Of Astronauts and Toy Trains

as told to J. B. Walsh

Even the grandeurs of space can pall. Pity the poor space farer, blasé from too much exposure to the milky way and too much condensed milk, bored by the sublime music of the spheres and the Texas twang of ground control. I heard a tale once of such an astronaut, who, at wits end for entertainment after months in a space station, decided to pass time by building and running a model train. This is his story.

It is not easy to run a model train in a space station. Aside from the space-wide shortage of HO gauge track, there is the problem of keeping the train on the rails in the absence of gravity. How he obtained the track and the Lionel engine must remain his secret, but we can reveal how he solved the problem of running a model train in free fall.

Let $\mathbf{R}(t)$ be the position vector of the train at time t. The tracks will be set in some curve in space, (there is no reason to confine them to the floor) and the train will follow that curve, so that $\mathbf{R}(t)$ will describe a curve that winds its way through the space station. Let $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ be its unit tangent, principal normal, and binormal vectors respectively. These are mutually orthogonal unit vectors, and the acceleration of the train is easily calculated in terms of them:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds}\frac{ds}{dt} = v\mathbf{T}; \tag{1}$$

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d(v\mathbf{T})}{dt} = \frac{dv}{dt}\mathbf{T} + v^2\frac{d\mathbf{T}}{ds}. \tag{2}$$

By the Frenet formulas, this is

$$\mathbf{A} = a\mathbf{T} + \kappa v^2\mathbf{N}. \tag{3}$$

where $v = ds/dt$ and $a = d^2s/dt^2$ are the tangential component of the velocity and acceleration of the train and κ is the curvature of the path. By Newton's law ($\mathbf{F} = m\mathbf{A}$), if $v > 0$ there will always be a component of force holding the train to the tracks as long as the curvature κ is greater than zero and $\mathbf{N}$ points upward. (This is from the toy engineer's point of view. The force $\mathbf{F}$ here is the force on the train, for it is this which causes it to accelerate. The force that the train exerts on the tracks is of course equal and opposite, so that as long as **A** points upward, the train will press down on the tracks.) Thus the astronaut's solution was this:

(a) arrange the tracks into a curve whose curvature is never zero, and

(b) bank them so that the principal normal always points straight up from the rails.

Consider the toy engineer, standing straight up in the cab and facing forward. If he extends his right arm horizontally and waves to the toy farmers in the toy fields as he passes, it will be pointing in the direction of the binormal vector **B**. (You can verify this by the right hand rule; there is only a moderate danger of spraining your wrist.) The Frenet frame **T**, **N**, and **B** will be a fixed set of coordinates as far as he is concerned: **T** points straight ahead, **N** points up, and **B** points to his right.

The astronaut was content to let the train run in simple plane curves for the first few days, circles and ellipses and such, and he found it amusing to see it loop-the-loop with so little effort. As time went on, he moved the tracks into more interesting curves, figure eights and spirals; he even tried a granny knot with some success, though he admitted later that a square knot was better.

Notice that as soon as the track moves out of a plane, it has to bank, which means it will twist, and the curve will have torsion. In fact the astronaut tried for some time to get the train to make a figure eight and to keep the tracks in the same plane. After a week's efforts, trying all possible radii and shapes of the eights, he finally gave up. He could have saved himself a lot of trouble had he stopped tinkering long enough to do a little mathematics. It turns out to be impossible for the tracks to form a figure eight and stay in the same plane, and this is not just because they must jog to miss each other when they cross. (Exercise: prove it. Remember that the curvature can never vanish.)

From the toy engineer's point of view, if the train follows a non-plane curve, it will bank and roll—even barrel-roll—as it travels along, rotating about its long axis. Suppose the train travels at unit speed. It is then easy to show that the rate of rotation about its long axis is exactly equal to the torsion of the curve. (Exercise: convince yourself that this is true. Was it easy? Sometimes mathematicians lie about things like that.)

Even toy trains can pall. The astronaut eventually became bored with the whole affair. (If he could get bored seeing the whole galaxy just outside of his window, he probably had a short attention span, so this isn't surprising.) In particular, he tired of watching the train going thru even the most intricate of four-in-hand knots in the corridors, and he decided to turn over the work of watching it to someone else. As the only other entity on board was the station's computer, it isn't hard to guess who got the job.

