Exponential Asymptotics and Convection-Diffusion-Reaction Models

Michael J. Ward

Introduction

The method of matched asymptotic expansions is a well-known and powerful method for systematically calculating asymptotic approximations to solutions of singularly perturbed problems. This method has been used successfully in a wide range of applications (cf. [**17**], [**18**], [**21**], [**29**], [**38**], [**50**]), and its theoretical foundations are rather well-developed.

However, there are certain classes of steady-state singularly perturbed boundary value problems where a straightforward application of this method fails to determine the solution uniquely. In particular, for problems where asymptotically exponentially small terms need to be resolved, a failure to asymptotically resolve such terms typically leads to a matched asymptotic approximation with undetermined constants. Two classical examples where this indeterminacy occurs are for linear turning point problems associated with boundary layer resonance [**1**] and for certain nonlinear autonomous boundary value problems with shock-type or spike-type internal layers [**9**].

From the viewpoint of matched asymptotic expansions, this indeterminacy suggests that exponential precision is needed in the asymptotic estimates in order to calculate a unique approximate solution to the differential equation. However, from the viewpoint of spectral theory, this indeterminacy in the matched asymptotic approximation arises as a direct consequence of an exponential ill-conditioning of a certain linearization of the full perturbed problem. By exponential ill-conditioning we mean that the spectrum of the eigenvalue problem associated with the linearization contains exponentially small eigenvalues. As a result of this exponential ill-conditioning, the solution to the steady-state perturbed problem is typically very sensitive to exponentially small changes in the coefficients of the differential operator. Moreover, for the corresponding time-dependent problem, this exponential ill-conditioning can lead to the occurrence of a phenomenon known as dynamic metastability whereby the time-dependent solution approaches a steady-state solution only over an asymptotically exponentially long time interval.

The goal of this paper is largely to illustrate and survey some results for metastable behavior and exponential ill-conditioning for various classes of linear and nonlinear singularly perturbed partial differential equations in one-spatial dimension. In each case, we show how this behavior can be analyzed asymptotically by using an asymptotic projection method, which supplements the method of

1991 *Mathematics Subject Classification.* Primary 35B25, 35K57; Secondary 35B40, 35Q99.

matched asymptotic expansions with certain spectral information associated with the linearized equation. This projection method exploits the existence of exponentially small eigenvalues by imposing limiting solvability conditions on the solution to the linearized equation.

The outline of this paper is as follows. In §1 we illustrate the basic ideas of the projection method applied in various simple situations. We also illustrate the typical features of metastable dynamics. In §2 we consider a linear convection-diffusion equation with a turning point that has metastable behavior. In §3 we consider some metastable nonlinear convection-diffusion equations with shock-type solutions, including Burgers equation and an equation describing the propagation of a flame-front interface in a vertical channel. In §4 we illustrate metastable behavior for various phase separation models arising from materials science applications. Finally, in §5 we analyze a nonlocal reaction diffusion equation arising from an activator-inhibitor system modeling morphogenesis.

1. The Projection Method and Metastable Behavior

The imposition of solvability conditions is very often employed in perturbation theory. In particular, solvability conditions are central to the calculation of eigenvalues for differential operators in weakly inhomogeneous media, in the suppression of secular terms in multi-scale expansions for oscillatory problems, and in the derivation of modulation equations in water wave theory, optics etc. Solvability conditions are associated with the usual Fredholm alternative condition arising in linear algebra and in the theory of linear two point boundary value problems. The projection method used throughout this paper is a limiting form of a solvability condition.

We first consider a simple problem from linear algebra to illustrate the main idea of the projection method. Let A_ϵ be an $n \times n$ symmetric matrix, depending on a small parameter $\epsilon > 0$, and consider the linear system

$$A_\epsilon \mathbf{x}_\epsilon = \mathbf{b}_\epsilon \, . \tag{1.1}$$

Here $\mathbf{x}_\epsilon \in \mathbb{R}^n$ and $\mathbf{b}_\epsilon \in \mathbb{R}^n$, depend on ϵ. Assume that $\mathbf{b}_\epsilon$ depends on m unknown parameters $q_1, .., q_m$, with $0 < m < n$. Let ϕ_j, λ_j for $j = 1, .., n$ denote the eigenpairs of A_ϵ, which are normalized by $(\phi_j, \phi_j) = 1$. Here the inner product dentoes the usual dot product. Then, we can formally express $\mathbf{x}_\epsilon$ as a spectral expansion

$$\mathbf{x}_\epsilon = \sum_{j=1}^{n} \frac{c_j}{\lambda_j} \phi_j \, , \qquad c_j = (\phi_j, \mathbf{b}_\epsilon) \; . \tag{1.2}$$

Suppose that $\lambda_j = 0$ for $j = 1, .., m$ for all values of ϵ, and that the remaining eigenvalues are non-zero for all values of ϵ. Then, for (1.1) to have a solution it is necessary and sufficient that the solvability conditions $(\phi_j, \mathbf{b}_\epsilon) = 0$ for $j = 1, .., m$ are satisfied. These conditions then yield algebraic equations for the m unknown parameters $q_1, .., q_m$.

Suppose, instead, that A_ϵ has m exponentially small eigenvalues of the order $\lambda_j = O(e^{-r_j/\epsilon})$ as $\epsilon \to 0$ for $j = 1, .., m$. Here $r_j > 0$ is independent of ϵ. Then, for (1.1) to have a solution that is defined in the limit $\epsilon \to 0$ we require that the following limiting solvability conditions be satisfied:

$$(\phi_j, \mathbf{b}_\epsilon) \to 0 \, , \qquad \text{as} \quad \epsilon \to 0 \, , \quad j = 1, .., m \, . \tag{1.3}$$

These conditions then give asymptotic equations valid as $\epsilon \to 0$ for the m unknown parameters $q_1, .., q_m$. Thus, we must eliminate the projection of the residual $\mathbf{b}$ against the eigenspace associated with the exponentially small eigenvalues.

In certain cases, a similar projection approach can be used to construct an asymptotic solution to singularly perturbed differential operators. We now outline the salient features of the method for

$$u_t = N(u) \equiv \epsilon^2 u_{xx} + F(x, u, u_x)\,, \quad -1 < x < 1\,, \quad t > 0\,, \tag{1.4}$$

$$u_x(\pm 1, t) = 0\,; \qquad u(x,0) = u_0(x)\,. \tag{1.5}$$

Here $u = u(x,t)$, F is smooth, and $\epsilon \to 0^+$. Most of the problems below in §2-5 can be cast in this form. The corresponding equilibrium problem is to determine U satisfying

$$N(U) = 0\,, \quad -1 < x < 1\,; \qquad U_x(\pm 1) = 0\,. \tag{1.6}$$

Typically, (1.6) is readily solved using the method of matched asymptotic expansions (cf. [**17**], [**18**], [**21**], [**29**], [**38**], [**50**]). However, in certain special cases this method yields an approximate solution to (1.6) with undetermined coefficients $\mathbf{q} = (q_1, .., q_m)$, where $\mathbf{q}$ lies in some subset $\mathbb{S}$ of $\mathbb{R}^m$. We label this approximate solution by

$$U(x) \sim \tilde{u}^{\epsilon}[x; \mathbf{q}]\,. \tag{1.7}$$

We assume that as $\epsilon \to 0$ the residual satisfies

$$N(\tilde{u}^{\epsilon}) = O\left(e^{-c/\epsilon}\right)\,; \qquad u_x^{\epsilon}[\pm 1; \mathbf{q}] = O\left(e^{-c/\epsilon}\right)\,, \tag{1.8}$$

uniformly for $\mathbf{q} \in \mathbb{S}$ and $x \in (-1, 1)$, where $c > 0$. Thus, we assume that $\tilde{u}^{\epsilon}$ fails to satisfy the equilibrium problem (1.6) by at most exponentially small terms for any $\mathbf{q} \in \mathbb{S}$. We refer to $\tilde{u}^{\epsilon}$ as a quasi-equilibrium solution. Thus, it is clear that exponential precision is needed in the asymptotic analysis to determine the correct value of $\mathbf{q}$ corresponding to a true equilibrium solution.

Let $\mathbf{q} \in \mathbb{S}$, and consider the linearization of (1.6) around $\tilde{u}^{\epsilon}$. We write

$$u(x) = \tilde{u}^{\epsilon}[x; \mathbf{q}] + v(x)\,, \tag{1.9}$$

where $v \ll \tilde{u}^{\epsilon}$, and obtain that v satisfies

$$\epsilon^2 v_{xx} + F^0_{u_x} v_x + F^0_u v = -N(\tilde{u}^{\epsilon})\,, \quad -1 < x < 1 \tag{1.10}$$

$$v_x(\pm 1) = -u_x^{\epsilon}[\pm 1; \mathbf{q}]\,. \tag{1.11}$$

Here F^0_u and $F^0_{u_x}$ denote the partial derivatives of F with respect to u and u_x evaluated at $\tilde{u}^{\epsilon}$, respectively. The corresponding eigenvalue problem is

$$L_{\epsilon}\phi \equiv \epsilon^2 \phi_{xx} + F^0_{u_x} \phi_x + F^0_u \phi = \lambda \phi\,, \quad -1 < x < 1 \tag{1.12}$$

$$\phi_x(\pm 1) = 0\,. \tag{1.13}$$

This eigenvalue problem can be cast in self-adjoint form by introducing a Liouville transformation to eliminate the ϕ_x term. This shows that the eigenvalues λ_j for $j \geq 0$ are real and that $\lambda_j \to -\infty$ as $j \to \infty$.

The associated linearized problem is very poorly conditioned. To see this, define

$$\tilde{\phi}_j \equiv \partial_{q_j} \tilde{u}^{\epsilon}[x; \mathbf{q}]\,, \quad j = 1, .., m\,. \tag{1.14}$$

Here q_j is the j^{th} coordinate of $\mathbf{q}$. We assume that these functions are independent. Then, upon differentiating (1.6) with respect to q_j, it is clear that $L_\epsilon(\tilde{\phi}_j)$ is exponentially small and that the boundary conditions (1.13) fail to be satisfied by only exponentially small terms as $\epsilon \to 0$ for any $\mathbf{q} \in \mathbb{S}$. Thus, zero is nearly an eigenvalue of multiplicty m for (1.12)-(1.13). For the problems considered below in §2-5, this leads to the existence of m exponentially small eigenvalues for (1.12)-(1.13). The corresponding eigenfunctions are given asymptotically by $\phi_j \sim \tilde{\phi}_j$. Typically, however, we must enforce the condition that the true eigenfunctions ϕ_j satisfy the boundary conditions in (1.13) exactly and not just asymptotically. Hence, we have to insert boundary layer correction terms to $\tilde{\phi}_j$ to determine ϕ_j. Having done so, we can use the resulting asymptotically accurate eigenfunctions in a Rayleigh quotient approach to give precise asymptotic estimates of the exponentially small eigenvalues.

