Time-reversal in hyperbolic s.p.d.e.'s

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1 Introduction

This paper was motivated by questions regarding changes of variables in stochastic partial differential equations (s.p.d.e.'s). To illustrate the issues, consider first the analogous question for a stochastic differential equation of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \qquad X_0 = x_0. \tag{1}$$

Given a smooth increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi'(u) > 0$, for all $u \geq 0$, set $Y_u = X_{\varphi(u)}$. Then (Y_u) is a (weak) solution of the following equation:

$$dY_u = b(\varphi^{-1}(u), Y_u)\varphi'(u)\, du + \sigma(\varphi^{-1}(u), Y_u)\sqrt{\varphi'(u)}\, d\tilde{B}(u),$$

for some Brownian motion $\tilde{B}$. That is, the change of variables $t = \varphi(u)$ affects equation (1) much as though it were an ordinary differential equation.

On the other hand, consider the change of variables $t = 1-u$, namely, time-reversal. It is well known [7] that the process $(\hat{X}_u = X_{1-u},\ 0 \leq u \leq 1)$ is a solution of the stochastic differential equation

$$d\hat{X}_u = \hat{b}(u, \hat{X}_u)du + \hat{\sigma}(u, \hat{X}_u)d\hat{B}_u, \qquad \hat{X}_0 = X_1,$$

where $\hat{B}$ is a Brownian motion independent of X_1, and $\hat{b}$ and $\hat{\sigma}$ are given by the formulas

$$\hat{\sigma}(u, x) = \sigma(1-u, x), \qquad \hat{b}(u, x) = -b(1-u, x) + \frac{\frac{\partial}{\partial x}(\rho_{1-u}(x)\sigma(1-u, x))}{\rho_{1-u}(x)}, \tag{2}$$

where $\rho_t(\cdot)$ is the density function of X_t. In the simplest case in which $b \equiv 0$, $\sigma \equiv 1$, X is a standard Brownian motion and these formulas give the following equation for $\hat{X}$:

$$d\hat{X}_u = -\frac{\hat{X}_u}{1-u}du + d\hat{B}_u.$$

As expected, the reversal of Brownian motion is a Brownian bridge.

These considerations have been considerably extended [5, 10], to include infinite systems of stochastic differential equations. Of course, the presence of the density of X_{1-u} and the derivative in x makes the extensions highly non-trivial, but under certains conditions, the formulas above (suitably reinterpreted) give an equation for the reversed process.

With s.p.d.e.'s, there is a much wider choice of changes of variables than with s.d.e.'s. However, the fundamental issue is similar to that of s.d.e.'s: if the change of variables respects the filtration, then the s.p.d.e. in the new variables is easily obtained from the s.p.d.e. in the old variables, almost as for deterministic p.d.e.'s (see Section 3). However, if the change of variables implies a change of filtration, then the situation is much more delicate. The aim of this paper is to examine this issue in the context of hyperbolic s.p.d.e.'s in two variables, driven by two-parameter white noise.

If one considers a change of variables such as time reversal in an s.p.d.e., one might be tempted to make use of the results of [5, 10]. Indeed, in an abstract sense, an s.p.d.e. can be interpreted as an infinite system of s.d.e.'s. However, the class of s.p.d.e.'s is only a small subset of the class of infinite systems of s.d.e.'s, and there is no reason to expect, given an infinite system of s.d.e.'s for the reversed process, that it will correspond to an s.p.d.e.

The outline of this paper is as follows. ...

2 Existence Theory for Hyperbolic SPDE's in the plane

Consider the (reduced) hyperbolic SPDE

$$\frac{\partial^2 X}{\partial s \partial t} + a_1(s,t)\frac{\partial X}{\partial s} + a_2(s,t)\frac{\partial X}{\partial t} + a_3(s,t,X) = a_4(s,t)\dot{W}, \tag{3}$$

with initial data

$$X(s,0) = X_0 + M^1_s\,, \qquad X(0,t) = X_0 + M^2_t\,.$$

Here, $\dot{W}$ is a space-time white noise. The coefficients $a_1, \ldots, a_4$ are deterministic functions: a_1 a_2 and a_3 are continuously differentiable and have bounded first partials, a_4 is bounded and continuous, and $a_3(s,t,X) = a_3(s,t,X(s,t))$. The boundary conditions X_0, M^1 and M^2 are (possibly) random, independent of the white noise $\dot{W}$, and M^1 and M^2 are continuous processes, with $M^1_0 = M^2_0 = 0$.

Equation (3) was studied in [4] using the theory of two-parameter processes. It was also studied in [11, 12], where it was formulated in mild form, using the Green's function, and it was shown that the two-parameter form, the mild form, and the weak form (see below) are equivalent, and have a unique solution.

To get the weak form of (3), multiply both sides by a test function $\phi \in C^{(2)}(I\!R^2)$, and integrate over the rectangle $R_{st} \stackrel{\text{def}}{=} [0,s] \times [0,t]$ to get

$$\begin{aligned} \iint_{R_{st}} \phi(u,v) \left(\frac{\partial^2 X}{\partial u \partial v} + a_1(u,v)\frac{\partial X}{\partial u} + a_2(u,v)\frac{\partial X}{\partial v} \right) du\,dv \\ = \iint_{R_{st}} \phi(u,v)\big[a_4(u,v)\,W(du\,dv) - a_3(u,v,X)\,du\,dv\big]\,. \end{aligned} \tag{4}$$

We use the integration by parts formula

$$
(5)\qquad \int_a^b dx \int_c^d dy f(x,y)\frac{\partial^2 g}{\partial x \partial y}
$$
$$
\begin{aligned}
&= f(b,d)g(b,d) - f(a,d)g(a,d) - f(b,c)g(b,c) + f(a,c)g(a,c) \\
&\quad - \int_a^b \left[\frac{\partial f}{\partial x}(x,d)g(x,d) - \frac{\partial f}{\partial x}(x,c)g(x,c)\right] dx \\
&\quad - \int_c^d \left[\frac{\partial f}{\partial x}(b,y)g(b,y) - \frac{\partial f}{\partial x}(a,y)g(a,y)\right] dy \\
&\quad + \int_a^b dx \int_c^d dy \frac{\partial^2 f}{\partial x \partial y} g(x,y),
\end{aligned}
$$

with $f = \phi$, $g = X$, to get all the derivatives onto ϕ:

$$
\begin{aligned}
(6)\quad & X(s,t)\phi(s,t) - X(s,0)\phi(s,0) - X(0,t)\phi(0,t) + X(0,0)\phi(0,0) \\
& - \int_0^s \Big(X(u,t)\Big[\frac{\partial \phi}{\partial u}(u,t) - a_2(u,t)\phi(u,t)\Big] - X(u,0)\Big[\frac{\partial \phi}{\partial u}(u,0) - a_2(u,0)\phi(u,0)\Big]\Big) du \\
& - \int_0^t \Big(X(s,v)\Big[\frac{\partial \phi}{\partial v}(s,v) - a_1(s,v)\phi(s,v)\Big] - X(0,v)\Big[\frac{\partial \phi}{\partial v}(0,v) - a_1(0,v)\phi(0,v)\Big]\Big) dv \\
& + \iint_{R_{st}} X(u,v)\Big[\frac{\partial^2 \phi}{\partial u \partial v}(u,v) - \frac{\partial}{\partial u}\big(a_1(u,v)\phi(u,v)\big) - \frac{\partial}{\partial v}\big(a_2(u,v)\phi(u,v)\big)\Big] du\, dv \\
& \qquad = \iint_{R_{st}} \phi(u,v)\big[a_4(u,v)\, W(du\, dv) - a_3(u,v,X)\, du\, dv\big].
\end{aligned}
$$

We say that a jointly measurable and locally integrable process $(X(s,t),\ (s,t) \in \mathbb{R}_+^2)$ is a *solution of* (3) if (6) holds a.s. for each $(s,t) \in \mathbb{R}_+^2$ and each function $\phi \in C^{(2)}(\mathbb{R}_+^2)$. A slight extension of [12, Theorem 1] (which only considers more restrictive initial conditions) shows that if $E(X_0^2) < \infty$, $E(\sup_{u \le s}(M_u^1)^2) < \infty$ and $E(\sup_{v \le t}(M_v^2)^2) < \infty$, then there exists a unique solution of (3) which has continuous sample paths, and which has the property that $\sup_{(u,v)\in \mathbb{R}_{st}} E(X(u,v)^2) < \infty$.

The solution of (3) has an integral representation using the Green's function for the problem. The Green's function and its properties are studied in [12] (Propositions 10 and 11): it is a function $\gamma(s,t;u,v)$ defined for $(s,t) \in \mathbb{R}_+^2$, $(u,v) \in \mathbb{R}_{st}$, which has the following properties.

(a) For fixed (S,T), for all $s \le S$ and $t \le T$, $\gamma(s,t;\cdot,\cdot)$ has continuous and uniformly bounded first derivatives and a continuous and uniformly bounded second order mixed derivative in R_{st}. For $u \le S$ and $v \le T$, $\gamma(\cdot,\cdot;u,v)$ has uniformly bounded first derivatives and a uniformly bounded second order mixed derivative in $R_{ST} \setminus R_{uv}$. (Note. The continuity statements are not made in [11, 12] because in those papers, a_1 and a_2 are not assumed to be C^1. However, under this assumption, they follow easily from the proof in [11, Proposition 3.2].

(b) For $(u,v) \in R_{st}$,

$$\gamma(s,t;u,v) = 1 - \int_v^t a_1(u,w)\gamma(s,t;u,w)\,dw - \int_u^s a_2(r,v)\gamma(s,t;r,v)\,dr\,;$$

(c) For $(u,v) \in R_{st}$,

$$\frac{\partial^2\gamma}{\partial u\partial v}(s,t;u,v) - \frac{\partial}{\partial u}\big(a_1(u,v)\gamma(s,t;u,v)\big) - \frac{\partial}{\partial v}\big(a_2(u,v)\gamma(s,t;u,v)\big) = 0\,,;$$

(d) $\dfrac{\partial\gamma}{\partial u}(s,t;u,t) - a_2(u,t)\gamma(s,t;u,t) = 0\,, \quad u \le s;$

(e) $\dfrac{\partial\gamma}{\partial v}(s,t;s,v) - a_1(s,v)\gamma(s,t;s,v) = 0\,, \quad v \le t;$

(f) $\gamma(s,t;s,t) = 1\,.$

Moreover, there exists a universal constant $C > 0$ such that

(g) $\displaystyle\sup_{(s,t)\in \mathbb{R}^2_+}\ \sup_{(u,v)\in R_{st}} |\gamma(s,t;u,v)| \le C\,;$

(h) $\displaystyle\sup_{\substack{s\ge u\vee r,\\ t\ge v\vee w}} |\gamma(s,t;u,v) - \gamma(s,t;r,w)| \le C\big(|u-r| + |v-w|\big)\,;$

(i) $\displaystyle\sup_{\substack{s\wedge r\ge u,\\ t\wedge w\ge v}} |\gamma(s,t;u,v) - \gamma(r,w;u,v)| \le C(|s-r| + |t-w|)\,.$

If we replace $\phi(u,v)$ by $\gamma(s,t;u,v)$ in (6) and use (c), (d) and (e), we get

$$\begin{aligned}(7)\quad X(s,t) = {} & \gamma(s,t;s,0)X(s,0) + \gamma(s,t;0,t)X(0,t) - \gamma(s,t,0,0)X(0,0)\\ & - \int_0^s X(u,0)\Big[\frac{\partial\gamma}{\partial u}(s,t;u,0) - a_2(u,0)\gamma(s,t;u,0)\Big]\,du\\ & - \int_0^t X(0,v)\Big[\frac{\partial\gamma}{\partial v}(s,t;0,v) - a_1(0,v)\gamma(s,t;0,v)\Big]\,dv\\ & + \iint_{R_{st}} \gamma(s,t;u,v)\big[a_4(u,v)\,W(du\,dv) - a_3(u,v,X)\,du\,dv\big]\,.\end{aligned}$$

Definition 2.1 *If* $\Delta =]a,b]\times]c,d] \subset \mathbb{R}^2_+$ *is a rectangle, the* planar increment of X over Δ *is*

$$X(\Delta) \overset{\text{def}}{=} X(b,d) - X(a,d) - X(b,c) + X(b,d)\,.$$

It is shown in [12] (Propositions 2.1 and 2.2) that the solution of (6) also satisfies (7). One can extend (6) to certain non-smooth ϕ, and in particular to indicator functions, as follows.

