THE EXACT $\frac{4}{3}$-VARIATION OF A PROCESS ARISING FROM BROWNIAN MOTION

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1. **Introduction.**

Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$, $B_0 = 0$, and define

$$A(t,x) \equiv \int_0^t I_{\{B_u \leq x\}}\, du = \int_{-\infty}^x L(t,y)\, dy,$$

where $\{L(t,x) : t \geq 0, x \in \mathbb{R}\}$ is the jointly continuous local time of B. The continuous adapted process $A(t, B_t)$ arose naturally in earlier work on the Brownian excursion filtration (see Rogers & Walsh [6], and particularly [7]), where it was important to know about the existence of a local time for $A(t, B_t)$. This would be a consequence of general results (Meyer [4], Yor [9]) if it were a semimartingale, but it is not; Rogers & Walsh [8] prove that the p-variation of

$$X_t \equiv A(t, B_t) - \int_0^t L(s, B_s)\, dB_s \tag{1.1}$$

is infinite if $p < \frac{4}{3}$, and is zero for $p > \frac{4}{3}$. Defining for $p, t > 0$

$$V_p^n(t) \equiv \sum_{j=1}^{[nt]} |X(j/n) - X((j-1)/n)|^p,$$

the sole aim of this paper is to prove the following result.

Theorem 1.1. *For each $t > 0$, $V_{4/3}^n(t)$ converges in probability and in L^2, the convergence is uniform for t in bounded sets, and the limit is given by*

$$\lim_{n\to\infty} V_{4/3}^n(t) = \gamma \int_0^t L(s, B_s)^{\frac{2}{3}}\, ds, \tag{1.2}$$

where

$$\gamma \equiv \frac{2^{\frac{2}{3}}}{\sqrt{\pi}}\, \Gamma\left(\frac{7}{6}\right) E\left[\left(\int_{-\infty}^{\infty} L(1,z)^2,\, dz\right)^{\frac{2}{3}}\right].$$

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The proof we present is long, though none of the steps is particularly difficult. We have to estimate the pth moment of $V^n_{4/3}(t) - \gamma\int_0^t L(s,B_s)^{\frac{2}{3}}\,ds$. If the random variables were Gaussian, this would be relatively easy, thanks to an elegant lemma of Kolmogorov (Lemma 1.3 below). However, they are not. In order to apply Kolmogorov's Lemma we approximate the increments by stochastic integrals and then notice that if we condition on the proper σ-field, these approximations are Gaussian. This requires a rather careful decomposition of the increments. To guide the reader, here is an outline of it. For brevity, we write $h = \frac{1}{n}$ throughout, and frequently suppress the dependence on n in the notation. We develop the increment

$$
\begin{aligned}
\Delta_h X_s &\equiv X_{s+h} - X_s \\
&= \int_{B_s}^{B_{s+h}} L(s,y)\,dy + \int_s^{s+h} I_{\{B_u \le B_{s+h}\}}\,du - \int_s^{s+h} L(u,B_u)\,dB_u \\
&= \int_{B_s}^{B_{s+h}} L(s,y)\,dy - \int_s^{s+h} L(s,B_u)\,dB_u + \bigg(\int_0^h I_{\{B_{s+u}-B_s \le B_{s+h}-B_s\}}\,du \\
&\qquad\qquad - \int_s^{s+h} (L(u,B_u) - L(s,B_u))\,dB_u\bigg) \\
&= \int_{B_s}^{B_{s+h}} L(s,y)\,dy - \int_s^{s+h} L(s,B_u)\,dB_u + \hat{X}_h, \qquad (1.3)
\end{aligned}
$$

where $\hat{X}$ is defined exactly as X, but in terms of the process $\hat{B}_t \equiv B_{t+s} - B_s$ (which is independent of $(B_u)_{u \le s}$). Now, as we shall see, the essential part of $\Delta_h X_s$ is the first two terms in (1.3); and if we write $f(x) \equiv L(s, x + B_s)$ we can rewrite these two terms as

$$\int_0^{\hat{B}_h} f(y)\,dy - \int_0^h f(\hat{B}_u)\,d\hat{B}_u$$

which, by a formal application of Itô's formula, is equal to

$$\tfrac{1}{2}\int_0^h f'(\hat{B}_u)\,du = \tfrac{1}{2}\int_{-\infty}^{\infty} f'(y)\hat{L}(h,y)\,dy\,,$$

where $\hat{L}(t,x) = L(s+t, x+B_s) - L(s, x+B_s)$ is the local time of $\hat{B}$. This is only formal, since f is not differentiable. Nonetheless, $(L(s,y))_{y\in\mathbb{R}}$ is a semimartingale in the excursion filtration (see Perkins [5], Jeulin [2], p. 261) and can be used for stochastic integration. An integration by parts leads us to interpret this expression as the stochastic integral $\fint_{-\infty}^{\infty} \frac{1}{2}\hat{L}(h, y - B_s)\,L(s,dy)$. We must be careful, though; this integral does not exist as an Itô integral. Even though the function $g(y) \equiv \hat{L}(h,y)$ is independent of the excursion filtration of

$(B_u)_{u\le s}$, the integrand $\hat{L}(h, y-B_s)$ will not be adapted for $y < B_s$, even if we augment the excursion filtration with the independent process g. We shall see that we must interpret it as a "split integral", and that the correct formulation of this argument leads to

$$\int_0^{\hat{B}_h} L(s, y+B_s)\,dy - \int_0^h L(s, \hat{B}_u + B_s)\,d\hat{B}_u = \tfrac{1}{2}\fint_{-\infty}^{\infty} \hat{L}(h, y-B_s) L(s, dy),$$

where $\fint$ denotes the integral split at B_s, running forward up from B_s to ∞, and backward down from B_s to $-\infty$; this will be made precise in §2.

For a subsequent stage of the argument, it turns out to be essential to approximate increments of X by (conditionally) Gaussian variables. The semimartingale decomposition of $(L(s,x))_{x\in\mathbb{R}}$ shows us what to expect.

Theorem 1.2 (Perkins [5]). *Let $(\mathcal{E}^s_x)_{x\in\mathbb{R}}$ be the excursion filtration of $(B_u)_{0\le u\le s}$. Then the process*

$$M_x \equiv L(s,x) - \int_{-\infty}^x \alpha(y)dy \tag{1.4}$$

is a continuous $(\mathcal{E}^s_x)$-martingale with quadratic-variation process

$$4\int_{-\infty}^x L(s,y)\,dy.$$

Here,

$$\begin{aligned}\alpha(y) = I_{\{y\ge \underline{B}_s\}} \Bigg[& 2I_{\{y\le 0\}} + 2I_{\{y\le B_s\}} \\ & + I_{\{y\le \bar{B}_s\}} L(s,y) \left(\frac{4I_{(y\ge B_s)}}{L(s,y)+2y^-} - \frac{L(s,y)+2y^-}{s-A(s,y)} \right) \Bigg]\end{aligned} \tag{1.5}$$

with $\bar{B}_s \equiv \sup\{B_u : u\le s\}$, $\underline{B}_s \equiv \inf\{B_u : u \le s\}$.

In view of this, we may write

$$\int_{B_s}^\infty \hat{L}(h, x-B_s)\,L(s,dx) = \int_{B_s}^\infty \hat{L}(h, x-B_s)\left(2\sqrt{L(s,x)}\,W_+(dx) + \alpha(x)\,dx\right)$$

and since the support of $\hat{L}(h,\cdot)$ is small, the stochastic integral with respect to W_+ turns out to be approximately

$$2\sqrt{L(s,B_s)}\int_{B_s}^\infty \hat{L}(h, x-B_s) W_+(dx), \tag{1.6}$$

which is Gaussian, conditional on the σ-field

$$\mathcal{A}_s \equiv \sigma\Big(B_s,\ L(s,B_s),\ (B_{s+u}-B_s)_{u\ge 0}\Big).$$

The sense in which the increment $\Delta_h X_s$ may be approximated by (1.6) and the analogous integral over $(-\infty, B_s)$ is explained and proved in §3.

The strategy for proving the theorem is to show that for any $\varepsilon > 0$, and $T > 0$, there exists $\delta > 0$ and $N \in \mathbb{N}$ such that if $0 \le s \le t \le T$, and $n \ge N$, and $t - s < \delta$, then

$$\left\| V^n_{\frac{4}{3}}(t) - V^n_{\frac{4}{3}}(s) - \gamma \int_s^t L(u, B_u)^{\frac{2}{3}}\, du \right\|_2 \le \varepsilon |t - s|. \tag{1.7}$$

From this, $\|V^n_{\frac{4}{3}}(t) - \gamma \int_0^t L(u, B_u)^{\frac{2}{3}}\, du\,\|_2 \le \varepsilon t$, and the theorem will follow. To estimate the L^2 norms appearing in (1.7), we have to know about the covariance of $|\Delta_h X_s|^{\frac{4}{3}}$ and $|\Delta_h X_t|^{\frac{4}{3}}$. It is hard to get hold of this for general random variables, but here we use the following result of Kolmogorov, which we learned from J. Bretagnolle. Let $\rho(X, Y) \equiv \operatorname{cov}(X, Y)/(\operatorname{var}(X)\operatorname{var}(Y))^{\frac{1}{2}}$ be the correlation coefficient of $X, Y \in L^2$.

Lemma 1.3 (Kolmogorov). *Let (X, Y) have a bivariate Gaussian distribution, and suppose that $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are such that $\varphi(X), \psi(Y) \in L^2$. Then*

$$|\rho(\varphi(X), \psi(Y))| \le |\rho(X, Y)|. \tag{1.8}$$

Moreover, if $E\{\varphi(X)(X - E\{X\})\} = 0$, then

$$|\rho(\varphi(X), \psi(Y))| \le \rho(X, Y)^2. \tag{1.9}$$

A proof of this pretty result appears in an appendix. It will allow us to estimate the correlation of (the conditionally Gaussian approximations to) $\Delta_h X_s$ and $\Delta_h X_t$. This final part of the proof occupies §4.