Next, we expand the solution to (1.10)-(1.11) in terms of the eigenfunctions of (1.12)-(1.13) as

$$v = \sum_{j=1}^{\infty} \frac{c_j}{\lambda_j} \phi_j \,. \tag{1.15}$$

The eigenfunctions are orthogonal in an inner product space with some weight $w > 0$. This orthogonality condition can be used to derive an explicit formula for the coefficients c_j in the form

$$c_j = -\left(\phi_j, N(\tilde{u}^\epsilon)\right)_w + b_j \tag{1.16}$$

Here the bracketed term indicates a weighted inner product with weight w, and b_j denotes a boundary term arising from the boundary conditions at $x = \pm 1$. Finally, since there are m eigenvalues λ_j that are exponentially small as $\epsilon \to 0$, we must enforce the limiting solvability conditions that the corresponding coefficients $c_j \to 0$ as $\epsilon \to 0$ for $j = 1, .., m$. This projection step, which is analogous to (1.3), yields m coupled algebraic equations for the unknown parameter $\mathbf{q} \in \mathbb{S}$.

Dynamic metastability occurs for the time-dependent problem (1.4)-(1.5) when the exponentially small eigenvalues are the *principal* eigenvalues associated with the linearization. In other words, we require that the eigenvalue problem (1.12)-(1.13) has no $O(1)$ positive eigenvalue. When this condition on the spectrum of the linearized problem holds we can construct a solution to (1.4)-(1.5) of the form

$$u(x,t) \sim \tilde{u}^\epsilon[x; \mathbf{q}(t)] \,, \tag{1.17}$$

where the vector parameter $\mathbf{q} \in \mathbb{S}$ now depends on t. To determine how $\mathbf{q}$ depends on t, we first introduce the quasi-steady linearization of (1.4)-(1.5)

$$u(x,t) = \tilde{u}^\epsilon[x; \mathbf{q}(t)] + v(x,t) \,, \tag{1.18}$$

where $v \ll \tilde{u}^\epsilon$ and $v_t \ll \partial_t \tilde{u}^\epsilon$. Next, we obtain that v satisfies (1.10)-(1.11) where we must add the term $q_j^{'} \partial_{q_j} \tilde{u}^\epsilon$ (sum on j) to the right side of (1.10). We then expand v as in (1.15) to derive formulae for the coefficients $c_j(t)$. Imposing the limiting solvability conditions then yields the following coupled system of differential equations for the components of the vector parameter $\mathbf{q}$:

$$q_i^{'} \left(\partial_{q_i} \tilde{u}^\epsilon, \phi_j\right)_w \sim \left(\phi_j, N(\tilde{u}^\epsilon)\right)_w - b_j \,, \quad j = 1, .., m \,. \tag{1.19}$$

The right side of this expression is exponentially small and can be evaluated asymptotically.

In §2-5 below, we give explicit details on how the projection approach can be successfully used to characterize metastable phenomena for various specific problems. In these problems, typically, $\mathbf{q}$ is a parameter that we can readily interpret in terms of the physical application under consideration. In particular, in §4 it denotes the locations of internal layers for the Allen-Cahn equation when $F(x,u,u_x) = 2(u - u^3)$.

2. A Linear Convection-Diffusion Equation

A very simple example where metastability occurs is for the following convection-diffusion equation for $u = u(x,t)$:

$$u_t = L_\epsilon u \equiv \epsilon u_{xx} - x u_x \,, \quad -1 < x < b \,, \quad t > 0 \,, \tag{2.1}$$

$$u(-1,t) = u_l \,, \quad u(b,t) = u_r \,; \qquad u(x,0) = u_0(x) \,. \tag{2.2}$$

Here $b > 0$, u_l and u_r are constants, $\epsilon \to 0^+$ and $u_0(x)$ is smooth. The spatial operator in (2.1) has a simple turning point at $x = 0$. The equilibrium problem corresponding to (2.1)-(2.2) and its associated eigenvalue problem arise in determining the exit time distribution for a Brownian particle confined by a potential well. This problem, and related turning point problems, have been studied in [**32**], [**33**], [**34**], and [**35**].

For $\epsilon \to 0$, a leading order boundary layer analysis for the equilibrium solution $U(x;\epsilon)$ to (2.1)-(2.2) has the form

$$U(x;\epsilon) \sim \tilde{u}^\epsilon\left[x; A_{0e}\right] \equiv A_{0e} + (u_r - A_{0e})\, e^{-b(b-x)/\epsilon} + (u_l - A_{0e})\, e^{-(1+x)/\epsilon} \,, \tag{2.3}$$

for some undetermined constant A_{0e}. Since $L_\epsilon \tilde{u}^\epsilon$ is exponentially small away from the boundary layer regions near $x = -1$ and $x = b$ for any choice of A_{0e}, the correct value of A_{0e} can only be determined by incorporating the effect of exponentially small terms into the asymptotic analysis. Singular perturbation problems of this type, where a conventional application of the method of matched asymptotic expansions fails to select certain constants uniquely, were first identified in [**1**] and later studied extensively in [**11**], [**16**], [**23**], [**31**], [**34**], and [**35**] (see also the references therein).

This apparent indeterminacy in selecting A_{0e} is associated with an exponential ill-conditioning of the underlying operator L_ϵ (see [**11**], [**23**], and [**31**]). More precisely, consider the eigenvalue problem associated with (2.1)-(2.2)

$$L_\epsilon \phi \equiv \epsilon \phi_{xx} - x\phi_x = \lambda \phi \,, \quad -1 < x < b \,; \qquad \phi(1) = \phi(b) = 0 \,, \tag{2.4}$$

$$(\phi,\phi)_w \equiv \int_{-1}^{b} \phi^2 w \, dx = 1 \,, \qquad w(x) \equiv e^{-x^2/2\epsilon} \,. \tag{2.5}$$

The eigenvalues λ_j for $j \geq 0$ are real with $\lambda_j < 0$ and the orthogonality relations $(\phi_j, \phi_k)_w = \delta_{jk}$ for $j,k = 0,1,..,$ hold. As discussed in §1, this problem has an eigenfunction ϕ that is well-approximated by the derivative of $\tilde{u}^\epsilon$ with respect to A_{0e}. Since this derivative yields a function that is strictly of one sign on the interval $-1 < x < b$, it must correspond to the principal eigenfunction ϕ_0. Hence, to leading order, ϕ_0 has the boundary layer form

$$\phi \sim M_0 \left(1 - e^{-(1+x)/\epsilon} - e^{-b(b-x)/\epsilon}\right) \,, \tag{2.6}$$

where M_0 is a normalization constant. From (2.4), we can derive the identity

$$\lambda_0 \left(1, \phi_0\right)_w = \epsilon w \phi_{0x} \Big|_{-1}^{b} , \tag{2.7}$$

which is used to estimate λ_0. Laplace's method is then used to estimate the integral $\left(1, \phi_0\right)_w$, whose dominant contribution arises from the region near $x = 0$. We get

$$\left(1, \phi_0\right)_w \sim \left(2\pi\epsilon\right)^{1/2} . \tag{2.8}$$

Substituting (2.6), (2.8) into (2.7), and using $w = e^{-x^2/2\epsilon}$, we obtain that the principal eigenvalue of (2.4)-(2.5) is exponentially small as $\epsilon \to 0$ and has the leading order asymptotic estimate

$$\lambda_0 \sim -\frac{1}{\left(2\pi\epsilon\right)^{1/2}} \left[b e^{-b^2/2\epsilon} + e^{-1/2\epsilon} \right] . \tag{2.9}$$

Asymptotic estimates for λ_0 for related turning point problems are given in (cf. [**11**], [**31**], [**32**], and [**35**]).

For the time-dependent problem we follow [**39**]. We seek a solution to (2.1)-(2.2) in the form

$$u(x,t) = \tilde{u}^{\epsilon}\left[x; A_0(t)\right] + v(x,t) , \tag{2.10}$$

where $\tilde{u}^{\epsilon}$ is defined in (2.3). Substituting (2.10) into (2.1)-(2.2), we obtain that $v(x,t)$ satisfies

$$v_t = L_{\epsilon} v - \tilde{u}_t^{\epsilon} + L_{\epsilon}\tilde{u}^{\epsilon} , \quad -1 < x < b , \quad t > 0 , \tag{2.11}$$

$$v(-1,t) = u_l - \tilde{u}^{\epsilon}\left[-1; A_0(t)\right] , \qquad v(b,t) = u_r - \tilde{u}^{\epsilon}\left[b; A_0(t)\right] . \tag{2.12}$$

$$v(x,0) = u_0(x) - \tilde{u}^{\epsilon}\left[x; A_0(0)\right] . \tag{2.13}$$

We then expand $v(x,t)$ in terms of the eigenfunctions ϕ_j of (2.4)-(2.5) as

$$v(x,t) = \sum_{j=0}^{\infty} c_j(t)\phi_j(x) . \tag{2.14}$$

Using the orthogonality properties of the eigenfunctions, we find that $c_j(t)$ satisfies the differential equation

$$c_j' - \lambda_j c_j = \left(\phi_j, L_{\epsilon}\tilde{u}^{\epsilon}\right)_w - \epsilon w v \phi_j' \Big|_{-1}^{b} - \left(\phi_j, \tilde{u}_t^{\epsilon}\right)_w , \tag{2.15}$$

together with the initial value

$$c_j(0) = \int_{-1}^{b} \left(u_0(x) - \tilde{u}^{\epsilon}[x; A_0(0)]\right) \phi_j w \, dx . \tag{2.16}$$

Since $\lambda_0 < 0$ and is exponentially small, it is necessary that $c_0(t) \equiv 0$ in order to ensure that $v \ll \tilde{u}^{\epsilon}$ over exponentially long time intervals . Therefore, the right sides of (2.15) and (2.16) must vanish when $j = 0$. This projection step yields the following implicit differential equation for $A_0(t)$:

$$\left(\phi_0, \tilde{u}_t^{\epsilon}\right)_w \sim \left(\phi_0, L_{\epsilon}\tilde{u}^{\epsilon}\right)_w - \epsilon w v \phi_0' \Big|_{-1}^{b} . \tag{2.17}$$