This raised another problem. Computers are not programmed to watch toy trains. Not that it had to physically watch the train going thru its paces. It had senses other than eyes, and could keep track of it in its own way. It just had to know where the train was at all times. The problem was to get that information to it. The astronaut first thought of putting relays on the rail ties to signal as the train passed over them, and then realized that that wouldn't be enough. If he moved the tracks, he would have to tell the computer where they were all over again. He needed something more flexible. He finally decided that the only satisfactory solution was to build an inertial guidance system and put it in the engine. How? Gyroscopes small enough to fit in a Lionel cab and accurate enough to control the position to four decimal places were expensive enough for NASA to keep them out of the hands of bored astronauts. On the other hand, there were miniature solid state devices in the station's spare parts stock—strain gauges and such—which could measure small forces with great accuracy. It would be possible to use these to measure the train's accelerations and then the computer could integrate these to find its position.

But there was still a problem. The device would be fixed in the cab of the train, so that it would see the coordinates $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ rather than x, y, and z. Any accelerations it measured would be in these coordinates, and one had to find the x y z position from these. This meant that it had to be possible to translate from $\mathbf{T}$ $\mathbf{N}$ $\mathbf{B}$ coordinates to x y z coordinates at all times. Here is how he solved the problem. He needed two small equal masses and enough gauges to measure the (vector) force on each of them. (Three gauges for each mass are enough if they can read both positive and negative forces: one for each of the $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ components of the force. As it turned out, he used fewer.)

We will neglect a couple of small effects. The first is the fact that the center of gravity of the engine doesn't quite coincide with the center of the tracks. The second effect is the coriolis acceleration, due to the motion of the station about the earth. Surprisingly enough, this is not negligeable, but the ship's computer, which did all the navigation for the station, could easily take it into account. (To see why it can't be neglected, think of a small object placed motionless with respect to the station's center of gravity, and a small distance to one side. This object is actually in its own orbit about the earth; the orbit has the same center—the earth's center of gravity—and the same radius and period, but it is in a slightly different plane than the orbit of the space-station's center of gravity. By symmetry, one half orbit later, the object will be in this same plane, the same distance from the station's center of gravity, but on the *other side*! After a full orbit, a matter of only several hours, it will be back in its original position. The observer in the station will see it oscillate back and forth slowly through the station's center of gravity, with a period equal to the period of the station's orbit, even though there is no visible force acting on it.)

The astronaut weighed the small masses carefully—their mass was m—and placed the first at the center of mass of the train and the second a carefully measured distance b directly to its right. The gauges were placed so that they could measure the $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ components of the forces on the masses. A small radio transmitter sent these measurements directly to the computer. Let the position of the first mass at time t be $\mathbf{R}(t)$, the position of the center of mass. The second mass will then be at

$$\mathbf{R}_2(t) = \mathbf{R}(t) + b\mathbf{B}(t). \tag{4}$$

The principal fact we will use is that the curve is determined by its curvature κ and its torsion τ. This might appear to be a rather abstract and theoretical piece of information, but it is the key. Once the computer has these two quantities, it can get the position of the train by doing what it does best: crunching numbers.

The gauges are set to measure the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ on the first and second masses respectively, so if $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ are the respective accelerations, we have

$$\mathbf{F}_i(t) = m\mathbf{A}_i(t), \quad \text{for} \quad i = 1, 2.$$

In order to bring the curvature and torsion into the picture, let us calculate the time derivatives of $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ with the aid of the Frenet formulas.

$$\begin{aligned}
\frac{d\mathbf{R}}{dt} &= v\mathbf{T} \\
\frac{d\mathbf{T}}{dt} &= v\kappa\mathbf{N} \\
\frac{d\mathbf{N}}{dt} &= -v\kappa\mathbf{T} + v\tau\mathbf{B}
\end{aligned}$$

$$\frac{d\mathbf{B}}{dt} = -v\tau\mathbf{N}.$$

We calculated the acceleration of the first mass back in equation (3). Let us use the above equations on it:

$$\mathbf{F}_1(t) = ma\mathbf{T} + m\kappa v^2\mathbf{N}.$$

If F_T^1, F_N^1, and F_B^1 are the $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ components of $\mathbf{F}_1$ respectively, then the tangential component of acceleration satisfies

$$\frac{d^2s}{dt^2} = a = \frac{1}{m}F_T^1. \tag{5}$$

The system can measure F_T^1 directly, so the computer, which is a genius at numerical integration, can compute the speed by integrating:

$$v(t) = v(0) + \frac{1}{m}\int_0^t F_T^1(u)du \tag{6}$$

Now that it knows v, it can get the curvature from the normal component of the force:

$$\kappa = \frac{1}{mv^2}F_N^1 \tag{7}$$

Next, it must get the torsion. For this, it looks at the other mass. Its velocity and acceleration can again be calculated by differerentiating equation (4) and using the Frenet formulas. The calculation is a little more complicated due to the extra term involving $\mathbf{B}$ in (4):

$$\mathbf{V}_2 = \frac{d\mathbf{R}_2}{dt} = \frac{d\mathbf{R}}{dt} + b\frac{d\mathbf{B}}{ds}\frac{ds}{dt} = v\mathbf{T} + bv(-\tau\mathbf{N}),$$

$$\mathbf{A}_2 = \frac{d\mathbf{V}_2}{dt} = a\mathbf{T} + v^2\frac{d\mathbf{T}}{ds} - b\frac{d(v\tau)}{dt}\mathbf{N} - bv^2\tau\frac{d\mathbf{N}}{ds}.$$