To conclude the introduction, we record a few simple results which we shall use repeatedly. Firstly, for $p \ge 1$, and positive a and b,

$$|\, b^p - a^p| \le p\,|b - a|\,(b^{p-1} \vee a^{p-1})\,, \tag{1.10}$$

$$|\sqrt{b} - \sqrt{a}| \le \sqrt{|b - a|}\,,$$

and secondly for any $p > 0$, there exists c_p such that for all $x \in \mathbb{R}$, and $t > 0$,

$$\|L(t, x) - L(t, 0)\|_p \le c_p(|x| \wedge t^{\frac{1}{2}} + (x^2 t)^{\frac{1}{4}}). \tag{1.11}$$

The inequalities in (1.10) are elementary, and (1.11) is proved in Rogers and Walsh [6]. From (1.11), it follows easily that for $x, y \in \mathbb{R}$, $t \geq 0$

$$\|L(t,x) - L(t,y)\|_p \leq c_p(|x-y| \wedge t^{\frac{1}{2}} + ((x-y)^2 t)^{\frac{1}{4}}). \tag{1.12}$$

Finally, let us note that the processes $\{B_t,\ t \geq 0\}$ and $\{cB_{t/c^2},\ t \geq 0\}$ have the same distribution, since they are both standard Brownian motions. Since A and L are both functions of the Brownian motion, we can compute them for both of these Brownian motions and compare to get the following scaling lemma, which we shall use repeatedly below.

Lemma 1.4. *The vector-valued processes* $\{(B_t,\ A(t,x),\ L(t,x)),\ t \geq 0,\ x \in \mathbb{R}\}$ *and* $\{(cB_{t/c^2},\ c^2 A(t/c^2, x/c),\ cL(t/c^2, x/c)),\ t \geq 0,\ x \in \mathbb{R}\}$ *have the same distribution.*

2. **Integral representation of increments of** X

Recall the decomposition (1.3) of the increment $\Delta_h X_s \equiv X_{s+h} - X_s$:

$$\Delta_h X_s = \int_{B_s}^{B_{s+h}} L(s,y)\,dy - \int_s^{s+h} L(s,B_u)\,dB_u + \hat{X}_h\,,$$

so that

$$\Delta_h X_s - \hat{X}_h = \lim_{\varepsilon \downarrow 0} \left[\int_{B_s}^{B_{s+h}} (\varphi_\varepsilon * L(s,\cdot))(y)\,dy - \int_s^{s+h} (\varphi_\varepsilon * L(s,\cdot))(B_u)\,dB_u \right], \tag{2.1}$$

where $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\epsilon)$ for some C^∞ function φ which is non-negative, supported in $[-1,1]$, and satisfies $\int \varphi(x)dx = 1$. The convergence of the first integral is almost sure. The second converges in L^2, as we see using (1.12). Hence by Itô's formula

$$\begin{aligned}
\Delta_h X_s - \hat{X}_h &= \lim_{\varepsilon \downarrow 0} \tfrac{1}{2} \int_s^{s+h} (\varphi_\varepsilon * L(s,\cdot))'(B_u)\,du \\
&= \lim_{\varepsilon \downarrow 0} \tfrac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s)(\varphi_\varepsilon * L(s,\cdot))'(y)dy \\
&= \lim_{\varepsilon \downarrow 0} \tfrac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s) \left(\int_{-\infty}^{\infty} \varphi_\varepsilon'(y-x) L(s,x)\,dx \right) dy\,.
\end{aligned}$$

By splitting the x-integral at B_s, we get two terms, one of which is

$$\tfrac{1}{2} \int_{-\infty}^{\infty} \hat{L}(h, y - B_s) \left(\int_{B_s}^{\infty} \varphi_\varepsilon'(y-x) L(s,x)dx \right) dy. \tag{2.2}$$

Now abbreviate

$$g(y) \equiv \hat{L}(h,y)$$

and notice that g is independent of $\{B_u : u < s\}$, so we may condition on $\{\hat{B}_u : u \geq 0\} \equiv \{B_{u+s} - B_s : u \geq 0\}$ and treat g as a deterministic function. As we recalled in Theorem 1.2, $L(s,x)$ is a semimartingale in the filtration $(\mathcal{E}_x^s)_{x\in\mathbb{R}}$ (if $\tau^s(t,x) \equiv \inf\{u < s : A(u,x) > t\} \wedge s$, then $\mathcal{E}_x^s$ is the σ-field generated by $(B(\tau^s(t,x)))_{t\geq 0}$—see Perkins [5], Jeulin [2]). Thus we may integrate by parts in (2.2) to give

$$\begin{aligned}
&\tfrac{1}{2}\int_{-\infty}^{\infty} g(y-B_s)\left(\varphi_\varepsilon(y-B_s)L(s,B_s) + \int_{B_s}^{\infty}\varphi_\varepsilon(y-x)L(s,dx)\right)dy \\
&\qquad = \tfrac{1}{2}L(s,B_s)\int_{-\infty}^{\infty}\varphi_\varepsilon(y-B_s)g(y-B_s)\,dy \qquad (2.3)\\
&\qquad\quad + \tfrac{1}{2}\int_{B_s}^{\infty}\left(\int_{-\infty}^{\infty} g(y-B_s)\varphi_\varepsilon(y-x)\,dy\right)L(s,dx).
\end{aligned}$$

The interchange of orders of integration is achieved by taking Riemann-sum approximations in (2.2) – which will be finite sums since g and φ_ε are of compact support – then interchanging sum and integral and finally letting the mesh go to zero, so that the integrands converge almost surely uniformly to $\int g(y-B_s)\varphi_\varepsilon(y-x)\,dy$, again by compact support. The representation (1.4)–(1.5) ensures the conclusion. Finally we let $\varepsilon \downarrow 0$ in (2.3) to get

$$\tfrac{1}{2}L(s,B_s)\,g(0) + \tfrac{1}{2}\int_{B_s}^{\infty} g(x-B_s)\,L(s,dx). \qquad (2.4)$$

The term coming from the x-integral over $(-\infty, B_s)$ is handled similarly, except that now we work in the filtration $(\check{\mathcal{E}}_x^s)_{x\in\mathbb{R}}$, where we define $\check{\mathcal{E}}_x^s$ to be the σ-field generated by $(B(\check{\tau}^s(t,-x)))_{t\geq 0}$, with

$$\check{\tau}^s(t,x) \equiv \inf\{u : u - A(u,x) > t\} \wedge s.$$

Thus $(\check{\mathcal{E}}_x^s)$ is simply the filtration $(\mathcal{E}_x^s)$ for the Brownian motion $-B$. If we write $\check{L}(s,x) \equiv L(s,-x)$, then $\check{L}(s,x)$ is a semimartingale in the filtration $(\check{\mathcal{E}}_x^s)$, and

$$\begin{aligned}
&\tfrac{1}{2}\int_{-\infty}^{\infty} g(y-B_s)\left(\int_{-\infty}^{B_s} dx\,\varphi_\varepsilon'(y-x)L(s,x)\right)dy \\
&\quad = \tfrac{1}{2}\int_{-\infty}^{\infty} g(y-B_s)\left\{-\varphi_\varepsilon(y-B_s)L(s,B_s) - \int_{-B_s}^{\infty}\varphi_\varepsilon(y+x)\check{L}(s,dx)\right\}dy \\
&\quad \to -\tfrac{1}{2}L(s,B_s)\,g(0) - \tfrac{1}{2}\int_{-B_s}^{\infty} g(-x-B_s)\,\check{L}(s,dx)
\end{aligned}$$

as $\varepsilon \downarrow 0$. Combining with (2.4), we have then that

$$\begin{aligned}
\Delta_h X_s - \hat{X}_h &= \tfrac{1}{2}\int_{B_s}^{\infty} g(x-B_s)L(s,dx) - \tfrac{1}{2}\int_{-B_s}^{\infty} g(-x-B_s)\check{L}(s,dx) \\
&\equiv \tfrac{1}{2}⨍_{-\infty}^{\infty} g(x-B_s)L(s,dx) \qquad (2.5)
\end{aligned}$$

as a piece of shorthand notation.

The aim now is to express (2.5) in terms of the semimartingale decomposition of L. From Theorem 1.1, we have that

$$M_x \equiv L(s,x) - \int_{-\infty}^{x} \alpha(y)\, dy$$
$$\check{M}_x \equiv \check{L}(s,x) - \int_{-\infty}^{x} \check{\alpha}(y)\, dy$$

are martingales in the filtrations $(\mathcal{E}^s_x)$, $(\check{\mathcal{E}}^s_x)$ respectively, where $\check{\alpha}$ is defined exactly as α, but in terms of $-B$:

$$\check{\alpha}(y) = I_{\{y \geq -\bar{B}_s\}} \Bigg[2I_{\{y\leq 0\}} + 2I_{\{y \leq -B_s\}} + I_{\{y\leq -\underline{B}_s\}} L(s,-y) \left(\frac{4I_{\{y\geq -\underline{B}_s\}}}{L(s,-y)+2y^-} - \frac{L(s,-y)+2y^-}{s - \int_{-y}^{\infty} L(s,v)\; dv} \right) \Bigg] .$$

We shall suppose that the probability space is enlarged if need be, and the filtrations $(\mathcal{E}^s_x)_{x\in\mathbb{R}}$, respectively $(\check{\mathcal{E}}^s_x)_{x\in\mathbb{R}}$, are enlarged so that there exist an $(\mathcal{E}^s_x)$-Brownian motion β^+_x and an $(\check{\mathcal{E}}^s_x)$-Brownian motion β^-_x which are independent of B and of each other. This is not essential, but simplifies the statements of results.