The initial condition $A_0(0)$ is obtained from

$$\int_{-1}^{b} \tilde{u}^{\epsilon}[x; A_0(0)]\phi_0 w \, dx = \int_{-1}^{b} u_0(x)\phi_0 w \, dx . \tag{2.18}$$

To obtain an explicit differential equation for A_0 we use the form for ϕ_0 given in (2.6) to evaluate the various terms in (2.17)-(2.18) asymptotically for $\epsilon \to 0$. Upon integrating by parts, we can show that

$$(\phi_0, L_\epsilon \tilde{u}^\epsilon)_w \sim \epsilon\,(u_l - A_0)\,w(-1)\phi_{0x}(-1) - \epsilon\,(u_r - A_0)\,w(b)\phi_{0x}(b)\,. \tag{2.19}$$

Since v is exponentially small at the endpoints, (2.19) dominates the second term on the right side of (2.17). Next, (2.3) yields

$$(\phi_0, \tilde{u}_t^\epsilon)_w \sim A_0^{'}\,(\phi_0, 1)_w\,. \tag{2.20}$$

Substituting (2.7) and (2.20) into (2.17) and neglecting the insignificant second term on the right side of this equation, we get

$$A_0^{'} \sim \lambda_0 A_0 + \frac{\epsilon}{(\phi_0, 1)_w}\left(u_l w(-1)\phi_{0x}(-1) - u_r w(b)\phi_{0x}(b)\right)\,. \tag{2.21}$$

Finally, we calculate ϕ_{0x} at the endpoints using (2.6) and we use (2.8) to estimate $(\phi_0, 1)_w$. Substituting these formulae into (2.21) we obtain the main result:

Proposition: *For $t \gg 1$, the solution $u(x,t)$ to (2.1) is given by $u(x,t) \sim \tilde{u}^\epsilon[x; A_0(t)]$, where $\tilde{u}^\epsilon$ is defined in (2.3). The function $A_0(t)$, representing the outer limit of $\tilde{u}^\epsilon$, satisfies the leading order asymptotic differential equation*

$$A_0^{'} \sim \lambda_0 A_0 + (2\pi\epsilon)^{-1/2}\left(u_l e^{-1/2\epsilon} + u_r b e^{-b^2/2\epsilon}\right)\,, \tag{2.22}$$

together with the initial condition

$$A_0(0) \sim (2\pi\epsilon)^{-1/2}\,(\phi_0 u_0, 1)_w \sim u_0(0)\,, \tag{2.23}$$

where $u_0(x) = u(x,0)$. The exponentially small eigenvalue $\lambda_0 < 0$ is estimated asymptotically in (2.9).

Since $\lambda_0 < 0$, the steady-state value $A_{0e} = A_0(\infty)$ is stable and is given by

$$A_{0e} = \frac{u_l + u_r b e^{-(b^2-1)/2\epsilon}}{1 + b e^{-(b^2-1)/2\epsilon}}\,. \tag{2.24}$$

Notice that A_{0e} is exponentially sensitive to perturbations in the endpoint location $x = b$, when b is close to $b = 1$. More specifically, as b is varied in an $O(\epsilon)$ region near $b = 1$, we sweep out all possible values of A_{0e} between u_l and u_r. This extreme sensitivity arises from the exponential ill-conditioning of the underlying spatial operator L_ϵ. In particular, for $\epsilon \to 0$, we have

$$A_{0e} = u_r \quad \text{for } b < 1 \tag{2.25}$$

$$A_{0e} = u_l \quad \text{for } b > 1 \tag{2.26}$$

$$A_{0e} = (u_l + u_r)/2 \quad \text{for } b = 1\,. \tag{2.27}$$

Hence, when $b < 1$, the limiting steady-state solution $\tilde{u}^\epsilon[x; A_{0e}]$ does not have a boundary layer at the right endpoint $x = b$. Alternatively, when $b > 1$, the limiting steady-state solution does not have a boundary layer at the left endpoint $x = -1$.

The solution to (2.22) with (2.23) is given by

$$A_0(t) = A_{0e} + (u_0(0) - A_{0e})\,e^{\lambda_0 t}\,. \tag{2.28}$$

where A_{0e} is given in (2.24). Thus, $A_0(t)$ approaches its steady-state value A_{0e} exponentially slowly as $\epsilon \to 0$. This characterizes the metastable behavior. In addition, since λ_0 is an exponential function of b and ϵ, the time needed to approach the steady-state solution depends very sensitively on the precise value of b when $\epsilon \ll 1$. This clearly illustrates the exponential sensitivity of a metastable problem.

Starting from arbitrary initial data, the approach to the equilibrium solution is exponentially slow. However, as shown in the finite element computational example by Adjerid et al. [**2**] in these proceedings, certain initial conditions will relax very quickly to the steady-state solution. To see this, suppose that $u_l = -u_r$ and that the initial data $u_0(x)$ is odd. Then, $A_{0e} = 0$, $u_0(0) = 0$, and hence $A_0(t) \equiv 0$ in (2.28). Thus, in this case, there is no metastable behavior.

3. Nonlinear Convection-Diffusion Equations

In this section we consider several different classes of nonlinear convection-diffusion equations on a finite interval that exhibit metastable behavior. In §3.1 a metastable shock-layer solution for Burgers equation is constructed using the Cole-Hopf transformation. In §3.2 the Cole-Hopf transformation is also used to analyze metastability for a forced Burgers equation. In §3.3, we use the projection method to study metastable dynamics for a class of nonlinear viscous shock problems. Finally, in §3.4 we give some metastability results for a convection-diffusion model that describes upward flame propagation in a vertical channel.

3.1. Burgers Equation on a Finite Interval. Consider the following initial boundary value problem for Burgers equation in the limit $\epsilon \to 0$:

$$u_t + uu_x = \epsilon\, u_{xx}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{3.1}$$

$$u(x,0) = u_0(x)\,; \qquad u(-1,t) = \alpha\,, \quad u(1,t) = -\alpha\,. \tag{3.2}$$

Here $\alpha > 0$ is a constant.

This problem was solved numerically in [**22**] where metastable behavior was observed. The computations of [**22**] showed that, starting from monotone decreasing initial data, a shock layer of width $O(\epsilon)$, connecting $u = \alpha$ and $u = -\alpha$, is formed on an $O(1)$ time scale. This shock layer is closely approximated by the stationary wave solution of (3.1)-(3.2)

$$u = -\alpha \tanh\left[\alpha\epsilon^{-1}(x - x_0(t))/2\right]\,. \tag{3.3}$$

The initial shock layer location $x_0(0)$ depends on the initial data. Once the shock layer has formed, it drifts towards the equilibrium solution, which has a shock layer at the midpoint of the domain, at an exceedingly slow rate. This equilibrium solution is stable. Thus we have two different time behaviors under (3.1)-(3.2): a transient $O(1)$ phase where the shock layer is formed, and an exponentially slow phase where the shock layer drifts towards its equilibrium location.

The goal below is to calculate an ordinary differential equation for the slow motion of the center of the shock layer as it approaches its equilibrium state. We define the center of the shock layer by the trajectory $x_0 = x_0(t)$ for which $u[x_0(t), t] = 0$. To derive an ODE for $x_0(t)$ we use the Cole-Hopf transformation, which allows us to solve (3.1)-(3.2) explicitly using the standard method of separation of variables. A different approach is taken in §3.3, where the projection method is used to analyze a general class of viscous shock problems.

The Cole-Hopf transformation is

$$u = -2\,\epsilon\,\frac{\psi_x}{\psi}\,. \tag{3.4}$$

In terms of ψ, (3.1)-(3.2) becomes

$$\psi_t = \epsilon \psi_{xx}, \qquad -1 < x < 1, \quad t > 0, \tag{3.5}$$

$$\psi_x(-1,t) = -\frac{\alpha}{2\epsilon}\psi(-1,t), \quad \psi_x(1,t) = \frac{\alpha}{2\epsilon}\psi(1,t), \tag{3.6}$$

$$\psi(x,0) = \psi_0(x) \equiv e^{\alpha\theta(x)/2\epsilon}, \qquad \theta'(x) = -u_0(x)/\alpha. \tag{3.7}$$

Next, we look for a separation of variables solution to (3.5)-(3.7) in the form

$$\psi(x,t) = \sum_{j=0}^{\infty} c_j\, e^{\sigma_j t}\, v_j(x). \tag{3.8}$$

The normalized eigenpairs v_j, σ_j for $j \geq 0$ are obtained from the Sturm-Liouville eigenvalue problem

$$\epsilon v'' - \sigma v = 0, \qquad -1 < x < 1, \tag{3.9}$$

$$v'(-1) = -\frac{\alpha}{2\epsilon} v(-1), \qquad v'(1) = \frac{\alpha}{2\epsilon} v(1). \tag{3.10}$$

The eigenvalues σ_j are ordered by $\sigma_j > \sigma_{j+1}$ with $\sigma_j \to -\infty$ as $j \to \infty$. Using orthogonality, we find that the normalized eigenfunctions $v_j(x)$ and the coefficients c_j satisfy

$$(v_j, v_k) \equiv \int_{-1}^{1} v_j(x) v_k(x)\, dx = \delta_{jk}, \qquad c_j \equiv (\psi_0, v_j). \tag{3.11}$$

In terms of these eigenfunctions, the solution to (3.1)-(3.2) is

$$u(x,t) = -2\epsilon \frac{\left[c_0 v_0'(x) + \sum_{j=1}^{\infty} c_j e^{(\sigma_j - \sigma_0)t} v_j'(x)\right]}{\left[c_0 v_0(x) + \sum_{j=1}^{\infty} c_j e^{(\sigma_j - \sigma_0)t} v_j(x)\right]}. \tag{3.12}$$

Thus the temporal behavior of u is determined by the difference $\sigma_j - \sigma_0$, where $\sigma_j - \sigma_0 < 0$ for $j \geq 1$. In addition, the shock layer trajectory $x_0 = x_0(t)$ defined by $u[x_0(t), t] = 0$ satisfies

$$c_0 v_0'[x_0(t)] = -\sum_{j=1}^{\infty} c_j e^{(\sigma_j - \sigma_0)t} v_j'[x_0(t)]. \tag{3.13}$$