Lemma 2.1 *Let* $0 < u_i < v_i$, $i = 1, 2$, *and set* $\Delta =]u_1, v_1] \times]u_2, v_2]$. *Then*

$$(8) \quad X(\Delta) - \iint_\Delta X(u,v)\Big[\frac{\partial a_1}{\partial u}(u,v) + \frac{\partial a_2}{\partial v}(u,v)\Big]\,du\,dv$$
$$+ \int_{u_1}^{v_1} \big[X(u,v_2)a_2(u,v_2) - X(u,u_2)a_2(u,u_2)\big]\,du$$
$$+ \int_{u_2}^{v_2} \big[X(v_1,v)a_1(v_1,v) - X(u_1,v)a_1(u_1,v)\big]\,dv$$
$$= \iint_\Delta \big[a_4(u,v)W(du\,dv) - a_3(u,v,X)\,du\,dv\big]\,.$$

Proof. Fix $s > v_1$, $t > v_2$. Let $\psi(x)$ be a non-negative smooth function with compact support, such that $\psi(0) > 0$ and $\int \psi(x)\,dx = 1$. Define

$$\phi_{i\varepsilon}(x) = \frac{1}{\varepsilon}\int_0^x \Big(\psi\Big(\frac{y-u_i}{\varepsilon}\Big) - \psi\Big(\frac{y-v_i}{\varepsilon}\Big)\Big)\,dy,$$

and let $\phi_\varepsilon(u,v) = \phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)$. If we put ϕ_ε into (6), the first three lines vanish if ε is small, and we get

$$(9) \quad \iint_{R_{st}} X(u,v)\Big[\phi'_{1\varepsilon}(u)\phi'_{2\varepsilon}(v) - a_1(u,v)\phi'_{1\varepsilon}(u)\phi_{2\varepsilon}(v) - a_2(u,v)\phi_{1\varepsilon}(u)\phi'_{2\varepsilon}(v)$$
$$- \phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)\Big(\frac{\partial a_1}{\partial u}(u,v) + \frac{\partial a_2}{\partial v}(u,v)\Big)\Big]\,du\,dv$$
$$= \iint_{R_{st}} \phi_{1\varepsilon}(u)\phi_{2\varepsilon}(v)\Big[a_4(u,v)\,W(du\,dv) - a_3(u,v,X)\,du\,dv\Big]\,.$$

Notice that as $\varepsilon \downarrow 0$, $\phi_{i\varepsilon}$ converges boundedly and pointwise to $I_{[u_i,v_i]}$ while $\phi'_{i\varepsilon}$ converges weakly to $\delta_{u_i} - \delta_{v_i}$. Since X, the a_i, and their first partials are continuous, the left-hand side of (9) converges to the left-hand side of (8). At the same time, the a_i are bounded and ϕ_ε converges pointwise and boundedly to the indicator function of Δ, so the right-hand side of (9) converges in L^2 to the right-hand side of (8), proving the lemma. ♣

2.1 Semimartingale Initial Data

We want to consider solutions with fairly regular initial values. In this context, "initial values" refers to the values of X on the boundary of $\mathbb{R}^2_+$, and "fairly regular" means that the boundary values should be well-behaved semimartingales.

Let $Y = (Y_t,\ t \geq 0)$ be a semimartingale with the decomposition $Y_t = M_t + V_t$, where M_t is a martingale (in some given filtration), and V_t is a process of locally finite variation. Let $\langle Y \rangle_t = \langle M \rangle_t$ be the predictable increasing process associated to Y.

Definition 2.2 *We say that a semimartingale Y is* smooth *if*

(i) M and V are continuous;

(ii) $t \mapsto \langle Y \rangle_t$ and $t \mapsto V_t$ are continuously differentiable;

(iii) $\frac{d\langle Y \rangle}{dt}$ is L^1-bounded in compact t-sets, and $\frac{dV}{dt}$ is L^2-bounded in compact t-sets.

Notice that a smooth semimartingale need not have smooth sample paths (quite the opposite, it will only have smooth sample paths if its martingale part is constant). It is the characteristics of the semimartingales, not the semimartingales themselves, which are smooth. One can think of a smooth semimartingale as the solution of a stochastic differential equation $dY = \sigma\, dW_t + \mu\, dt$, where $\sigma(x,t)$ and $\mu(x,t)$ are Lipschitz continuous.

Remark 2.2 *It is straightforward to show that if f is a bounded, continuous, adapted process and Y is a smooth semimartingale, then $Z_t \stackrel{\text{def}}{=} \int_0^t f(s)\, dY_s$ is also a smooth semimartingale.*

Assumption A *Let $Y_u^1 = X(u,0)$ and $Y_v^2 = X(0,v)$. $(Y_u^1,\ u \geq 0)$ and $(Y_v^2,\ v \geq 0)$ are smooth semimartingales (in their respective natural filtrations) which are independent of $\dot{W}$, with semimartingale decomposition $Y_u^i = M_u^i + V_u^i$, $i = 1,2$.*

Under this assumption, denote

$$\begin{aligned} \sigma_i^2(u) &\stackrel{\text{def}}{=} \frac{d\langle Y^i \rangle_u}{du}, \quad i = 1,2, \\ \mu_i(u) &\stackrel{\text{def}}{=} \frac{dV_u^i}{du}, \quad i = 1,2, \end{aligned}$$

and for $(s,t) \in \mathbb{R}_+^2$, set

$$\mathcal{F}_{s,t} = \sigma(Y_u^1,\ Y_v^2,\ \dot{W}_{u,v},\ u \leq s,\ v \leq t).$$

Lemma 2.3 *Under Assumption A, for any $(s,t) \in \mathbb{R}_+^2$ the processes $(X(u,t),\ 0 \leq u \leq s)$ and $(X(s,v),\ 0 \leq v \leq t)$ are smooth semimartingales (in the respective filtrations $(\mathcal{F}_{u,t},\ 0 \leq u \leq s)$ and $(\mathcal{F}_{s,v},\ 0 \leq v \leq t)$). Then the L^p-bounds on their characteristics are uniform for (s,t) in compact sets.*

Moreover, if $\Delta = (u_1, v_1] \times (u_2, v_2]$, (8) can be written

$$(10) \quad X(\Delta) + \int_{u_2}^{v_2} dv \int_{u_1}^{v_1} a_1(u,v)\, X(du,v) + \int_{u_1}^{v_1} du \int_{u_2}^{v_2} a_2(u,v)\, X(u,dv) = \iint_\Delta \left[a_4(u,v)\, W(du\, dv) - a_3(u,v,X)\, du\, dv \right].$$

Proof. Since $X(u,0) = Y_u^1$ and $X(0,v) = Y_v^2$ are semimartingales, we can integrate by parts in the first two integrals on the left-hand side of (7) to get

$$
\begin{aligned}
(11) \quad X(s,t) &= \gamma(s,t;0,0)X(0,0) + \int_0^s \gamma(s,t;u,0)\, dY_u^1 + \int_0^t \gamma(s,t;0,v)\, dY_v^2 \\
&\quad + \int_0^s Y_u^1 a_2(u,0)\gamma(s,t;u,0)\, du + \int_0^t Y_v^2 a_1(0,v)\gamma(s,t;0,v)\, dv \\
&\quad + \iint_{R_{st}} \gamma(s,t;u,v)\Big[a_4(u,v)W(du\,dv) - a_3(u,v,X)\, du\, dv\Big] \\
&\stackrel{\text{def}}{=} I_1(s,t) + \cdots + I_6(s,t)\,.
\end{aligned}
$$

The integrals with respect to dY_u^1 and dY_v^2 are stochastic integrals relative to semimartingales. One can show that each of them has a version which is continuous in (s,t), and we will always take that version.

We will show that if we fix s or t, $I_1, \ldots, I_6$ are smooth semimartingales in the remaining variable. By symmetry, it is enough to fix t. Let us decompose $I_1, \ldots, I_6$ into their martingale and bounded variation parts in s.

Note that I_1, I_3, I_4, and I_5 are each $C^{(1)}$ and have no martingale part, so $\langle I_1 \rangle = \langle I_3 \rangle = \langle I_4 \rangle = \langle I_5 \rangle \equiv 0$. Indeed, this is clear for I_1 and I_5 thanks to the differentiability of $s \mapsto \gamma(s,t;u,v)$ (property (a) above). In I_3, one can differentiate (with care!) inside the stochastic integral to see that

$$
(12) \qquad \frac{\partial}{\partial s} I_3(s,t) = \int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\, dY_v^2\,,
$$

which is continuous in s by (a). For I_4, write

$$
\gamma(s,t;u,v) = \gamma(u,t;u,v) + \int_u^s \frac{\partial}{\partial r}\gamma(r,t;u,v)\, dr
$$

and use Fubini's theorem:

$$
I_4(s,t) = \int_0^s Y_u^1 a_2(u,0)\gamma(u,t;u,0)\, du + \int_0^s dr \int_0^r Y_u^1 a_2(u,0)\frac{\partial}{\partial r}\gamma(r,t;u,0)\, du\,,
$$

which is clearly differentiable in s. The same idea can be used in I_2 and I_6, although one has to use Fubini's Theorem for mixed stochastic/Riemann integrals [13]:

$$
\begin{aligned}
I_2(s,t) &= \int_0^s \gamma(u,t;u,0)\, dY_u^1 + \int_0^s dr \int_0^r \frac{\partial}{\partial r}\gamma(r,t;u,0)\, dY_u^1 \\
&= \int_0^s \gamma(u,t;u,0)\, dM_u^1 + \int_0^s \gamma(u,t;u,0)\mu_1(u)\, du + \int_0^s dr \int_0^r \frac{\partial}{\partial r}\gamma(r,t;u,0)\, dY_u^1
\end{aligned}
$$

and

$$I_6(s,t) = \iint_{R_{st}} \gamma(u,t;u,v)a_4(u,v)\,W(du\,dv) + \int_0^s dr \iint_{R_{rt}} \frac{\partial}{\partial r}\gamma(r,t;u,v)a_4(u,v)\,W(du\,dv)\,.$$

This gives us the semimartingale decomposition of I_2 and I_6—so that we have the decomposition of all the I_j—and we see that

$$\begin{aligned} \frac{\partial}{\partial s}\langle I_2\rangle_{st} &= \gamma^2(s,t;s,0)\sigma_1^2(s)\,, \\ \frac{\partial}{\partial s}\langle I_6\rangle_{st} &= \int_0^t \gamma^2(s,t;s,v)a_4^2(s,v)\,dv\,. \end{aligned}$$

Now, a_4 is bounded by hypothesis and γ is bounded by property (g), and both are continuous and deterministic. Further, $\sigma_1^2(s)$ is continuous and locally L^1-bounded by the smoothness of Y^1. So we conclude that the derivatives of the $\langle I_j\rangle$ are all continuous and L_1-bounded, and the bound is uniform for (s,t) in bounded sets.

We must also check that the $\frac{\partial}{\partial s}(V_s^j)$ are continuous and L^2-bounded, and that the bound is uniform for (s,t) in bounded sets. Since we have explicit formulas for $V^1,\dots,V^6$, this is straightforward. We will just check I_3, which contains a stochastic integral, and leave the rest to the reader. Fix $R_{s_0t_0}$.

Since I_3 is $C^{(1)}$, $V_3 = I_3$ and from (12) we must bound

$$A(s,t) \stackrel{\text{def}}{=} E\Big\{\Big(\frac{\partial}{\partial s}I_3\Big)^2\Big\} = E\Big\{\Big(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\,dY_v^2\Big)^2\Big\}\,,\ (s,t)\in R_{s_0t_0}\,.$$

Now $dY_v^2 = dM_v^2 + \mu_2(v)\,dv$ so

$$A(s,t) \le 2E\Big\{\Big(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\,dM_v^2\Big)^2\Big\} + 2E\Big\{\Big(\int_0^t \frac{\partial}{\partial s}\gamma(s,t;0,v)\mu_2(v)\,dv\Big)^2\Big\}.$$

The first expectation equals $2E\{\int_0^t(\frac{\partial}{\partial s}\gamma(s,t;0,v))^2\sigma_2^2(v)\,dv\}$. If $(u,v)\prec(s,t)\in R_{s_0t_0}$, there is a constant $K = K_{s_0t_0}$ such that $|\frac{\partial}{\partial s}\gamma(s,t;0,v)|\le K$ by (a). The second expectation is bounded by $2K^2E\{\big(\int_0^{t_0}|\mu_2(v)|\,dv\big)^2\}$. Thus by the Schwartz inequality, if $(s,t)\in R_{s_0t_0}$,

$$A(s,t) \le 2K^2 t_0\left(\sup_{v\le t_0} E\{\sigma_2^2(v)\} + \sup_{v\le t_0} E\{\mu_2^2(v)\}\right)\,.$$

Now Y is a smooth semimartingale, so this is bounded independently of (s,t), hence $A(s,t)$ is uniformly bounded for $(s,t)\in R_{s_0t_0}$, as claimed.

