\]

\[
= C_p \int_0^1 dt u^{p/2} t^{(p-1)/2} (1 - t)^{(p+1)/2} (s - u + ut)^{-p}.
\]

Now using the fact that \( g(y - B_s) \equiv \hat{L}(h, y - B_s) \leq \hat{L}^*(h) \overset{\mathcal{D}}{=} \sqrt{h} \hat{L}^*(1) \), we deduce that

\[
E \{|J_2|^p\} \leq C_p h^{p - \frac{1}{2}} \int_0^\infty y^p E \left\{ \left| \frac{L(s,y)}{A(s,y)} \right|^p : y < B_s \right\} dy
\]

\[
= C_p h^{p - \frac{1}{2}} \int_0^\infty y^p dy \int_0^s \frac{ye^{-y^2/2(s-u)}}{\sqrt{2\pi(s-u)^3}} du
\]

\[
\int_0^1 dt u^{\frac{p-1}{2}} t^{\frac{p-1}{2}} (1 - t)^{\frac{p+1}{2}} (s - u + ut)^{-p}
\]

\[
= C_p h^{p - \frac{1}{2}} \int_0^s du (s - u)^{\frac{p-1}{2}} \int_0^1 dt u^{\frac{p-1}{2}} t^{\frac{p-1}{2}} (s - u + ut)^{-p}
\]

\[
= C_p h^{p - \frac{1}{2}} \sqrt{s} \int_0^1 \int_0^1 dw dt (1 - w)^{\frac{p-1}{2}} w^{\frac{p-1}{2}} t^{\frac{p-1}{2}} (1 - t)^{\frac{p+1}{2}} (1 - w + wt)^{-p}
\]

\[
= C_p h^{p - \frac{1}{2}} \sqrt{s} \int_0^1 \int_0^1 dw dt \left| \frac{(1 - w)wt}{(1 - w + wt)^2} \right|^{\frac{p}{2}} (1 - w)^{-\frac{3}{2}} t^{-\frac{3}{2}} (1 - t)^{\frac{p+1}{2}}
\]

\[
= C_p h^{p - \frac{1}{2}} \sqrt{s}.
\]

Hence we have

\[
\|J_2\|_p \leq C_p h^{\frac{p}{2}}.
\]

(3.8)

Assembling the bounds on \( J_1 \) and \( J_2 \), we see that for small \( h \),

\[
\|I_3\|_p \leq C_p h^{\frac{p}{2}}.
\]

Finally, we estimate

\[
\hat{X}_h = \int_0^h I_{\{B_u \leq B_h\}} du - \int_0^h \hat{L}(u, \hat{B}_u) d\hat{B}_u.
\]
The first term is at most \( h \). For the second, taking \( p \geq 2 \) we see that

\[
E \left\{ \left| \int_0^h \dot{L}(u, B_u) \, dB_u \right|^p \right\} \leq C_p E \left\{ \left| \int_0^h (u, B_u)^2 \, du \right|^{p/2} \right\}
\]
\[
\leq h^{p/2 - 1} C_p E \left\{ \left| \int_0^h (u, B_u)^p \, du \right| \right\}
\]
\[
\leq C_p h^p,
\]
so that

\[
\| \tilde{X}_h \|_p \leq C_p h.
\]

In short, then, for every \( p \geq 1 \)

\[
\| \Delta_h X_s - I_1 \|_p \leq C_p h^\frac{p}{2}.
\]

(3.9)

This completes the analysis of \( \Delta_h X_s \); we now analyse the joint behavior of the two increments. Let \( t > s \) and let us consider \( \Delta_h X_t \). The arguments for the decomposition of this increment are broadly similar to the foregoing, but with one refinement which will be needed at a later stage; we cut \((0, t] \) into \( (0, s] \cup (s, s + h] \cup (s + h, t] \), so that the contribution from the last interval will be independent of the contribution from the first. If we simply cut \((0, t] \) into \((0, s] \cup (s, t] \), this will not be true.

To begin, then, from (1.3)

\[
\Delta_h X_t = \int_{B_t}^{B_{t+h}} L(t, y) \, dy - \int_t^{t+h} L(t, B_u) \, dB_u + \tilde{X}_h
\]

where \( \tilde{X} \) is defined exactly as \( X \), (see (1.1)) but in terms of the Brownian motion \( \tilde{B}_u \equiv B_{t+u} - B_t \). We split the first two terms as

\[
\left[ \int_{B_t}^{B_{t+h}} L(s, y) \, dy - \int_t^{t+h} L(s, B_u) \, dB_u \right]
\]
\[
+ \left[ \int_{B_t}^{B_{t+h}} (L(s + h, y) - L(s, y)) \, dy - \int_t^{t+h} (L(s + h, B_u) - L(s, B_u)) \, dB_u \right]
\]
\[
+ \left[ \int_{B_t}^{B_{t+h}} (L(t, y) - L(s + h, y)) \, dy - \int_t^{t+h} (L(t, B_u) - L(s + h, B_u)) \, dB_u \right]
\]
\[
\equiv K_{[0,s]} + K_{[s,s+h]} + K_{(s+h,t]},
\]
say. Taking the terms one by one, the analysis of \( K_{[0,s]} \) is very similar to the analysis of \( \Delta h X_s \). Following the steps (2.2)—(2.5) we see