We now calculate σ_j and $v_j(x)$ explicitly. The first two eigenvalues and eigenfunctions of (3.9)-(3.10) are given by

$$v_0(x) = A_0 \cosh(\mu_0 x), \quad \text{where} \quad \sigma_0 = \epsilon\mu_0^2, \quad \tanh(\mu_0) = \frac{\alpha}{2\epsilon\mu_0}, \tag{3.14}$$

$$v_1(x) = A_1 \sinh(\mu_1 x), \quad \text{where} \quad \sigma_1 = \epsilon\mu_1^2, \quad \tanh(\mu_1) = \frac{2\epsilon\mu_1}{\alpha}. \tag{3.15}$$

Here A_0 and A_1 are normalization constants. For $\epsilon \to 0$, we can solve the transcendental equations for μ_0 and μ_1 to obtain

$$\mu_0 \sim \frac{\alpha}{2\epsilon}\left[1 + 2e^{-\alpha/\epsilon}\right], \qquad \sigma_0 \sim \frac{\alpha^2}{4\epsilon}\left[1 + 4e^{-\alpha/\epsilon}\right], \tag{3.16}$$

$$\mu_1 \sim \frac{\alpha}{2\epsilon}\left[1 - 2e^{-\alpha/\epsilon}\right], \qquad \sigma_1 \sim \frac{\alpha^2}{4\epsilon}\left[1 - 4e^{-\alpha/\epsilon}\right]. \tag{3.17}$$

The higher eigenvalues and eigenfunctions of (3.9)-(3.10) are given by

$$v_{2n}(x) = A_{2n}\cos(\mu_{2n}x)\,, \qquad \sigma_{2n} = -\epsilon\,\mu_{2n}^2\,, \;\; n = 1,2,\ldots\,, \tag{3.18}$$

$$v_{2n+1}(x) = A_{2n+1}\sin(\mu_{2n+1}x)\,, \qquad \sigma_{2n+1} = -\epsilon\,\mu_{2n+1}^2\,, \;\; n = 1,2,\ldots\,. \tag{3.19}$$

Here, μ_{2n} is the n^{th} positive root of $\tan(z) = -\alpha/(2\epsilon z)$ and μ_{2n+1} is the n^{th} positive root of $\tan(z) = 2\epsilon z/\alpha$. For $\epsilon \to 0$, with $\epsilon n \ll 1$, it follows that

$$\sigma_{2n} \sim -(2n-1)^2\pi^2\epsilon/4\,, \qquad \sigma_{2n+1} \sim -n^2\pi^2\epsilon\,. \tag{3.20}$$

Finally, the normalization coefficients A_j for $j \geq 0$ are found to be

$$A_0^2 = \frac{2\,\mu_0}{\sinh(2\mu_0) + 2\mu_0}\,, \quad A_1^2 = \frac{2\,\mu_1}{\sinh(2\mu_1) - 2\mu_1}\,, \tag{3.21}$$

$$A_{2n}^2 = \frac{2\,\mu_{2n}}{2\mu_{2n} + \sin(2\mu_{2n})}\,, \quad A_{2n+1}^2 = \frac{2\,\mu_{2n+1}}{2\mu_{2n+1} - \sin(2\mu_{2n+1})}\,. \tag{3.22}$$

The critical property of the spectrum of the Sturm-Liouville problem (3.9)-(3.10) is that for $\epsilon \to 0$,

$$\sigma_1 - \sigma_0 \sim -2\alpha^2\epsilon^{-1}\,e^{-\alpha/\epsilon}\,, \qquad \sigma_j - \sigma_0 = O(\epsilon)\,, \quad j = 2,\ldots \tag{3.23}$$

Thus, $\sigma_1 - \sigma_0$ is exponentially small as $\epsilon \to 0$, and the Sturm-Liouville problem is nearly degenerate. We refer to (3.23) as the spectral gap property.

As a consequence of the spectral gap property, when $\epsilon t \gg 1$ we can neglect all the terms in the infinite sums in (3.12) and (3.13) starting from $j = 2$. This allows us to easily characterize the metastable dynamics describing the slow approach of the time-dependent solution $u(x,t)$ towards the equilibrium solution $u_e(x)$ defined by

$$u_e(x) = -2\epsilon\frac{v_0'(x)}{v_0(x)} = -2\epsilon^{-1}\mu_0\tanh(\mu_0 x) \sim -\alpha\tanh\left(\frac{\alpha x}{2\epsilon}\right)\,. \tag{3.24}$$

When $\epsilon t \gg 1$ and $\epsilon \ll 1$, we neglect the terms in (3.13) corresponding to $j \geq 2$ to get that the shock layer trajectory $x_0 = x_0(t)$ satisfies

$$c_0 A_0\mu_0\sinh(\mu_0 x_0) = -c_1 A_1\mu_1 e^{-t/t_s}\cosh(\mu_1 x_0)\,, \tag{3.25}$$

where t_s is given by

$$t_s = -\frac{1}{\sigma_1 - \sigma_0} \sim \frac{\epsilon}{2\alpha^2}e^{\alpha/\epsilon}\,. \tag{3.26}$$

Thus, the time scale t_s for metastable motion is determined by the reciprocal of the exponentially small gap width $\sigma_1 - \sigma_0$. Using (3.16), (3.17) and (3.21) for μ_0, μ_1, A_0 and A_1, we find that (3.25) reduces for $\epsilon \ll 1$ to

$$\tanh\left(\frac{\alpha x_0}{2\epsilon}\right) \sim -\frac{c_1}{c_0}e^{-t/t_s}\,. \tag{3.27}$$

Here c_1/c_0 is the ratio of the inner products

$$\frac{c_1}{c_0} \sim \frac{\left(\sinh\left[\frac{\alpha}{2\epsilon}x\right], \psi_0(x)\right)}{\left(\cosh\left[\frac{\alpha}{2\epsilon}x\right], \psi_0(x)\right)}\,. \tag{3.28}$$

Finally, to determine an approximation for $u(x,t)$ when $\epsilon t \gg 1$ and $\epsilon \ll 1$, we neglect the terms in (3.12) starting from $j = 2$ and use $\mu_0 \sim \mu_1 \sim \alpha\epsilon^{-1}/2$ and $A_0 \sim A_1$. In this way we get the following result as obtained previously in [**42**] (see also [**26**]):

Proposition: (Metastability) *For $\epsilon t \gg 1$ and $\epsilon \ll 1$ the solution $u(x,t)$ to (3.1)-(3.2) satisfies*

$$u(x,t) \sim -\alpha \tanh\left[\frac{\alpha\epsilon^{-1}}{2}(x - x_0(t))\right], \tag{3.29}$$

where $x_0(t)$ satisfies (3.27).

By differentiating (3.27) with respect to t we get the following ODE for $x_0(t)$:

Corollary: (Metastability) *For $\epsilon t \gg 1$ and $\epsilon \ll 1$, the shock layer trajectory $x_0 = x_0(t)$ defined by $u[x_0(t), t] = 0$ satisfies the nonlinear asymptotic ordinary differential equation*

$$x_0' \sim \alpha\left(e^{-\alpha(1+x_0)/\epsilon} - e^{-\alpha(1-x_0)/\epsilon}\right). \tag{3.30}$$

The initial condition $x_0^0 \equiv x_0(0)$ for (3.30) is given by

$$\tanh\left[\frac{\alpha}{2\epsilon}x_0^0\right] = -\frac{c_1}{c_0}, \tag{3.31}$$

where c_1/c_0, which satisfies $-1 < c_1/c_0 < 1$ is defined in (3.28). This result was obtained in [**26**] and [**42**]. Notice that $x_0(t) \to 1/2$ as $t \to \infty$ for any initial condition $x_0^0 \in (-1,1)$.

In §3.3 below, we will use the projection method to derive a similar differential equation for the shock layer trajectory of a general class of viscous shock problems. However, we will not be able to determine an analytical formula for the initial condition for the differential equation as was obtained in (3.31) for the special case of Burgers equation.

We now use (3.31) to evaluate x_0^0 for $\epsilon \ll 1$ in terms of the initial data $u_0(x)$. Substituting $\psi_0(x) = \exp\left[\alpha\epsilon^{-1}\theta(x)/2\right]$ into (3.28), we can write c_1/c_0 as

$$\frac{c_1}{c_0} = \frac{1 - I_-/I_+}{1 + I_-/I_+}, \tag{3.32}$$

where

$$I_\pm \equiv \int_{-1}^{1} e^{\lambda h_\pm(x)}\,dx, \qquad h_\pm(x) \equiv \theta(x) \pm x, \qquad \lambda \equiv \frac{\alpha}{2\epsilon} \gg 1. \tag{3.33}$$

Here we recall that $\theta'(x) \equiv -u_0(x)/\alpha$. We consider initial data $u_0(x)$ with $u_0(-1) = \alpha$, $u_0(1) = -\alpha$ and $u_0'(x) < 0$ for $x \in [-1,1]$. Thus $\theta(x)$ satisfies $\theta'(-1) = -1$, $\theta'(1) = 1$ and $\theta''(x) > 0$ for $x \in [-1,1]$. Since $|\theta'(x)| < 1$ for $x \in (-1,1)$, it follows that $h_+'(x) > 0$ on $(-1,1]$ and $h_-'(x) < 0$ on $[-1,1)$. Therefore, for $\lambda \to \infty$ ($\epsilon \to 0$), the dominant contributions to I_+ and I_- arise from the regions near $x = 1$ and $x = -1$, respectively. Since $h_\pm'' > 0$, we can then evaluate the two integrals $I_\pm$ asymptotically as $\lambda \to \infty$ ($\epsilon \to 0$) by using integration by parts. We obtain that

$$I_+ \sim \frac{e^\lambda}{2\lambda}e^{\lambda\theta(1)}\left[1 + \frac{\theta''(1)}{4\lambda} + \cdots\right], \qquad I_- \sim \frac{e^\lambda}{2\lambda}e^{\lambda\theta(-1)}\left[1 + \frac{\theta''(-1)}{4\lambda} + \cdots\right]. \tag{3.34}$$

Substituting (3.34) into (3.32) and noting (3.31), we obtain that x_0^0 is determined in terms of the initial data $u_0(x)$ by

$$x_0^0 = \frac{1}{2\alpha}\int_{-1}^{1} u_0(x)\,dx + \frac{\epsilon^2}{2\alpha^3}\left[u_0'(1) - u_0'(-1)\right] + \cdots, \qquad \epsilon \to 0. \tag{3.35}$$

3.2. A Forced Burgers Equation. Now consider the forced Burgers equation

$$u_t + u u_x = \epsilon u_{xx} + \epsilon^{-2} f(x)\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{3.36}$$

$$u(x,0) = u_0(x)\,; \qquad u(-1,t) = 0\,, \quad u(1,t) = 0\,. \tag{3.37}$$

Here $f(x)$ satisfies

$$f(0) = f(\pm 1) = 0\,, \quad f(x) = -f(-x) \;\text{ for }\; -1 < x < 1\,, \tag{3.38}$$

$$f^{'}(-1) > 0\,, \quad f(x) > 0 \;\text{ for }\; -1 < x < 0\,. \tag{3.39}$$

Thus $\int_{-1}^{1} f(x)\, dx = 0$. The numerical study of [**22**] showed that (3.36)-(3.37) forms a shock layer in finite time and that this layer drifts slowly towards $x = 0$ as $t \to \infty$.