Proposition 2.1. *There is an $(\mathcal{E}^s_x)$-Brownian motion W^+ independent of $\mathcal{A}_s \equiv \sigma\left(\{B_s, L(s,B_s), (B_{s+u}-B_s)_{u\geq 0}\}\right)$ such that*

$$L(s, y\vee B_s) - L(s,B_s) = 2\int_{B_s}^{y\vee B_s} \sqrt{L(s,x)}\; dW^+_x + \int_{B_s}^{y\vee B_s} \alpha(x)\, dx. \tag{2.6}$$

Similarly, there is an $(\check{\mathcal{E}}^s_x)$-Brownian motion W^- independent of $\mathcal{A}_s$ such that

$$\check{L}(s,(-B_s)\vee y) - \check{L}(s,-B_s) = 2\int_{-B_s}^{(-B_s)\vee y} \sqrt{\check{L}(s,x)}\, dW^-_x + \int_{-B_s}^{(-B_s)\vee y} \check{\alpha}(x)\, dx. \tag{2.7}$$

Moreover, W^+ and W^- are independent.

Proof. We begin by defining

$$dW^+_x = I_{\{x\leq B_s\}}\, d\beta^+_x + (2\sqrt{L(s,x)})^{-1} I_{\{B_s\leq x<\bar{B}_s\}}\, dM_x + I_{\{x\geq \bar{B}_s\}}\, d\beta^+_x,$$
$$dW^-_x = I_{\{x\leq -B_s\}}\, d\beta^-_x + (2\sqrt{L(s,-x)})^{-1} I_{\{-B_s<x\leq -\underline{B}_s\}}\, d\check{M}_x + I_{\{x\geq -\underline{B}_s\}}\, d\beta^-_x,$$
$$W^+_0 = W^-_0 = 0.$$

It is clear that W^+ (respectively, W^-) is a Brownian motion in $(\mathcal{E}^s_x)$ (respectively, $(\check{\mathcal{E}}^s_x)$), and that (2.6), (2.7) hold by construction. To prove the independence properties, take $f_+, f_- \in C_K^\infty$, $\xi \in b\mathcal{A}_s$, and compute

$$E\left\{\exp\left(i\xi\int f_+(x)\,dW_x^+ + i\xi\int f_-(x)\,dW_x^-\right)\right\}$$
$$= E\left\{\exp\left(i\xi\int_{-\infty}^{\infty} f_+(x)\,dW_x^+ + i\xi\int_{-B_s}^{\infty} f_-(x)\,dW_x^- - \frac{\xi^2}{2}\int_{-\infty}^{-B_s} f_-(x)^2 dx\right)\right\},$$

since β^- is independent of β^+ and B;

$$= E\left\{\exp\left(-\frac{\xi^2}{2}\int_{B_s}^{\infty} f_+(x)^2\,dx + i\xi\int_{-\infty}^{B_s} f_+(x)\,dW_x^+ \right.\right.$$
$$\left.\left. + i\xi\int_{-B_s}^{\infty} f_-(x)\,dW_x^- - \frac{\xi^2}{2}\int_{-\infty}^{-B_s} f_-(x)^2 dx\right)\right\},$$

by conditioning on $\mathcal{E}_{B_s}^s \vee \sigma(\beta^-)$, since $\int_{-B_s}^{\infty} f_-(x)\,dW_x^-$ is measurable on this σ-field and W^+ is a Brownian motion in the filtration $(\mathcal{E}_x^s \vee \sigma(\beta^-))$;

$$= E\left\{\exp\left(-\frac{\xi^2}{2}\int f_+(x)^2\,dx + i\xi\int_{-B_s}^{\infty} f_-(x)\,dW_x^- - \frac{\xi^2}{2}\int_{-\infty}^{-B_s} f_-(x)^2 dx\right)\right\},$$

since β^+ is independent of B, β^-;

$$= E\left\{\exp\left(-\frac{\xi^2}{2}\int f_+(x)^2\,dx - \frac{\xi^2}{2}\int f_-(x)^2 dx\right)\right\},$$

by conditioning on $\check{\mathcal{E}}_{-B_s}^s \supseteq \mathcal{A}_s$. This proves that $W^\pm$ are independent Brownian motions, independent of $\mathcal{A}_s$. ✠

In view of this, then, we can re-express (2.5) as

$$\Delta_h X_s - \hat{X}_h = ⨍_{-\infty}^{\infty} g(x-B_s)\sqrt{L(s,x)}\,dW_x + \tfrac{1}{2}⨍_{-\infty}^{\infty} g(x-B_s)\,\alpha(x)\,dx\,, \tag{2.8}$$

where the two split integrals are defined by

$$⨍_{-\infty}^{\infty} g(x-B_s)\sqrt{L(s,x)}\,dW_x$$
$$= \int_{B_s}^{\infty} g(x-B_s)\sqrt{L(s,x)}\,dW_x^+ - \int_{-B_s}^{\infty} g(-x-B_s)\sqrt{L(s,-x)}\,dW_x^- \tag{2.9}$$

and

$$⨍_{-\infty}^{\infty} g(x-B_s)\,\alpha(x)\,dx = \int_{B_s}^{\infty} g(x-B_s)\,\alpha(x)\,dx - \int_{-B_s}^{\infty} g(-x-B_s)\,\check{\alpha}(x)\,dx\,. \tag{2.10}$$

Note that the splitting point of the split integral depends on the value of B_s. We suppress this from the notation since in most cases the splitting point will be clear from the context.

Let $f(x) = g(x - B_s)\sqrt{L(s,x)}$ and apply Burkholder's inequality to each stochastic integral in (2.9) to see that the L^p norm of the split integral in (2.8) satisfies

$$\left\| \fint_{-\infty}^{\infty} f(y)\, dW_y \right\|_p \leq C_p \left\| \int_{-\infty}^{\infty} f(y+B_s)^2 dy \right\|_{p/2}^{1/2} . \tag{2.11}$$

3. **The main estimates**

We now take $0 < s < t \leq 1$ and estimate $\Delta_h X_s$ and $\Delta_h X_t$ in turn. We decompose $\Delta_h X_s$ (using (2.8), (2.9)) as

$$\begin{aligned} \Delta_h X_s = &\fint_{-\infty}^{\infty} g(x - B_s)\sqrt{L(s,B_s)}\, dW_x \qquad (3.1)\\ &+ \fint_{-\infty}^{\infty} g(x-B_s)\left(\sqrt{L(s,x)} - \sqrt{L(s,B_s)}\right) dW_x \\ &+ \tfrac{1}{2}\fint_{-\infty}^{\infty} g(x-B_s)\alpha(x)\, dx + \hat{X}_h \\ \equiv{}& I_1 + I_2 + I_3 + \hat{X}_h \,, \end{aligned}$$

say. Firstly we take I_2. By (2.11)

$$\|I_2\|_p \leq C_p \left\| \int_0^{\infty} g(y)^2 |L(s, B_s + y) - L(s,B_s)|\, dy \right\|_{p/2}^{\frac{1}{2}} . \tag{3.2}$$

Write the norm in terms of expectations and use Hölder's inequality with $p/2$ and $p/(p-2)$; this is

$$\leq C_p \left(E\left\{ \int_0^{\infty} g(y)^2 |L(s,B_s+y) - L(s,B_s)|^{p/2}\, dy \left(\int_0^{\infty} g(x)^2 dx\right)^{p/2-1} \right\}\right)^{1/p} ;$$

change order and use the fact that g is independent of $L(s,\cdot)$:

$$\leq C_p \left(\int_0^{\infty} E\left\{ |L(s,y) - L(s,0)|^{p/2}\right\} E\left\{ g(y)^2 \left(\int_0^{\infty} g(x)^2 dx\right)^{p/2-1} \right\} dy \right)^{1/p} ;$$

now use (1.11) on the first expectation and the Schwartz inequality on the second:

$$\leq C_p \left(\int_0^{\infty} (y \wedge \sqrt{s} + y^{\frac{1}{2}} s^{\frac{1}{4}})^{p/2} E\left\{g(y)^4\right\}^{\frac{1}{2}} \left(E\left\{ \left(\int_0^{\infty} g(x)^2 dx\right)^{p-2} \right\}\right)^{\frac{1}{2}} dy \right)^{1/p} ,$$

using (1.11) on the first term. Now recall that $g(y) = \hat{L}(h,y) \stackrel{\mathcal{D}}{=} \sqrt{h}\hat{L}(1, y/\sqrt{h})$ so that

$$E\left\{g(y)^4\right\} = h^2 E\left\{\hat{L}(1, y/\sqrt{h})^4\right\}$$
$$\leq h^2 P\left\{H_{y/\sqrt{h}} < 1\right\} E\left\{\hat{L}(1,0)^4\right\},$$

where $H_x = \inf\{u : B_u = x\}$

$$\leq ch^2 \bar{\Phi}(y/\sqrt{h})$$

where $\bar{\Phi}$ is the tail of the standard normal distribution. Moreover,

$$E\left\{\left(\int_0^\infty g(x)^2 dx\right)^{p-2}\right\} = h^{p-2} E\left\{\left(\int_0^\infty \hat{L}(1, y/\sqrt{h})^2\, dy\right)^{p-2}\right\}$$
$$= h^{3(p-2)/2} E\left\{\left(\int_0^\infty \hat{L}(1,y)^2\, dy\right)^{p-2}\right\}$$
$$\leq Ch^{3(p-2)/2},$$

since by the results of Barlow and Yor [1], $L^*(1) \in \bigcap_p L^p$, where $L^*(s) \equiv \sup_y L(s,y)$. Thus

$$\|I_2\|_p \leq C_p h^{\frac{3}{4}} \left(\int_0^\infty (y \wedge \sqrt{s} + y^{\frac{1}{2}} s^{\frac{1}{4}})^{p/2} \bar{\Phi}(y/\sqrt{h})^{\frac{1}{2}}\, h^{-\frac{1}{2}}\, dy\right)^{1/p}$$
$$\leq C_p h^{\frac{7}{8}} \tag{3.3}$$

since s is bounded.