\[
K_{[0,s]} = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{t}^{t+h} (\phi_\varepsilon \ast L(s, \cdot))'(B_u) \, du
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{-\infty}^{\infty} \tilde{L}(h, y - B_t)(\phi_\varepsilon \ast L(s, \cdot))'(y) \, dy
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(y - B_s) \left( \int_{-\infty}^{\infty} dx \phi_\varepsilon'(y - x)L(s, x) \right) \, dy,
\]

where \( \tilde{g}(x) \equiv \tilde{L}(h, x-B_t+B_s) \equiv \tilde{L}(t-s+h, x)-\tilde{L}(t-s, x) \) is measurable with respect to \( \sigma(\{\hat{B}_u : u \geq 0\}) \) and therefore is independent of \( \{B_u : u \leq s\} \). The analysis of \( \Delta h X_s \) in (2.1)—(2.5) goes through \textit{mutatis mutandis} to give the analogue of (2.5):

\[
K_{[0,s]} = \frac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(x - B_s)L(s, dx).
\]

We apply Proposition 2.1 again to obtain the analogue of (2.8):

\[
K_{[0,s]} = \int_{-\infty}^{\infty} \tilde{g}(x - B_s)\sqrt{L(s, x)} \, dW_x + \frac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \alpha_x \, dx
\]

\[
\equiv K_1 + K_2
\]

say, where the second split integral is defined by

\[
\int_{-\infty}^{\infty} \tilde{g}(x - B_s) \alpha_x \, dx = \int_{B_s}^{\infty} \tilde{g}(x - B_s) \alpha_x \, dx - \frac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(-x - B_s) \alpha_x \, dx.
\]

Recall that \( \tilde{g} \) has its support in a small neighborhood of 0, so that we can (almost) factor out the square root from \( K_1 \). In fact

\[
K_1 - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x
\]

\[
= \left[ \sqrt{L(s, B_t)} - \sqrt{L(s, B_s)} \right] \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x
\]

\[
+ \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \left( \sqrt{L(s, x)} - \sqrt{L(s, B_t)} \right) \, dW_x,
\]

\[
\equiv M_1 + M_2.
\]

We will see that \( M_1 \) and \( M_2 \) are small. Now

\[
\|M_1\|_p \leq \left\| \sqrt{L(s, B_t)} - \sqrt{L(s, B_s)} \right\|_{2p} \left\| \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x \right\|_{2p}
\]
and by (1.10)
\[
\left\| \sqrt{L(s, B_t)} - \sqrt{L(s, B_s)} \right\|_{2p} \leq \left\| L(s, B_t) - L(s, B_s) \right\|_{p}^{\frac{2}{p}}
\]
\[
= (E\{|L(s, Z) - L(s, 0)|^p\})^{1/2p},
\]
where \(Z \sim N(0, t - s)\) is independent of \(B\); let \(pt(\cdot)\) be the \(N(0, t)\) density:
\[
\leq C_p \left( \int pt-s(x) (|x| \land \sqrt{s} + (x^2s)^{\frac{1}{4}})^p \, dx \right)^{1/2p},
\]
using the estimate (1.11). This is
\[
\leq C_p (\int pt-s(x) (|x| \land \sqrt{s} + (x^2s)^{\frac{1}{4}})^p \, dx)^{1/2p},
\]
for all \(0 \leq s \leq t \leq 1\). Next we estimate
\[
\left\| \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x \right\|_p \leq C_p \left( \int_{-\infty}^{\infty} \tilde{g}(x - B_s)^2 \, dx \right)^{1/2} + \left( \int_{-\infty}^{B_s} \tilde{g}(x - B_s)^2 \, ds \right)^{1/2}
\]
\[
\leq C_p \left( \int_{-\infty}^{\infty} L(h, y)^2 \, dy \right)^{1/2} \leq C_p h^\frac{3}{4},
\]
since \(L(t, x) \equiv cL(t/c^2, x/c)\) by Lemma 1.4. Thus
\[
\|M_1\|_p \leq C_p |t - s|^{\frac{1}{8}} h^\frac{3}{4}. \quad (3.12)
\]

The second term on the right of (3.11), \(M_2\), is handled exactly as was \(I_2\): following (3.2)–(3.3) for \(p \geq 2\), we get
\[
\left\| \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \left( \sqrt{L(s, x)} - \sqrt{L(s, B_t)} \right) \, dW_x \right\|_p
\]
\[
\leq \left\| \int_{-\infty}^{\infty} \tilde{g}(x - B_s)^2 |L(s, x) - L(s, B_t)| \, dx \right\|_{p/2}^{\frac{1}{2}} \leq C_p h^{7/8}. \quad (3.13)
\]

We combine (3.12) and (3.13) into the estimate
\[
\left\| K_1 - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x \right\|_p \leq C_p \left( |t - s|^{\frac{1}{8}} h^\frac{3}{4} + h^\frac{7}{8} \right). \quad (3.14)
\]
Next we turn to $K_2$. This term is analogous to $I_3$, and we can follow that analysis. We break it into the sum of two similar integrals, $K_2^+ + K_2^-$, and consider $K_2^+$, which satisfies (see (3.4))

$$2K_2 = \int_{B_s}^\infty \tilde{g}(y - B_s) \left[ 2I_{\{y \leq 0\}} + I_{\{y \leq B_s\}} \frac{L(s, y)(L(s, y) + 2y^-)}{s - \int_y^\infty L(s, v) \, dv} \right] \, dy.$$