To analyze (3.36)-(3.37) we use the Cole-Hopf transformation $u = -2\,\epsilon\,\psi_x/\psi$, which transforms (3.36)-(3.37) to

$$\psi_t = \epsilon\,\psi_{xx} - \epsilon^{-2} F(x)\psi\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{3.40}$$

$$\psi_x(-1,t) = 0\,, \quad \psi_x(1,t) = 0\,, \tag{3.41}$$

$$\psi(x,0) = \psi_0(x) \equiv e^{\alpha\theta(x)/2\epsilon}\,, \qquad \theta^{'}(x) = -u_0(x)/\alpha\,. \tag{3.42}$$

Here $F(x)$ is defined by

$$F(x) = \frac{1}{2}\int_{-1}^{x} f(\eta)\, d\eta\,. \tag{3.43}$$

Next, we look for a separation of variables solution to (3.40)-(3.42) in the form

$$\psi(x,t) = \sum_{j=0}^{\infty} c_j\, e^{-\lambda_j t/\epsilon}\, v_j(x)\,. \tag{3.44}$$

The normalized eigenpairs v_j, λ_j for $j \geq 0$ are obtained from the Sturm-Liouville eigenvalue problem

$$\epsilon^2 v^{''} + \left[\lambda - \epsilon^{-2} F(x)\right] v = 0\,, \qquad -1 < x < 1\,, \tag{3.45}$$

$$v^{'}(-1) = 0\,, \qquad v^{'}(1) = 0\,. \tag{3.46}$$

Using (3.38), (3.39) and (3.43), it follows that $F(x)$ is an even function (i. e. $F(x) = F(-x)$), which also satisfies

$$F(\pm 1) = F^{'}(\pm 1) = 0\,, \tag{3.47}$$

$$F^{''}(\pm 1) = 2\beta\,, \qquad \beta \equiv f^{'}(\pm 1)/4 \tag{3.48}$$

$$F(0) = \max_{-1 \leq x \leq 1} F(x) \equiv F^* = \frac{1}{2}\int_{-1}^{0} f(x)\, dx\,. \tag{3.49}$$

Thus, $F(x)$ is a symmetric double-well potential. The eigenvalues λ_j are ordered by $\lambda_j < \lambda_{j+1}$ with $\lambda_j \to \infty$ as $j \to \infty$. Using orthogonality, we find that the relation (3.11) holds and that the solution to (3.36)-(3.37) is given by

$$u(x,t) = -2\epsilon \frac{\left[c_0 v_0^{'}(x) + \sum_{j=1}^{\infty} c_j e^{(\lambda_0 - \lambda_j)t/\epsilon} v_j^{'}(x)\right]}{\left[c_0 v_0(x) + \sum_{j=1}^{\infty} c_j e^{(\lambda_0 - \lambda_j)t/\epsilon} v_j(x)\right]}\,. \tag{3.50}$$

The shock layer trajectory $x_0 = x_0(t)$, defined by $u[x_0(t), t] = 0$, satisfies

$$c_0 v_0'[x_0(t)] = -\sum_{j=1}^{\infty} c_j e^{(\lambda_0-\lambda_j)t/\epsilon} v_j'[x_0(t)]\,. \tag{3.51}$$

The eigenvalue problem (3.45)-(3.46) for a symmetric double-well potential arises in quantum mechanics and is associated with the tunneling of a particle through a large symmetric barrier. The asymptotic properties of the eigenvalue problem are well-known (e. g. see [**46**]). The eigenspace of (3.45) can be decomposed into even and odd eigenfunctions. In particular, v_0 is even and v_1 is odd, and the gap width $\lambda_0 - \lambda_1 < 0$ is exponentially small as $\epsilon \to 0$. By modifying the analysis in [**46**] to account for the Neumann boundary conditions in (3.45) at the minima of the potential wells, we can readily derive the following spectral gap property as $\epsilon \to 0$:

$$\lambda_0 - \lambda_1 \sim -\frac{16\beta^{1/2}}{\pi^{1/2}} e^{-1/2} \exp\left(-\frac{1}{\epsilon} \int_{x_1(\epsilon)}^{0} \left[F(x) - 4\epsilon^2 \beta^{1/2} \right]^{1/2} dx \right)\,, \tag{3.52}$$

$$\lambda_0 - \lambda_j = o(\lambda_0 - \lambda_1)\,, \quad j \geq 2\,. \tag{3.53}$$

Here $x_1(\epsilon)$ is the root of $F(x) = 4\epsilon^2\beta^{1/2}$ for which $x_1(\epsilon) \to 2\epsilon\beta^{-1/4}$ as $\epsilon \to 0$.

Hence for t sufficiently large, we can neglect the terms in the sums in (3.50) and (3.51) starting from $j = 2$ to get the following main result:

Proposition: (Metastability) *For t sufficiently large and $\epsilon \ll 1$ the solution $u(x,t)$ to (3.36)-(3.37) is given by*

$$u(x,t) \sim -2\epsilon \frac{d}{dx} \left[\log\left(v_1'(x_0(t)) v_0(x) - v_0'(x_0(t)) v_1(x) \right) \right]\,. \tag{3.54}$$

The shock layer trajectory $x_0 = x_0(t)$ satisfies

$$\frac{v_0'[x_0(t)]}{v_1'[x_0(t)]} \sim -\frac{c_1}{c_0} e^{-t/t_s}\,, \tag{3.55}$$

where t_s and c_1/c_0 are defined by

$$t_s \sim -\frac{1}{\lambda_0 - \lambda_1}\,, \qquad \frac{c_1}{c_0} \sim \frac{(v_1(x), \psi_0(x))}{(v_0(x), \psi_0(x))}\,, \tag{3.56}$$

Here $\psi_0(x)$ is defined in (3.42) and v_0 and v_1 are the first two normalized eigenpairs of (3.45)-(3.46).

Since v_0 is even it follows that $x_0(t) \to 0$ as $t \to \infty$. The key observation is that the metastable time scale t_s is determined by the reciprocal of the exponentially small gap width between the first two eigenvalues of (3.45)-(3.46).

3.3. A Viscous Shock Problem.

Next, we consider the more general viscous shock problem

$$u_t + [f(u)]_x = \epsilon\, u_{xx}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{3.57}$$

$$u(-1,t) = \alpha_- > 0\,, \quad u(1,t) = \alpha_+ < 0\,; \qquad u(x,0) = u_0(x)\,. \tag{3.58}$$

Here $\alpha_\pm$ are constants, $u_0(x)$ is monotone decreasing with $u_0(\pm 1) = \alpha_\pm$ and $\epsilon \to 0^+$. Here $f(u)$ is a smooth convex function that satisfies

$$f(\alpha_+) = f(\alpha_-)\,, \qquad f(0) = f'(0) = 0\,, \qquad uf'(u) > 0 \quad \text{for} \quad u \neq 0\,. \tag{3.59}$$

The special case $f(u) = u^2/2$ yields Burgers equation, which was studied in §3.1.

The key condition $f(\alpha_+) = f(\alpha_-)$ ensures that (3.57) has a stationary wave solution $u_c(x/\epsilon)$ on the infinite line connecting α_+ and α_-. Here $u_c(z)$, also called the viscous shock profile, is the unique solution to

$$u_c'(z) = f\left[u_c(z)\right] - f(\alpha_+)\,, \quad -\infty < z < \infty\,; \quad u_c(\pm\infty) = \alpha_\pm\,, \quad u_c(0) = 0\,, \tag{3.60}$$

with $u_c' < 0$. This solution has the far-field behavior

$$u_c(z) \sim \alpha_\pm \pm a_\pm e^{\mp\nu_\pm z}\,, \quad \text{as} \;\; z \to \pm\infty\,; \qquad \nu_\pm \equiv \mp f'(\alpha_\pm) > 0\,, \tag{3.61}$$

for some positive constants $a_\pm > 0$ (cf. [**42**]). In particular, for Burgers equation with $\alpha_\pm = \mp 1$, we have $u_c(z) = -\tanh(z/2)$, $\nu_\pm = 1$ and $a_\pm = 2$.

A matched asymptotic expansion analysis shows that (3.57)-(3.58) has an equilibrium shock-layer solution $U(x;\epsilon)$ of the form $U(x;\epsilon) \sim u_c\left[(x - x_{0e})/\epsilon\right]$, for some undetermined $x_{0e} \in (-1,1)$. Since $u_c(z)$ decays exponentially as $z \to \pm\infty$, it follows that $u_c\left[(x - x_{0e})/\epsilon\right]$ satisfies the boundary conditions at $x = \pm 1$ to within exponentially small terms as $\epsilon \to 0$ for any shock-layer location $x_{0e} \in (-1,1)$. This suggests that the problem of determining x_{0e} is exponentially ill-conditioned. When $f(u)$ is even, it is clear by symmetry that $x_{0e} = 0$. For more general $f(u)$, the correct value $x_{0e} = (\nu_+ - \nu_-)/(\nu_+ + \nu_-) + O(\epsilon)$ can be obtained analytically by using a spectral projection method (cf. [**42**]), or an extension of the method of matched asymptotic expansions (cf. [**25**], [**27**]).

The time-dependent problem (3.57)-(3.58) admits metastable shock-layer solutions that have the same qualitative features as for Burgers equation described following (3.1)-(3.2) above. A thin shock layer, which connects $u = \alpha_+$ and $u = \alpha_-$, is formed quickly in time from the initial data $u_0(x)$. This shock layer, which is closely approximated by the viscous profile $u_s[(x - x_0^0)/\epsilon]$ for some x_0^0 depending on $u_0(x)$, then translates exceedingly slowly towards the equilibrium shock layer solution centered at $x = 0$. This equilibrium solution is stable, but the principal eigenvalue λ_0 associated with the linearization around this solution is exponentially small as $\epsilon \to 0$. Metastable shock layer motion for (3.57)-(3.58) is analyzed in [**42**] using a spectral projection method and in [**25**] using an extension of the method of matched asymptotic expansions. The extreme sensitivity of the metastable shock layer motion to small changes in the coefficients of the differential operator or to the boundary conditions is analyzed in [**27**] and [**44**]. A related class of metastable viscous shock problems is studied in [**53**].

