ASYMPTOTICS BEYOND ALL ORDERS FOR

A LOW REYNOLDS NUMBER FLOW

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Abstract

The solution for slow incompressible flow past a circular cylinder involves terms in powers of $1/\log\epsilon$, ϵ times powers of $1/\log\epsilon$, etc., where ϵ is the Reynolds number. Previously we showed how to determine the sum of all terms in powers of $1/\log\epsilon$. Now we show how to go beyond all those terms to find the sum of all terms containing ϵ times a power of $1/\log\epsilon$. The first sum gives the drag coefficient and represents a symmetric flow in the Stokes region near the cylinder. The second term reveals the asymmetry of the flow near the body. This problem is studied using a hybrid method which combines numerical computation and asymptotic analysis.

1. Introduction

The steady low Reynolds number flow of an incompressible viscous fluid past a circular cylinder was studied by Stokes [9]. He linearized the governing equations about a state of rest, and found that the resulting boundary value problem had no solution. Oseen [6] observed that this *Stokes paradox* was due to an incorrect treatment of the flow far from the cylinder and could be avoided by linearizing about the flow at infinity. Lamb [5] solved the resulting Oseen equations to obtain a first approximation for the flow and the drag. Much later Proudman and Pearson [7] and Kaplun [2] showed that this problem could be solved to any order by a systematic use of the method of

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matched asymptotic expansions. The inner expansion involved the Stokes linearization and the outer expansion involved the Oseen linearization. The method and results are described by Van Dyke [11].

Both the inner and the outer expansions involve a series of positive powers of $1/\log\epsilon$, where ϵ is the Reynolds number of the flow. In addition, there are terms in powers of ϵ and in powers of ϵ multiplied by powers of $1/\log\epsilon$. These terms are smaller than all positive powers of $1/\log\epsilon$, so they are said to be 'beyond all orders' of $1/\log\epsilon$, or to be 'transcendentally small terms'. Skinner [8] showed how to calculate a few of these terms. He pointed out, however, that they are negligible compared to the terms of orders $(1/\log\epsilon)^4$ and higher, which were unknown. Only the first three terms in powers of $1/\log\epsilon$ had been found, the last one by Kaplun [2].

We have developed a method to evaluate the sum of all the terms in powers of $1/\log\epsilon$ in certain singular perturbation problems (Ward, Henshaw and Keller [12]), and we have applied it to low Reynolds number flow (Kropinski, Ward and Keller [3]). In §3 we shall show how to extend our method to determine terms beyond all orders of $1/\log\epsilon$. We shall also show how our results determine the asymmetry of the flow and the drag coefficient of the cylinder.

Before treating the flow problem, in §2 we shall illustrate our method by applying it to a model problem proposed by Lagerstrom (see Lagerstrom [4] and the references therein). It is analogous to the flow problem, but is much simpler.

2. Lagerstrom's Problem

We now analyze the following version of Lagerstrom's problem as $\epsilon \to 0$:

$$u'' + R^{-1}u' + uu' = 0\,, \qquad R \geq \epsilon\,, \tag{2.1a}$$

$$\epsilon u' - \kappa u = 0\,, \qquad R = \epsilon\,, \tag{2.1b}$$

$$u \sim 1\,, \qquad R \to \infty\,. \tag{2.1c}$$

Here κ is a positive constant. To find $u(R,\epsilon)$ we first seek an inner expansion by defining r and $w(r,\epsilon)$ as follows:

$$r = \epsilon^{-1}R\,, \qquad w(r,\epsilon) = u(\epsilon r,\epsilon)\,. \tag{2.2}$$

Then $(2.1a,b)$ become the following equations for w:

$$\begin{aligned} w'' + r^{-1}w' &= -\epsilon w w'\,, \qquad r \geq 1\,, \\ w' - \kappa w &= 0\,, \qquad r = 1\,. \end{aligned} \tag{2.3}$$

Following our previous analysis (see [3]), we introduce the notation

$$z = \epsilon d\,, \qquad d = e^{-1/\kappa}\,, \qquad \nu(z) = -1/\log z\,. \tag{2.4}$$

Now we assume that $w(r,\epsilon)$ has an asymptotic expansion in powers of ϵ with coefficients W_i which are functions of r and $\nu(z)$:

$$w(r,\epsilon) \sim W_1[r,\nu(z)] + \epsilon W_2[r,\nu(z)] + \cdots . \tag{2.5}$$

Substituting (2.5) into (2.3) and equating coefficients of ϵ^0 and ϵ^1 we obtain

$$W_1'' + r^{-1} W_1' = 0\,, \quad r \ge 1\,; \qquad W_1' - \kappa W_1 = 0\,, \quad r = 1\,, \tag{2.6}$$

$$W_2'' + r^{-1} W_2' = -W_1 W_1'\,, \quad r \ge 1\,; \qquad W_2' - \kappa W_2 = 0\,, \quad r = 1\,. \tag{2.7}$$

The solution of (2.6) for W_1 can be written in the form

$$W_1[r,\nu(z)] = A[\nu(z)] \log\left(r/d\right)\,. \tag{2.8}$$

Here $A[\nu(z)]$ is undetermined so far, and $d = e^{-1/\kappa}$ was defined in (2.4). Now we use (2.8) to evaluate the term $W_1 W_1'$ in (2.7) and solve (2.7) to get

$$W_2[r,\nu(z)] = -\left(A[\nu(z)]\right)^2 \left(r \log\left(r/d\right) - 2r + 2 + \kappa^{-2} - 2\kappa^{-1}\right) + B[\nu(z)] \log\left(r/d\right)\,. \tag{2.9}$$

Again $B[\nu(z)]$ is not yet determined.

We now use (2.2) in (2.5) with $r = \epsilon^{-1} R$ to get

$$u(R,\epsilon) = w(\epsilon^{-1}R,\epsilon) \sim W_1[\epsilon^{-1}R,\nu(z)] + \epsilon W_2[\epsilon^{-1}R,\nu(z)] + \cdots . \tag{2.10}$$

To obtain the form of $u(R,\epsilon)$ for $\epsilon^{-1}R$ large, we rewrite W_1 and W_2, given by (2.8) and (2.9), and use the results in (2.10):

$$\begin{aligned} u(R,\epsilon) \sim\ & A\left(\log R + \nu^{-1}\right) - A^2 R\left(\log R - 2 + \nu^{-1}\right) \\ & + \epsilon\left[B\left(\log R + \nu^{-1}\right) - A^2\left(2 + \kappa^{-2} - 2\kappa^{-1}\right)\right] + \cdots . \end{aligned} \tag{2.11}$$

Here $A = A[\nu(z)]$, $B = B[\nu(z)]$ and $\nu = \nu(z)$. Equation (2.11) determines the behavior of $u(R,\epsilon)$ as $R \to 0$.