As with the estimation of $I_3$, the integral involving $I_{\{y \leq 0\}}$ is at most $h$, so causes no trouble. What remains is the sum of two terms, the first equal in distribution to

$$\tilde{J}_1 \equiv \int_{B_s}^0 \tilde{g}(-y)L(s, y)^2 \frac{dy}{A(s, y)},$$

the second equal in distribution to

$$\tilde{J}_2 \equiv I_{\{B_s > 0\}} \int_0^{B_s} \tilde{L}(h, B_t - y) \frac{2yL(s, y)}{A(s, y)} \, dy.$$

We handle the estimation of $\tilde{J}_1$ as before, in (3.5)—(3.6): for $p \geq 1$,

$$|\tilde{J}_1|^p \leq h^{p-1} \int_{B_s}^0 \tilde{g}(-y) \left| \frac{L(s, y)^2}{A(s, y)} \right|^p \, dy \equiv h^{p-1} \int_{B_s}^0 \tilde{L}(h, -y - B_s) \left| \frac{L(s, y)^2}{A(s, y)} \right|^p \, dy$$

from which, as before,

$$\|\tilde{J}_1\|_p \leq C_p h.$$  (3.16)

The estimation of $\tilde{J}_2$ is exactly like the estimation of $J_2$ in (3.7)—(3.8), except that $\tilde{L}(h, y - B_s)$ is replaced by $\tilde{L}(h, B_t - x)$. The estimation of $J_2$ involved the bound $\tilde{L}(h, y - B_s) \leq \tilde{L}^*(h) \overset{p}{=} \sqrt{h}\tilde{L}^*(1)$, and the analogous bound is good for $\tilde{L}(h, B_t - x)$. We thus obtain

$$\|\tilde{J}_2\|_p \leq C_p h^{\frac{p}{2}}.$$  (3.17)

Assembling the last two equations gives

$$\|K_2\|_p \leq C_p h^{\frac{p}{2}}.$$  (3.15)

Putting (3.10), (3.14), and (3.15) together, we get

$$\left\| K_{(0,s]} - \sqrt{L(s, B_s)} \int_{-\infty}^\infty \tilde{g}(x - B_s) \, dW_x \right\|_p \leq C_p \left\{ |t - s|^\frac{p}{2} h^{\frac{p}{2}} + h^{\frac{p}{2}} \right\}.$$  (3.16)

Next we must estimate $K_{(s,s+h]}$, an easier task. We leave the reader to confirm that

$$\|K_{(s,s+h]}\|_p \leq C_p h.$$  (3.17)
Finally, we deal with $K_{[s+h,t]}$. If we define $v \equiv t - s - h$, and $B'_u \equiv B_{u+s+h} - B_{s+h}$, we have then that

$$K_{[s+h,t]} = \int_{B'_v}^{B_{v+h}} L'(v, x) \, dx - \int_{v}^{v+h} L'(v, B'_u) \, dB'_u$$

so that the analysis of this term is exactly the same as the analysis of $\Delta h X_s - \hat{X}_h$. Thus, following the steps (2.1)–(2.8) we obtain the analogue of (2.8):

$$K_{[s+h,t]} = \int_{-\infty}^{\infty} g'(x - B'_v) \sqrt{L'(v, x)} \, dW'_x + \frac{1}{2} \int_{-\infty}^{\infty} g'(x - B'_v) \alpha'_v \, dx$$

(3.18)

where $L'$, $\alpha'$, $W'^+$, $W'^-$ are defined in terms of $B'$ exactly as $L$, $\alpha$, $\bar{\alpha}$, $W^+$, $W^-$ were defined in terms of $B$, and $g'(x) = \bar{L}(h, x)$. The stochastic integral splits at $B'_v$, of course.

It is important to realise that everything which appears in (3.18) is determined by $B'$, and so is independent of $\{B_u : u \leq s + h\}$, and hence of $W_+$ and $W_-$. The point of this is that if we mimic the estimation of $\Delta h X_s$ which led to (3.9) we obtain

$$\left\| K_{[s+h,t]} - \int_{-\infty}^{\infty} g'(x - B'_v) \sqrt{L'(v, B'_v)} \, dW'_x \right\|_p \leq C_p h^{\frac{1}{2}}$$

(3.19)

which says that $K_{[s+h,t]}$ is very nearly equal to a random variable whose law, given $B'_v$, $L'(v, B'_v)$ and $(B'_{v+u} - B'_v)_{u \geq 0}$, is Gaussian, and which is independent of $\{B_u : u \leq s + h\}$, and therefore independent of $\Delta h X_s$.