Next we estimate I_3, a more delicate task. Write it as the sum of two integrals, $\int_{\underline{B}_s}^\infty + \int_{-\infty}^{\underline{B}_s} = I_3^+ + I_3^-$, say. By symmetry we only have to treat the first one. By (1.5) we have

$$2I_3^+ = \int_{B_s}^\infty g(y - B_s)\left(2I_{\{y\leq 0\}} + I_{(y\leq \bar{B}_s)} L(s,y)\, \frac{L(s,y) + 2y^-}{s - A(s,y)}\right)\, dy\,.$$

The integral $\int g(y - B_s) I_{\{y\leq 0\}}\, dy$ is at most h, so that will cause no trouble. What remains is the sum of two terms,

$$\int_{B_s}^{\bar{B}_s} g(y - B_s)\frac{L(s,y)^2}{s - A(s,y)}\, dy + 2I_{\{B_s<0\}} \int_{B_s}^0 g(y - B_s)\frac{|y| L(s,y)}{s - A(s,y)}\, dy\,.$$

If we make the replacements $B_u \mapsto -B_{s-u} + B_s$ in the first term and $B_u \mapsto -B_u$ in the second, we see that they are equal in distribution to

$$J_1 \equiv \int_{\underline{B}_s}^0 g(y) L(s,y)^2 \frac{dy}{A(s,y)}\,,$$

and

$$J_2 \equiv I_{\{B_s>0\}} \int_0^{B_s} g(y - B_s) \, \frac{2yL(s,y)}{A(s,y)} \, dy$$

respectively. Now by Jensen's inequality for $p \geq 1$

$$|J_1|^p \leq h^{p-1} \int_{\underline{B}_s}^0 g(y) \left| \frac{L(s,y)^2}{A(s,y)} \right|^p \, dy. \tag{3.4}$$

For $y \in (\underline{B}_s, 0)$, $L(s,y)^2/A(s,y) \stackrel{\mathcal{D}}{=} L(s - H_y, 0)^2/A(s - H_y, 0)$, and, by scaling, $L(t,0)^2/\ A(t,0) \stackrel{\mathcal{D}}{=} L(1,0)^2/A(1,0)$. Thus we have the estimate

$$E\left\{|J_1|^p\right\} \leq h^p E\left\{L(1,0)^{2p}/A(1,0)^p\right\} \equiv h^p \gamma_p, \tag{3.5}$$

say, and according to Karatzas and Shreve [3] who compute the joint distribution of $(B_t, L(t,0),\ A(t,0))$,

$$P\left\{L(1,0) \in db, A(1,0) \in ds\right\} = \frac{b}{\pi(s(1-s))^{\frac{3}{2}}} \ \exp(-b^2/2s(1-s)) \, db \, ds,$$

hence

$$\begin{aligned}
\gamma_p &= \int_0^\infty db \int_s^1 ds \frac{b^{2p}}{s^p} \, \frac{b}{\pi(s(1-s))^{\frac{3}{2}}} \ \exp(-b^2/2s(1-s)) \\
&= \int_0^\infty dv \int_0^1 ds (2v(1-s))^p \, \frac{e^{-v}}{\pi\sqrt{s(1-s)}} \\
&= \Gamma(p+1) 2^p \int_0^1 ds (1-s)^p / \sqrt{s(1-s)} \\
&< \infty.
\end{aligned}$$

To summarize, then,

$$\|J_1\|_p \leq C_p h. \tag{3.6}$$

The estimation of J_2 proceeds along similar lines: for $p \geq 1$

$$|J_2|^p \leq h^{p-1} \int_0^\infty g(y - B_s) \left| \frac{2yL(s,y)}{A(s,y)} \right|^p I_{\{y<B_s\}} \, dy\,; \tag{3.7}$$

$$E\left\{\left|\frac{L(s,y)}{A(s,y)}\right|^p;\ y < B_s\right\} = \int_0^s \frac{ye^{-y^2/2(s-u)}}{\sqrt{2\pi(s-u)^3}} E\left\{\left|\frac{L(u,0)}{s-u+A(u,0)}\right|^p; B_u > 0\right\} du\,.$$

Using once again the result of Karatzas and Shreve,

$$
\begin{aligned}
&E\left\{\left|\frac{L(u,0)}{s-u+A(u,0)}\right|^p; B_u>0\right\}\\
&\quad=\int_0^\infty db\int_0^u dv\,\frac{b}{\pi(v^3(u-v))^{\frac12}}\;\exp\left(-\frac{b^2}{2}\,\frac{u}{v(u-v)}\right)\left|\frac{b}{s-u+v}\right|^p\\
&\quad=C_p\int_0^u dv\,v^{(p-1)/2}(u-v)^{(p+1)/2}u^{-(p+2)/2}(s-u+v)^{-p}\\
&\quad=C_p\int_0^1 dt\,u^{p/2}t^{(p-1)/2}(1-t)^{(p+1)/2}(s-u+ut)^{-p}.
\end{aligned}
$$

Now using the fact that $g(y-B_s)\equiv\hat L(h,y-B_s)\le\hat L^*(h)\overset{\mathcal D}{=}\sqrt h\hat L^*(1)$, we deduce that

$$
\begin{aligned}
E\{|J_2|^p\}&\le C_ph^{p-\frac12}\int_0^\infty y^pE\left\{\left|\frac{L(s,y)}{A(s,y)}\right|^p; y<B_s\right\}dy\\
&=C_ph^{p-\frac12}\int_0^\infty y^p\,dy\int_0^s\frac{ye^{-y^2/2(s-u)}}{\sqrt{2\pi(s-u)^3}}\,du\\
&\qquad\qquad\int_0^1 dt\,u^{\frac p2}t^{\frac{p-1}{2}}(1-t)^{\frac{p+1}{2}}(s-u+ut)^{-p}\\
&=C_ph^{p-\frac12}\int_0^s du(s-u)^{\frac{p-1}{2}}\int_0^1 dt\,u^{\frac p2}t^{\frac{p-1}{2}}(s-u+ut)^{-p}\\
&=C_ph^{p-\frac12}\sqrt s\int_0^1\int_0^1 dw\,dt\,(1-w)^{\frac{p-1}{2}}w^{\frac p2}t^{\frac{p-1}{2}}(1-t)^{\frac{p+1}{2}}(1-w+wt)^{-p}\\
&=C_ph^{p-\frac12}\sqrt s\int_0^1\int_0^1 dw\,dt\left|\frac{(1-w)wt}{(1-w+wt)^2}\right|^{\frac p2}(1-w)^{-\frac12}t^{-\frac12}(1-t)^{\frac{p+1}{2}}\\
&=C_ph^{p-\frac12}\sqrt s\,.
\end{aligned}
$$

Hence we have

$$\|J_2\|_p\le C_ph^{\frac78}. \tag{3.8}$$

Assembling the bounds on J_1 and J_2, we see that for small h,

$$\|I_3\|_p\le C_ph^{\frac78}.$$

Finally, we estimate

$$\hat X_h=\int_0^h I_{\{\hat B_u\le\hat B_h\}}\,du-\int_0^h\hat L(u,\hat B_u)\,d\hat B_u.$$

The first term is at most h. For the second, taking $p \geq 2$ we see that

$$E\left\{\left|\int_0^h \hat{L}(u,\hat{B}_u)\,d\hat{B}_u\right|^p\right\} \leq C_p E\left\{\left|\int_0^h L(u,B_u)^2\,du\right|^{p/2}\right\}$$
$$\leq h^{p/2-1} C_p E\left\{\int_0^h L(u,B_u)^p\,du\right\}$$
$$\leq C_p h^p,$$

so that

$$\|\hat{X}_h\|_p \leq C_p h.$$

In short, then, for every $p \geq 1$

$$\|\Delta_h X_s - I_1\|_p \leq C_p h^{\frac{7}{8}}. \tag{3.9}$$

This completes the analysis of $\Delta_h X_s$; we now analyse the joint behavior of the two increments. Let $t > s$ and let us consider $\Delta_h X_t$. The arguments for the decomposition of this increment are broadly similar to the foregoing, but with one refinement which will be needed at a later stage; we cut $(0,t]$ into $(0,s] \cup (s,s+h] \cup (s+h,t]$, so that the contribution from the last interval will be *independent of* the contribution from the first. If we simply cut $(0,t]$ into $(0,s] \cup (s,t]$, this will not be true.