In the outer region where $R = O(1)$, we seek an asymptotic expansion for $u(R,\epsilon)$ in the form

$$u(R,\epsilon) \sim U_1[R,\nu(z)] + \epsilon U_2[R,\nu(z)] + \cdots . \tag{2.12}$$

Substituting (2.12) into (2.1a,c) and into the left side of (2.11), and equating coefficients of ϵ^0 and ϵ^1, we obtain the following problems for $U_1[R,\nu(z)]$ and $U_2[R,\nu(z)]$:

$$U_1'' + R^{-1} U_1' + U_1 U_1' = 0\,, \qquad R > 0\,, \tag{2.13a}$$

$$U_1 \sim A\left(\log R + \nu^{-1}\right) - A^2 R\left(\log R - 2 + \nu^{-1}\right), \qquad R \to 0, \tag{2.13b}$$

$$U_1 \sim 1, \qquad R \to \infty, \tag{2.13c}$$

$$U_2'' + R^{-1} U_2' + \left(U_1 U_2\right)' = 0, \qquad R > 0, \tag{2.14a}$$

$$U_2 \sim B\left(\log R + \nu^{-1}\right) - A^2\left(2 + \kappa^{-2} - 2\kappa^{-1}\right), \qquad R \to 0, \tag{2.14b}$$

$$U_2 \to 0, \qquad R \to \infty. \tag{2.14c}$$

We now show how to determine $A = A[\nu(z)]$ and $B = B[\nu(z)]$ in (2.13) and (2.14) from the numerical solution to the following hybrid problem introduced in [3]:

$$U_H'' + R^{-1} U_H' + U_H U_H' = 0, \qquad R > 0, \tag{2.15a}$$

$$U_H = S \log R + C + o(1), \qquad R \to 0, \tag{2.15b}$$

$$U_H \sim 1, \qquad R \to \infty. \tag{2.15c}$$

Here S is a positive parameter. The solution, $U_H = U_H(R, S)$ and $C = C(S)$ to (2.15) when S varies over a range of values near $S = 0$, can be computed using the boundary value solver COLSYS (see Ascher et al. [1]).

Comparing (2.15) with (2.13), we see that $A = S$ and that $\nu = S/C(S)$. This gives a parametric representation for $A = A(\nu)$. It is then easy to show that the solution U_1 to (2.13a) with $U_1 = A \log R + A\nu^{-1} + o(1)$ as $R \to 0$, is such that as $R \to 0$ the difference $U_1 - A\log R - A\nu^{-1}$ agrees with the term in (2.13b) proportional to A^2. To determine B we first observe that $\partial_S u_H$ solves (2.14a, c) and has a logarithmic singularity of the form $\log R + C'(S)$ at the origin. Thus, $U_2 = B\partial_S u_H$ solves (2.14a, c) and has the asymptotic form $U_2 \sim B \log R + BC'(S)$ as $R \to 0$. Therefore $BC'(S)$ must agree with the $O(1)$ term in (2.14b). Thus, we have

$$B = -S^2\left(2 + \kappa^{-2} - 2\kappa^{-1}\right)\left(C'(S) - \nu^{-1}\right)^{-1}, \qquad A = S. \tag{2.16}$$

Since $S = A[\nu(z)]$, this gives $B = B[\nu(z)]$.

With A and B known, W_1 and W_2 in (2.8) and (2.9) are determined uniquely. Then, from (2.10), a two-term inner expansion of u is

$$w(r, \epsilon) \sim A \log(r/d) + \epsilon\left[-A^2\left(r\log(r/d) - 2r + 2 + \kappa^{-2} - 2\kappa^{-1}\right) + B\log(r/d)\right] + \cdots. \tag{2.17}$$

From (2.12) and (2.15) a two-term outer expansion is

$$u(R, \epsilon) \sim A u_H(R, S) + \epsilon \partial_S u_H(R, S) + \cdots, \tag{2.18}$$

where $S = S(\nu)$ is obtained from $\nu = S/C(S)$. Further terms in the inner and outer expansions in powers of ϵ can be obtained in a similar way.

The 'flux' Q, defined below, is given in terms of A and B by

$$Q = \epsilon \frac{du}{dR}\Big|_{R=\epsilon} = A[\nu(\epsilon d)] + \epsilon \left[\left(A[\nu(\epsilon d)]\right)^2 \left(1 - \kappa^{-1}\right) + B[\nu(\epsilon d)] \right] + O(\epsilon^2) \, . \tag{2.19}$$

Since A depends only on the product ϵd, all the logarithmic terms in (2.19) for Q display an analogue of Kaplun's equivalence principle. However, this principle does not hold for the transcendentally small $O(\epsilon)$ term in (2.19), which involves ϵ itself.

To compute the numerical solution to (2.15) we first calculate

$$u_H = S \log R + C - S^2 R \log R + R \left(2S^2 - SC\right) + \cdots \, , \qquad R \to 0 \, . \tag{2.20}$$

Next, we decompose u_H as

$$u_H = 1 - SE_1(R) + u^* \, , \qquad E_1(y) = \int_y^\infty \eta^{-1} e^{-\eta} \, d\eta \, . \tag{2.21}$$

Substituting (2.21) into (2.15) and using (2.20) we find that u^* satisfies

$$u^{*''} + \left(R^{-1} + 1\right) u^{*'} = \left(SE_1(R) - u^*\right) \left(SR^{-1}e^{-R} + u^{*'}\right) \, , \qquad R > 0 \, , \tag{2.22a}$$

$$u^* = C - 1 - S\gamma - S^2 R \log R + R \left(S + 2S^2 - SC\right) + \cdots \, , \qquad R \to 0 \, , \tag{2.22b}$$

$$u^{*'} = -S^2 \log R + S + S^2 - SC + \cdots \, , \qquad R \to 0 \, , \tag{2.22c}$$

$$u^* \to 0 \, , \qquad R \to \infty \, . \tag{2.22d}$$

Here γ is Euler's constant. We truncate the infinite domain to a finite domain by imposing the artificial boundary condition $u^{*'} + \left(R^{-1} + 1\right) u^* = 0$ at some $R = R_\infty \gg 1$. The singularity conditions $(2.22b, c)$ are imposed at some $R = \mu \ll 1$. The unknowns in (2.22) are $u^* = u^*(R, S)$ and $C = C(S)$. We then use COLSYS to compute the solution to (2.22) as S is varied, and u_H is obtained from (2.21).

In Fig. 1 we plot the curves $C(S)$ and $C'(S)$ computed from (2.22). In Fig. 2 we plot curves of A and B versus $-1/\log(\epsilon d)$ obtained by the procedure described following (2.15) for $\kappa = 1$ and $d = e^{-1}$. In Fig. 3 we compare the full numerical result for Q computed from (2.1) using COLSYS for $\kappa = 1$ with the asymptotic result (2.19) for Q. The asymptotic result is shown with and without the $O(\epsilon)$ term in (2.19).