We now outline the projection method used in [**42**]. We first construct the quasi-steady linearization of (3.57)-(3.58) around the viscous shock profile, where the shock-layer trajectory $x_0 = x_0(t)$ is to be determined. Thus, in (3.57)-(3.58), we set

$$u(x,t) = u_c\left(\left[x - x_0(t)\right]/\epsilon\right) + v(x,t)\,, \tag{3.62}$$

where $v \ll u_c$ and $v_t \ll \partial_t u_c$. Linearizing (3.57)-(3.58) around u_c, and using (3.61), we obtain the following quasi-steady problem for $v(x,t)$:

$$L_\epsilon v \equiv \epsilon v_{xx} - \left[f'(u_c)v\right]_x = -\epsilon^{-1} x_0' u_c'\,, \tag{3.63}$$

$$v(-1,t) \sim a_- e^{-\nu_-(1+x_0)/\epsilon}\,, \qquad v(1,t) \sim -a_+ e^{-\nu_+(1-x_0)/\epsilon}\,. \tag{3.64}$$

Here $x_0' = dx_0/dt$. Let $x_0 \in (-1,1)$ be fixed. The corresponding eigenvalue problem is

$$L_\epsilon \phi = \lambda \phi \quad -1 < x < 1\,; \qquad \phi(\pm 1) = 0\,. \tag{3.65}$$

The eigenvalue problem for the adjoint operator $L_\epsilon^\dagger$ is readily seen to be

$$L_\epsilon^\dagger \Phi \equiv \epsilon \Phi_{xx} - \Psi' \Phi_x = \Lambda \Phi\,, \quad -1 < x < 1\,; \qquad \Phi(\pm 1) = 0\,, \tag{3.66}$$

$$(\Phi, \Phi)_w \equiv \int_{-1}^{1} \Phi^2 w\, dx = 1\,, \qquad w = e^{-\Psi/\epsilon}\,. \tag{3.67}$$

Here $\Psi' = \Psi'(x;\epsilon)$ and the weight function $w = w(x;\epsilon) > 0$ are defined by

$$\Psi'(x;\epsilon) = -f'\left(u_c\left[(x - x_0)/\epsilon\right]\right)\,, \qquad w(x;\epsilon) = -u_c'\left[(x - x_0)/\epsilon\right]\,. \tag{3.68}$$

The eigenvalues Λ_j for $j \geq 0$ are real with $\Lambda_j > 0$ for $j \geq 0$ and $(\Phi_j, \Phi_k)_w = \delta_{jk}$ for $j,k = 0,1,\ldots$ Let (λ_j, ϕ_j) and (Λ_j, Φ_j) for $j \geq 0$ be the eigenpairs of (3.65) and (3.66), respectively. Then, it easy to show that we can relate these (un-normalized) eigenpairs by

$$\phi_j(x) = -u_c'\left[(x - x_0)/\epsilon\right] \Phi_j(x)\,, \qquad \lambda_j = \Lambda_j\,, \quad j = 0,1,\ldots \tag{3.69}$$

We now outline a key property of the spectra of (3.65) and (3.66). From the properties of $f(u)$ and $u_c(z)$, it follows that Ψ in (3.66) has a global minimum on $[-1,1]$ at the point $x = x_0$, where $\Psi' = 0$ and $\Psi'' > 0$. Therefore, (3.66) is very similar in form to the eigenvalue problem (2.4) for the linear convection-diffusion equation considered in §2.1. Thus, with this analogy, we expect that Λ_0 and, hence λ_0, are exponentially small as $\epsilon \to 0$. The corresponding eigenfunction Φ_0 has a boundary layer approximation similar in form to that given by the right side of (2.6). Since $\Psi' \sim -f'(\alpha_\pm) = \pm\nu_\pm$ at $x = \pm 1$, we find that

$$\Phi_0 \sim M_0 \left(1 - e^{-\nu_-(x+1)/\epsilon} - e^{-\nu_+(1-x)/\epsilon}\right)\,, \tag{3.70}$$

as $\epsilon \to 0$, where M_0 is a normalization constant. Then, upon integrating by parts in (3.66), we find that Λ_0 can be estimated as $\epsilon \to 0$ from the identity

$$\Lambda_0 \left(\Phi_0, 1\right)_w = \epsilon e^{-\Psi/\epsilon}\, \Phi_0' \Big|_{-1}^{1}\,. \tag{3.71}$$

Finally, substituting (3.70) and (3.71) we obtain the following estimate for the exponentially small eigenvalue as derived in [**42**]:

$$\Lambda_0 = \lambda_0 \sim -\frac{1}{\epsilon(\alpha_- - \alpha_+)} \left[a_+ \nu_+^2\, e^{-\nu_+(1-x_0)/\epsilon} + a_- \nu_-^2\, e^{-\nu_-(1+x_0)/\epsilon}\right]\,. \tag{3.72}$$

From (3.69) and (3.70), we observe that ϕ_0 is proportional to $\partial_{x_0} u_c\left[(x - x_0)/\epsilon\right]$ away from the boundary layer regions near $x = \pm 1$. This was anticipated from the discussion in §1. Thus, away from these boundary layers, ϕ_0 is asymptotically close to the translation eigenfunction associated with (3.57) on the infinite line. The finite domain in (3.57) breaks the translation invariance and, together with the exponential decay behavior (3.61), leads to the exponentially small eigenvalue λ_0.

Next, we expand the solution $v(x,t)$ to (3.63)-(3.64) as

$$v(x,t) = \sum_{j=0}^{\infty} \frac{c_j(t)}{\lambda_j} \phi_j(x)\,. \tag{3.73}$$

Using Green's identity and the orthogonality condition $(\phi_j, \Phi_k) = 0$ for $j \neq k$, we obtain

$$c_j = -\epsilon^{-1} x_0^{'} \left(\Phi_j, u_c^{'} \right) + \epsilon \Phi_j^{'} v \Big|_{-1}^{1} . \tag{3.74}$$

Since $\lambda_0 \to 0$ exponentially as $\epsilon \to 0$, we must impose the limiting solvability condition that $c_0 \to 0$ as $\epsilon \to 0$. In this way, the following metastability result was obtained in [**42**]:

Proposition: (Metastability) *For $t \gg 1$ and $\epsilon \ll 1$ the solution $u(x,t)$ to (3.57)–(3.58) is given by*

$$u(x,t) \sim u_c \left[(x - x_0(t))/\epsilon \right] , \tag{3.75}$$

where the shock-layer trajectory $x_0 = x_0(t)$ satisfies the nonlinear asymptotic differential equation

$$x_0^{'} \sim \frac{1}{(\alpha_- - \alpha_+)} \left[a_- \nu_- \, e^{-\nu_- (1+x_0)/\epsilon} - a_+ \nu_+ \, e^{-\nu_+ (1-x_0)/\epsilon} \right] . \tag{3.76}$$

Here $a_\pm$ and $\nu_\pm$ are defined in (3.61).

This result is also derived in [**25**] using a different method. Clearly $x_0(t) \to x_{0e}$ as $t \to \infty$, where

$$x_{0e} = \frac{\nu_+ - \nu_-}{\nu_+ + \nu_-} + \frac{\epsilon}{\nu_+ + \nu_-} \log \left(\frac{a_- \nu_-}{a_+ \nu_+} \right) . \tag{3.77}$$

As a remark, consider the special case of Burgers equation where $f(u) = u^2/2$. Then, from (3.68) we obtain

$$u_c = -\alpha \tanh \left[\alpha \epsilon^{-1} (x - x_0)/2 \right] = -\Psi^{'}(x; \epsilon) . \tag{3.78}$$

We transform the eigenvalue problem (3.66) using the Liouville transformation

$$\phi = e^{\Psi/(2\epsilon)} \psi , \qquad \Psi(x;\epsilon) = 2\epsilon \log \left[\cosh \left(\alpha \epsilon^{-1} (x - x_0)/2 \right) \right] . \tag{3.79}$$

This yields the following equivalent eigenvalue problem:

$$L_s \psi \equiv \epsilon^2 \psi_{xx} + \frac{\alpha^2}{4} \left(-1 + 2 \operatorname{sech}^2 \left[\frac{\alpha}{2\epsilon} (x - x_0) \right] \right) \psi = \epsilon \Lambda \psi , \quad -1 < x < 1 \tag{3.80}$$

$$\psi(-1) = 0 , \quad \psi(1) = 0 . \tag{3.81}$$

Notice that $L_s \tilde{\psi}_0 = 0$, where $\tilde{\phi}_0 = \operatorname{sech} \left[\frac{\alpha}{2\epsilon} (x - x_0) \right]$. In addition, $\tilde{\psi}_0$ is of one sign and fails to satisfy the boundary conditions (3.81) by only exponentially small terms as $\epsilon \to 0$. Hence, we expect that the principal eigenfunction ψ_0 of (3.80)-(3.81) is exponentially close to $\tilde{\psi}_0$ as $\epsilon \to 0$. The principal eigenvalue Λ_0 of this problem is exponentially small and has the estimate (3.72) where $a_\pm = 2$ and $\nu_\pm = 1$. Thus, for the case of Burgers equation, the eigenvalue problem (3.66) can be transformed into a more conventional eigenvalue problem involving a sech^2 potential. Eigenvalue problems, involving a sech^2 potential well that are very similar in form to (3.80)-(3.81) are central to the metastability analysis given in §4 and §5 below.

3.4. Metastable Flame-Front Propagation in a Vertical Channel. As discussed in [**40**], under certain conditions flame-front interfaces for upward flame propagation in vertical channels can assume a parabolic-type shape. Under various physical assumptions, a model equation describing the evolution of such a parabolic-shaped interface was derived in [**40**] using a weakly nonlinear analysis.

In dimensionless variables, this model for the flame-front interface $y = y(x,t)$ is given by

$$y_t - \frac{1}{2}y_x^2 = \epsilon y_{xx} + y - \int_0^1 y\,dx\,, \qquad 0 < x < 1\,, \quad t > 0\,, \tag{3.82}$$

$$y_x(0,t) = y_x(1,t) = 0\,; \qquad y(x,0) = y_0(x)\,, \tag{3.83}$$

where $0 < \epsilon \ll 1$ is a small parameter.