Assembling (3.16), (3.17), and (3.19), we have that

$$\left\| \Delta h X_t - \sqrt{L(s), B_s} \int_{-\infty}^{\infty} \bar{g}(x - B_s) \, dW_x - \sqrt{L'(v, B'_v)} \int_{-\infty}^{\infty} g'(x - B'_v) \, dW'_x \right\|_p$$

$$\leq C_p \left( |t - s|^{\frac{1}{2}} h^{\frac{1}{2}} + h^{\frac{1}{2}} \right).$$

(3.20)

4. The $L^2$ convergence

Let us summarize the main point of the estimation so far, with a view to clarifying the steps still to come. Our strategy is to approximate the increments $\Delta h X_s$ by stochastic integrals, and then to use the fact that these integrals are conditionally Gaussian to bound their moments. At this point we have developed the estimates to justify this approximation. The central results are (3.9) and (3.20). In order to unify our notation, let us define $\Delta L_u(x) = L(u + h, x + B_u) - L(u, x + B_u)$. Note that, in the notation of the previous section, $\hat{L}(h, x) = g(x) = \Delta L_s(x)$,
\[ \tilde{g}(x) = \Delta L_t(x + B_s - B_t), \text{ and } g'(x) = \Delta L_t(x). \] We proved in (3.9) that
\[ \left\| \Delta_h X_s - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \Delta L_t(x - B_s) \, dW_x \right\|_p \leq C_p h^{\frac{7}{8}}, \] (4.1)
and we also proved (3.20) that if \( 0 < s < t \) and \( v = t - s - h \), that
\[ \left\| \Delta_h X_t - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \Delta L_t(x + B_s + h - B_t) \, dW_x \right\|_p \leq C_p \left( |t - s|^{\frac{7}{8}} h^{\frac{7}{8}} + h^{\frac{7}{8}} \right). \] (4.2)

Now we aim to prove Theorem 1: for \( 0 \leq t \leq 1 \), if \( h = \frac{1}{n} \)
\[ \left( \sum_{k=0}^{[nt]-1} \left| \Delta_h X_{kh} \right|^{\frac{4}{3}} - \gamma \int_0^t L(s, B_s)^{\frac{2}{3}} \, ds \right)^{\frac{3}{2}} \to 0 \]
as \( n \to \infty \). This follows once we can prove that
\[ \sum_{k=0}^{[nt]-1} \left( \left| \Delta_h X_{kh} \right|^{\frac{4}{3}} - \gamma h L(kh, B_{kh})^{\frac{2}{3}} \right)^{\frac{3}{2}} \to 0. \] (4.3)

Let us transform this slightly. Recall that \( \gamma = \beta E \right\{ \left| \int_{-\infty}^{\infty} L(1, x)^2 \, dx \right|^{\frac{2}{3}} \}, \) with \( \beta \equiv 2^{\frac{1}{6}} \pi^{-\frac{1}{6}} \Gamma(7/6) = E\{ |B_1|^{\frac{4}{3}} \}. \) Hold \( n \) fixed. The process
\[ S_j \equiv \sum_{k=0}^{j} L(kh, B_{kh})^{\frac{4}{3}} \left( \gamma h - \beta \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2 \, dy \right|^{\frac{2}{3}} \right), \; j = 1, 2 \ldots \]
is a martingale relative to the filtration \( \mathcal{F}_{(j+1)h} \); this is because \( \Delta L_{kh}(\cdot) \) is independent of \( \mathcal{F}_{kh} \) and because
\[ E \left\{ \left| \int_{-\infty}^{\infty} \Delta L_{kh}(x)^2 \, dx \right|^{\frac{2}{3}} \right\} = E \left\{ \left| \int_{-\infty}^{\infty} L(h, x)^2 \, dx \right|^{\frac{2}{3}} \right\} \]
\[ = h E \left\{ \left| \int_{-\infty}^{\infty} L(1, x)^2 \, dx \right|^{\frac{2}{3}} \right\}, \; \text{ using Lemma 1.4} \]
\[ = \gamma h / \beta \] (4.4)
by definition of \( \gamma \) and \( \beta \). Moreover
\[ E\{ S_j^2 \} = \sum_{k=0}^{j} E \left\{ L(kh, B_{kh})^{\frac{4}{3}} \right\} E \left\{ \left( \gamma h - \beta \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2 \, dy \right|^{\frac{2}{3}} \right)^2 \right\} \]
\[ \leq C_j h^2. \]
Accordingly, \( E\{S_{[nt]}^2\} \leq Cth \). This reduces the proof of Theorem 1 to the proof that
\[
\sum_{k=0}^{[nt]-1} \left\{ \left| \Delta_h X_{kh} \right|^\frac{4}{3} - \beta L(kh, B_{kh}) \right\} \to 0.
\]
(4.5)

Let us abbreviate
\[
\alpha_{kn} \equiv \beta L(kh, B_{kh}) \left| \int_{-\infty}^\infty \Delta L_{kh}(y)^2 \, dy \right|^\frac{2}{3},
\]
and define
\[
U_n(t) \equiv \sum_{k=0}^{[nt]-1} \left( \left| \Delta_h X_{kh} \right|^\frac{4}{3} - \alpha_{kn} \right).
\]