To begin, then, from (1.3)

$$\Delta_h X_t = \int_{B_t}^{B_{t+h}} L(t,y)\,dy - \int_t^{t+h} L(t,B_u)\,dB_u + \tilde{X}_h$$

where $\tilde{X}$ is defined exactly as X, (see (1.1)) but in terms of the Brownian motion $\tilde{B}_u \equiv B_{t+u} - B_t$. We split the first two terms as

$$\left[\int_{B_t}^{B_{t+h}} L(s,y)\,dy - \int_t^{t+h} L(s,B_u)\,dB_u\right]$$
$$+\left[\int_{B_t}^{B_{t+h}} (L(s+h,y) - L(s,y))\,dy - \int_t^{t+h} (L(s+h,B_u) - L(s,B_u))\,dB_u\right]$$
$$+\left[\int_{B_t}^{B_{t+h}} (L(t,y) - L(s+h,y))\,dy - \int_t^{t+h} (L(t,B_u) - L(s+h,B_u))\,dB_u\right]$$
$$\equiv K_{(0,s]} + K_{(s,s+h]} + K_{(s+h,t]}\,,$$

say. Taking the terms one by one, the analysis of $K_{(0,s]}$ is very similar to the analysis of $\Delta_h X_s$. Following the steps (2.2)—(2.5) we see

$$\begin{aligned} K_{(0,s]} &= \lim_{\varepsilon\downarrow 0} \tfrac{1}{2} \int_t^{t+h} (\phi_\varepsilon * L(s,\cdot))'(B_u)\,du \\ &= \lim_{\varepsilon\downarrow 0} \tfrac{1}{2} \int_{-\infty}^{\infty} \tilde{L}(h, y - B_t)(\phi_\varepsilon * L(s,\cdot))'(y)\,dy \\ &= \lim_{\varepsilon\downarrow 0} \tfrac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(y - B_s) \left(\int_{-\infty}^{\infty} dx\, \phi'_\varepsilon(y-x) L(s,x) \right\} dy\,, \end{aligned}$$

where $\tilde{g}(x) \equiv \tilde{L}(h, x - B_t + B_s) \equiv \hat{L}(t-s+h, x) - \hat{L}(t-s, x)$ is measurable with respect to $\sigma(\{\hat{B}_u : u \geq 0\})$ and therefore is independent of $\{B_u : u \leq s\}$. The analysis of $\Delta_h X_s$ in (2.1)—(2.5) goes through *mutatis mutandis* to give the analogue of (2.5):

$$K_{(0,s]} = \tfrac{1}{2} \int_{-\infty}^{\infty} \tilde{g}(x - B_s) L(s, dx)\,.$$

We apply Proposition 2.1 again to obtain the analogue of (2.8):

$$\begin{aligned} K_{(0,s]} &= \fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\sqrt{L(s,x)}\,dW_s + \tfrac{1}{2}\fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\alpha_x\,dx \\ &\equiv K_1 + K_2 \end{aligned} \tag{3.10}$$

say, where the second split integral is defined by

$$\fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\,\alpha_x\,dx = \int_{B_s}^{\infty} \tilde{g}(x - B_s)\,\alpha_x\,dx - \tfrac{1}{2}\int_{-B_s}^{\infty} \tilde{g}(-x - B_s)\,\check{\alpha}_x\,dx\,.$$

Recall that $\tilde{g}$ has its support in a small neighborhood of 0, so that we can (almost) factor out the square root from K_1. In fact

$$\begin{aligned} K_1 - \sqrt{L(s,B_s)} \fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\,dW_x & \\ &= \left[\sqrt{L(s,B_t)} - \sqrt{L(s,B_s)}\right] \fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\,dW_x \\ &\qquad + \fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\left(\sqrt{L(s,x)} - \sqrt{L(s,B_t)}\right) dW_x\,, \\ &\equiv M_1 + M_2\,. \end{aligned} \tag{3.11}$$

We will see that M_1 and M_2 are small. Now

$$\|M_1\|_p \leq \left\|\sqrt{L(s,B_t)} - \sqrt{L(s,B_s)}\right\|_{2p} \left\|\fint_{-\infty}^{\infty} \tilde{g}(x - B_s)\,dW_x\right\|_{2p}$$

and by (1.10)

$$\left\| \sqrt{L(s,B_t)} - \sqrt{L(s,B_s)} \right\|_{2p} \leq \|L(s,B_t) - L(s,B_s)\|_p^{\frac{1}{2}}$$
$$= (E\{|L(s,Z) - L(s,0)|^p\})^{1/2p} \ ,$$

where $Z \sim N(0, t-s)$ is independent of B; let $p_t(\cdot)$ be the $N(0,t)$ density:

$$= \left(\int p_{t-s}(x)\, E\{|L(s,x) - L(s,0)|^p\} dx \right)^{1/2p} ,$$
$$\leq C_p \left(\int p_{t-s}(x)\, (|x| \wedge \sqrt{s} + (x^2 s)^{\frac{1}{4}})^p \, dx \right)^{1/2p} ,$$

using the estimate (1.11). This is

$$= C_p \left(\int p_1(x) \, \left(|x\sqrt{t-s}| \wedge \sqrt{s} + (t-s)^{\frac{1}{4}} (x^2 s)^{\frac{1}{4}} \right)^p \, dx \right)^{\frac{1}{2p}} ,$$
$$\leq C_p (t-s)^{\frac{1}{8}}$$

for all $0 \leq s \leq t \leq 1$. Next we estimate

$$\left\| \int_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x \right\|_p \leq C_p \left(\left\| \int_{B_s}^{\infty} \tilde{g}(x-B_s)^2 \, dx \right\|_{p/2}^{\frac{1}{2}} + \left\| \int_{-\infty}^{B_s} \tilde{g}(x - B_s)^2 \, ds \right\|_{p/2}^{\frac{1}{2}} \right)$$
$$\leq C_p \left\| \int_{-\infty}^{\infty} L(h,y)^2 \, dy \right\|_{p/2}^{\frac{1}{2}}$$
$$\leq C_p h^{\frac{3}{4}} ,$$

since $L(t,x) \equiv cL(t/c^2, x/c)$ by Lemma 1.4. Thus

$$\|M_1\|_p \leq C_p |t-s|^{\frac{1}{8}} h^{\frac{3}{4}} . \tag{3.12}$$

The second term on the right of (3.11), M_2, is handled exactly as was I_2: following (3.2)—(3.3) for $p \geq 2$, we get

$$\left\| ⨍_{-\infty}^{\infty} \tilde{g}(x - B_s) \left(\sqrt{L(s,x)} - \sqrt{L(s,B_t)} \right) \, dW_x \right\|_p$$
$$\leq \left\| \int_{-\infty}^{\infty} \tilde{g}(x-B_s)^2 \, |L(s,x) - L(s,B_t)| \, dx \right\|_{p/2}^{\frac{1}{2}} \leq C_p h^{7/8} . \tag{3.13}$$

We combine (3.12) and (3.13) into the estimate

$$\left\| K_1 - \sqrt{L(s,B_s)} ⨍_{-\infty}^{\infty} \tilde{g}(x - B_s) \, dW_x \right\|_p \leq C_p \left(|t-s|^{\frac{1}{8}} h^{\frac{3}{4}} + h^{\frac{7}{8}} \right) . \tag{3.14}$$

Next we turn to K_2. This term is analogous to I_3, and we can follow that analysis. We break it into the sum of two similar integrals, $K_2^+ + K_2^-$, and consider K_2^+, which satisfies (see (3.4))

$$2K_2 = \int_{B_s}^{\infty} \tilde{g}(y - B_s) \left[2I_{\{y\le 0\}} + I_{\{y\le \bar{B}_s\}} \frac{L(s,y)(L(s,y)+2y^-)}{s - \int_{-\infty}^{y} L(s,v)\,dv} \right] dy.$$

As with the estimation of I_3, the integral involving $I_{(y\le 0)}$ is at most h, so causes no trouble. What remains is the sum of two terms, the first equal in distribution to

$$\tilde{J}_1 \equiv \int_{\underline{B}_s}^{0} \tilde{g}(-y) L(s,y)^2 \frac{dy}{A(s,y)} \ ,$$

the second equal in distribution to

$$\tilde{J}_2 \equiv I_{\{B_s>0\}} \int_0^{B_s} \tilde{L}(h, B_t - y) \frac{2yL(s,y)}{A(s,y)} \ dy.$$

We handle the estimation of $\tilde{J}_1$ as before, in (3.5)—(3.6): for $p \ge 1$,

$$\begin{aligned} |\tilde{J}_1|^p &\le h^{p-1} \int_{\underline{B}_s}^{0} \tilde{g}(-y) \left| \frac{L(s,y)^2}{A(s,y)} \right|^p dy \\ &\equiv h^{p-1} \int_{\underline{B}_s}^{0} \tilde{L}(h, -y - B_t + B_s) \left| \frac{L(s,y)^2}{A(s,y)} \right|^p dy \end{aligned}$$

from which, as before,

$$\|\tilde{J}_1\|_p \le C_p h.$$

The estimation of $\tilde{J}_2$ is exactly like the estimation of J_2 in (3.7)—(3.8), except that $\hat{L}(h, y - B_s)$ is replaced by $\tilde{L}(h, B_t - x)$. The estimation of J_2 involved the bound $\hat{L}(h, y - B_s) \le \hat{L}^*(h) \overset{\mathcal{D}}{=} \sqrt{h}\hat{L}^*(1)$, and the analogous bound is good for $\tilde{L}(h, B_t - x)$. We thus obtain

$$\|\tilde{J}_2\|_p \le C_p h^{\frac{7}{8}} \ .$$

Assembling the last two equations gives

$$\|K_2\|_p \le C_p h^{\frac{7}{8}} \ . \tag{3.15}$$

Putting (3.10), (3.14), and (3.15) together, we get

$$\left\| K_{(0,s]} - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \tilde{g}(x - B_s)\, dW_x \right\|_p \le C_p \left\{ |t-s|^{\frac{1}{8}} h^{\frac{3}{4}} + h^{\frac{7}{8}} \right\} . \tag{3.16}$$

Next we must estimate $K_{(s,s+h]}$, an easier task. We leave the reader to confirm that

$$\|K_{(s,s+h]}\|_p \le C_p h. \tag{3.17}$$

Finally, we deal with $K_{(s+h,t]}$. If we define $v \equiv t-s-h$, and $B'_u \equiv B_{u+s+h} - B_{s+h}$, we have then that

$$K_{(s+h,t]} = \int_{B'_v}^{B'_{v+h}} L'(v,x)\,dx - \int_v^{v+h} L'(v,B'_u)\,dB'_u$$

so that the analysis of this term is *exactly the same as the analysis of* $\Delta_h X_s - \hat{X}_h$! Thus, following the steps (2.1)–(2.8) we obtain the analogue of (2.8):

$$K_{(s+h,t]} = \int_{-\infty}^{\infty} g'(x-B'_v)\sqrt{L'(v,x)}\,dW'_x + \tfrac{1}{2}\int_{-\infty}^{\infty} g'(x-B'_v)\alpha'_x\,dx \tag{3.18}$$

where L', α', $\check{\alpha}'$, W'^{+}, W'^{-} are defined in terms of B' exactly as L, α, $\check{\alpha}$, W^{+}, W^{-} were defined in terms of B, and $g'(x) = \tilde{L}(h,x)$. The stochastic integral splits at B'_v, of course.