To get (10), integrate by parts in the first double integral on the left-hand side of (8). ♣

The following is a direct consequence of Lemma 2.3.

Corollary 2.4 *Let $\Delta = (s-h, s] \times (t-k, t]$. Under Assumption A, for any $(s_0, t_0) \in I\!R_+^2$ there exists a constant $C = C_{s_0 t_0}$ such that if $(s,t) \in R_{s_0 t_0}$,*

$$E\left\{X\big(\Delta)\big)^2\right\} \le Chk\,, \tag{13}$$

$$E\left\{\big(X(s,t) - X(s-h, t-k)\big)^2\right\} \le C(h+k)\,. \tag{14}$$

2.2 Planar Quadratic Variation

We recall some basic notions. Let $\Delta_{ij} = (\frac{i-1}{2^n}, \frac{i}{2^n}] \times (\frac{j-1}{2^n}, \frac{j}{2^n}]$. Define the *planar quadratic variation* Q of X by

$$Q_n(s,t) = \sum_{i \le [2^n s],\ j \le [2^n t]} |X(\Delta_{ij})|^2,$$

and

$$Q(s,t) = \limsup_{n\to\infty} Q_n(s,t)\,.$$

At the same time, we can define the linear quadratic variation in one variable s or t: set

$$\begin{aligned} Q_n^1(s,t) &= \sum_{i \le [2^n s]} \Big(X\big(\frac{i}{2^n}, t\big) - X\big(\frac{i-1}{2^n}, t\big)\Big)^2, \\ Q_n^2(s,t) &= \sum_{j \le [2^n t]} \Big(X\big(s, \frac{j}{2^n}\big) - X\big(s, \frac{j-1}{2^n}\big)\Big)^2, \end{aligned}$$

and let

$$Q^1(s,t) = \limsup_{n\to\infty} Q_n^1(s,t), \qquad Q^2(s,t) = \limsup_{n\to\infty} Q_n(s,t)$$

be the *quadratic 1-variation* and the *quadratic 2-variation* respectively. Imkeller [8] has shown CHECK that

Lemma 2.5 *(i) If the quadratic variations of X satisfy $Q^i(s,t) < \infty$, $i = 1, 2$, for all (s,t), and if $Q^i(s,t) \equiv 0$ for all (s,t) for either $i = 1$ or $i = 2$, then the planar quadratic variation $Q(s,t)$ vanishes identically. NOT QUITE PROVED in [8].*

(ii) If X has finite planar quadratic variation and if Y has zero planar quadratic variation, then the planar quadratic variation of $X + Y$ equals the planar quadratic variation of X.

Proposition 2.6 *Under Assumption A, the planar quadratic variation of the solution X of* (6) *is*

$$\langle X\rangle_{st} = \iint_{R_{st}} a_4^2(u,v)\,du\,dv\,. \tag{15}$$

Proof. From Lemma 2.3,

$$\begin{aligned} X(\Delta_{ij}) = {} & \iint_{\Delta_{ij}} a_4(u,v)\,W(du\,dv) - \iint_{\Delta_{ij}} a_3(u,v,X)\,du\,dv \\ & - \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} dv \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} a_1(u,v)\,X(du,v) - \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} du \int_{u_2}^{v_2} a_2(u,v)\,X(u,dv)\,. \end{aligned} \tag{16}$$

Let us define $Z_1(s,t) = \iint_{R_{st}} a_4(u,v)\,W(du\,dv)$, $Z_2(s,t) = \iint_{R_{st}} a_3(u,v,X)\,du\,dv$, $Z_3(s,t) = \iint_{R_{st}} a_1(u,v)\,X(du,v)\,dv$, and $Z_4(s,t) = \iint_{R_{st}} a_2(u,v)\,du\,X(u,dv)$. Then (16) is equivalent to

$$X(\Delta_{ij}) = Z_1(\Delta_{ij}) - Z_2(\Delta_{ij}) - Z_3(\Delta_{ij}) - Z_4(\Delta_{ij})\,.$$

Notice that Z_2 is a differentiable function of both s and t, so it has finite variation in s and also in t, hence it has zero planar quadratic variation by Lemma 2.5 (i). Next, $t \mapsto Z_3(s,t)$ is $C^{(1)}$, so Z_3 has finite 1-variation, and hence zero quadratic 1-variation. At the same time, $s \mapsto Z_3(s,t)$ is a semimartingale, as one can see by reversing the order of integration, and thus has finite quadratic 2-variation. Thus Z_3 has zero planar quadratic variation, again by Lemma 2.5 (i). The same is true for Z_4. Thus, the quadratic variation of X is equal to the quadratic variation of Z_1 by Lemma 2.5 (ii). But Z_1 is a stochastic integral with respect to white noise, and (15) follows from [**?**]. ♣

Remark 2.7 *Proposition 2.6 implies that* $|a_4(s,t)|$ *can be determined from the sample paths of* $(X(s,t))$*. In fact, as long as* a_4 *is never zero, one can determine the sign of* a_4 *as well, since* $\dot W$ *is then* X*-measurable and* $\langle X, W\rangle_{st} = \iint_{R_{st}} a_4(u,v)\,du\,dv$*.*

3 Naive Changes of Variables

Ito integrals depend in a fundamental way on the underlying filtration. In a one parameter setting the filtration is usually obvious, and can be fixed once and for all. However, there may be many reasonable filtrations related to a given s.p.d.e., and the one chosen may depend on the particular coordinate system. A change of variables can involve an implicit change of filtration. For instance, the usual filtration for the Brownian sheet is a two-parameter filtration which depends strongly on the coordinates: the "past" at point (s,t) is generally taken to be $\mathcal{P}_{s,t} = R_{s,t}$, and one sets $\mathcal{F}_{s,t} = \sigma\{\dot W_{u,v},\ (u,v) \in \mathcal{P}_{s,t}\}$. However, a rotation by 45° changes the Brownian sheet into a solution of the stochastic wave equation [13], and the most natural filtration for such an evolution equation may be a one-parameter filtration $(\hat{\mathcal{F}}_t)$ ordered by time: the "past" at time t and position x is $\mathcal{P}_{t,x} = \{(s,y) : s \le t,\ y \in I\!R\}$, and the sigma-field $\hat{\mathcal{F}}_t$ is generated by the white noise in $\mathcal{P}_{t,x}$. So a change of variables which includes a change of filtration may involve a delicate transformation of stochastic integrals. However, if the integrand of a stochastic integral is deterministic, it is adapted to any filtration, and its stochastic integral is independent of the filtration, so to a certain extent at least, one may ignore the question of filtrations.

When we speak of a *naive* change of variables in an SPDE's, we mean that the new filtration is the image of the old one: if $\mathcal{F}_t = \sigma(\dot{W}_{u,v},\ (u,v) \in \mathcal{P}_t)$, then $\hat{\mathcal{F}}_t = \sigma(\dot{\hat{W}}_{u,v},\ (u,v) \in \zeta^{-1}(\mathcal{P}_t))$. We will see that naive changes of variables of SPDE's work as expected. It is only when the filtrations change that we find new phenomena.

3.1 Changing variables in stochastic integrals

Let O be an open set and let ζ be a one-to-one $C^{(\infty)}$ map of O onto an open set $D \subset I\!R^2$. Suppose the Jacobian J of ζ never vanishes. Then for a Borel subset $A \subset D$ and an integrable f on A,

$$\int_A f(z)\,dz = \int_{\zeta^{-1}(A)} f(\zeta(\xi))J(\xi)\,d\xi \tag{17}$$

If W is a white noise on D, define a set function $\hat{W}$ on O by

$$\hat{W}(B) = \int_{\zeta(B)} \frac{1}{\sqrt{J(\zeta^{-1}(z))}}\,W(dz)\,. \tag{18}$$

Lemma 3.1 *$\hat{W}(B)$ is a standard white noise on O, and if A is a Borel subset of D and if f is a deterministic square-integrable function on A,*

$$\int_A f(z)W(dz) = \int_{\zeta^{-1}(A)} f(\zeta(\xi))\,\sqrt{J(\xi)}\,\hat{W}(d\xi)\,. \tag{19}$$

Proof. $\hat{W}(B)$ is clearly a mean zero Gaussian random variable (if finite) and from (18) and (17)

$$\begin{aligned} E\{\hat{W}(B)^2\} &= \int_{\zeta(B)} J(\zeta^{-1}(z))^{-1}\,dz \\ &= \int_B J(\xi)^{-1}J(\xi)\,d\xi \\ &= |B| \end{aligned}$$

which shows that $\hat{W}$ is defined and has the correct variance on sets of finite Lebesgue measure. Moreover, if A and B are disjoint subsets of O, $\zeta(A)$ and $\zeta(B)$ are disjoint in D, so $\hat{W}(A)$ and $\hat{W}(B)$ are independent, being stochastic integrals of W over disjoint sets.

Equation (19) holds by (18) if f is of the form $f(z) = 1_B(z)$, hence it holds for simple f by linearity, and for square-integrable f by the usual functional completion argument. ♣

3.2 Changing variables in SPDE's

Let X be a solution of (3). Let $D_1 = \partial/\partial s$, $D_2 = \partial/\partial t$, and set

$$L = D_1 D_2 + a_1 D_1 + a_2 D_2 \,,$$

so that the formal adjoint of L is

$$L^* \phi = D_1 D_2 \phi - D_1(a_1 \phi) - D_2(a_2 \phi)\,.$$

Then for $\phi \in C^{(2)}(\mathbb{R}^2_+)$, X will satisfy (6), which we write in the form:

$$(20) \quad (X\phi)(R_{st}) + \oint_{\partial R_{st}} X(z)\big[\nabla\phi(z) - \phi(z)(a_2(z)\hat{\mathbf{i}} - a_1(z)\hat{\mathbf{j}})\big] \cdot \mathbf{T}\, ds \\ + \int_{R_{st}} \big(X(z)L^*\phi(z) + a_3(z,X)\phi(z)\big)\, dz = \int_{R_{st}} \phi(z) a_4(z)\, W(dz)\,,$$

where $\mathbf{T}$ is the unit tangent vector, $\hat{\mathbf{i}} = (1,0)$, $\hat{\mathbf{j}} = (0,1)$, ds is the element of arc length, and $X = X_0$ on the boundary of $\mathbb{R}^2_+$, where $X_0(s,0) \stackrel{\text{def}}{=} X_0 + M^1_s$, $X_0(0,t) = X_0 + M^2_t$, as in (3).

Let ζ be a $C^{(\infty)}$ homeomorphism of an open set O onto an open set $D \supset \mathbb{R}^2_+$. We suppose that $D \supset \mathbb{R}^2_+$, and let $\hat{D} = \zeta^{-1}(\mathbb{R}^2_+)$. Let $\hat{W}$ be the white noise on O which is related to W by (18). If $\hat{\phi}(\xi) \stackrel{\text{def}}{=} \phi(\zeta(\xi))$, then a straightforward calculation gives us a differential operator $\hat{L}^*$ on $\hat{D}$ for which

$$(L^*\phi)(\zeta(\xi)) = \hat{L}^*\hat{\phi}(\xi)\,.$$

We let $\hat{L}$ be the formal adjoint of $\hat{L}^*$, define

$$\hat{X}(\xi) \stackrel{\text{def}}{=} X(\zeta(\xi)),$$

and for $i = 3, 4$, we set $\hat{a}_i(\xi, x) = a_i(\zeta(\xi), x)$.