We claim that \( U_n(t) \to 0 \) in \( L^2 \). This will prove (4.5), and hence the theorem. Let \( s < t \), and write
\[
\|U_n(t) - U_n(s)\|^2 = E \left\{ \sum_{j=[ns]}^{[nt]-1} \left( \left| \Delta_h X_{jh} \right|^\frac{4}{3} - \alpha_{jn} \right)^2 \right\}
\]
\[
= \sum_{j=[ns]}^{[nt]-1} E \left\{ \left( \left| \Delta_h X_{jh} \right|^\frac{4}{3} - \alpha_{jn} \right)^2 \right\}
\]
\[
+ 2 \sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} E \left\{ \left( \left| \Delta_h X_{jh} \right|^\frac{4}{3} - \alpha_{jn} \right) \left( \left| \Delta_h X_{kh} \right|^\frac{4}{3} - \alpha_{kn} \right) \right\}
\]
\[
\equiv V_1 + V_2.
\]

The steps remaining in the proof are now the following:

(i) approximate the \( \Delta_h X_{jh} \) in the diagonal terms \( V_1 \) (4.6) by stochastic integrals with respect to \( L \), using (4.1);

(ii) approximate both \( \Delta_h X_{jh} \) and \( \Delta_h X_{kh} \) by stochastic integrals in \( V_2 \), using (4.2) for the latter;

(iii) show that the stochastic integrals are conditionally Gaussian, so that we can apply Kolmogorov’s Lemma;

(iv) assemble the estimates to give (4.3).

**Step (i).** Let
\[
\xi_j \equiv \Delta_h X_{jh},
\]
\[
I_j = \sqrt{L(jh, B_{jh})} \int_{-\infty}^\infty \Delta L_{jh}(x - B_{jh}) \, dW_x,
\]
\[
\eta_j = I_j - \xi_j.
\]
We claim that the first term on the right-hand side of (4.6) is

\[ V_1 = \sum_{j=[ns]}^{[nt]-1} E \left\{ \left( |I_j|^\frac{4}{3} - \alpha \right)^2 \right\} + O \left( (t-s)h^\frac{9}{8} \right). \] (4.7)

We know from (4.1) that \( \| \eta_j \|_p \leq C_p h^{\frac{7}{8}} \). Exploiting the fact (Proposition 2.1) that \( W \) is independent of both \( \hat{L}(h, x - B_{jh}) \) and \( L(jh, B_{jh}) \), we have by Burkholder’s inequality that

\[ \| I_j \|_p \leq C_p \left\| \left( \int L(h, x)^2 \, dx \right)^{\frac{1}{2}} \right\|_p \leq C_p h^{\frac{7}{8}}, \] (4.8)

by Lemma 1.4, at least for all \( s \leq 1 \), which is all that concerns us here. Hence for small \( h \), \( \| \xi_j \|_p \leq C_p h^{\frac{7}{8}} \). Now using the elementary bound (1.10) and Hölder’s inequality, we estimate

\[ \| |I_j|^p - |\xi_j|^p \|_2 \leq C_p \| \eta_j \|_p \left\| |\xi_j|^{p-1} + |I_j|^{p-1} \right\|_2 \leq C_p \| \eta_j \|_p \left\| |\xi_j|^{p-1} + |I_j|^{p-1} \right\|_{2\alpha'}, \] (4.9)

where \( \frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \). Now take \( p = 4/3 \), the case of interest to us:

\[ \leq C_p h^{\frac{7}{8} + \frac{3(4-1)}{4}} = C_p h^{\frac{4}{3}} \] (4.10)

Consider the \( j \)th term of \( V_1 (4.6) \), which we write \( E \left\{ (|\xi_j|^\frac{4}{3} - \alpha)^2 \right\} \).

The difference between this and what we want is

\[ \left| E \left\{ (|\xi_j|^\frac{4}{3} - \alpha)^2 \right\} - E \left\{ (|I_j|^\frac{4}{3} - \alpha)^2 \right\} \right| = \left| E \left\{ (|\xi_j|^\frac{4}{3} - |I_j|^\frac{4}{3} \right) (|\xi_j|^\frac{4}{3} + |I_j|^\frac{4}{3} - 2\alpha) \right\} \right| \leq \| |\xi_j|^\frac{4}{3} - |I_j|^\frac{4}{3} \|_2 \| |\xi_j|^\frac{4}{3} + |I_j|^\frac{4}{3} - 2\alpha \|_2 \leq C_p h^{\frac{47}{48}}, \]

where we have used (4.8), (4.4) and (4.10) to bound the norms of the various terms. The sum contains at most \( (t-s)/h \) terms, so the total contribution is at most \( C_p (t-s)h^{\frac{47}{48}} \).
Step (ii). The argument is similar to that of Step (i). Let \( j < k \) and define

\[
J_{jk} = \sqrt{L(jh, B_{jh})} \int_{-\infty}^{\infty} \Delta L_{kh}(x - B_{kh}) \, dW_x
\]

\[
K_{jk} = \sqrt{L'((k - j - 1)h, B'_{(k - j - 1)h})} \int_{-\infty}^{\infty} \Delta L'_{(k - j - 1)h}(x - B'_{(k - j - 1)h}) \, dW',
\]

where \( B'_u \equiv B_{u+jh+h} - B_{jh+h} \) is the Brownian motion from \((j + 1)h\) onward, and \( L' \) is its local time. (These definitions allow us to abbreviate (4.1) and (4.2) to \( \|\eta_j\|_p \leq C_p h^\frac{3}{4} \), \( \|\xi_k - J_{jk} - K_{jk}\|_p \leq C_p \|t-s\|^{\frac{1}{3}} h^\frac{3}{4} + h^\frac{3}{2} \) respectively.) We have \( \|\xi_j\|_p + \|I_j\|_p + \|\xi_k\|_p + \|J_{jk} + K_{jk}\|_p \leq C_p h^\frac{3}{2} \).