3. Slow Viscous Flow Past a Circular Cylinder

Let $\psi(r, \theta, \epsilon)$ be the dimensionless stream function for the steady flow of an incompressible viscous fluid around a circular cylinder. Here ϵ is the Reynolds number based upon the cylinder radius and r is measured in units of this radius. Then ψ satisfies the equations

$$\triangle^2 \psi + \epsilon J\left(\psi, \triangle\psi\right) = 0\,, \qquad r > 1\,, \tag{3.1a}$$

$$\psi(1, \theta, \epsilon) = 0\,, \qquad \psi_r(1, \theta, \epsilon) = 0\,, \tag{3.1b}$$

$$\psi(r, \theta, \epsilon) \sim r \sin\theta\,, \qquad \text{as} \quad r \to \infty\,. \tag{3.1c}$$

Here $J(u, v) \equiv r^{-1}\left(u_r v_\theta - u_\theta v_r\right)$ is the Jacobian.

Following Skinner [8] we seek an inner expansion for ψ of the form

$$\psi(r, \theta, \epsilon) \sim \psi_1(r, \theta, \delta) + \epsilon\psi_2(r, \theta, \delta) + \epsilon^2\psi_3(r, \theta, \delta) + \cdots\,. \tag{3.2}$$

Here $\delta \equiv (\log 4 - \gamma - \log\epsilon + 1/2)^{-1} = [\log(3.7026/\epsilon)]^{-1}$, where γ is Euler's constant. By substituting (3.2) into (3.1a, b), and equating coefficients of powers of ϵ, we find that each ψ_j satisfies (3.1b). In addition,

$$\triangle^2\psi_1 = 0\,, \qquad r > 1\,, \tag{3.3}$$

$$\triangle^2\psi_2 = -J(\psi_1, \triangle\psi_1)\,, \qquad r > 1\,, \tag{3.4}$$

$$\triangle^2\psi_3 = -J(\psi_1, \triangle\psi_2) - J(\psi_2, \triangle\psi_1)\,, \qquad r > 1\,. \tag{3.5}$$

In view of (3.1c) we seek a solution ψ_1 of the form

$$\psi_1(r, \theta, \delta) = Z_1(r, \delta)\sin\theta\,. \tag{3.6}$$

Substitution of (3.6) into (3.3) yields a fourth order homogeneous linear ordinary differential equation for Z_1. Two constants of integration are determined by (3.1b). Another is determined by setting the coefficient of r^3 in the solution equal to zero to permit matching with the outer solution. The fourth constant is a multiplicative factor $a(\delta)$. Thus, Z_1 is found to be

$$Z_1(r, \delta) = a(\delta)\left(r\log r - \frac{r}{2} + \frac{r^{-1}}{2}\right)\,. \tag{3.7}$$

When (3.6) is used for ψ_1 in (3.4), the inhomogeneous term J is found to be proportional to $\sin 2\theta$. Therefore, we seek ψ_2 in the form

$$\psi_2(r, \theta, \delta) = Z_2(r, \delta)\sin 2\theta\,. \tag{3.8}$$

A fourth order equation for Z_2 results. Again the highest power of r in the solution must have a coefficient equal to zero to permit matching, while (3.1b) determines two other constants. Thus, one constant $b(\delta)$ remains undetermined, and the solution is

$$Z_2(r,\delta) = \frac{a^2(\delta)}{32}\left\{r^2\left(2\log^2 r - \log r + \frac{1}{4}\right) - \frac{r^{-2}}{4}\right\} + \frac{b(\delta)}{8}\left(r^2 - 2 + r^{-2}\right). \tag{3.9}$$

By continuing in the same way, Skinner [8] determined Ψ_3, which contains three undetermined constants $c(\delta)$, $d(\delta)$ and $e(\delta)$.

Upon using the preceding results in (3.2) we obtain the inner expansion of ψ:

$$\begin{aligned}\psi(r,\theta,\epsilon) &= Z_1(r,\delta)\sin\theta + \epsilon Z_2(r,\delta)\sin 2\theta + O(\epsilon^2)\\ &= a(\delta)\left(r\log r - \frac{r}{2} + \frac{r^{-1}}{2}\right)\sin\theta\\ &+ \epsilon\left[\frac{a^2(\delta)}{32}\left\{r^2\left(2\log^2 r - \log r + \frac{1}{4}\right) - \frac{r^{-2}}{4}\right\} + \frac{b(\delta)}{8}\left(r^2 - 2 - r^{-2}\right)\right]\sin 2\theta + O(\epsilon^2).\end{aligned} \tag{3.10}$$

To determine the two constants $a(\delta)$ and $b(\delta)$ in (3.10), we first introduce the outer variable $R = \epsilon r$, and rewrite (3.10) in terms of it to get

$$\begin{aligned}\epsilon\psi(\epsilon^{-1}R,\theta,\epsilon) &\sim \left[aR\log R - a\left(\log\epsilon + \frac{1}{2}\right)R + O(\epsilon^2)\right]\sin\theta\\ &+ \left[\frac{a^2}{16}(R\log R)^2 - \frac{a^2}{8}(\log\epsilon + \frac{1}{4})R^2\log R + \left\{\frac{a^2}{32}(2\log^2\epsilon + \log\epsilon + \frac{1}{4}) + \frac{b}{8}\right\}R^2 + O(\epsilon^2)\right]\sin 2\theta.\end{aligned} \tag{3.11}$$

We shall use (3.11) to match the inner expansion to an outer expansion, and by doing so we shall find a and b.