Theorem 3.2 *The process $\hat{X}$ is a solution of the stochastic partial differential equation*

$$(21) \qquad \hat{L}(J\hat{X}) + \hat{a}_3 J = \hat{a}_4 \sqrt{J}\, \dot{\hat{W}}\,,$$

with boundary values $\hat{X}(\xi) = X_0(\zeta(\xi))$ on $\zeta^{-1}(\partial\mathbb{R}^2_+)$. (Note. Formally, equation (21) is interpreted as equation (23).)

Proof. The map ζ is a smooth homeomorphism on a neighborhood of $\mathbb{R}^2_+$, so its restriction is a smooth homeomorphism of $\hat{D}$ onto the closed set $\mathbb{R}^2_+$, which takes the boundary of $\mathbb{R}^2_+$ onto a (possibly proper) subset of the boundary of $\hat{D}$. Clearly $\hat{X}$ has the correct boundary values, so we need only check that the (21) holds in the interior. For this, we check the weak form of the equation for $\phi \in C_K^{(\infty)}(\mathbb{R}^2_+)$ whose support is in the interior of $\mathbb{R}^2_+$. If we choose (s,t) large enough so that the support of ϕ is in the interior of R_{st}, the boundary terms of (20) drop out and we are left with

$$(22)\qquad \int_{R_{st}} \big(X(z)L^*\phi(z) + a_3(z,X)\phi(z)\big)\,dz = \iint_{R_{st}} \phi(z)a_4(z)\,W(dz)\,.$$

The left-hand side is a Riemann integral and transforms under the mapping ζ in the usual way, while the right-hand side is a stochastic integral which transforms according to Lemma 11. Since the homeomorphism induces a one-to-one map of X to $\hat{X}$, there is a function $\hat{a}_3$ such that $\hat{a}_3(\xi,\hat{X}) = a_3(\zeta(\xi),X)$. So, setting $\xi = \zeta^{-1}(z)$, we have $X(z) = \hat{X}(\xi)$, $L^*\phi(z) = \hat{L}^*\hat{\phi}(\xi)$, and (22) becomes:

$$(23)\qquad \int_{\hat{D}} \big(\hat{X}(\xi)\hat{L}^*\hat{\phi}(\xi) + \hat{a}_3(\xi,\hat{X})\hat{\phi}(z)\big)J(\xi)\,d\xi = \int_{\hat{D}} \hat{\phi}(\xi)\hat{a}_4(\xi)\sqrt{J(\xi)}\;\hat{W}(d\xi),$$

which is the weak form of (21). ♣

Example 3.1 *Assume that $a_i = a_i(s,t)$, $i = 1,\dots,4$, and $O = D = \mathbb{R}^2_+$. Let $\zeta(x,y) = (s(x),t(y))$, where*

$$s(x) = \frac{e^{2a} - e^{2a(1-x)}}{2a}\,, \qquad t(y) = \frac{e^{2b} - e^{2b(1-y)}}{2b}\,.$$

Suppose that $X(s,t)$ satisfies

$$(24)\qquad \frac{\partial^2 X}{\partial s \partial t} + a_1\frac{\partial X}{\partial s} + a_2\frac{\partial X}{\partial t} + a_3 X = a_4\dot{W},$$

with initial conditions $X(s,0) \equiv X(0,t) \equiv 0$. Set $\hat{a}_i(x,y) = a_i(s(x),t(y))$. With the notations above, $J(x,y) = s'(x)t'(y)$,

$$(L^*\phi)(s(x),t(y)) = \frac{\partial^2\phi}{\partial s\partial t}(s(x),t(y)) - \frac{\partial(a_1\phi)}{\partial s}(s(x),t(y)) - \frac{\partial(a_2\phi)}{\partial t}(s(x),t(y))$$

and

$$\hat{L}^*\hat{\phi}(x,y) = \frac{1}{s'(x)t'(y)}\frac{\partial^2\hat{\phi}}{\partial x\partial y}(x,y) - \frac{1}{s'(x)}\frac{\partial(\hat{a}_1\hat{\phi})}{\partial x}(x,y) - \frac{1}{t'(y)}\frac{\partial(\hat{a}_2\hat{\phi})}{\partial y}(x,y).$$

Therefore,

$$\hat{L}\hat{\phi} = \frac{\partial^2}{\partial x \partial y}\left(\frac{1}{s'(x)t'(y)}\hat{\phi}\right) + \hat{a}_1 \frac{\partial}{\partial x}\left(\frac{1}{s'(x)}\hat{\phi}\right) + \hat{a}_2 \frac{\partial}{\partial y}\left(\frac{1}{t'(y)}\hat{\phi}\right).$$

Let $\hat{X}(x,y) = X(s(x,y), t(x,y))$. Then $\hat{X}$ satisfies

$$\frac{\partial^2 \hat{X}}{\partial x \partial y} + \hat{a}_1 e^{2b(1-y)} \frac{\partial \hat{X}}{\partial x} + \hat{a}_2 e^{2a(1-x)} \frac{\partial \hat{X}}{\partial y} + \hat{a}_3 \hat{X} e^{2a(1-x)+2b(1-y)} = \hat{a}_4 e^{a(1-x)+b(1-y)} \dot{\hat{W}}\,,$$

with initial conditions $\hat{X}(s,0) \equiv \hat{X}(0,t) \equiv 0$.

Example 3.2 *Assume a_1 and a_2 are constants, $a_3 = a_3(s,t,x)$ and $a_4 = a_4(s,t,x)$. Suppose $O = (-\infty, 1]^2$ and $\zeta(u,v) = (1-u, 1-v)$. Suppose that $X(s,t)$ satisfies*

$$\frac{\partial^2 X}{\partial s \partial t} + a_1 \frac{\partial X}{\partial s} + a_2 \frac{\partial X}{\partial t} + a_3 X = a_4 \dot{W},$$

with initial conditions $X(s,0) \equiv X(0,t) \equiv 0$. With the notations above, $J(u,v) \equiv 1$, and

$$L^*\phi(1-u, 1-v) = \frac{\partial^2 \phi}{\partial s \partial t}(1-u,1-v) - a_1 \frac{\partial \phi}{\partial s}(1-u,1-v) - a_2 \frac{\partial \phi}{\partial t}(1-u,1-v),$$

so

$$\hat{L}\hat{\phi} = \frac{\partial^2 \hat{\phi}}{\partial u \partial v} - a_1 \frac{\partial \hat{\phi}}{\partial u} - a_2 \frac{\partial \hat{\phi}}{\partial v}.$$

Therefore, $\hat{X}(u,v) = X(1-u, 1-v)$ satisfies

$$\hat{L}\hat{X}(u,v) + \hat{a}_3(u,v,\hat{X}) = \hat{a}_4(u,v,\hat{X})\dot{\hat{W}},$$

with boundary conditions $\hat{X}(u,1) \equiv \hat{X}(1,v) \equiv 0$. This statement should be compared with the very different conclusion of Theorem 6.3, in which the change of variables is the same but the underlying filtration is different.

4 SPDEs as Distributional PDE's

Let us specialize to the linear case, where $a_3(s,t,X) = a_3(s,t)X(s,t)$. There is a meta-theorem which states that linear SPDE's are simply random PDE's with distribution values. We will illustrate this.

Let $L = \sum_{i,j} a_{ij} D_i D_j + \sum_i b_i D_i + c$ be a partial differential operator on a domain $D \subset \mathbb{R}^2_+$, whose coefficients a_{ij}, b_i, and c are deterministic Lipschitz functions, with

$a_{ij} \in C^{(2)}(D)$, $b_i \in C^{(1)}(D)$ and $c \in C(D)$. Let $F \in L^1(D)$, $G \in L^2(D)$ be deterministic functions, and consider the SPDE in D

$$LX = F + G\dot{W}\,. \tag{25}$$

We say that X is a *weak solution* of (25) in D if for each $\phi \in C_K^{(\infty)}(D)$

$$\int_D X(z)(L^*\phi)(x)\,dz = \int_D \phi(x)[\big[G(z)\,W(dz) + F(x)\,dz\big] \tag{26}$$

with probability one.

Proposition 4.1 *If $X = \{X(z),\ z \in D\}$ is a weak solution of* (26) *with continuous sample paths, and if $\hat{D}$ is an open, relatively compact subdomain of D, then X defines a random distribution on $\hat{D}$. With probability one it is a distributional solution of* (25) *on $\hat{D}$.*

PROOF. To say X is a distribution is to say it is a continuous linear functional on a nuclear space. Let us choose the nuclear space to be the completion of $C_K^{(\infty)}(\hat{D})$ in the vector space topology generated by the seminorms

$$F_n(\phi) = \|\phi\|_2^2 + \sum_{i=1}^n \sum_{j=1}^n \|D_1^i D_2^j \phi\|_2^2\,,$$

where $\|\phi\|$ is the norm of ϕ in $L^2(\hat{D})$.

Let $L^*\phi = \sum_{ij} D_i D_j(a_{ij}\phi) - \sum_i D_i(b_i\phi) + c\phi$ be the formal adjoint of L. If ω is such that $z \mapsto X(z,\omega)$ is continuous on D, $X(\cdot,\omega)$ defines a distribution on $\hat{D}$ by

$$X(\phi,\omega) = \int_{\hat{D}} \phi(z) X(z,\omega)\,dz\,,$$

and LX is also a distribution: $LX(\phi) = X(L^*\phi)$.

On the right hand side of (26), $F + G\dot{W}$ also defines a distribution: $(F + G\dot{W})(\phi) = \int_{\hat{D}} \phi(z)[F(x)\,dz + G(z)\,W(dz)]$ a.s. for each ϕ (see [13], chapter 4). Then (26) says that for a fixed $\phi \in C_K^{(\infty)}$,

$$(LX)(\phi,\omega) = (F + G\dot{W})(\phi,\omega) \tag{27}$$

for a.e. ω. This is true simultaneously for a countable dense set of ϕ, hence for all ϕ by continuity, since both sides are distributions. ♣

We chose a particularly simple space of distributions to avoid having to discuss the boundary behavior of X. It should be clear that one can extend this to include boundaries.

In other words, equation (26), and even equation (3), is an equation in distribution space which holds for a.e. ω. Consequently, all operations which are legal on such equations are legal on this one—*as long as they do not change the definition of the stochastic integral in (6).* It is interesting to consider the previous section from this point of view. In particular, if we multiply X by a deterministic $C^{(\infty)}$ function, we can just use the usual calculus to see what SPDE it satisfies.

Corollary 4.2 *Suppose that $A = (a_{ij})$ and $f \in \mathcal{C}^{(2)}(D)$, $f > 0$. Let $L_1 = L - c$. If X satisfies (25) and if $f\tilde{X} = X$, then $\tilde{X}$ satisfies*

$$L_1\tilde{X} + \frac{1}{f}\nabla f \cdot (A + A^T)\nabla^T \tilde{X} + \frac{Lf}{f}\tilde{X} = \frac{1}{f}(F + f^{-1}G\dot{W})\,. \tag{28}$$

Example 4.1 *Suppose X satisfies* (24)*. Let $Y(s,t) = e^{as+bt}X(s,t)$. Then $Y(s,t)$ satisfies*

$$\frac{\partial^2 Y}{\partial s \partial t} + (a_1 - b)\frac{\partial Y}{\partial s} + (a_2 - a)\frac{\partial Y}{\partial t} + (a_3 - ab - a_1 a - a_2 b)Y = e^{as+bt}a_4\,\dot{W}\,. \tag{29}$$

5 Changing filtrations: final values as initial conditions

We now want to consider some changes of variables which involve changes of filtration, namely time-reversal.

Consider the linear form of (3):

$$\frac{\partial^2 X}{\partial s \partial t} + a_1(s,t)\frac{\partial X}{\partial s} + a_2(s,t)\frac{\partial X}{\partial t} + a_3(s,t)X(s,t) = a_4(s,t)\dot{W} \tag{30}$$

where the initial values $X(s,0)$ and $X(0,t)$ are given, and satisfy Assumption A.

Remark 5.1 *We'll want some kind of discussion of the relation of time-reversal and filtrations, either here or in the intro.*

Thus, let X be a solution of (30). We will consider two fundamental types of time reversal.

• Reversal in one coordinate: $(s,t) \mapsto (1-s,t)$.

Let $\hat{X}(s,t) = X(1-s,t)$, $0 \le s \le 1$, and let $\hat{\mathcal{F}}_s$ be the one-parameter filtration $\hat{\mathcal{F}}_s = \sigma\{\hat{X}(u,v) : u \le s\} = \sigma\{X(u,v) : u \ge 1-s\}$.