We claim that the double sum in (4.6) is

\[
V_2 = 2 \sum_{j=\lfloor n t \rfloor}^{\lfloor n t \rfloor - 1} \sum_{k=j+1}^{\lfloor n t \rfloor - 1} E \left\{ (|I_j|^{\frac{3}{4}} - \alpha_{jn}) \left( |J_{jk} + K_{jk}|^{\frac{3}{4}} - \alpha_{kn} \right) \right\}
\]

\[+ O((t-s)^2(h^{\frac{3}{4}} + |t-s|^{\frac{3}{2}})). \tag{4.11} \]

Indeed, let us estimate the difference between \( V_2 \) and what we would have if we replaced each \( \xi_j \) by the stochastic integrals. Set

\[
Y = |\xi_j|^{\frac{3}{4}} - \alpha_{jn}, \quad Y' = |I_j|^{\frac{3}{4}} - \alpha_{jn}
\]

\[
Z = |\xi_k|^{\frac{3}{4}} - \alpha_{kn}, \quad Z' = |J_{jk} + K_{jk}|^{\frac{3}{4}} - \alpha_{kn}.
\]

Then the difference is

\[
|E\{YZ - Y'Z'\}| \leq E\{|Y(Z - Z')| \} + E\{|Z'(Y - Y')| \}
\]

\[
\leq \|Z - Z'\|_2 \|Y\|_2 + \|Y'\|_2 \|Z'\|_2
\]

\[
\leq ch \left( \|\xi_k|^{\frac{3}{4}} - |J_{jk} + K_{jk}|^{\frac{3}{4}} \|_2 + ch \left( \|\xi_j|^{\frac{3}{4}} - |I_j|^{\frac{3}{4}} \|_2 \right) \right),
\]

since \( \|\alpha_{jn}\|_p \leq c_p h \). Now we estimate the other terms as we did in (4.9); we obtain

\[
\|\xi_j|^{\frac{3}{4}} - |I_j|^{\frac{3}{4}} \|_2 \leq ch^{\frac{3}{2}},
\]

\[
\|\xi_k|^{\frac{3}{4}} - |J_{jk} + K_{jk}|^{\frac{3}{4}} \|_2 \leq ch^{\frac{3}{2}} \left( |t-s|^{\frac{1}{3}} h^{\frac{3}{4}} + h^{\frac{3}{2}} \right).
\]

Thus

\[
|E\{YZ - Y'Z'\}| \leq ch^2 \left( |t-s|^{\frac{1}{3}} h^{\frac{3}{4}} + h^{\frac{3}{2}} \right),
\]

There are at most \((t-s)^2/h^2\) terms in the double sum, so the error introduced in (4.6) by replacing the \( |\Delta X|^{\frac{3}{4}} \) throughout by the stochastic integrals is at worst

\[
c \left[ (t-s)h^{\frac{3}{4}} + (t-s)^2(h^{\frac{3}{4}} + |t-s|^{\frac{3}{2}}) \right]. \tag{4.12} \]
Evidently, \( \|U_n(t) - U_n(s)\|_2 = 0 \) if \( 0 \leq s \leq t \leq s + h \), so we assume that \( (t - s) \geq h \), which implies that \( (t - s)h \leq (t - s)^2 \) and we see that the upper bound (4.12) translates into

\[
c(t - s)^2 \left( h \frac{1}{2} + |t - s\frac{1}{2}) \right). \tag{4.13}
\]

Thus we have proved that

\[
\|U_n(t) - U_n(s)\|_2^2 = \sum_{j=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} E \left\{ (|I_j|^{\frac{4}{3}} - \alpha_{jn})^2 \right\}
+ 2 \sum_{j=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 2} \sum_{k=j+1}^{\lfloor nt \rfloor - 1} E \left\{ (|I_j|^{\frac{4}{3}} - \alpha_{jn})(|J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn}) \right\}
+ O((t - s)^2(h \frac{1}{2} + |t - s\frac{1}{2})). \tag{4.14}
\]

The proof is completed by analysing the two sums on the right of this equation, which is the work of the final step.

**Step (iv).** By Prop. 2.1, \( W \) is independent of \( \mathcal{A}_j \equiv \sigma \{ (B_{jh}, L(jh, B_{jh}), \Delta L_{jh}(\cdot)) \} \), so that if we condition on \( \mathcal{A}_j \), the integrand in \( I_j \) is effectively deterministic; so is the square root multiplying the stochastic integral. The stochastic integral of a deterministic function with respect to a Brownian motion is Gaussian, so we conclude that, given \( \mathcal{A}_j \), the random variable \( I_j \) is conditionally Gaussian, with mean zero and variance

\[
\sigma_j^2 \equiv L(jh, B_{jh}) \int_{-\infty}^{\infty} \Delta L_{jh}(x)^2 dx \equiv (\beta^{-1}\alpha_{jn})^{\frac{3}{2}}. \tag{4.15}
\]