To obtain the outer expansion we set $\Psi(R,\theta,\epsilon) = \epsilon\psi(\epsilon^{-1}R,\theta,\epsilon)$ and we expand

$$\Psi(R,\theta,\epsilon) = \Psi_0(R,\theta) + \epsilon^2\Psi_1(R,\theta,\delta) + \cdots. \tag{3.12}$$

Then (3.1a,c) become the following equations for Ψ_0:

$$\triangle^2\Psi_0 + J(\Psi_0, \triangle\Psi_0) = 0, \tag{3.13a}$$

$$\Psi_0 \sim R\sin\theta, \qquad \text{as} \quad R\to\infty. \tag{3.13b}$$

Instead of imposing the boundary conditions (3.1b) upon Ψ_0, we require that Ψ_0 match with (3.11) as $R\to 0$. This requirement shows that Ψ_0 must have the singularity

$$\Psi_0 \sim aR\log R\,\sin\theta, \qquad \text{as} \quad R\to 0. \tag{3.14}$$

The conditions $(3.13a, b)$ and (3.14) determine a unique solution Ψ_0 for each choice of a, which we computed numerically in Kropinski, Ward and Keller [3]. To match that solution with (3.11) we compute its two Fourier sine coefficients

$$\Psi_{0j}(R) = \frac{2}{\pi} \int_0^\pi \Psi_0(R,\theta) \sin j\theta \, d\theta \,, \qquad j = 1, 2 \,. \tag{3.15}$$

From (3.11) we see that these coefficients must have the following forms as $R \to 0$:

$$\Psi_{01}(R) \sim aR \log R + C_{00} R \,, \qquad R \to 0 \,, \tag{3.16}$$

$$\Psi_{02}(R) \sim C_{01} \left(R \log R\right)^2 + C_{02} R^2 \log R + C_{03} R^2 \,, \qquad R \to 0 \,. \tag{3.17}$$

For each choice of a, Ψ_0 can be computed, the Ψ_{0j} evaluated, and then the C_{0j} can be found. The $C_{0j}(a)$ are functions of a, since the solution $\Psi_0(R,\theta)$ is determined by a.

Now matching requires that $\Psi_{01}(R)$ match with the coefficient of $\sin\theta$ in (3.11) and Ψ_{02} match with the coefficient of $\sin 2\theta$. The first match yields the equation

$$C_{00}(a) = -a \left(\log \epsilon + \frac{1}{2} \right) \,. \tag{3.18}$$

This equation determines a as a function of $\log \epsilon$, and therefore as a function of $\delta = [\log{(3.7026/\epsilon)}]^{-1}$. The conditions $C_{01}(a) = a^2/16$ and $C_{02}(a) = -a^2(\log \epsilon + 1/4)/8$ must be identities. The fourth matching condition gives an equation for b which has the solution

$$b(\delta) = 8C_{03}(a) - \frac{a^2}{4} \left(2\log^2 \epsilon + \log \epsilon + \frac{1}{4} \right) \,, \tag{3.19}$$

where $\log \epsilon = \log(3.7026) - \delta^{-1}$. Previously in [3], we obtained a by using a somewhat cruder form of (3.18). The numerical method of [3], which we extend to calculate $b(\delta)$, is summarized in Appendix A. In Fig. 4 we plot the curve $C_{00} = C_{00}(a)$ obtained by this method.

In this way we have completed the determination of the term of order ϵ in the inner expansion (3.10). This term is 'beyond all orders' of δ. All orders of δ are contained in the first term $a(\delta)$, and (3.19) determines $b(\delta)$ to all orders. Kaplun [2] obtained the first two non-zero terms in the expansion for $a(\delta)$ and Skinner [8] obtained the first two non-zero terms in $b(\delta)$. For $\delta \to 0$, they found that

$$a(\delta) \sim \delta - 0.8669\,\delta^3 + \cdots \,, \qquad b(\delta) \sim -1/2 + \delta/4 + \cdots \,. \tag{3.20}$$

In Fig. 5 we compare the two-term expansions (3.20) with the corresponding results for $a[\delta(\epsilon)]$ and $b[\delta(\epsilon)]$ computed from the hybrid method. This plot shows that (3.20) becomes inaccurate for a when $\epsilon \approx 0.5$ and for b when $\epsilon \approx 0.2$.

With a and b known, Z_1 and Z_2 in (3.7) and (3.9) are determined uniquely. From (3.10), a two-term inner expansion for ψ is

$$\psi(r,\theta,\epsilon) = [Z_1(r,\delta) + 2\epsilon Z_2(r,\delta)\cos\theta]\sin\theta + O(\epsilon^2)\,. \tag{3.21}$$

The streamline $\psi(r,\theta,\epsilon)=0$ consists of the circle $r=1$, the axis $\theta=0$ and $\theta=\pi$, and the outer boundary of the eddy, when there is an eddy. The equation of this boundary, obtained by using (3.21) for ψ without the $O(\epsilon^2)$ term is

$$\sec\theta = -2\epsilon Z_2(r,\delta)/Z_1(r,\delta)\,. \tag{3.22}$$

As ϵ increases, the eddy will first appear at the smallest value of ϵ for which the right side of (3.22) attains the value one. Since this will occur at $r=1$, where the right side is not defined, we define the limit of the right side as $r\to 1$ to be $\mu(\epsilon)$. Substituting (3.7) and (3.9) into (3.22) and using L'Hopital's rule we obtain

$$\mu(\epsilon) = -\epsilon b[\delta(\epsilon)]/a[\delta(\epsilon)]\,. \tag{3.23}$$

The parameter $\mu(\epsilon)$ gives a measure of the upstream-downstream asymmetry of the flow field near the cylinder.

From the leading term of a and the two-term expansion for b, Skinner [8] derived

$$\mu(\epsilon) = \left(\epsilon\delta^{-1}/2\right)\left[1-\delta/2+O(\delta^2)\right]\,. \tag{3.24}$$

The unknown $O(\delta^2)$ term in (3.24) involves the $O(\delta^3)$ term in $a(\delta)$ and the unknown $O(\delta^2)$ term in $b(\delta)$. In Fig. 6 we compare the curve $\mu(\epsilon)$ versus ϵ obtained from (3.23) with the corresponding two-term result (3.24). This plot shows that (3.24) is accurate only for ϵ very small and that $\mu(\epsilon)$ from (3.23) is a monotone increasing function of ϵ. It follows from (3.22) that flow separation is initiated at the minimum value of ϵ for which $\mu(\epsilon)=1$ (see [11] p. 236). Using (3.23), we find that this critical value of ϵ is roughly $\epsilon\approx 1.2$ (see Fig. 6).

The drag coefficient C_D is given by (Skinner [8])

$$C_D = \frac{4\pi}{\epsilon}\left[a(\delta) - \frac{\epsilon^2}{4}e(\delta) + O(\epsilon^4)\right]\,. \tag{3.25}$$

Here $e(\delta)$, which occurs in ψ_3, is given by $e(\delta)=\frac{1}{8}\left[1-\delta/2+O(\delta^2)\right]$. By using this value in (3.25) we get

$$C_D = \frac{4\pi}{\epsilon}\left[a(\delta) - \frac{\epsilon^2}{32}\left(1-\frac{\delta}{2}\right) + O(\epsilon^2\delta^2)\right]\,. \tag{3.26}$$

In Fig. 7 we compare the hybrid result (3.26) for C_D, neglecting the $O(\epsilon^2)$ term, with experimental results of Tritton [10], with full numerical results computed in [3] and with the two-term expansion of Kaplun $C_D \sim 4\pi\epsilon^{-1}\left(\delta - 0.8669\delta^3\right)$ (see [2]). In this plot, the hybrid result for C_D is virtually indistinguishable from the full numerical result even when ϵ ranges over values near unity. Finally, in Fig. 8 we compare the hybrid result (3.26) with and without the $O(\epsilon^2)$ correction term. It is clear from Fig. 7 and Fig. 8 that the transcendentally small term of $O(\epsilon^2)$ does not do much to either improve or worsen the prediction of the drag coefficient when ϵ is near unity. However, these figures do show that the crucial factor for accurately predicting the drag coefficient is to obtain all the terms in $a(\delta)$, as the hybrid method does, rather than only a few terms in $a(\delta)$. The result (3.26) for C_D, neglecting the $O(\epsilon^2)$ term, also holds for a cylinder of arbitrary shape when the constant d, introduced in [3], is used to adjust ϵ.