By symmetry, results for this will translate directly to the reversal $(s,t) \mapsto (s,1-t)$.

• Reversal in two coordinates: $(s,t) \mapsto (1-s,1-t)$.

Let $\hat{X}(s,t) = X(1-s, 1-t)$, $0 \le s \le 1$, $0 \le t \le 1$, and give this the one-parameter filtration $\hat{\mathcal{F}}_s = \sigma\{\hat{X}(u,v) : u+v \le s\sqrt{2}\} = \sigma\{X(u,v) : u \ge 1-s, v \ge 1-t\}$.

Let us suppose that $Y = \hat{X}$ is the solution of an SPDE of the form

$$\frac{\partial^2 Y}{\partial s \partial t} + \hat{a}_1(s,t)\frac{\partial Y}{\partial s} + \hat{a}_2(s,t)\frac{\partial Y}{\partial t} + \hat{a}_3(s,t)Y = \hat{a}_4(s,t)\dot{\hat{W}}, \quad s \ge 0,\ t \ge 0\,. \tag{31}$$

where the initial values for Y are specified on the axes of $\mathbb{R}^2_+$: in the case of one parameter reversal, $Y(0,t) = X(1,t)$, $Y(s,0) = X(1-s,0)$, and in the case of two-parameter reversal, $Y(0,t) = X(1,1-t)$, $Y(s,0) = X(1-s,1)$; and $\hat{a}_1, \dots, \hat{a}_4$ satisfy the smoothness conditions of Section 2 and $\dot{\hat{W}}$ is a white noise relative to the new filtration $(\hat{F}_t)$ (IMPLICITELY, independent of the boundary values of Y).

The first question we shall ask is this: "If the reversed process actually is the solution of (31), what can we say about the coefficients $\hat{a}_1, \dots, \hat{a}_4$?"

Let us first establish a property of the original solution, which clarifies the independence of the solution and the white noise. Let

$$\begin{aligned} \mathcal{G}_{st} &= \sigma\{X(u,v) : u \le s \ \text{ or } \ v \le t\}\,; \\ \mathcal{H}_{st} &= \sigma\{W(A) : \text{Borel } A \subset (s,\infty) \times (t,\infty)\}\,. \end{aligned}$$

Note that $\mathcal{H}_{st}$ represents information in the strict future of (s,t), while $\mathcal{G}_{st}$ represents information in the wide-sense past, which is roughly everything not in the strict future.

Proposition 5.2 *Let X be a solution of* (30)*. Then for each $s \ge 0$, $t \ge 0$, $\mathcal{G}_{st}$ and $\mathcal{H}_{st}$ are independent.*

PROOF. From (7), $X(s,t)$ is measurable with respect to $\mathcal{F}^0_{st} \stackrel{\text{def}}{=} \sigma\{Y^1_u,\ u \le s\} \vee \sigma\{Y^2_v,\ v \le t\} \vee \sigma\{W([0,u] \times [0,v]),\ u \le s\ v \le t\}$.

If $A \subset (s,\infty) \times (t,\infty)$, and either $u \le s$ or $v \le t$, then $W(A)$ is independent of $W([0,u] \times [0,v])$. White noise is a Gaussian process, so it follows that $\mathcal{H}_{st}$ is independent of $\sigma\{W([0,u] \times [0,v]),\ u \le s\ v \le t\}$. Since the Y^i are independent of the white noise, it follows that $\mathcal{H}_{st}$ is independent of $\vee_{u \le s, v \le t} \mathcal{F}^0_{uv} \supset \mathcal{G}_{st}$. ♣

Set $\Delta = [s-h, s] \times [t-h, t]$ and $\hat{\Delta} = [1-s, 1-s+h] \times [1-t, 1-t+h]$, and consider a two-parameter reversal. If $\hat{X}$ is a solution of (31), Proposition 5.2 implies that $\dot{\hat{W}}|_{\hat{\Delta}}$ is independent of

$$\hat{\mathcal{G}}_{1-s,1-t} \stackrel{\text{def}}{=} \sigma\{Y(u,v),\ u \le 1-s \text{ or } v \le 1-t\} = \sigma\{X(u,v),\ u \ge s \text{ or } v \ge t\}\,. \tag{32}$$

Proposition 5.3 *Consider reversal in two coordinates, and set $\hat{s} = 1-s$, $\hat{t} = 1-t$. Suppose that the reversed process $Y = \hat{X}$ is a solution of* (31) *in the above sense. Then* *the a_i and $\hat a_i$ are related as follows:*

$$\hat a_4(\hat s, \hat t\,) = a_4(s,t); \tag{33}$$

$$E\Big\{\int_\Delta a_4(s,t)\,W(ds\,dt) \mid \hat{\mathcal{G}}_{\hat s\hat t}\Big\} \tag{34}$$
$$= \big(a_1(s,t)+\hat a_1(\hat s,\hat t\,)\big)\,\big(X(s,t)-X(s-h,t)\big)h$$
$$+ \big(a_2(s,t)+\hat a_2(\hat s,\hat t\,)\big)\,\big(X(s,t)-X(s,t-h)\big)h$$
$$+ \big(a_3(s,t)-\hat a_3(\hat s,\hat t\,)\big)\,X(s,t)h^2$$
$$+ \mathcal{E}(s,t;h),$$

where

$$E\{\mathcal{E}(s,t;h)^2\} \le Ch^4\,. \tag{35}$$

Remark 5.4 *If we consider reversal in one coordinate , then we would set $Y(s,t) = \hat X(s,t) = X(1-s,t)$, $\hat s = 1-s$, $\hat t = t$, and*

$$\hat{\mathcal{G}}_{1-s,t} \stackrel{\text{def}}{=} \sigma\{Y(u,v),\ u\le 1-s \text{ or } v\le t\} = \sigma\{X(u,v),\ u\ge s \text{ or } v\le t\}\,.$$

Then $\hat\Delta = [1-s, 1-s+h]\times[t-h,t]$ and Prop 5.2 implies that $\dot{\hat W}|_{\hat\Delta}$ is independent of $\hat{\mathcal{G}}_{\hat s\hat t}$. So with these definitions, formula (34) remains valid.

Proof. Equality (33) follows from Proposition 2.6. ¿From (10),

$$\hat X(\Delta) + \iint_{\hat\Delta} \hat a_1(u,v)\,\hat X(du,v)\,dv + \iint_{\hat\Delta} \hat a_2(u,v)\,du\,\hat X(u,dv) \tag{36}$$
$$+ \iint_{\hat\Delta} a_3(u,v)\hat X(u,v)\,du\,dv = \iint_{\hat\Delta} \hat a_4(u,v)\,\hat W(du\,dv)\,.$$

On the other hand,

$$X(\Delta) + \iint_{\Delta} a_1(u,v)\,X(du,v)\,dv + \iint_{\Delta} a_2(u,v)\,du\,X(u,dv) \tag{37}$$
$$+ \iint_{\Delta} a_3(u,v)X(u,v)\,du\,dv = \iint_{\Delta} a_4(u,v)\,W(du\,dv)\,.$$

By definition, $\hat X(\hat\Delta) = X(\Delta)$, and $\hat X(\hat s,\hat t\,) = X(s,t)$. Subtract these equations to see that:

$$(38)\quad \iint_{\Delta} a_4(u,v)\, W(du\, dv) - \iint_{\hat\Delta} \hat a_4(u,v)\, \hat W(du\, dv)$$

$$= \iint_{\Delta} a_1(u,v)\, X(du,v)\, dv - \iint_{\hat\Delta} \hat a_1(u,v)\, \hat X(du,v)\, dv$$

$$+ \iint_{\Delta} a_2(u,v)\, du\, X(u,dv) - \iint_{\hat\Delta} \hat a_2(u,v)\, du\, \hat X(u,dv)$$

$$+ \iint_{\Delta} a_3(u,v) X(u,v)\, du\, dv - \iint_{\hat\Delta} \hat a_3(u,v) \hat X(u,v)\, du\, dv\,.$$

Approximate $a_i(u,v)$ and $X(u,v)$ by $a_i(s,t)$ and $X(s,t)$ to see that:

$$(39)\quad \iint_{\Delta} a_4(u,v)\, W(du\, dv) - \iint_{\hat\Delta} \hat a_4(u,v)\, \hat W(du\, dv)$$

$$= a_1(s,t)\big(X(s,t) - X(s-h,t)\big)h - \hat a_1(\hat s,\hat t)\big(\hat X(\widehat{s-h},\hat t) - \hat X(\hat s,\hat t\,)\big)h + \mathcal{E}_1 - \hat{\mathcal{E}}_1$$

$$+ a_2(s,t)\big(X(s,t) - X(s,t-h)\big)h - \hat a_2(\hat s,\hat t\,)\big(\hat X(\hat s,\widehat{t-h}) - \hat X(\hat s,\hat t\,)\big)h + \mathcal{E}_2 - \hat{\mathcal{E}}_2$$

$$+ \big(a_3(s,t) - \hat a_3(\hat s,\hat t\,)\big)X(s,t)h^2 + \mathcal{E}_3 - \hat{\mathcal{E}}_3\,,$$

where the $\mathcal{E}_i$ and $\hat{\mathcal{E}}_i$ are the errors in the respective approximations. Now condition on $\hat{\mathcal{G}}_{\hat s\hat t}$. Note that $\hat W$ is a white noise with respect to the reversed filtration and that $Y = \hat X$ is a solution of (31), so Proposition 5.2 implies that the white noise on $\hat\Delta$ is independent of $\hat{\mathcal{G}}_{\hat s,\hat t}$, and therefore

$$E\Big\{\iint_{\hat\Delta} \hat a_4(u,v)\, \hat W(du\, dv) \mid \hat{\mathcal{G}}_{\hat s,\hat t}\Big\} = 0\,.$$

On the other hand, all the terms on the right-hand side except the errors are $\hat{\mathcal{G}}_{\hat s,\hat t}$-measurable, so that we get (34) with $\mathcal{E}(s,t;h) = \sum_{i=1}^{3} E\{\mathcal{E}_i - \hat{\mathcal{E}}_i \mid \hat{\mathcal{G}}_{\hat s,\hat t}\}$.

In order to finish the proof of the proposition, we need only show that there exists $C > 0$ such that $E\{\mathcal{E}_i^2\} \le Ch^4$ and $E\{\hat{\mathcal{E}}_i^2\} \le Ch^4$ for $i = 1,\dots,3$. Consider

$$(40)\quad \mathcal{E}_1 = \iint_{\Delta} \big(a_1(u,v) - a_1(s,t)\big)\, X(du,v)\, dv$$

$$+ a_1(s,t) \int_{t-h}^{t} \Big(X(s,v) - X(s-h,v) - \big(X(s,t) - X(s-h,t)\big)\Big)\, dv$$

$$\stackrel{\text{def}}{=} I_1 + I_2\,.$$

Let $X(du,v) = dM_u^v + dV_u^v$ be the semimartingale decomposition of $X(\cdot,v)$ and write

$$I_1 = \int_{t-h}^{t} \int_{s-h}^{s} \big(a_1(u,v) - a_1(s,t)\big)\big(dM_u^v + dV_u^v\big)\, dv$$

and use the Schwartz inequality:

$$(41)\quad E\{I_1^2\} \le 2hE\Big\{\int_{t-h}^{t}\int_{s-h}^{s}\big(a_1(u,v)-a_1(s,t)\big)^2\big(d\langle M^v\rangle_s\,dv\Big\}$$
$$+\,2hE\Big\{\int_{t-h}^{t}\bigg(\int_{s-h}^{s}\big(a_1(u,v)-a_1(s,t)\big)\,dV_u^v\bigg)^2\,dv\Big\}\,.$$

By Lemma 2.3, $d\langle M^v\rangle_u = \sigma^2(u,v)\,du$ and $dV_u^v = \mu(u,v)\,du$. Moreover, a_1 has uniformly bounded derivatives, so $|a_1(u,v)-a_1(s,t)| \le C(|s-u|+|t-v|) \le 2Ch$. Thus this is

$$(42)\quad \le 8Ch^3\int_{t-h}^{t}\int_{s-h}^{s}E\{\sigma^2(u,v)\}\,du\,dv$$
$$+\,8Ch^3\int_{t-h}^{t}E\left\{\bigg(\int_{s-h}^{s}|\mu(u,v)|\,du\bigg)^2\right\}\,dv\,.$$

By Proposition 2.3, $E\{\mu^2(u,v)\}$ and $E\{\sigma^2(u,v)\}$ are bounded for (u,v) in compact sets, so there is a constant C' for which $E\{I_1^2\}$ is bounded by $C'h^5$.