It is important to realise that everything which appears in (3.18) is determined by B', and so is *independent of* $\{B_u : u \le s+h\}$, and hence of W_+ and W_-. The point of this is that if we mimic the estimation of $\Delta_h X_s$ which led to (3.9) we obtain

$$\left\| K_{(s+h,t]} - \int_{-\infty}^{\infty} g'(x-B'_v)\sqrt{L'(v,B'_v)}\,dW'^{+}_x \right\|_p \le C_p h^{\frac{7}{8}}\,, \tag{3.19}$$

which says that $K_{(s+h,t]}$ is very nearly equal to a random variable whose law, given B'_v, $L'(v,B'_v)$ and $(B'_{v+u}-B'_v)_{u\ge 0}$, is *Gaussian*, and which is independent of $\{B_u : u \le s+h\}$, and therefore *independent of* $\Delta_h X_s$.

Assembling (3.16), (3.17), and (3.19), we have that

$$\left\| \Delta_h X_t - \sqrt{L(s,B_s)}\int_{-\infty}^{\infty}\tilde{g}(x-B_s)\,dW_x - \sqrt{L'(v,B'_v)}\int_{-\infty}^{\infty} g'(x-B'_v)\,dW'_x \right\|_p \le C_p\left(|t-s|^{\frac{1}{8}}h^{\frac{3}{4}} + h^{\frac{7}{8}}\right)\,. \tag{3.20}$$

4. The L^2 convergence

Let us summarize the main point of the estimation so far, with a view to clarifying the steps still to come. Our strategy is to approximate the increments $\Delta_h X_s$ by stochastic integrals, and then to use the fact that these integrals are conditionally Gaussian to bound their moments. At this point we have developed the estimates to justify this approximation. The central results are (3.9) and (3.20). In order to unify our notation, let us define $\Delta L_u(x) = L(u+h, x+B_u) - L(u, x+B_u)$. Note that, in the notation of the previous section, $\hat{L}(h,x) = g(x) = \Delta L_s(x)$,

$\tilde{g}(x) = \Delta L_t(x + B_s - B_t)$, and $g'(x) = \Delta L_t(x)$. We proved in (3.9) that

$$\left\| \Delta_h X_s - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \Delta L_s(x - B_s)\, dW_x \right\|_p \le C_p h^{\frac{7}{8}}\,, \tag{4.1}$$

and we also proved (3.20) that if $0 < s < t$ and $v = t - s - h$, that

$$\Big\| \Delta_h X_t - \sqrt{L(s, B_s)} \int_{-\infty}^{\infty} \Delta L_t(x - B_s)\, dW_x - \sqrt{L'(v, B'_v)} \int_{-\infty}^{\infty} \Delta L_t(x + B_{s+h} - B_t)\, dW'_x \Big\|_p \le C_p \left(|t - s|^{\frac{1}{8}} h^{\frac{3}{4}} + h^{\frac{7}{8}} \right). \tag{4.2}$$

Now we aim to prove Theorem 1: for $0 \le t \le 1$, if $h = \frac{1}{n}$

$$\left(\sum_{k=0}^{[nt]-1} |\Delta_h X_{kh}|^{\frac{4}{3}} - \gamma \int_0^t L(s, B_s)^{\frac{2}{3}}\, ds \right) \xrightarrow{L^2} 0$$

as $n \to \infty$. This follows once we can prove that

$$\sum_{k=0}^{[nt]-1} \left(|\Delta_h X_{kh}|^{\frac{4}{3}} - \gamma h L(kh, B_{kh})^{\frac{2}{3}} \right) \xrightarrow{L^2} 0. \tag{4.3}$$

Let us transform this slightly. Recall that $\gamma = \beta E \left\{ \left| \int_{-\infty}^{\infty} L(1, x)^2 dx \right|^{\frac{2}{3}} \right\}$, with $\beta \equiv 2^{\frac{2}{3}} \pi^{-\frac{1}{2}} \Gamma(7/6) = E\{|B_1|^{\frac{4}{3}}$. Hold n fixed. The process

$$S_j \equiv \sum_{k=0}^{j} L(kh, B_{kh})^{\frac{2}{3}} \left(\gamma h - \beta \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2\, dy \right|^{\frac{2}{3}} \right),\ j = 1,\ 2\ \ldots$$

is a martingale relative to the filtration $(\mathcal{F}_{(j+1)h})$; this is because $\Delta L_{kh}(\cdot)$ is independent of $\mathcal{F}_{kh}$ and because

$$\begin{aligned} E\left\{ \left| \int_{-\infty}^{\infty} \Delta L_{kh}(x)^2 dx \right|^{\frac{2}{3}} \right\} &= E\left\{ \left| \int_{-\infty}^{\infty} L(h, x)^2\, dx \right|^{\frac{2}{3}} \right\} \\ &= h\, E\left\{ \left| \int_{-\infty}^{\infty} L(1, x)^2\, dx \right|^{\frac{2}{3}} \right\}, \quad \text{using Lemma 1.4} \\ &= \gamma h / \beta \end{aligned} \tag{4.4}$$

by definition of γ and β. Moreover

$$\begin{aligned} E\{S_j^2\} &= \sum_{k=0}^{j} E\left\{ L(kh, B_{kh})^{\frac{4}{3}} \right\} E\left\{ \left(\gamma h - \beta \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2\, dy \right|^{\frac{2}{3}} \right)^2 \right\} \\ &\le C j h^2. \end{aligned}$$

Accordingly, $E\{S_{[nt]}^2\} \le Cth$. This reduces the proof of Theorem 1 to the proof that

$$\sum_{k=0}^{[nt]-1} \left\{ |\Delta_h X_{kh}|^{\frac{4}{3}} - \beta L(kh, B_{kh})^{\frac{2}{3}} \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2 \, dy \right|^{\frac{2}{3}} \right\} \xrightarrow{L^2} 0 \, . \tag{4.5}$$

Let us abbreviate

$$\alpha_{kn} \equiv \beta L(kh, B_{kh})^{\frac{2}{3}} \left| \int_{-\infty}^{\infty} \Delta L_{kh}(y)^2 \, dy \right|^{\frac{2}{3}} ,$$

and define

$$U_n(t) \equiv \sum_{k=0}^{[nt]-1} \left(|\Delta_h X_{kh}|^{\frac{4}{3}} - \alpha_{kn} \right) .$$

We claim that $U_n(t) \longrightarrow 0$ in L^2. This will prove (4.5), and hence the theorem. Let $s < t$, and write

$$\begin{aligned} \|U_n(t) - U_n(s)\|_2^2 &= E\left\{ \left(\sum_{j=[ns]}^{[nt]-1} \left(|\Delta_h X_{jh}|^{\frac{4}{3}} - \alpha_{jn} \right) \right)^2 \right\} \\ &= \sum_{j=[ns]}^{[nt]-1} E\left\{ \left(|\Delta_h X_{jh}|^{\frac{4}{3}} - \alpha_{jn} \right)^2 \right\} \\ &\qquad + 2 \sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} E\left\{ \left(|\Delta_h X_{jh}|^{\frac{4}{3}} - \alpha_{jn} \right) \left(|\Delta_h X_{kh}|^{\frac{4}{3}} - \alpha_{kn} \right) \right\} \\ &\equiv V_1 + V_2 \, . \end{aligned} \tag{4.6}$$

The steps remaining in the proof are now the following:

(i) *approximate the $\Delta_h X_{jh}$ in the diagonal terms V_1 (4.6) by stochastic integrals with respect to L, using (4.1);*

(ii) *approximate both $\Delta_h X_{jh}$ and $\Delta_h X_{kh}$ by stochastic integrals in V_2, using (4.2) for the latter;*

(iii) *show that the stochastic integrals are conditionally Gaussian, so that we can apply Kolmogorov's Lemma;*

(iv) *assemble the estimates to give (4.3).*

Step (i). Let

$$\begin{aligned} \xi_j &\equiv \Delta_h X_{jh} \, , \\ I_j &= \sqrt{L(jh, B_{jh})} \fint_{-\infty}^{\infty} \Delta L_{jh}(x - B_{jh}) \, dW_x \, , \\ \eta_j &= I_j - \xi_j \, . \end{aligned}$$

We claim that the first term on the right-hand side of (4.6) is

$$V_1 = \sum_{j=[ns]}^{[nt]-1} E\left\{\left(|I_j|^{\frac{4}{3}} - \alpha_{jn}\right)^2\right\} + O\left((t-s)h^{\frac{9}{8}}\right) . \tag{4.7}$$