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Appendix A: The Numerical Method

We now summarize the numerical method of [3] used to solve the hybrid problem (3.13) with the singularity condition (3.14). We also describe our improved method for computing C_{00} in (3.16) and our extension of the method of [3] to calculate C_{03} in (3.17). In terms of C_{00} and C_{03}, ϵ and b are calculated from (3.18) and (3.19), respectively.

The solution to (3.13) is decomposed as

$$\Psi_0(R,\theta) = R\sin\theta + a\Psi_{os}(R,\theta) + \Psi_0^*(R,\theta)\,. \tag{A.1}$$

Here $\Psi_{os}(R,\theta)$ is the linearized Oseen solution given by

$$\begin{aligned} \Psi_{os}(R,\theta) &= -\sum_{n=1}^{\infty} \frac{R}{n}\phi_n(R/2)\sin n\theta\,, \\ \phi_n(\xi) &= K_0(\xi)I_{n-1}(\xi) + K_0(\xi)I_{n+1}(\xi) + 2K_1(\xi)I_n(\xi)\,, \end{aligned} \tag{A.2}$$

where $I_n(\xi)$ and $K_n(\xi)$ are the modified Bessel functions.

Substituting $(A.1)$ into (3.13) and (3.14), and exploiting the symmetry of the flow field, we obtain that Ψ_0^* is regular as $R \to 0$ and satisfies

$$L_{0s}\Psi_0^* = -J\left[a\Psi_{os} + \Psi_0^*, a\triangle\Psi_{os} + \triangle\Psi_0^*\right]\,, \qquad R > 0\,, \quad 0 < \theta < \pi\,, \tag{A.3a}$$

$$\partial_R\Psi_0^* \to 0\,, \quad \text{as} \quad R \to \infty\,; \qquad \Psi_0^*(R,\theta) = -\Psi_0^*(R,-\theta)\,, \tag{A.3b}$$

$$\Psi_0^*(R,\theta) = \Psi_{01}^*(R)\sin\theta + \Psi_{02}^*(R)\sin 2\theta + \cdots\,, \qquad R \to 0\,. \tag{A.3c}$$

Here L_{os} is the linearized Oseen operator defined by

$$L_{os}\Psi_{os} \equiv \triangle^2\Psi_{os} + \left(R^{-1}\sin\theta\,\partial_\theta - \cos\theta\,\partial_R\right)\,\triangle\Psi_{os}\,. \tag{A.4}$$

In $(A.3c)$, $\Psi^*_{01}(R)$ and $\Psi^*_{02}(R)$ are the Fourier sine coefficients of $\Psi^*_0(R,\theta)$, defined for $R \to 0$ by

$$\Psi^*_{0j}(R) = \frac{2}{\pi}\int_0^\pi \Psi^*_0(R,\theta)\,\sin j\theta\,d\theta\,, \qquad j = 1, 2\,. \tag{A.5}$$

To solve $(A.3)$, we introduce a logarithmic radial stretching function $\tau = \log(1+R)$ in $(A.3)$ and we solve the resulting transformed problem by a finite difference scheme with an appropriate artificial boundary condition imposed at some large value of R (see [3] for details). To avoid evaluating the singular derivatives of Ψ_{os} at $R = 0$ in the Jacobian given in $(A.3a)$, the first gridline in R in the finite difference scheme is offset slightly from $R = 0$ (see [3]). At this first gridline we evaluate $\Psi^*_{0j}(R)$ in $(A.5)$ by using the trapezoidal rule. From (3.15), $(A.1)$ and $(A.2)$, we obtain for $R \to 0$ that

$$\Psi_{01}(R) = -aR\phi_1(R/2) + R + \Psi^*_{01}(R)\,, \tag{A.6}$$

$$\Psi_{02}(R) = -\frac{aR}{2}\phi_2(R/2) + \Psi^*_{02}(R)\,. \tag{A.7}$$

We then substitute the right sides of (3.16) and (3.17) into the left sides of $(A.6)$ and $(A.7)$ and we use the local behavior of $\phi_1(R/2)$ and $\phi_2(R/2)$ as $R \to 0$ to evaluate the first terms on the right sides of $(A.6)$ and $(A.7)$, respectively. Then, by evaluating $(A.6)$ and $(A.7)$ on the first gridline we obtain a numerical values for the unknowns C_{00} and C_{03}, respectively. Finally, with C_{00} and C_{03} known, we calculate ϵ and b from (3.18) and (3.19), respectively.

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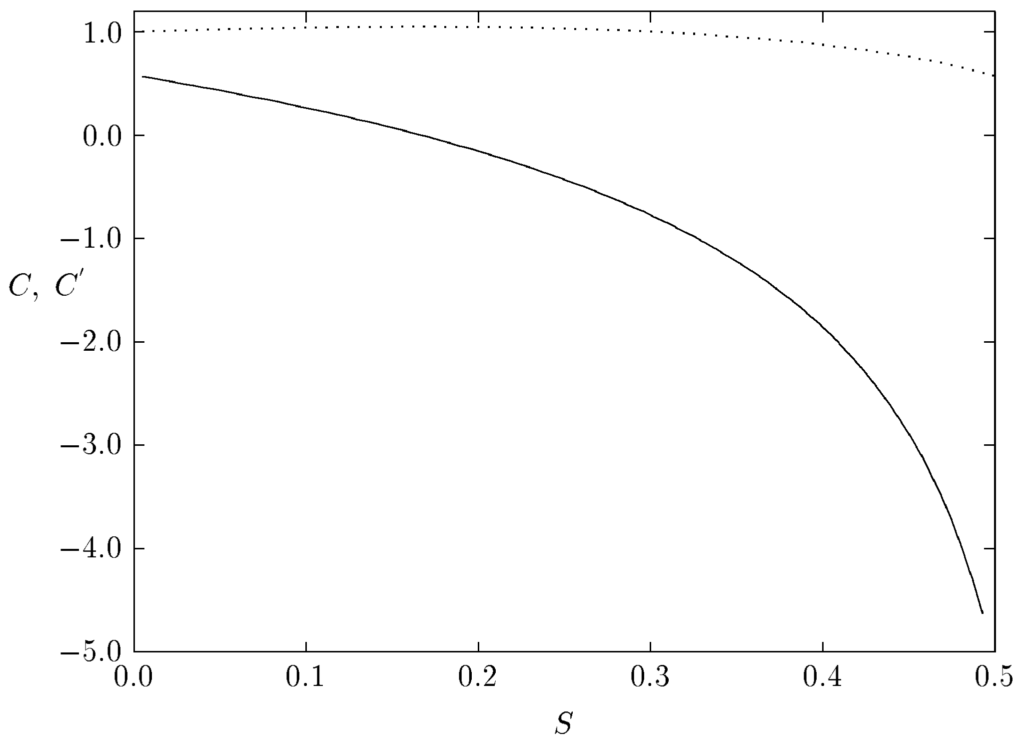