Let $Z(v) = X\big((s-h,s]\times(v,t]\big)$ and note that

$$I_2 = a_1(s,t)\int_{t-h}^{t}Z(v)\,dv\,,$$

so that

$$\begin{aligned} E\{I_2^2\} &= a_1^2(s,t)\int_{t-h}^{t}\int_{t-h}^{t}E\big\{Z(u)Z(v)\big\}\,du\,dv \\ &\le a_1^2(s,t)\int_{t-h}^{t}\int_{t-h}^{t}E\big\{Z^2(u)\big\}^{1/2}E\big\{Z^2(v)\big\}^{1/2}\,du\,dv\,. \end{aligned}$$

From Corollary 2.4, $E\{Z^2(u)\} \le Ch(t-v) \le Ch^2$, so this is bounded by, say, $C''a_1^2(s,t)h^4$. Thus, for small h,

$$E\{\mathcal{E}_1^2\} \le 2E\{I_1^2\} + 2E\{I_2^2\} \le C'h^5 + C''h^4 \le Ch^4$$

for a suitable constant C which depends only on $s+t$, the coefficients a_i, and the smoothness of the initial semimartingales Y^i. The errors $\hat{\mathcal{E}}_1$, $\mathcal{E}_2$, and $\hat{\mathcal{E}}_2$ are similar.

Moving to $\mathcal{E}_3$, we have

$$\begin{aligned} \mathcal{E}_3 &= \iint_\Delta\big(a_3(u,v)X(u,v)-a_3(s,t)X(s,t)\big)\,du\,dv \\ &= \iint_\Delta\big(a_3(u,v)-a_3(s,t)\big)X(u,v)\,du\,dv + a_3(s,t)\iint_\Delta\big(X(u,v)-X(s,t)\big)\,du\,dv \\ &\stackrel{\text{def}}{=} J_1 + J_2\,. \end{aligned}$$

Using the same reasoning as above, we see that

$$E\{J_1^2\} \le E\left\{\left(\iint_\Delta ChX(u,v)\,du\,dv\right)^2\right\} = C^2h^2\int_{\Delta\times\Delta} E\{X(u,v)X(u'v')\}\,du\,dv\,du'\,dv'\,.$$

But $E\{X(u,v)X(u'v')\} \le \sup_{(u,v)\in\Delta} E\{X^2(u,v)\} \le C'$, so $E\{J_1^2\} \le Ch^6$.

$$E\{J_2^2\} = a_3^2(s,t)\int_{\Delta\times\Delta} E\big\{\big(X(u,v)-X(s,t)\big)\big(X(u',v')-X(s,t)\big)\big\}\,du\,dv\,du'\,dv'\,,$$

while $\sup_{(u,v)\in\Delta} E\{\big(X(u,v)-X(s,t)\big)^2\} \le 2Ch$ by Corollary 2.4. Thus this is bounded by $2Ca_3^2(s,t)h^5$. The same bound holds for $\hat{\mathcal{E}}_3$ by symmetry.

Adding the errors together, we see $E\{\mathcal{E}^2(s,t;h)\} \le Ch^4$ for small h. ♣

Remark 5.5 *The only error term above which has order as large as $O(h^4)$ is I_2. The others are all $O(h^5)$ or smaller.*

6 Reversals of the Brownian sheet

6.1 Reversal in one coordinate

Theorem 6.1 *Let $(W(s,t))$ be a standard Brownian sheet. Set $Y(s,t) = W(1-s,t)$. Then there is a standard Brownian sheet $(B(s,t))$ independent of $(W(1,t),\ t\ge 0)$ such that $(Y(s,t))$ is the solution on $[0,1[\,\times \mathbb{R}_+$ of*

$$\frac{\partial^2 Y}{\partial s\partial t} + \frac{1}{1-s}\frac{\partial Y}{\partial t} = \frac{\partial^2 B}{\partial s\partial t}, \tag{43}$$

with initial conditions $Y(0,t) = W(1,t)$, $Y(s,0) = 0$.

Remark 6.2 *One easily checks (with $\hat{\mathcal{G}}_{\hat{s},\hat{t}}$ as in Remark 5.4), that*

$$E(W(\Delta)\mid \hat{\mathcal{G}}_{\hat{s},\hat{t}}) = \frac{h}{s}(X(s,t)-X(s,t-h)),$$

so from (34), we guess that $\hat{a}_1(\hat{s},\hat{t}) = 1/s$, i.e. $\hat{a}_1(s,t) = 1/(1-s)$, and $\hat{a}_2 \equiv \hat{a}_3 \equiv 0$. Therefore, that (43) should hold is suggested by Proposition 5.3.

Proof of Theorem 6.1. According to Lemma 2.1, with (8) written as in (10) (would NEED (8) implies (6)), it suffices to show that the integral of the left-hand side of (43) over $[0,s]\times[0,t]$ is a Brownian sheet. The integral equals

$$Y(s,t) - Y(s,0) - Y(0,t) + Y(0,0) + \int_0^s \frac{du}{1-u}\,(Y(u,t)-Y(u,0)).$$

Replace $Y(s,t)$ by $W(1-s,t)$ and do the change of variables $u \mapsto 1-u$ to see that this expression equals

$$W(1-s,t) - W(1,t) + \int_{1-s}^{1} \frac{du}{u} W(u,t).$$

Do the change of variables $x = 1/u$ to get

$$W(1-s,t) - W(1,t) + \int_{1}^{\frac{1}{1-s}} \frac{dx}{x} W(1/x,t). \tag{44}$$

Let $\xi(s,t) = sW(1/s,t)$. Then $(\xi(s,t))$ is a standard Brownian sheet [13] and the expression above can be written

$$(1-s)\xi\left(\frac{1}{1-s}, t\right) - \xi(1,t) + \int_{1}^{\frac{1}{1-s}} \frac{dx}{x^2} \xi(x,t).$$

Integrate by parts to see that this expression is equal to

$$\int_{1}^{\frac{1}{1-s}} \xi(dx,t)\,\frac{1}{x} = \int_{1}^{\frac{1}{1-s}} \int_{0}^{t} \xi(dx,dy)\,\frac{1}{x} \stackrel{\text{def}}{=} B(s,t).$$

It is not difficult to check that $(B(s,t))$ is a Brownian sheet. For instance, if $s < s'$ and $t < t'$, then

$$E(B(s,t)B(s',t')) = \int_{1}^{\frac{1}{1-s}} dx \int_{0}^{t} dy\,\frac{1}{x^2} = t\,\frac{1}{x}\bigg|_{\frac{1}{1-s}}^{1} = t(1-(1-s)) = st,$$

while if $s < s'$ and $t' < t$, this covariance is st'.

We now check that $(B(s,t))$ is independent of $(W(1,t),\ t \geq 0)$. Fix $a \geq 1$ and $b \geq 0$. If $0 \leq s \leq 1$ and $t \leq b$, then we use the fact that $B(s,t)$ is equal to the expression in (44) to write

$$\begin{aligned} E(B(s,t)W(a,b)) &= E\left(\left[W(1-s,t) - W(1,t) + \int_{1-s}^{1} \frac{du}{u} W(u,t)\right] W(a,b)\right) \\ &= -st + \int_{1-s}^{1} \frac{du}{u}\, ut \\ &= -st + st \\ &= 0. \end{aligned}$$

If $t \geq b$, then

$$E(B(s,t)W(a,b)) = -sb + \int_{1-s}^{1} \frac{du}{u}\, ub = -sb + sb = 0.$$

This proves the desired independence and completes the proof. ♣

6.2 Reversal in two coordinates

Theorem 6.3 *Let $(W(s,t))$ be a standard Brownian sheet. Set*

$$Y(s,t) = W(1-s, 1-t).$$

Then there is a standard Brownian sheet $(B(s,t))$ independent of $(W(x,1),\ W(1,x),\ 0 \leq x \leq 1)$ such that $(Y(s,t))$ is a solution on $[0,1[^2$ of

$$\frac{\partial^2 Y}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial Y}{\partial s} + \frac{1}{1-s}\frac{\partial Y}{\partial t} + \frac{1}{(1-s)(1-t)} Y(s,t) = \frac{\partial^2 B}{\partial s \partial t}, \tag{45}$$

with initial conditions $Y(0,x) = W(1, 1-x)$, $Y(x,0) = W(1-x, 1)$, $0 \leq x \leq 1$.

Remark 6.4 *With Δ and $\hat{\mathcal{G}}_{\hat{s},\hat{t}}$ defined as in (32), it is not difficult to check that*

$$E(W(\Delta) \mid \hat{\mathcal{G}}_{\hat{s},\hat{t}}) = \frac{h}{t}(X(s,t) - X(s-h,t)) + \frac{h}{s}(X(s,t) - X(s,t-h)) + \frac{h^2}{st} X(s,t)$$

(this formula also can be obtained from [3, Theorem 4.2]). From (34), we guess that

$$\hat{a}_1(\hat{s},\hat{t}) = \frac{1}{t}, \qquad \hat{a}_2(\hat{s},\hat{t}) = \frac{1}{s}, \qquad \hat{a}_3(\hat{s},\hat{t}) = \frac{1}{st}.$$

Proposition 5.3 suggests, therefore, that equation (45) should hold.

Proof of Theorem 6.3. Again according to Lemma 2.1, with (8) written as in (10) (would NEED (8) implies (6)), it suffices to show that the integral of the left-hand side of (45) over $[0,s] \times [0,t]$ is a Brownian sheet, with the desired independence properties. The integral is equal to

$$\begin{aligned} Y(s,t) - Y(s,0) - Y(0,t) + Y(0,0) + \int_0^t \frac{dv}{1-v}\,(Y(s,v) - Y(0,v)) \\ + \int_0^s \frac{du}{1-u}\,(Y(u,t) - Y(u,0)) + \int_0^s \frac{du}{1-u} \int_0^t \frac{dv}{1-v}\, Y(u,v). \end{aligned}$$

Replace $Y(s,t)$ by $W(1-s,1-t)$ and do the change of variables $(u,v) \mapsto (1-u, 1-v)$ to get

$$\begin{aligned} & W(1-s,1-t) - W(1-s,1) - W(1,1-t) + W(1,1) \\ & \quad + \int_{1-t}^1 \frac{dv}{v}\,(W(1-s,v) - W(1,v)) + \int_{1-s}^1 \frac{du}{u}\,(W(u,1-t) - W(u,1)) \\ & \quad + \int_{1-s}^1 \frac{du}{u} \int_{1-t}^1 \frac{dv}{v}\, W(u,v). \end{aligned} \tag{46}$$

Now do the change of variables $x = 1/u$, $y = 1/v$, to see that this equals

$$W(]1-s,1]\times]1-t,1]) + \int_1^{\frac{1}{1-t}} \frac{dy}{y}\,(W(1-s,1/y) - W(1,1/y))$$
$$+ \int_1^{\frac{1}{1-s}} \frac{dx}{x}\,(W(1/x,1-t) - W(1/x,1)) + \int_1^{\frac{1}{1-s}} \frac{dx}{x} \int_1^{\frac{1}{1-t}} \frac{dy}{y}\, W\left(1/x,1/y\right).$$

Let $\xi(s,t) = stW(1/s,1/t)$. Then $(\xi(s,t))$ is a standard Brownian sheet, and the expression above can be written

$$(1-s)(1-t)\xi\left(\frac{1}{1-s},\frac{1}{1-t}\right) - (1-s)\xi\left(\frac{1}{1-s},1\right) - (1-t)\xi\left(1,\frac{1}{1-t}\right) + \xi(1,1)$$
$$- \int_1^{\frac{1}{1-t}} \left[(1-s)\frac{-1}{y^2}\xi(\frac{1}{1-s},y) - \frac{-1}{y^2}\xi(1,y)\right] dy$$
$$- \int_1^{\frac{1}{1-s}} \left[(1-t)\frac{-1}{x^2}\xi(x,\frac{1}{1-t}) - \frac{-1}{x^2}\xi(x,1)\right] dx$$
$$+ \int_1^{\frac{1}{1-s}} dx \int_1^{\frac{1}{1-t}} dy \frac{1}{x^2y^2}\,\xi(x,y).$$

Using the formula for integration by parts (5), with $f(x,y) = \xi(x,y)$ and $g(x,y) = 1/(xy)$, we see that this equals

$$\int_1^{\frac{1}{1-s}} dx \int_1^{\frac{1}{1-t}} dy\, \frac{1}{xy}\,\xi(dx,dy) \overset{\text{def}}{=} B(s,t).$$

It is now straightforward to check that $(B(s,t))$ so defined is a standard Brownian sheet. For instance, if $s < s'$ and $t' < t$, then

$$E(B(s,t)B(s',t')) = \int_1^{\frac{1}{1-s}} dx \int_1^{\frac{1}{1-t'}} dy \frac{1}{x^2y^2} = \left.\frac{-1}{x}\right|_1^{\frac{1}{1-s}} \cdot \left.\frac{-1}{y}\right|_1^{\frac{1}{1-t'}} = st'.$$

This proves that (45) holds.