For the time-dependent problem, the computational results of [**36**] showed that a concave parabolic-shaped flame-front interface can develop from initial data that are concave. Their results showed that the location $x_0 = x_0(t)$ of the tip, or nose, of this interface, defined by $y_x[x_0(t),t] = 0$, drifts very slowly in time towards one of the endpoints of the interval. A plot of the numerical solution to (3.82)-(3.83) at different times for a particular value of ϵ is shown in Fig. 1. Experimental results showing a seemingly stable flame-front interface in a vertical channel are shown in [**30**].

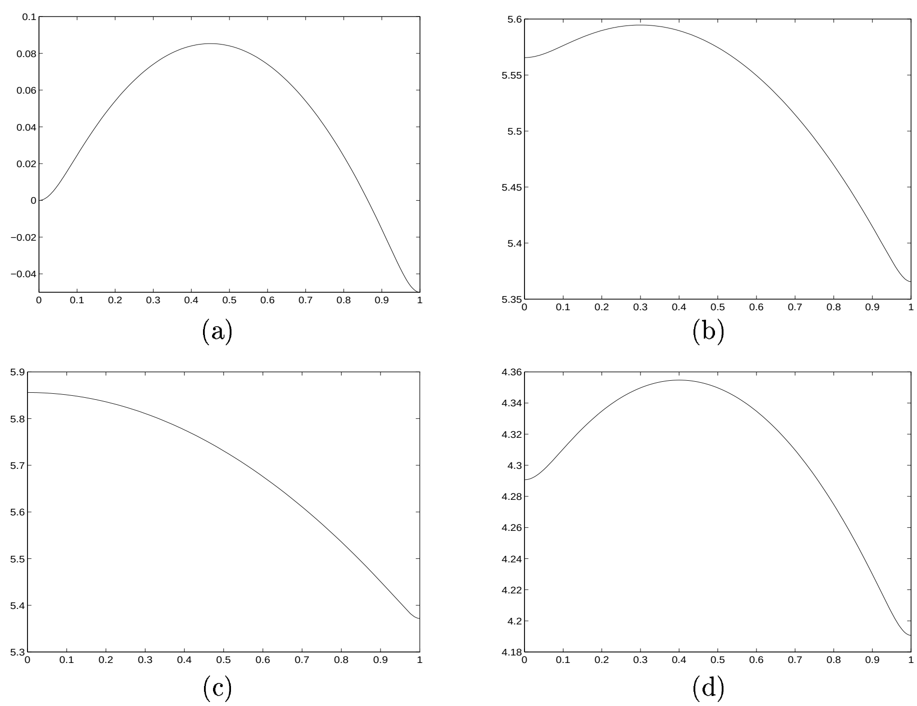

FIGURE 1. Plots of $y(x,t)$ versus x with $\epsilon = .0115$ computed numerically from (3.82)-(3.83). (*a*) initial condition $y(x,0)$ with tip location at $x_0 = 0.45$; (*b*) solution at $t = 90.05$ where $x_0 = 0.4$; (*c*) solution at $t = 113.69$ where $x_0 = 0.3$; (*d*) final stable equilibrium solution at $t \geq 117.07$.

The analysis of [**7**] proved that this interface is dynamically metastable in the sense that the tip location $x_0(t)$ remains in a small neighborhood of its initial value for an asymptotically exponentially long time interval as $\epsilon \to 0$. In [**47**], this metastable behavior was analyzed asymptotically using the projection method and an explicit ODE for $x_0(t)$ was derived.

To analyze (3.82)-(3.83) it is very convenient to introduce the new variable u by $u(x,t) = -y_x(x,t)$. This leads to the following problem for $u(x,t)$ (cf. [**36**]):

$$u_t + uu_x - u = \epsilon\, u_{xx}\,, \qquad 0 < x < 1\,, \quad t > 0\,, \tag{3.84}$$

$$u(0,t) = u(1,t) = 0\,; \qquad u(x,0) = u_0(x)\,. \tag{3.85}$$

As shown in [**7**], this problem has several equilibrium solutions. However, only the equilibrium solution that corresponds to a concave interface is metastable. For $\epsilon \to 0$, the leading order boundary layer analysis of [**47**] showed that the equilibrium solution $U(x;\epsilon)$ for (3.84)-(3.85) corresponding to a concave parabolic-shaped flame-front interface $Y(x;\epsilon)$ has the form

$$U(x;\epsilon) \sim \tilde{u}^{\epsilon}[x;x_{0e}] \equiv x - x_{0e} + u_l[x/\epsilon;x_{0e}] + u_r[(1-x)/\epsilon;x_{0e}]\,, \tag{3.86}$$

for some undetermined $x_{0e} \in (0,1)$. Here the boundary layer functions u_l and u_r are defined by

$$u_l(y;\alpha) \equiv \alpha\left(1 - \tanh\left(\alpha y/2\right)\right)\,, \quad u_r(y;\alpha) \equiv (\alpha - 1)\left(1 - \tanh\left[(1-\alpha)y/2\right]\right)\,. \tag{3.87}$$

We refer to $\tilde{u}^{\epsilon}[x;x_0]$ as the quasi-equilibrium solution.

In Fig. 2, we plot $\tilde{u}^{\epsilon}[x;x_{0e}]$ for a fixed $x_{0e} \in (0,1)$. Since $U(x_{0e};\epsilon) = O(e^{-c/\epsilon})$ for some $c > 0$ and $Y_x = -U$, it follows that x_{0e} asymptotically represents the tip, or nose, location for the parabolic-shaped equilibrium interface $Y = Y(x:\epsilon)$. By symmetry we expect that the correct value for x_{0e} is $x_{0e} = 1/2$. However, this value for x_{0e} still cannot be determined by the method of matched asymptotic expansions even after one performs a higher order boundary layer analysis near the endpoints $x = 0$ and $x = 1$. To explain this apparent indeterminacy in x_{0e}, we note that $\tilde{u}^{\epsilon}_x - 1 = O(e^{-c/\epsilon})$ in the outer region $O(\epsilon) \ll x \ll 1 - O(\epsilon)$. Thus, in this region, the equation $\epsilon u_{xx} - u(u_x - 1) = 0$ is satisfied by $\tilde{u}^{\epsilon}$ to within exponentially small terms as $\epsilon \to 0$ for any choice $x_{0e} \in (0,1)$. Hence, this problem is exponentially ill-conditioned and exponential precision is required to determine x_{0e}.

We now briefly outline the analysis and the results given in [**47**]. The metastable dynamics for (3.84)-(3.85) is represented by

$$u(x,t) = \tilde{u}^{\epsilon}[x;x_0(t)] + v(x,t)\,, \tag{3.88}$$

where $\tilde{u}^{\epsilon}$ is defined in (3.86) and $v \ll \tilde{u}^{\epsilon}$ and $v_t \ll \partial_t \tilde{u}^{\epsilon}$. Here $x_0 = x_0(t)$ is the unknown trajectory of the tip of the flame-front interface. The quasi-steady linearization of (3.84)-(3.85) around $\tilde{u}^{\epsilon}$ yields

$$L_{\epsilon} v \equiv \epsilon v_{xx} - \left(\tilde{u}^{\epsilon} v\right)_x + v = -R_{\epsilon}(x;x_0) + x_0^{'} \partial_{x_0} \tilde{u}^{\epsilon}\,, \qquad 0 < x < 1\,, \tag{3.89}$$

$$v(0,t) = -\tilde{u}^{\epsilon}[0;x_0]\,, \qquad v(1,t) = -\tilde{u}^{\epsilon}[1;x_0]\,. \tag{3.90}$$

Here $x_0^{'} \equiv dx_0/dt$ and the residual R_{ϵ} is defined by

$$R_{\epsilon}(x;x_0) \equiv \epsilon \partial_{xx} \tilde{u}^{\epsilon} - \tilde{u}^{\epsilon}\left(\partial_x \tilde{u}^{\epsilon} - 1\right)\,. \tag{3.91}$$

Let $x_0 \in (0,1)$ be fixed, and consider the associated eigenvalue problem

$$L_{\epsilon}\phi = \lambda\phi\,, \quad 0 < x < 1\,; \qquad \phi(0) = \phi(1) = 0\,, \tag{3.92}$$

$$(\phi,\phi)_w = \int_0^1 \phi^2 w\, dx = 1\,, \qquad w \equiv \exp\left(-\epsilon^{-1}\int_{x_0}^{x} \tilde{u}^{\epsilon}[z;x_0]\, dz\right)\,. \tag{3.93}$$

The eigenvalues λ_j for $j \geq 0$ are real and the orthogonality relation $(\phi_j,\phi_k)_w = \delta_{jk}$ for $j,k = 0,1,..,$ holds.

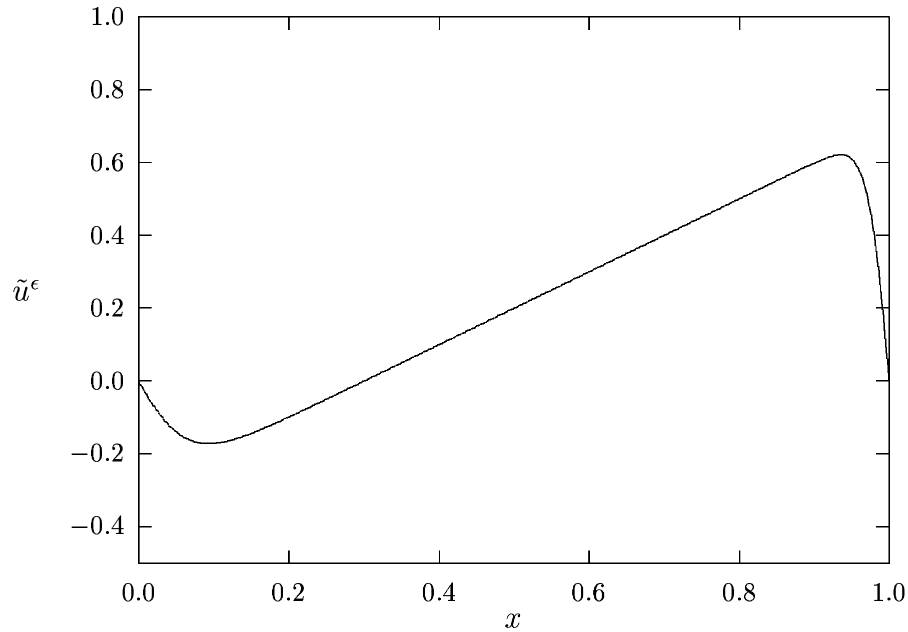

FIGURE 2. Plot of $\tilde{u}^\epsilon$ versus x when $x_{0e} = 0.3$ and $\epsilon = 0.01$.