Figure 1: Plots of C and C' versus S obtained from the numerical solution to the hybrid problem (2.22). The dotted line is $C(S)$ and the solid line is $C'(S)$.

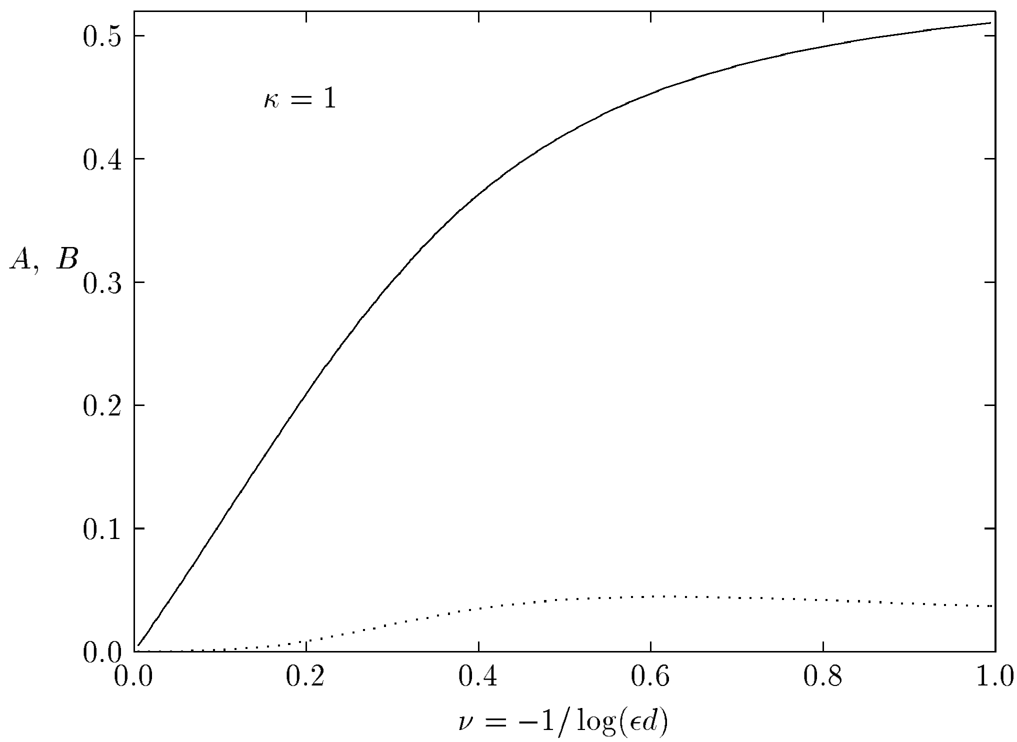

Figure 2: Plots of A and B versus $\nu = -1/\log(\epsilon d)$ obtained from (2.16) when $\kappa = 1$ and $d = e^{-1}$. The solid line is A and the dotted line is B.

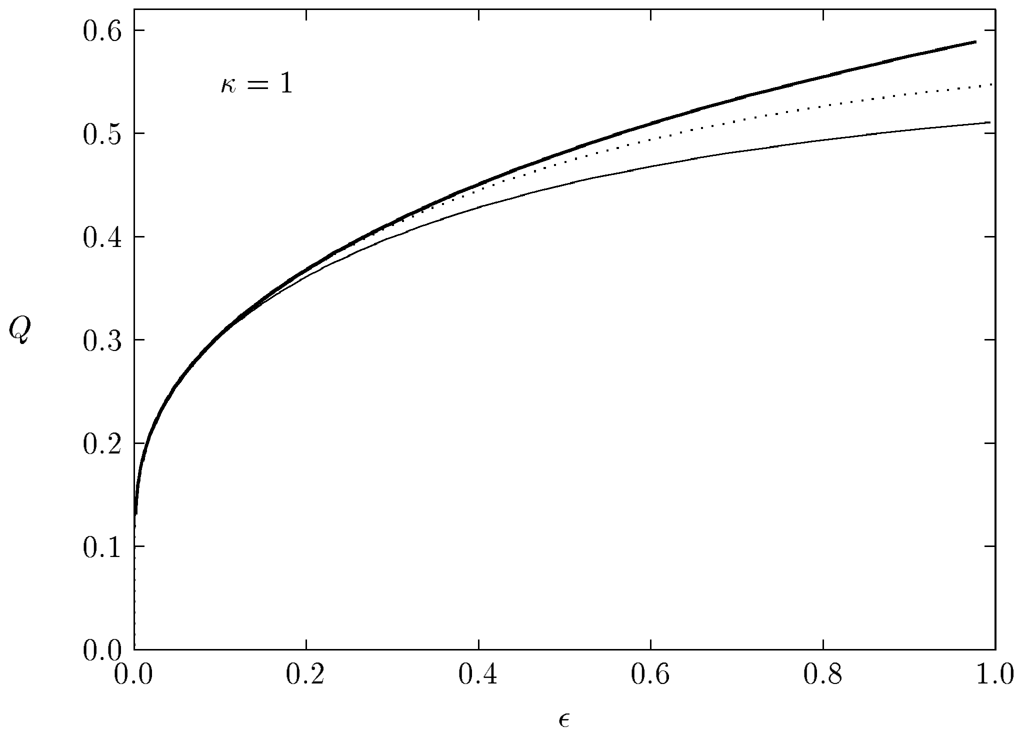

Figure 3: Plots of Q versus ϵ when $\kappa = 1$. The top curve (heavy solid line) is the full numerical result for Q. The bottom curve (solid line) is given by (2.19) without the $O(\epsilon)$ term. The middle curve (dotted line) is (2.19) with the $O(\epsilon)$ term.