Thus

\[E \left\{ |I_j|^{\frac{4}{3}} \mid \mathcal{A}_j \right\} = \alpha_{jn}\]

by definition of \( \beta \), and

\[E \left\{ (|I_j|^{\frac{4}{3}} - \alpha_{jn})^2 \mid \mathcal{A}_j \right\} = \text{const } (\sigma_j^2)^{\frac{4}{3}},\]

where the constant is unimportant. We now take expectations of the first sum on the right-hand side of (4.14) by conditioning firstly on the \( \mathcal{A}_j \), to obtain the upper bound

\[
V_1 \leq c(t - s)h^{-1} \cdot h^2 = c(t - s)h. \tag{4.16}
\]

The second sum is more subtle, and needs Kolmogorov’s Lemma. If we fix \( j \) and \( k \) in (4.14) and look at the joint representation of \( I_j, J_{jk}, \) and \( K_{jk} \) as stochastic integrals, we see that \( W \) and \( W' \) are independent
Brownian motions, independent of $B_{jh}$, $L(jh, B_{jh})$, $(B_{jh+u} - B_{jh})_{0 \leq u \leq h}$, $B'_v$, $L'(v, B'_v)$, and $(B_{kh+u} - B_{kh})_{u \geq 0}$. Thus, if we condition by $A_{jk} \equiv \sigma\{B_{jk}, L(jh, B_{jh}), (B_{u+jh} - B_{jh})_{0 \leq u \leq h}, B'_{(k-j-1)h}, L'(k-j-1)h, B'_{(k-j-1)h}, (B_{u+kh} - B_{kh})_{u \geq 0}\}$, the integrands and the square roots in $I_j$, $J_{jk}$, and $K_{jk}$ are effectively deterministic; stochastic integrals of deterministic functions with respect to independent Brownian motions are jointly Gaussian, so that these random variables are (conditionally) jointly Gaussian, with mean zero and covariance matrix

$$
\begin{pmatrix}
\sigma^2_j & \theta_{jk} & 0 \\
\theta_{jk} & \sigma^2_{jk} & 0 \\
0 & 0 & \sigma^2_{jk}\n\end{pmatrix}
$$

where $\sigma^2_j$ is as at (4.15), and

$$
\sigma^2_{jk} \equiv L(jh, B_{jh}) \int \Delta L_{kh}(x)^2 dx,
$$

$$
\theta_{jk} \equiv L(jh, B_{jh}) \int \Delta L_{kh}(x) \Delta L_{jh}(x) dx,
$$

$$
\sigma^2_{jk} \equiv L'(k-j-1)h, B_{(k-j-1)h} \int \Delta L'_{(k-j-1)h}(x)^2 dx.
$$

Using equation (1.9) from Kolmogorov's Lemma we obtain

$$
E\left\{ \left( |I_j|^{\frac{4}{3}} - \alpha_{jn} \right)^2 \left( |J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn} \right)^2 \middle| A_{jk} \right\}
\leq c \sigma^2_j \left( \sigma^2_{jk} + \sigma^2_{jk} \right)^{\frac{4}{3}} \cdot \frac{\theta^2_{jk}}{\sigma^2_j \sigma^2_{jk}} \left( \sigma^2_j + \sigma^2_{jk} \right)
\leq c \theta^4_{jk}.
$$

Now we estimate

$$
E\left\{ \theta^4_{jk} \right\} = E\left\{ L(jh, B_{jh})^{\frac{4}{3}} \right\} E\left\{ \left( \int \Delta L_{kh}(x) \Delta L_{jh}(x) dx \right)^{\frac{4}{3}} \right\}_{(4.17)}
\leq c E\left\{ \int \Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x) dx \right\} h^{\frac{4}{3}},
$$
using Jensen’s inequality. If \( m = k - j - 1 \), we have
\[
\int E \left\{ \Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x) \right\} \, dx = \int E \left\{ \tilde{L}(h, x)^{\frac{4}{3}} [L((m + 1)h, x) - L(mh, x)] \right\} \, dx,
\]
where \( \tilde{L} \) and \( L \) are the local time processes of independent Brownian motions.

Thus
\[
\sum_{j=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 2} \sum_{k=j+1}^{\lfloor nt \rfloor - 1} \int E \left\{ \Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x) \right\} \, dx
\]
\[
= \int E \left\{ \tilde{L}(h, x)^{\frac{4}{3}} \right\} \sum_{j=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 2} E\{L((\lfloor nt \rfloor - j - 1)h, x)\} \, dx
\]
\[
\leq \int dx \, E \left\{ \tilde{L}(h, x)^{\frac{4}{3}} \right\} \frac{1}{h} \int_{0}^{t-s} E\{L(u, x)\} \, du
\]
\[
\leq \frac{t-s}{h} \int E \left\{ \tilde{L}(h, x)^{\frac{4}{3}} \right\} \, dx
\]
\[
\leq c \left( \int E \left\{ \tilde{L}(h, x)^{\frac{4}{3}} \right\} \, dx \right)^{\frac{1}{2}}
\]
\[
\leq c(t-s)h^{-\frac{1}{12}}
\]
by the scaling (Lemma 1.4) of local time. Returning to (4.17) gives
\[
V_2 \leq c \sum_{j=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 2} \sum_{k=j+1}^{\lfloor nt \rfloor - 1} E\left\{ \theta_{jk}^{\frac{4}{3}} \right\} \leq c(t-s)h^{\frac{7}{2}}. \tag{4.18}
\]