The computer uses the Frenet formulas again on the derivatives of $\mathbf{T}$ and $\mathbf{N}$ to see that

$$\mathbf{A}_2 = (a + bv^2\kappa\tau)\mathbf{T} + \left(v^2\kappa - b\frac{d(v\tau)}{dt}\right)\mathbf{N} - bv^2\tau^2\mathbf{B}.$$

This is also equal to $(1/m)\mathbf{F}_2$ by Newton's law. The expression may look complicated, but it only has to find τ, and the $\mathbf{T}$ component is enough for this. It sets the $\mathbf{T}$-components of $\mathbf{A}$ and $\frac{1}{m}\mathbf{F}_2$ equal to see that $(a+bv^2\kappa\tau) = (1/m)F_T^2$. It then solves for τ and uses (6) and (7) to get everything in terms of the forces:

$$\tau = \frac{F_T^2 - F_T^1}{bF_N^1} \tag{8}$$

It has now determined κ and τ at all times. (Note that it wasn't necessary to measure all the components of the forces on the two masses. Only the tangential and normal components of the first mass and the tangential component of the second were necessary, so the astronaut only needed to filch three strain gauges from the storeroom. This turned out to be a good thing too, for...but that's another story.) The computer now has all the information it needs to calculate the position of the train at any time, or it will as soon as it knows the initial conditions. All that remains to be done is to divulge how the computer actually computes the whereabouts of the train from this. It is simple (for a computer). The astronaut gives it the initial position and and the initial orientation of $\mathbf{T}$, $\mathbf{N}$ nd $\mathbf{B}$ and starts the train from rest. All the computer has to do is to integrate the five following equations numerically:

$$\begin{aligned}
\frac{dv}{dt} &= a \\
\frac{d\mathbf{T}}{dt} &= v\kappa\mathbf{N} \\
\frac{d\mathbf{N}}{dt} &= -v\kappa\mathbf{T} + v\tau\mathbf{B} \\
\frac{d\mathbf{B}}{dt} &= -v\tau\mathbf{N} \\
\frac{d\mathbf{R}}{dt} &= v\mathbf{T}.
\end{aligned}$$

It does this in discrete steps. Let Δ be the step size. The computer is given the values at $t = 0$, and then determines them successively at $t = \Delta$, $t = 2\Delta$,... and so forth as follows. Suppose it knows the values of v, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ and $\mathbf{R}$ at time $t = n\Delta$. In order to get the values at $t+\Delta$ it measures $F_T^1(t)$, $F_N^1(t)$, and $F_T^2(t)$ directly from the gauges, and then uses equations (6), (7), (8) and (9) to compute $a(t)$, $\kappa(t)$, and $\tau(t)$. Then it uses the discrete version of the five equations above to get

$$
\begin{aligned}
v(t+\Delta) &= v(t) + a(t)\Delta \\
\mathbf{T}(t+\Delta) &= \mathbf{T}(t) + v(t)\kappa(t)\mathbf{N}(t)\Delta \\
\mathbf{N}(t+\Delta) &= \mathbf{N}(t) - v(t)\kappa(t)\mathbf{N}(t)\Delta + v(t)\tau(t)\mathbf{B}(t)\Delta \\
\mathbf{B}(t+\Delta) &= \mathbf{B}(t) - v(t)\tau(t)\mathbf{N}(t)\Delta.
\end{aligned}
$$

Finally, the new position is

$$\mathbf{R}(t+\Delta) = \mathbf{R}(t) + v(t)\mathbf{T}(t)\Delta.$$

Of course, the computer would use a more sophisticated numerical integration scheme, but that is only a detail. It is interesting to note that even tho it only wants to find $\mathbf{R}(t)$, it has to find $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ as well. This seems extravagant, but that's the way of the frame.

That is just about all there is to the story. The astronaut was rotated home shortly after, got deeply into transcendental meditation and tofu, and forgot all about toy trains. The computer who told me the story, on the other hand, is still delighted with its toy, and keeps an active subprogram playing with it to this day.