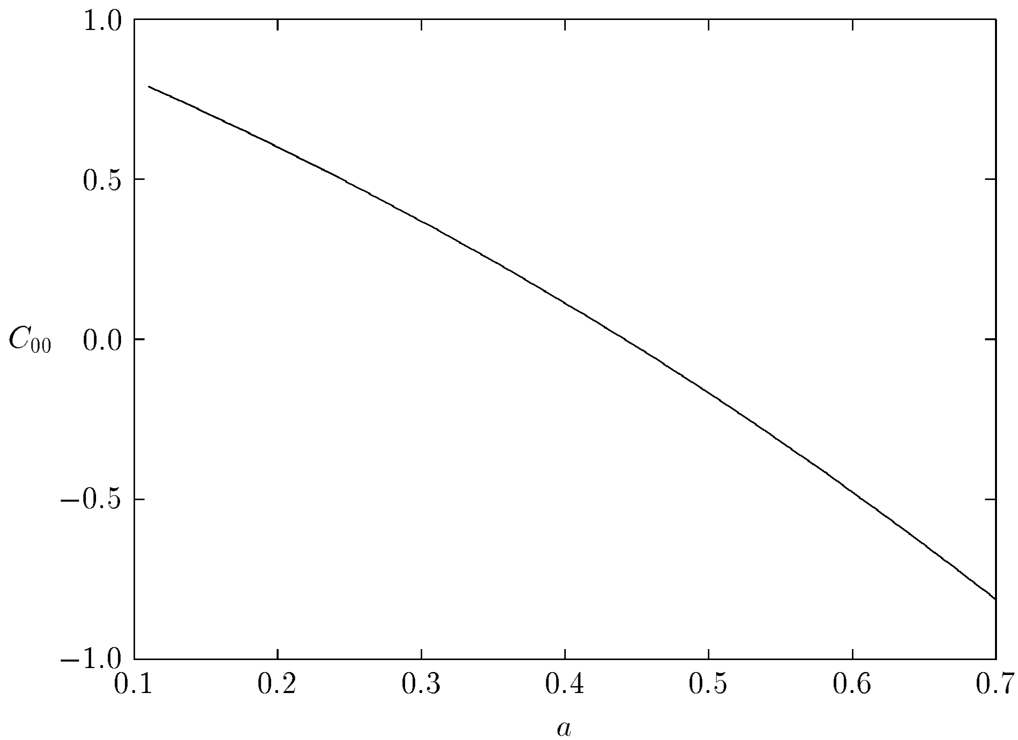

Figure 4: Plot of C_{00} versus a obtained from the numerical solution to the hybrid problem (3.13).

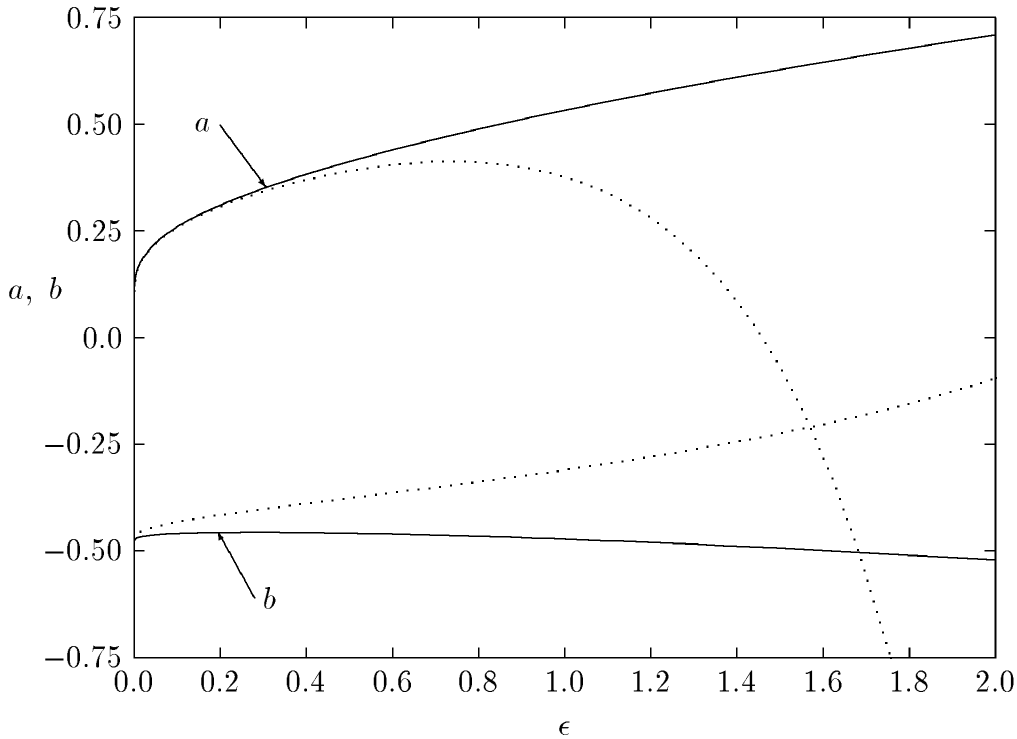

Figure 5: Plots of $a[\delta(\epsilon)]$ and $b[\delta(\epsilon)]$ versus ϵ obtained from the numerical solution to the hybrid problem (3.13) (solid lines). The two-term approximations (3.20) for a and b are also shown (dotted lines).