It remains to prove that $(B(s,t))$ is independent of $(W(a,1),\ W(1,a),\ 0 \le a \le 1)$. For this, it suffices to compute the covariance between the expression in (46) and $W(a,1)$, then $W(1,a)$. We omit the second computation and do the first.

From the covariance of the Brownian sheet and elementary geometric considerations, using the fact that $B(s,t)$ is equal to the expression in (46), we see that for $a \le 1-s$,

$$E(B(s,t)W(a,1)) = \int_{1-s}^1 \frac{du}{u}\,(-at) + \int_{1-s}^1 \frac{du}{u} \int_{1-t}^1 \frac{dv}{v}\,(av) = 0,$$

and for $1-s \le a \le 1$,

$$\begin{aligned} E(B(s,t)W(a,1)) &= (a-1+s)t + \int_{1-t}^{1} \frac{dv}{v}(-v(a-1+s)) \\ &\quad + \int_{1-s}^{a} \frac{du}{u}(-ut) + \int_{a}^{1} \frac{du}{u}(-at) \\ &\quad + \int_{1-s}^{a} \frac{du}{u}\int_{1-t}^{1} \frac{dv}{v}(uv) + \int_{a}^{1} \frac{du}{u}\int_{1-t}^{1} \frac{dv}{v}(av) \\ &= 0. \end{aligned}$$

This completes the proof. ♣

Remark 6.5 *Equation (45) is reminiscent of the equation for a Brownian bridge* $(X_s,\ 0 \le s \le 1)$*:*

$$dX_s + \frac{X_s}{1-s} = dB_s,$$

where (B_s) *is a standard Brownian motion. The law of the reversed process* $(B(1-s),\ 0 \le s \le 1)$*, is the same as the law of* (Y_t)*, where*

$$Y_t = (1-t)Z + X_t,$$

and Z *is a standard Normal random variable independent of the Brownian bridge* (X_t)*. A similar identity in law occurs for the Brownian sheet, as is shown in the following theorem. MENTION articles [1, 9].*

Theorem 6.6 *Let* $(W(s,t))$ *and* $(X(s,t))$ *be standard Brownian sheets. Set*

$$\begin{aligned} (47)\quad U(s,t) &= X(s,t) - sX(1,t) - tX(s,1) + stX(1,1), \\ (48)\quad Z(s,t) &= (1-s)W(1,1-t) + (1-t)W(1-s,1) - (1-s)(1-t)W(1,1). \end{aligned}$$

Then U *and* Z *are independent, and* $Y = U + Z$ *has the same law as* $(W(1-s,1-t),\ (s,t) \in [0,1]^2)$*. In particular,* Y *is a (weak) solution of (45) with initial conditions* $Y(0,x) = W(1,1-x)$*,* $Y(x,0) = W(1-x,1)$*,* $0 \le x \le 1$*.*

The proof of this theorem relies on two lemmas.

Lemma 6.7 Z *is a solution of the equation*

$$(49)\qquad \frac{\partial^2 Z}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial Z}{\partial s} + \frac{1}{1-s}\frac{\partial Z}{\partial t} + \frac{1}{(1-s)(1-t)}Z(s,t) = 0.$$

Proof. Again according to Lemma 2.1, with (8) written as in (10) (would NEED (8) implies (6)), it suffices to show that the integral of the left-hand side of (49) over $\Delta = [0,s] \times [0,t]$ vanishes. This integral is

$$Z(\Delta) + \int_0^t \frac{dv}{1-v}\,(Z(s,v) - Z(0,v)) \\ + \int_0^s \frac{du}{1-u}\,(Z(u,t) - Z(u,0)) + \int_0^s \frac{du}{1-u} \int_0^t \frac{dv}{1-v}\, Z(u,v).$$

Using formula (48), this equals

$$\begin{aligned}
&(1-s)X(1,1-t) + (1-t)X(1-s,1) - (1-s)(1-t)X(1,1) \\
&\quad -X(1,1-t) - X(1-s,1) + X(1,1) \\
&\quad + \int_0^t \frac{dv}{1-v}\,((1-s)X(1,1-v) + (1-v)X(1-s,1) \\
&\quad -(1-s)(1-v)X(1,1) - X(1,1-v)) \\
&\quad + \int_0^s \frac{du}{1-u}\,((1-t)X(1-u,1) + (1-u)X(1,1-t) \\
&\quad -(1-u)(1-t)X(1,1) - X(1-u,1)) \\
&\quad + \int_0^s \frac{du}{1-u} \int_0^t \frac{dv}{1-v}\,((1-u)X(1,1-v) + (1-v)X(1-u,1) \\
&\quad -(1-u)(1-v)X(1,1)).
\end{aligned}$$

This expression is easily seen to simplify to 0. ♣

Lemma 6.8

$$E(Z(s,t)Z(s',t')) = (1 - (s \wedge s')(t \wedge t'))(1 - s \vee s')(1 - t \vee t'), \tag{50}$$

and

$$E(U(s,t)U(s',t')) = (s \wedge s')(t \wedge t')(1 - s \vee s')(1 - t \vee t'). \tag{51}$$

PROOF. Using elementary algebra, one checks that

$$\begin{aligned} Z(s,t) &= (1-s+1-t-(1-s)(1-t))X(1-s,1-t) \\ &\quad +(1-s-(1-s)(1-t))(X(1,1-t)-X(1-s,1-t)) \\ &\quad +(1-t-(1-s)(1-t))(X(1-s,1)-X(1-s,1-t)) \\ &\quad -(1-s)(1-t)X([1-s,1]\times[1-t,1]) \\ &= (1-st)X(1-s,1-t) \\ &\quad +t(1-s)(X(1,1-t)-X(1-s,1-t)) \\ &\quad +s(1-t)(X(1-s,1)-X(1-s,1-t)) \\ &\quad -(1-s)(1-t)X([1-s,1]\times[1-t,1]). \end{aligned}$$

The four terms in the last expression are independent. It is now a tedious but elementary calculation, using the covariance of the Brownian sheet, to check that $E(Z(s,t)Z(s',t'))$ is given by formula (50). These calculations are left to the reader. Similarly, the tedious but elementary calculations that establish formula (51) are left to the reader. ♣

PROOF OF THEOREM 6.6. From Lemma 6.8, we see that $Y = U + Z$ has the same covariance, hence the same law, as $(W(1-s,1-t))$. This of course implies that there is a white noise $\dot{B}$ such that Y is the solution of equation (45), but we prefer to give a direct derivation. By Lemma 6.7, it suffices to check that there is a Brownian sheet $(\xi(s,t))$ such that $(U(s,t))$ is the solution of

$$\frac{\partial^2 U}{\partial s \partial t} + \frac{1}{1-t}\frac{\partial U}{\partial s} + \frac{1}{1-s}\frac{\partial U}{\partial t} + \frac{1}{(1-s)(1-t)}U(s,t) = \frac{\partial^2 \xi}{\partial s \partial t}. \tag{52}$$

Set

$$W(s,t) = X([1-s,1]\times[1-t,1]),$$

so that

$$X(s,t) = W([1-s,1]\times[1-t,1]). \tag{53}$$

Because U vanishes on the axes, the double integral of the left-hand side of (52) over $\Delta = [0,s]\times[0,t]$ is equal to

$$U(s,t) + \int_0^t \frac{dv}{1-v}\,U(s,v) + \int_0^s \frac{du}{1-u}\,U(u,t) + \int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v}\,U(u,v).$$

Replace $U(\cdot,\cdot)$ by its expression in terms of X given in (47) to get

$$\begin{aligned}
&X(s,t) - sX(1,t) - tX(s,1) + stX(1,1)\\
&\quad + \int_0^t \frac{dv}{1-v}\,(X(s,v) - sX(1,v) - vX(s,1) + svX(1,1))\\
&\quad + \int_0^s \frac{du}{1-u}\,(X(u,t) - uX(1,t) - tX(u,1) + utX(1,1))\\
&\quad + \int_0^s \frac{du}{1-u}\int_0^t \frac{dv}{1-v}\,(X(u,v) - uX(1,v) - vX(u,1) + uvX(1,1)).
\end{aligned}$$

Rearrange the terms and simplify to get

$$\begin{aligned}
(54)\quad &X(s,t) + \int_0^t \frac{dv}{1-v}\,(X(s,v) - X(s,1))\\
&+ \int_0^s \frac{du}{1-u}\,(X(u,t) - X(1,t))\\
&+ \int_0^s du \int_0^t dv\,\Bigg(X(1,1) - \frac{X(1,v)}{1-v} + \frac{vX(1,1)}{1-v} - \frac{X(u,1)}{1-u} + \frac{uX(1,1)}{1-u}\\
&\qquad\qquad + \frac{X(u,v) - uX(1,v) - vX(u,1) + uvX(1,1)}{(1-u)(1-v)}\Bigg).
\end{aligned}$$

The integrand in the double integral simplifies to

$$\frac{X(u,v) - X(1,v) - X(u,1) + X(1,1)}{(1-u)(1-v)}.$$

Now replace $X(\cdot,\cdot)$ by its expression in terms of W given in (53) and do the changes of variables $u \mapsto 1-u$, $v \mapsto 1-v$, to see that (54) is equal to

$$\begin{aligned}
&W(1-s,1-t) - W(1-s,1) - W(1,1-t) + W(1,1)\\
&\quad + \int_{1-t}^1 \frac{dv}{v}\,(W(1-s,v) - W(1,v)) + \int_{1-s}^1 \frac{du}{u}\,(W(u,1-t) - W(u,1))\\
&\quad + \int_{1-s}^1 \frac{du}{u}\int_{1-t}^1 \frac{dv}{v}\,W(u,v).
\end{aligned}$$

This is exactly the expression in (46), and we have shown in the lines that follow (46) that this expression is a standard Brownian sheet, that is independent of $(W(1-x,1),\ W(1,1-x),\ 0 \le x \le 1)$. ♣

7 Reversal in hyperbolic s.p.d.e.'s

We shall consider the reversal in two coordinates of the the solution of the hyperbolic equation with constant coeficients

$$\frac{\partial^2 X}{\partial s \partial t} + a_1 \frac{\partial X}{\partial s} + a_2 \frac{\partial X}{\partial t} + a_3 X(s,t) = \dot{W}, \tag{55}$$

with vanishing initial conditions $X(s,0) = 0$, $X(0,t) = 0$. The reversal in one coordinate could be done similarly, and in fact, more simply. In this equation, the case $a_3 \neq a_1 a_2$ corresponds to the telegraph equation [6, Chap.IV, §43], whereas in the special case where $a_3 = a_1 a_2$, equation (55) can be transformed into the wave equation by a change of variables and parameters. We shall restrict to this special case.