Combining (4.16) and (4.18) with (4.14), we see that there exists a constant \( C \) such that
\[
\|U_n(t) - U_n(s)\|_2 \leq C \left( (t-s)(h + h^{1/4}) + (t-s)^2h^{1/8} + (t-s)^{17/8} \right).
\]
Now \( h = 1/n \); if we let \( \Delta = t/m \) for some integer \( m \), and note that \( U_n(0) \equiv 0 \), we see
\[
\|U_n(t)\|_2 \leq \sum_{k=0}^{m-1} \|U_n((k+1)\Delta) - U_n(k\Delta)\|_2
\]
\[
\leq \frac{c}{\Delta} \left( \Delta/n + \Delta/n^{1/4} + \Delta^2/n^{1/8} + \Delta^{17/8} \right)^{\frac{1}{2}},
\]
so that
\[
\limsup_{n \to \infty} \|U_n(t)\| \leq c\Delta^{1/16}.
\]
Since $\Delta$ is arbitrary, we must have $\lim \|U_n(t)\|_2 = 0$. This shows that $V_{4/3}^n(t)$ converges in $L^2$ and in probability. To see that the $L^2$ convergence is uniform in $t$, just note that $t \mapsto V_{4/3}^n(t)$ is an increasing function, and that the limit is a continuous increasing function of $t$. It then follows immediately that the $L^2$ convergence is uniform in $t$ for $t$ in bounded intervals. This completes the proof.

**Appendix A. Proof of Kolmogorov’s Lemma**

Here is a proof of Lemma 1.3 in the spirit of this paper. Let $(X_t)_{0 \leq t \leq 1}$ and $(Y_t)_{0 \leq t \leq 1}$ be a pair of standard Brownian motions on the same filtration, having correlation $\rho$, so that $d[X,Y]_t = \rho dt$. Assume without loss of generality that $E\{\phi(X_1)\} = E\{\psi(Y_1)\} = 0$, and let $p_t(x,y) = (2\pi t)^{-\frac{1}{2}} \exp((y-x)^2/2t)$ be the Brownian transition density. Write $P_t\phi(x) = \int p_t(x,y)\phi(y) \, dt$. Consider

$$M_t = P_{1-t}\phi(X_t), \quad 0 \leq t \leq 1,$$

and note that $M_0 = E\{\phi(X_1)\} = 0$ and $M_1 = \phi(X_1)$. By Itô’s Lemma for $t < 1$,

$$dM_t = \frac{\partial}{\partial x} P_{1-t}\phi(X_t) \, dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} P_{1-t}\phi(X_t) \, dt + \frac{\partial}{\partial t} P_{1-t}\phi(X_t) \, dt. \quad (A.1)$$

The last two terms cancel since $p_t$ satisfies the heat equation, so that $M$ is a stochastic integral—we could also have seen this by noticing from the Markov property that $M$ is a martingale—hence

$$\phi(X_1) = \int_0^1 \frac{\partial}{\partial x} P_{1-t}\phi(X_t) \, dX_t. \quad (A.2)$$

Thus, as $d[X,Y] = \rho dt$,

$$|E\{\phi(X_1)\psi(Y_1)\}| = |\rho| \left| E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t)(P_{1-t}\psi)'(Y_t) \, dt \right\} \right|$$

$$\leq |\rho| E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t)^2 \, dt \right\}^{\frac{1}{2}} E\left\{ \int_0^1 (P_{1-t}\psi)'(Y_t)^2 \, dt \right\}^{\frac{1}{2}}$$

$$= |\rho| E\{\phi(X_1)\} E\{\psi(Y_1)\}^{\frac{1}{2}},$$

which implies the first statement of Kolmogorov’s Lemma.

To prove the second statement, suppose $E\{X_1\phi(X_1)\} = 0$. Using (A.2),

$$0 = E\{X_1\phi(X_1)\} = E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t) \, dt \right\}. \quad (A.4)$$
But the Brownian semigroup commutes with differentiation with respect to $x$, so that $E\{(P_{1-t}\phi)'(X_t)\} = P_t\{(P_{1-t}\phi)'\} = (P_tP_{1-t}\phi)' = (P_1\phi)'$, independent of $t$. Its common value must be zero by (A.4). Thus we can apply (A.3)—on the interval $[0,t]$ rather than $[0,1]$—to $(P_{1-t}\phi)'$ and $(P_{1-t}\psi)'$:

$$E\{(P_{1-t}\phi)'(X_t)(P_{1-t}\psi)'(Y_t)\} \leq \rho E\{(P_{1-t}\phi)'(X_t)^2\}^{\frac{1}{2}} E\{(P_{1-t}\psi)'(Y_t)^2\}^{\frac{1}{2}}.$$  

Putting this back into (A.3) gives us the factor of $\rho^2$ in (1.8) and completes the proof.

**References**


L. C. G. Rogers  
School of Mathematical Sciences  
Queen Mary & Westfield College  
Mile End Rd  
London E1 4NS  

John B. Walsh  
Department of Mathematics  
University of British Columbia