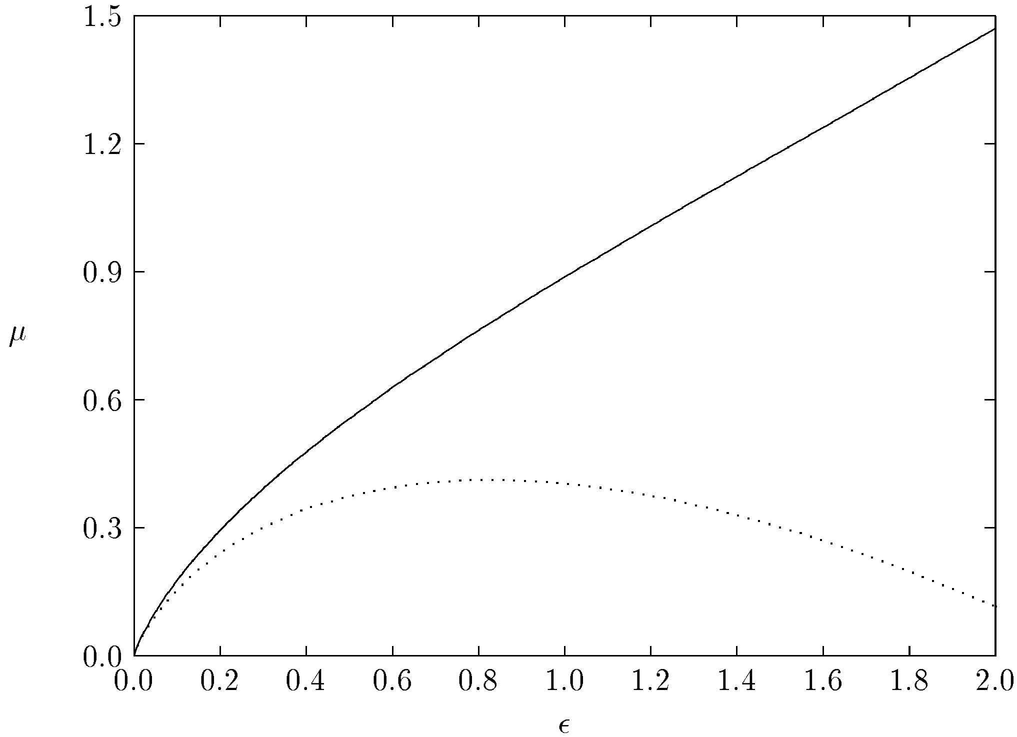

Figure 6: Plot of the asymmetry parameter $\mu(\epsilon)$ versus ϵ given by (3.23). The two-term approximation (3.24) for $\mu(\epsilon)$ is also shown (dotted line).

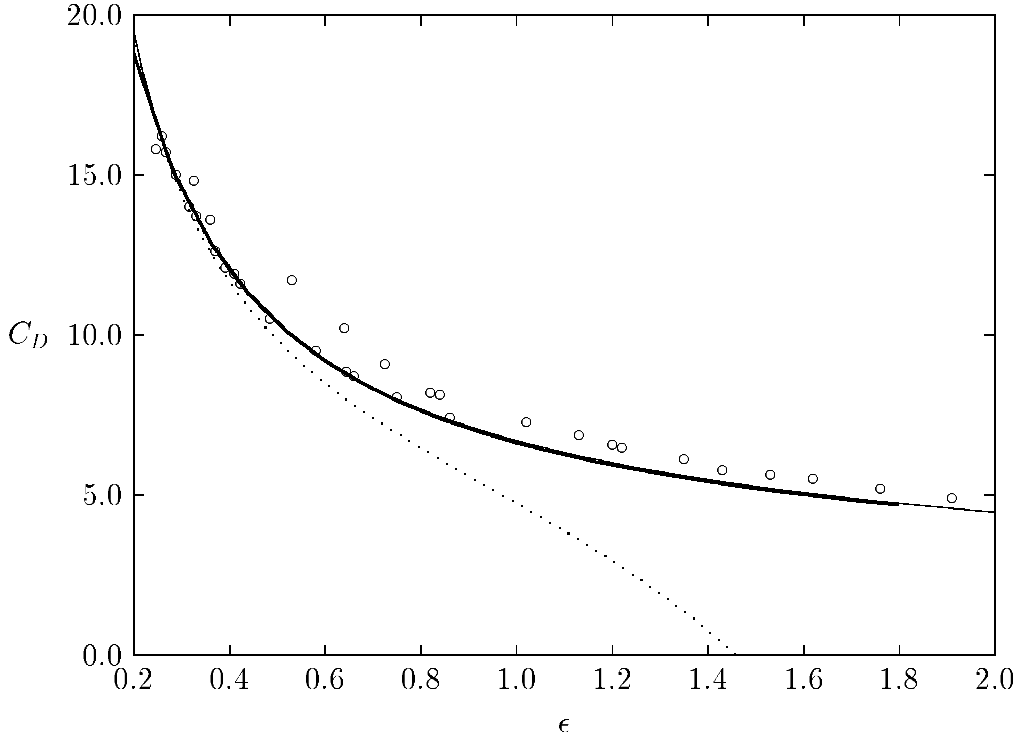

Figure 7: The hybrid result (3.26) for C_D without the $O(\epsilon^2)$ term (solid line) is compared with experimental results (discrete points), with full numerical results (heavy solid line) and with the two-term result $C_D \sim 4\pi\epsilon^{-1}\,(\delta - 0.8669\,\delta^3)$ (dotted line). Notice that the hybrid results and the full numerical results for C_D are virtually indistinguishable.

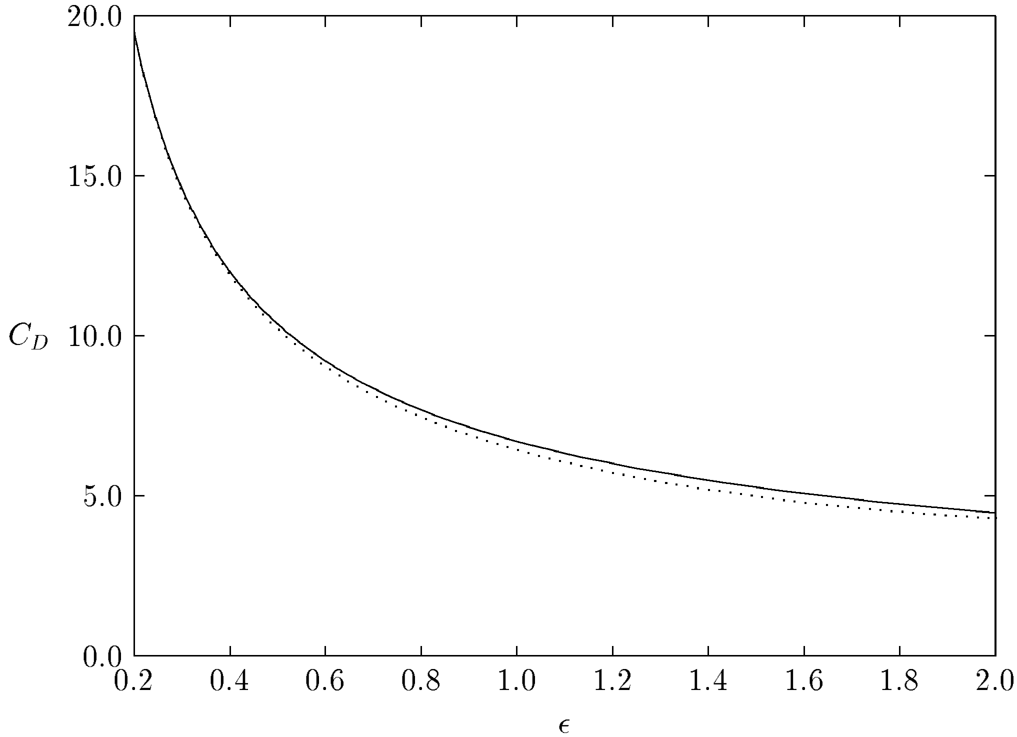

Figure 8: The hybrid result (3.26) for C_D without the $O(\epsilon^2)$ term (solid line) is compared with the hybrid result (3.26) with the $O(\epsilon^2)$ term (dotted line).