We know from (4.1) that $\|\eta_j\|_p \le C_p h^{\frac{7}{8}}$. Exploiting the fact (Proposition 2.1) that W is independent of both $\hat{L}(h, x - B_{jh})$ and $L(jh, B_{jh})$, we have by Burkholder's inequality that

$$\begin{aligned} \|I_j\|_p &\le C_p \left\| \left(L(jh, B_{jh}) \int \hat{L}(h,x)^2\,dx \right)^{\frac{1}{2}} \right\|_p \\ &\le C_p h^{\frac{3}{4}} , \end{aligned} \tag{4.8}$$

by Lemma 1.4, at least for all $s \le 1$, which is all that concerns us here. Hence for small h, $\|\xi_j\|_p \le C_p h^{\frac{3}{4}}$. Now using the elementary bound (1.10) and Hölder's inequality, we estimate

$$\begin{aligned} \|\,|I_j|^p - |\xi_j|^p\,\|_2 &\le C_p \left\| \eta_j \left(|\xi_j|^{p-1} + |I_j|^{p-1} \right) \right\|_2 \\ &\le C_p \|\eta_j\|_{2\alpha} \left\| \,|\xi_j|^{p-1} + |I_j|^{p-1} \right\|_{2\alpha'} \end{aligned} \tag{4.9}$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Now take $p = 4/3$, the case of interest to us:

$$\begin{aligned} &\le C_p h^{\frac{7}{8}} h^{3(p-1)/4} \\ &= C_p h^{\frac{9}{8}} \end{aligned} \tag{4.10}$$

Consider the j^{th} term of V_1 (4.6), which we write $E\left\{(|\xi_j|^{\frac{4}{3}} - \alpha)^2\right\}$. The difference between this and what we want is

$$\begin{aligned} \left| E\left\{\left(|\xi_j|^{\frac{4}{3}} - \alpha\right)^2\right\} - E\left\{\left(|I_j|^{\frac{4}{3}} - \alpha\right)^2\right\} \right| &= \left| E\left\{ \left(|\xi_j|^{\frac{4}{3}} - |I_j|^{\frac{4}{3}}\right) \left(|\xi_j|^{\frac{4}{3}} + |I_j|^{\frac{4}{3}} - 2\alpha\right) \right\} \right| \\ &\le \left\| |\xi_j|^{\frac{4}{3}} - |I_j|^{\frac{4}{3}} \right\|_2 \left\| |\xi_j|^{\frac{4}{3}} + |I_j|^{\frac{4}{3}} - 2\alpha \right\|_2 \\ &\le C_p h^{\frac{17}{8}} , \end{aligned}$$

where we have used (4.8), (4.4) and (4.10) to bound the norms of the various terms. The sum contains at most $(t-s)/h$ terms, so the total contribution is at most $C_p(t-s)h^{\frac{9}{8}}$.

Step (ii). The argument is similar to that of Step (i). Let $j < k$ and define

$$J_{jk} = \sqrt{L(jh, B_{jh})} \int_{-\infty}^{\infty} \Delta L_{kh}(x - B_{kh})\, dW_x$$
$$K_{jk} = \sqrt{L'((k-j-1)h, B'_{(k-j-1)h})} \int_{-\infty}^{\infty} \Delta L'_{(k-j-1)h}(x - B'_{(k-j-1)h})\, dW'_x\,,$$

where $B'_u \equiv B_{u+jh+h} - B_{jh+h}$ is the Brownian motion from $(j+1)h$ onward, and L' is its local time. (These definitions allow us to abbreviate (4.1) and (4.2) to $\|\eta_j\|_p \le C_p h^{\frac{7}{8}}$, $\|\xi_k - J_{jk} - K_{jk}\|_p \le C_p[|t-s|^{\frac{1}{8}} h^{\frac{3}{4}} + h^{\frac{7}{8}}]$ respectively.) We have $\|\xi_j\|_p + \|I_j\|_p + \|\xi_k\|_p + \|J_{jk} + K_{jk}\|_p \le C_p h^{\frac{3}{4}}$.

We claim that the double sum in (4.6) is

$$V_2 = 2 \sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} E\left\{ \left(|I_j|^{\frac{4}{3}} - \alpha_{jn} \right) \left(|J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn} \right) \right\}$$
$$+ O\Big((t-s)^2 (h^{\frac{1}{8}} + |t-s|^{\frac{1}{8}}) \Big)\,. \quad (4.11)$$

Indeed, let us estimate the difference between V_2 and what we would have if we replaced each ξ_j by the stochastic integrals. Set

$$Y = |\xi_j|^{\frac{4}{3}} - \alpha_{jn}, \quad Y' = |I_j|^{\frac{4}{3}} - \alpha_{jn}$$
$$Z = |\xi_k|^{\frac{4}{3}} - \alpha_{kn}, \quad Z' = |J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn}\,.$$

Then the difference is

$$\begin{aligned} |E\{YZ - Y'Z'\}| &\le E\{|Y(Z - Z')|\} + E\{|Z'(Y - Y')|\} \\ &\le \|Z - Z'\|_2 \|Y\|_2 + \|Y - Y'\|_2 \|Z'\|_2 \\ &\le ch \left\| |\xi_k|^{\frac{4}{3}} - |J_{jk} + K_{jk}|^{\frac{4}{3}} \right\|_2 + ch \left\| |\xi_j|^{\frac{4}{3}} - |I_j|^{\frac{4}{3}} \right\|_2\,, \end{aligned}$$

since $\|\alpha_{jn}\|_p \le c_p h$. Now we estimate the other terms as we did in (4.9); we obtain

$$\left\| |\xi_j|^{\frac{4}{3}} - |I_j|^{\frac{4}{3}} \right\|_2 \le c h^{\frac{9}{8}}\,,$$
$$\left\| |\xi_k|^{\frac{4}{3}} - |J_{jk} + K_{jk}|^{\frac{4}{3}} \right\|_2 \le c h^{\frac{1}{4}} \left(|t-s|^{\frac{1}{8}} h^{\frac{3}{4}} + h^{\frac{7}{8}} \right)\,.$$

Thus

$$|E\{YZ - Y'Z'\}| \le c h^2 \left(|t-s|^{\frac{1}{8}} + h^{\frac{1}{8}} \right)\,,$$

There are at most $(t-s)^2/h^2$ terms in the double sum, so the error introduced in (4.6) by replacing the $|\Delta X|^{\frac{4}{3}}$ throughout by the stochastic integrals is at worst

$$c\left[(t-s) h^{\frac{9}{8}} + (t-s)^2 (h^{\frac{1}{8}} + |t-s|^{\frac{1}{8}}) \right]\,. \quad (4.12)$$

Evidently, $\|U_n(t) - U_n(s)\|_2 = 0$ if $0 \le s \le t \le s + h$, so we assume that $(t - s) \ge h$, which implies that $(t - s)h \le (t - s)^2$ and we see that the upper bound (4.12) translates into

$$c(t-s)^2 \left(h^{\frac{1}{8}} + |t-s|^{\frac{1}{8}}\right) . \tag{4.13}$$

Thus we have proved that

$$\begin{aligned} \|U_n(t) - U_n(s)\|_2^2 = &\sum_{j=[ns]}^{[nt]-1} E\left\{(|I_j|^{\frac{4}{3}} - \alpha_{jn})^2\right\} \\ &+ 2 \sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} E\left\{(|I_j|^{\frac{4}{3}} - \alpha_{jn})(|J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn})\right\} \\ &+ O\Big((t-s)^2(h^{\frac{1}{8}} + |t-s|^{\frac{1}{8}})\Big) . \end{aligned} \tag{4.14}$$

The proof is completed by analysing the two sums on the right of this equation, which is the work of the final step.

Step (iv). By Prop. 2.1, W is independent of $\mathcal{A}_j \equiv \sigma\{\, (B_{jh},\ L(jh, B_{jh}),\ \Delta L_{jh}(\cdot))\,\}$, so that if we condition on $\mathcal{A}_j$, the integrand in I_j is effectively deterministic; so is the square root multiplying the stochastic integral. The stochastic integral of a deterministic function with respect to a Brownian motion is Gaussian, so we conclude that, given $\mathcal{A}_j$, the random variable I_j is conditionally Gaussian, with mean zero and variance

$$\sigma_j^2 \equiv L(jh, B_{jh}) \int_{-\infty}^{\infty} \Delta L_{jh}(x)^2 dx \equiv (\beta^{-1}\alpha_{jn})^{\frac{3}{2}} . \tag{4.15}$$

Thus

$$E\left\{|I_j|^{\frac{4}{3}} \;\middle|\; \mathcal{A}_j\right\} = \alpha_{jn}$$

by definition of β, and

$$E\left\{(|I_j|^{\frac{4}{3}} - \alpha_{jn})^2 \;\middle|\; \mathcal{A}_j\right\} = \text{const } (\sigma_j^2)^{\frac{4}{3}} ,$$

where the constant is unimportant. We now take expectations of the first sum on the right-hand side of (4.14) by conditioning firstly on the $\mathcal{A}_j$, to obtain the upper bound

$$V_1 \le c(t-s)h^{-1} \cdot h^2 = c(t-s)h. \tag{4.16}$$

The second sum is more subtle, and needs Kolmogorov's Lemma. If we fix j and k in (4.14) and look at the joint representation of I_j, J_{jk}, and K_{jk} as stochastic integrals, we see that *W and W' are independent*