Theorem 7.1 *Fix $a_1, a_2, a_3 \in \mathbb{R}$ and suppose $a_3 = a_1 a_2 \neq 0$. Let $(X(s,t))$ be the solution of (55) with vanishing initial conditions, and set $\hat{X}(s,t) = X(1-s, 1-t)$. Then there is a Brownian sheet $(B(s,t))$ independent of $(X(u,1),\ X(1,u),\ 0 \le u \le 1)$ such that $(\hat{X}(s,t))$ is the solution on $[0,1[^2$ of*

$$\frac{\partial^2 \hat{X}}{\partial s \partial t} + \hat{a}_1(s,t) \frac{\partial \hat{X}}{\partial s} + \hat{a}_2(s,t) \frac{\partial \hat{X}}{\partial t} + \hat{a}_3(s,t) \hat{X}(s,t) = \frac{\partial^2 B}{\partial s \partial t}, \tag{56}$$

with initial conditions $\hat{X}(s,0) = X(1-s,1)$, $\hat{X}(0,t) = X(1,1-t)$, where

$$\hat{a}_1(s,t) = \frac{2a_1 e^{2a_1(1-s)}}{e^{2a_1(1-s)} - 1} - a_1, \qquad \hat{a}_2(s,t) = \frac{2a_2 e^{2a_2(1-s)}}{e^{2a_2(1-s)} - 1} - a_2, \tag{57}$$

and $\hat{a}_3(s,t) = \hat{a}_1(s,t)\hat{a}_2(s,t)$.

Remark 7.2 *The case $a_1 = a_2 = 0$ has been discussed in Theorem 6.3. In order to recover this case from the theorem above, it is not possible to set $a_i = 0$ in (57), but there is no problem in taking the limit as $a_i \to 0$. Doing this for $i = 1, 2$ leads to equation (45).*

Proof of Theorem 7.1. Define $\tilde{X}(s,t) = e^{a_2 s + a_1 t} X(s,t)$. From Example 4.1, we see that $\tilde{X}$ satisfies the equation

$$\frac{\partial^2 \tilde{X}}{\partial s \partial t} = e^{a_2 s + a_1 t} \dot{W}.$$

Therefore, there is a Brownian sheet $\tilde{W}$ such that

$$\tilde{X}(s,t) = \tilde{W}\left(\frac{e^{2a_2 s} - 1}{2a_2}, \frac{e^{2a_1 t} - 1}{2a_1}\right),$$

and therefore,

$$X(s,t) = e^{-a_2 s - a_1 t}\, \tilde{W}\left(\frac{e^{2a_2 s} - 1}{2a_2}, \frac{e^{2a_1 s} - 1}{2a_1}\right)$$

and

$$\hat{X}(s,t) = e^{-a_2(1-s)-a_1(1-t)}\,\tilde{W}\left(\frac{e^{2a_2(1-s)}-1}{2a_2}, \frac{e^{2a_1(1-t)}-1}{2a_1}\right).$$

Set

$$Z(s,t) = \tilde{W}\left(\frac{e^{2a_2}-1}{2a_2} - s, \frac{e^{2a_1}-1}{2a_1} - t\right).$$

Then

$$\hat{X}(s,t) = e^{-a_2(1-s)-a_1(1-t)} Z\left(\frac{e^{2a_2}-e^{2a_2(1-s)}}{2a_2}, \frac{e^{2a_1}-e^{2a_1(1-t)}}{2a_1}\right).$$

By Theorem 6.3, $(Z(s,t))$ is a solution of the equation

$$\frac{\partial^2 Z}{\partial s \partial t} + f(a_1,t)\frac{\partial Z}{\partial s} + f(a_2,s)\frac{\partial Z}{\partial t} + f(a_1,s)f(a_2,t)Z = \dot{B},$$

where B is a Brownian sheet independent of

$$\left(Z\left(\frac{e^{2a_2}-1}{2a_2}, x\right),\ Z\left(x, \frac{e^{2a_1}-1}{2a_1}\right), \qquad 0 \le x \le 1\right),$$

and

$$f(a,x) = \left(\frac{e^{2a}-1}{2a} - x\right)^{-1}.$$

Let

$$Y(s,t) = Z\left(\frac{e^{2a_2}-e^{2a_2(1-s)}}{2a_2}, \frac{e^{2a_1}-e^{2a_1(1-t)}}{2a_1}\right).$$

¿From Example 3.1, we conclude that $(Y(s,t))$ is a solution of the equation

$$\frac{\partial^2 Y}{\partial s \partial t} + g(a_1,x)\frac{\partial Y}{\partial s} + g(a_2,x)\frac{\partial Y}{\partial t} + g(a_1,s)g(a_2,t)Y = e^{a_2(1-s)+a_1(1-t)}\text{white noise},$$

where

$$g(a,x) = \frac{2ae^{2a(1-x)}}{e^{2a(1-x)}-1}.$$

Again by Example 4.1, we conclude that $(\hat{X}(s,t))$ solves equation (56). This proves the theorem. ♣

8 Reversal with initial conditions

Theorem 8.1 *Let X_0 be a $N(0,\sigma^2)$ random variable, (M_s^1) and (M_t^2) be Gaussian martingales such that $M_0^1 = M_0^2 = 0$ and $E((M_u^i)^2) = f_i(u)$, $i = 1,2$. We assume that X_0, (M_s^1) and (M_t^2) are independent.*

Let X be the solution of the s.p.d.e.

$$\frac{\partial^2 X}{\partial s \partial t} = \dot{W}, \tag{58}$$

with the initial conditions $X(s,0) = X_0 + M_0^1$ and $X(0,t) = X_0 + M_t^2$, $s,t \geq 0$. Then there exists a Brownian sheet $(B(s,t))$ such that $\hat{X}(s,t) = X(1-s, 1-t)$ satisfies an s.p.d.e. of the form

$$\frac{\partial^2 \hat{X}}{\partial s \partial t} + a_1(s,t)\frac{\partial \hat{X}}{\partial s} + a_2(s,t)\frac{\partial \hat{X}}{\partial t} + a_3(s,t)\hat{X} = a_4(s,t)\dot{B} \tag{59}$$

if and only if $a_4 \equiv 1$ and there are real numbers $T_1 > 0$ and $T_2 > 0$ such that $f_i(u) = T_{3-i}\,u$ and $T_1 T_2 = \sigma^2$. In other words, X can be embedded into a Brownian sheet $\tilde{W}$ as follows:

$$X(s,t) = \tilde{W}(T_1 + s, T_2 + t), \qquad (s,t) \in I\!R_+^2. \tag{60}$$

Proof. We know from Theorem 6.3 that the reversal of $\tilde{W}$ in both coordinates does satisfy an s.p.d.e. of the form (59). So we assume that $\hat{X}$ satisfies such an s.p.d.e. and show that X can be embedded into a Brownian sheet (the fact that a_4 must be identically 1 follows immediately from Proposition 2.6).

Fix s,t such that $s + t = 2 - r$, and let $\Delta = [s-h, s] \times [t-h, t]$. According to (34),

$$\begin{aligned} E(W(\Delta) \mid \hat{\mathcal{F}}(r)) \;=\; & a_1(s,t)h(X(s,t) - X(s-h,t)) \\ & + a_2(s,t)h(X(s,t) - X(s,t-h)) + a_3(s,t)h^2 X(s,t) \\ & + \varepsilon(s,t;h), \end{aligned}$$

or, equivalently, for $u + v \geq s + t$,

$$\begin{aligned} E([W(\Delta) - a_1 h(X(s,t) - X(s-h,t)) \\ - a_2 h(X(s,t) - X(s,t-h)) - a_3 h^2 X(s,t) - \varepsilon] X(u,v)) = 0. \end{aligned} \tag{61}$$

Because X solves (58), Lemma 2.3 implies that

$$X(s,t) = X_0 + M_s^1 + M_t^2 + W(s,t),$$

and therefore,

$$\begin{aligned} X(s,t) - X(s-h,t) &= M_s^1 - M_{s-h}^1 + W(s,t) - W(s-h,t), \\ X(s,t) - X(s,t-h) &= M_t^2 - M_{t-h}^2 + W(s,t) - W(s,t-h). \end{aligned}$$

Write (61) for $u \leq s - h$ to get

$$-a_2 h(hf_2'(t) + o(h) + uh) - a_3 h^2(\sigma^2 + f_1(u) + f_2(t) + ut) - E(\varepsilon X(u,v)) = 0, \tag{62}$$

for $u \geq s$ and $v \geq t$ to get

$$\begin{aligned} h^2 - a_1 h(hf_1'(s) + o(h) + ht) - a_2 h(hf_2'(t) + o(h) + hs) \\ - a_3 h^2(\sigma^2 + f_1(s) + f_2(t) + st) - E(\varepsilon X(u,v)) = 0, \end{aligned} \tag{63}$$

and for $v \leq t - h$ to get

$$-a_1 h(hf_1'(s) + o(h) + hv) - a_3 h^2(\sigma^2 + f_1(s) + f_2(v) + sv) + E(\varepsilon X(u,v)) = 0. \tag{64}$$

Divide the three equations by h^2, let $h \downarrow 0$ and use the fact that Var $\varepsilon(s,t;h) = o(h^4)$ to get the three equations

$$-a_2(f_2'(t) + u) - a_3(\sigma^2 + f_1(u) + f_2(t) + ut) = 0, \tag{65}$$

$$1 - a_1(f_2'(s) + t) - a_2(f_2'(t) + s) - a_3(\sigma^2 + f_1(s) + f_2(t) + st) = 0, \tag{66}$$

$$-a_1(f_1'(s) + v) - a_3(\sigma^2 + f_1(s) + f_2(v) + sv) = 0 \tag{67}$$

(the first equation is valid for $u \leq s$, the third for $v \leq t$). ¿From (65), we get

$$f_1(u) = -f_2(t) - \frac{a_2}{a_3} f_2'(t) - \sigma^2 - u\left(t + \frac{a_2}{a_3}\right), \tag{68}$$

and from (67), we get

$$f_2(v) = -f_1(s) - \frac{a_1}{a_3} f_1'(s) - \sigma^2 - v\left(s + \frac{a_1}{a_3}\right). \tag{69}$$

Therefore, f_1 and f_2 are affine functions of u and v, respectively. Because $f_1(0) = f_2(0) = 0$, there are numbers $T_1 > 0$ and $T_2 > 0$ such that

$$f_1(u) = T_2 u, \qquad f_2(v) = T_1 v. \tag{70}$$

Identifying coefficients in (68) and (69) with those in (70), we see that

$$T_1 = -s - a_1/a_3, \qquad T_2 = -t - a_2/a_3,$$

and these expressions cannot depend on s and/or t. In addition,

$$-f_2(t) - \frac{a_2}{a_3} f_2'(t) - \sigma^2 = 0,$$

and from (70), the left-hand side is equal to

$$-T_1 t - \frac{a_2}{a_3} T_1 - \sigma^2 = 0.$$

Because $-t - a_2/a_3 = T_2$, we conclude that

$$T_1 T_2 = \sigma^2.$$

This completes the proof. ♣

Remark 8.2 *Non-locality of the reversal of the wave equation.*

Remark 8.3 *Requesting that the reversal $\hat{X}$ of the solution to (58) satisfy a linear equation is natural, since $\hat{X}$ is Gaussian. On the other hand, it is the fact that the terms in (59) are local (i.e. only depend on $X(s,t)$ and its derivatives at (s,t)) that prevents $\hat{X}$ from satisfying such an equation unless X is a Brownian sheet. It is interesting to point out that even in the setting of d-dimensional diffusions, with d* decoupled *equations, most kinds of initial conditions will lead to* coupled *equations for the reversed process. The simplest exampe, suggested to the first author by E. Mayer-Wolf and O. Zeitouni, is the following. Let $(B^1, \dots, B^d)$ be a d-dimensional Brownian motion,*

$$dX_t^i = dB_t^i, \qquad X_0^i = Y^i, \qquad i = 1, \dots, d,$$

where $(Y^1, \dots, Y^d)$ is an $\mathbb{R}^d$-valued and centered Gaussian random variable with covariance matrix Ξ. Then the law of X_t is $N(0, \Xi + uI)$, where I is the $d \times d$ identity matrix. According to the d-dimensional version of (2), the system of diffusion equations for $\hat{X}_u = (X_{1-u}^1, \dots, X_{1-u}^d)$ is

$$d\hat{X}_u^i = d\hat{B}_u^i - \sum_{j=1}^{d} a_{i,j}(u) \hat{X}_u^j \, du, \tag{71}$$

where $(a_{i,j}(u)) = (\Xi + (1-u)I)^{-1}$. Unless Ξ is diagonal (that is, $Y^1, \dots, Y^d$ are independent), the drift in (71) is "non-local," in that it depends on all components of $\hat{X}_u^j$.

This example and Theorem 8.1 suggest that the only type of equation that the reversal of (58) may satisfy is an equation with non-local coefficients. This should motivate the development of an existence theory for such equations.

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