Brownian motions, independent of B_{jh}, $L(jh, B_{jh})$, $(B_{jh+u} - B_{jh})_{0 \le u \le h}$, B'_v, $L'(v, B'_v)$, and $(B_{kh+u} - B_{kh})_{u \ge 0}$. Thus, if we condition by

$$\begin{aligned}\mathcal{A}_{jk} \equiv \sigma(\{B_{jk}, L(jh, B_{jh}),\, (B_{u+jh} - B_{jh})_{0 \le u \le h},\, B'_{(k-j-1)h}, \\ L'((k-j-1)h, B'_{(k-j-1)h}), (B_{u+kh} - B_{kh})_{u \ge 0}\})\,,\end{aligned}$$

the integrands and the square roots in I_j, J_{jk}, and K_{jk} are effectively deterministic; stochastic integrals of deterministic functions with respect to independent Brownian motions are jointly Gaussian, so that these random variables are (conditionally) jointly Gaussian, with mean zero and covariance matrix

$$\begin{pmatrix} \sigma_j^2 & \theta_{jk} & 0 \\ \theta_{jk} & \sigma_{jk}^2 & 0 \\ 0 & 0 & \sigma'^2_{jk} \end{pmatrix}$$

where σ_j^2 is as at (4.15), and

$$\begin{aligned}\sigma_{jk}^2 &\equiv L(jh, B_{jh}) \int \Delta L_{kh}(x)^2 dx, \\ \theta_{jk} &\equiv L(jh, B_{jh}) \int \Delta L_{kh}(x) \Delta L_{jh}(x)\, dx, \\ \sigma'^2_{jk} &\equiv L'((k-j-1)h, B_{(k-j-1)h}) \int \Delta L'_{(k-j-1)h}(x)^2\, dx.\end{aligned}$$

Using equation (1.9) from Kolmogorov's Lemma we obtain

$$\begin{aligned}&\left| E\left\{ \left(|I_j|^{\frac{4}{3}} - \alpha_{jn}\right)^2 \left(|J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn}\right)^2 \;\Big|\; \mathcal{A}_{jk} \right\} \right| \\ &\quad \le E\left\{ \left(|I_j|^{\frac{4}{3}} - \alpha_{jn}\right)^2 \;\Big|\; \mathcal{A}_{jk} \right\}^{\frac{1}{2}} E\left\{ \left(|J_{jk} + K_{jk}|^{\frac{4}{3}} - \alpha_{kn}\right)^2 \;\Big|\; \mathcal{A}_{jk} \right\}^{\frac{1}{2}} \rho(I_j, J_{jk} + K_{jk})^2 \\ &\quad \le c\sigma_j^{\frac{4}{3}} \left(\sigma_{jk}^2 + \sigma'^2_{jk}\right)^{\frac{2}{3}} \cdot \theta_{jk}^2 / \sigma_j^2 \left(\sigma_{jk}^2 + \sigma'^2_{jk}\right) \\ &\quad \le c\, \frac{\theta_{jk}^2}{\sigma_j^{\frac{2}{3}} \sigma_{jk}^{\frac{2}{3}}} \\ &\quad \le c\theta_{jk}^{\frac{4}{3}}\,.\end{aligned}$$

Now we estimate

$$\begin{aligned}E\left\{\theta_{jk}^{\frac{4}{3}}\right\} &= E\left\{L(jh, B_{jh})^{\frac{4}{3}}\right\} E\left\{\left(\int \Delta L_{kh}(x) \Delta L_{jh}(x)\, dx\right)^{\frac{4}{3}}\right\} \\ &\le cE\left\{\int \Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x)\, dx\right\} h^{\frac{1}{3}}\,,\end{aligned} \tag{4.17}$$

using Jensen's inequality. If $m = k - j - 1$, we have

$$\int E\left\{\Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x)\right\} dx = \int E\left\{\bar{L}(h,x)^{\frac{4}{3}}[L((m+1)h,x) - L(mh,x)]\right\} dx,$$

where $\bar{L}$ and L are the local time processes of independent Brownian motions;

$$= \int E\left\{\bar{L}(h,x)^{\frac{4}{3}}\right\} E\{L((m+1)h,x) - L(mh,x)\} \, dx\,.$$

Thus

$$\begin{aligned}
\sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} & \int E\left\{\Delta L_{jh}(x)^{\frac{4}{3}} \Delta L_{kh}(x)\right\} dx \\
&= \int E\left\{\bar{L}(h,x)^{\frac{4}{3}}\right\} \sum_{j=[ns]}^{[nt]-2} E\{L(([nt]-j-1)h,x)\} \, dx \\
&\le \int dx \, E\left\{\bar{L}(h,x)^{\frac{4}{3}}\right\} \frac{1}{h} \int_0^{t-s} E\{L(u,x)\} \, du \\
&\le \frac{t-s}{h} \int E\left\{\bar{L}(h,x)^{\frac{4}{3}}\right\} E\{L(1,x)\} \, dx \\
&\le c \, \frac{t-s}{h} \left(\int E\left\{\bar{L}(h,x)^{\frac{8}{3}}\right\} dx \right)^{\frac{1}{2}} \\
&\le c(t-s)h^{-\frac{1}{12}}
\end{aligned}$$

by the scaling (Lemma 1.4) of local time. Returning to (4.17) gives

$$V_2 \le c \sum_{j=[ns]}^{[nt]-2} \sum_{k=j+1}^{[nt]-1} E\left\{\theta_{jk}^{\frac{4}{3}}\right\} \le c(t-s)h^{\frac{1}{4}}\,. \tag{4.18}$$

Combining (4.16) and (4.18) with (4.14), we see that there exists a constant C such that

$$\|U_n(t) - U_n(s)\|_2 \le C\left((t-s)(h + h^{1/4}) + (t-s)^2 h^{1/8} + (t-s)^{17/8}\right)\,.$$

Now $h = 1/n$; if we let $\Delta = t/m$ for some integer m, and note that $U_n(0) \equiv 0$, we see

$$\begin{aligned}
\|U_n(t)\|_2 &\le \sum_{k=0}^{m-1} \|U_n((k+1)\Delta) - U_n(k\Delta)\|_2 \\
&\le \frac{c}{\Delta}\left(\Delta/n + \Delta/n^{1/4} + \Delta^2/n^{1/8} + \Delta^{17/8}\right)^{\frac{1}{2}}\,,
\end{aligned}$$

so that

$$\limsup_{n\to\infty} \|U_n(t)\| \le c\Delta^{1/16}\,.$$

Since Δ is arbitrary, we must have $\lim \|U_n(t)\|_2 = 0$. This shows that $V^n_{4/3}(t)$ converges in L^2 and in probability. To see that the L^2 convergence is uniform in t, just note that $t \mapsto V^n_{4/3}(t)$ is an increasing function, and that the limit is a continuous increasing function of t. It then follows immediately that the L^2 convergence is uniform in t for t in bounded intervals. This completes the proof. ✠

Appendix A. **Proof of Kolmogorov's Lemma**

Here is a proof of Lemma 1.3 in the spirit of this paper. Let $(X_t)_{0\le t\le 1}$ and $(Y_t)_{0\le t\le 1}$ be a pair of standard Brownian motions on the same filtration, having correlation ρ, so that $d[X,Y]_t = \rho dt$. Assume without loss of generality that $E\{\phi(X_1)\} = E\{\psi(Y_1)\} = 0$, and let $p_t(x,y) = (2\pi t)^{-\frac{1}{2}} \exp((y-x)^2/2t)$ be the Brownian transition density. Write $P_t\phi(x) = \int p_t(x,y)\phi(y)\,dt$. Consider

$$M_t = P_{1-t}\phi(X_t), \quad 0 \le t \le 1,$$

and note that $M_0 = E\{\phi(X_1)\} = 0$ and $M_1 = \phi(X_1)$. By Itô's Lemma for $t < 1$,

$$dM_t = \frac{\partial}{\partial x} P_{1-t}\phi(X_t)\,dX_t + \tfrac{1}{2}\frac{\partial^2}{\partial x^2} P_{1-t}\phi(X_t)\,dt + \frac{\partial}{\partial t} P_{1-t}\phi(X_t)\,dt\,. \tag{A.1}$$

The last two terms cancel since p_t satisfies the heat equation, so that M is a stochastic integral—we could also have seen this by noticing from the Markov property that M is a martingale—hence

$$\phi(X_1) = \int_0^1 \frac{\partial}{\partial x} P_{1-t}\phi(X_t)\,dX_t\,. \tag{A.2}$$

Thus, as $d[X,Y] = \rho\,dt$,

$$\begin{aligned} |E\{\phi(X_1)\psi(Y_1)\}| &= |\rho| \left| E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t)(P_{1-t}\psi)'(Y_t)\,dt \right\} \right| \\ &\le |\rho| E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t)^2\,dt \right\}^{\frac{1}{2}} E\left\{ \int_0^1 (P_{1-t}\psi)'(Y_t)^2\,dt \right\}^{\frac{1}{2}} \qquad \text{(A.3)} \\ &= |\rho| E\{\phi(X_1)\}^{\frac{1}{2}} E\{\psi(Y_1)\}^{\frac{1}{2}}\,, \end{aligned}$$

which implies the first statement of Kolmogorov's Lemma.

To prove the second statement, suppose $E\{X_1\phi(X_1)\} = 0$. Using (A.2),

$$0 = E\{X_1\phi(X_1)\} = E\left\{ \int_0^1 (P_{1-t}\phi)'(X_t)\,dt \right\}\,. \tag{A.4}$$

But the Brownian semigroup commutes with differentiation with respect to x, so that $E\{(P_{1-t}\phi)'(X_t)\} = P_t\{(P_{1-t}\phi)'\} = (P_t P_{1-t}\phi)' = (P_1\phi)'$, independent of t. Its common value must be zero by (A.4). Thus we can apply (A.3)—on the interval $[0,t]$ rather than $[0,1]$—to $(P_{1-t}\phi)'$ and $(P_{1-t}\psi)'$:

$$E\left\{(P_{1-t}\phi)'(X_t)(P_{1-t}\psi)'(Y_t)\right\} \le |\rho| E\left\{(P_{1-t}\phi)'(X_t)^2\right\}^{\frac{1}{2}} E\left\{(P_{1-t}\psi)'(Y_t)^2\right\}^{\frac{1}{2}} .$$

Putting this back into (A.3) gives us the factor of ρ^2 in (1.8) and completes the proof.

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