As discussed in §1, since $\tilde{u}^\epsilon[x; x_0]$ almost satisfies the equilibrium problem for any $x_0 \in (0,1)$, then $L_\epsilon\left(\partial_{x_0}\tilde{u}^\epsilon\right)$ is uniformly small on $(0,1)$. Hence, (3.92)-(3.93) should have an eigenfunction that is close to $\partial_{x_0}\tilde{u}^\epsilon$. Since $\partial_{x_0}\tilde{u}^\epsilon$ is of one sign, the principal eigenvalue of (3.92)-(3.93) has the form $\phi_0 \sim M_0\partial_{x_0}\tilde{u}^\epsilon$, where M_0 is a normalization constant (cf. [**47**]).

Using this form for ϕ_0 as an asymptotically accurate trial function for a Rayleigh quotient, it was shown in [**47**] that the corresponding eigenvalue λ_0 is exponentially small as $\epsilon \to 0$ and has the estimate

(3.94)
$$\lambda_0 \sim \frac{1}{\epsilon}\left[x_0\left(x_0 - \epsilon^{1/2}c\right)e^{-x_0^2/2\epsilon} + (1-x_0)\left((1-x_0) - \epsilon^{1/2}c\right)e^{-(1-x_0)^2/2\epsilon}\right],$$

where $c = \sqrt{8/\pi}$.

To derive an equation of motion for $x_0(t)$, we expand $v(x,t)$ in terms of the eigenfunctions ϕ_j of (3.92)-(3.93) as

$$v(x,t) = \sum_{j=0}^{\infty}\frac{c_j}{\lambda_j}\phi_j\,. \tag{3.95}$$

Upon integrating by parts, we find that the coefficient c_j is given by

$$c_j = x_0^{'}\left(\phi_j, \partial_{x_0}\tilde{u}^\epsilon\right)_w - \left(\phi_j, R_\epsilon\right)_w + \epsilon w v \phi_j^{'}\Big|_0^1\,, \tag{3.96}$$

where w and the inner product are defined in (3.93). Here $x_0' \equiv dx_0/dt$. Since $\lambda_0 \to 0$ exponentially as $\epsilon \to 0$, we impose the limiting solvability condition $c_0 \to 0$ as $\epsilon \to 0$. This condition yields a differential equation for $x_0(t)$. After a lengthy calculation of evaluating the inner product $(\phi_0, R_\epsilon)_w$ as $\epsilon \to 0$, the following metastability result was obtained in [**47**]:

Proposition: (Metastability) *For initial data corresponding to a concave interface and for $t \gg 1$, $\epsilon \ll 1$ the solution $u(x,t)$ to (3.84)–(3.85) is given by*

$$u(x,t) \sim \tilde{u}^\epsilon[x; x_0(t)] \equiv x - x_0(t) + u_l[x/\epsilon; x_0(t)] + u_r[(1-x)/\epsilon; x_0(t)]\,, \tag{3.97}$$

where u_l and u_r were defined in (3.87). The tip $x_0 = x_0(t)$ of the flame-front interface satisfies the nonlinear asymptotic differential equation

$$x_0' \sim \left(\frac{2}{\pi\epsilon}\right)^{1/2} \left[\left((1-x_0)^2 + \frac{\pi^2\epsilon}{3}\right) e^{-(1-x_0)^2/2\epsilon} - \left(x_0^2 + \frac{\pi^2\epsilon}{3}\right) e^{-x_0^2/2\epsilon}\right]. \tag{3.98}$$

The initial condition $x_0(0)$ for (3.98) is found from a transient analysis describing the formation of the interface from initial data. The ODE (3.98) differs significantly in form from the corresponding result (3.76) for the viscous shock problem. From (3.98), the (unstable) equilibrium value for x_0 is $x_{0e} = 1/2$. Also note that $x_0(t)$ will collapse against the endpoint $x = 1$ ($x = 0$) on an exponentially long time interval when $x_0(0) > 1/2$ ($x_0(0) < 1/2$). This qualitative behavior is observed in the computational results of [**36**]. In [**47**], the asymptotic results (3.94) and (3.98) are favorably compared with full numerical results.

4. Phase Separation Models

In §4.2 we consider some reaction-diffusion problems in one spatial dimension modeling the phase separation of a binary material. Here the main feature is the occurrence of metastable dynamics of a collection of interfaces, or internal layers. The projection method is used to derive a coupled system of differential equations characterizing the slow dynamics of the interfaces. In §4.1 we consider a nonlinear differential equation introduced in [**9**] known as the Carrier-Pearson (CP) problem. The metastable dynamics of the phase separation problems is shown to be very closely related to the CP problem.

4.1. The Carrier-Pearson Problem.

One of the first examples of a nonlinear singular perturbation problem having a matched asymptotic expansion solution with undetermined coefficients is the CP problem for $u = u(x)$ introduced in [**9**]

$$\epsilon^2 u'' + 2(u - u^3) = 0\,, \quad -1 < x < 1 \tag{4.1}$$

$$u'(-1) = u'(1) = 0\,. \tag{4.2}$$

A phase plane analysis shows that (4.1)-(4.2) has many types of solutions when $\epsilon \ll 1$. We first try to construct a solution for which $u \sim -1$ for $-1 < x < x_0$ and $u \sim 1$ for $x_0 < x < 1$. Here x_0, called the shock-layer location, satisfies $-1 < x_0 < 1$. In the neighborhood of x_0 the solution changes very quickly on a scale of $O(\epsilon)$. In this region we let $z = \epsilon^{-1}(x - x_0)$ and $u \sim u_s(z)$ where the shock layer profile $u_s(z)$ satisfies

$$u_s'' + 2(u_s - u_s^3) = 0\,, \quad -\infty < z < \infty\,; \qquad u_s(\pm\infty) = \pm 1\,, \quad u_s(0) = 0\,. \tag{4.3}$$

The solution is $u_s(z) = \tanh(z)$. Therefore, the matched asymptotic approximation to a one-shock layer solution to (4.1)-(4.2) is

$$u(x) \sim \tilde{u}^{\epsilon}[x; x_0] \equiv \tanh\left[\epsilon^{-1}(x - x_0)\right] . \tag{4.4}$$

Notice that $\tilde{u}^{\epsilon}$ satisfies (4.1) exactly, but fails to satisfy the boundary conditions (4.2) by exponentially small terms as $\epsilon \to 0$ for any $x_0 \in (-1, 1)$. Thus, determining the correct value $x_0 = 0$ within the framework of the method of matched asymptotic expansions requires exponential precision.

One method to resolve this indeterminacy in selecting x_0 is to use the projection method as was done in [**51**]. Other methods to calculate x_0 are found in the references of [**51**]. We first linearize (4.1)-(4.2) about $\tilde{u}^{\epsilon}$ by writing

$$u(x) = \tilde{u}^{\epsilon}[x; x_0] + v(x) , \tag{4.5}$$

where $v \ll \tilde{u}^{\epsilon}$. We obtain that v satisfies

$$\epsilon^2 v_{xx} + \left(-4 + 6\,\mathrm{sech}^2\left[\epsilon^{-1}(x - x_0)\right]\right) v = 0 , \quad -1 < x < 1 , \tag{4.6}$$

$$v_x(-1, t) = -\tilde{u}^{\epsilon}_x(-1; x_0) , \qquad v_x(1, t) = -\tilde{u}^{\epsilon}_x(1; x_0) . \tag{4.7}$$

The corresponding eigenvalue problem is

$$L_{\epsilon}\phi \equiv \epsilon^2 \phi_{xx} + \left(-4 + 6\,\mathrm{sech}^2\left[\epsilon^{-1}(x - x_0)\right]\right) \phi = \lambda\phi , \quad -1 < x < 1 , \tag{4.8}$$

$$\phi_x(\pm 1) = 0 , \qquad (\phi, \phi) \equiv \int_{-1}^{1} \phi^2 \, dx = 1 . \tag{4.9}$$

Notice that this eigenvalue problem is very similar in form to the eigenvalue problem (3.80)-(3.81) derived in §3.3 for Burgers equation.

Let $x_0 \in (-1, 1)$ and define $\tilde{\phi}_0 \equiv \partial_{x_0} \tanh\left[\epsilon^{-1}(x - x_0)\right]$. Since $L_{\epsilon}(\tilde{\phi}_0) = 0$, $\tilde{\phi}_0$ is of one sign and $\tilde{\phi}_0$ fails to satisfy the boundary conditions in (4.9) by only exponentially small terms, then the principal eigenvalue λ_0 of (4.8)-(4.9) is exponentially small and the corresponding eigenfunction ϕ_0 is asymptotically close to $\tilde{\phi}_0$. A boundary layer analysis is then used to satisfy the boundary conditions in (4.9) and we obtain

$$\phi_0 \sim M_0 \left(\tilde{\phi}_0 - 4\epsilon^{-1} e^{-2\epsilon^{-1}(1 - x_0)} e^{-2\epsilon^{-1}(1 - x)} - 4\epsilon^{-1} e^{-2\epsilon^{-1}(1 + x_0)} e^{-2\epsilon^{-1}(1 + x)} \right) , \tag{4.10}$$

where M_0 is a normalization constant. To estimate λ_0 we use the following identity, which is readily derived by integration by parts:

$$\lambda_0 \left(\phi_0, \tilde{\phi}_0 \right) \sim -\epsilon^2 \phi_0 \tilde{\phi}_{0x} \Big|_{-1}^{1} . \tag{4.11}$$

The dominant contribution to the inner product arises from the region near $x = x_0$. We calculate,

$$\left(\phi_0, \tilde{\phi}_0 \right) \sim M_0 \epsilon^{-1} \int_{-\infty}^{\infty} \left[\mathrm{sech}(z)\right]^4 \, dz = 4 M_0 \epsilon^{-1}/3 . \tag{4.12}$$

Finally, substituting (4.12) and (4.10) into (4.11) we obtain the estimate for λ_0

$$\lambda_0 \sim 48 \left(e^{-4\epsilon^{-1}(1 - x_0)} + e^{-4\epsilon^{-1}(1 + x_0)} \right) . \tag{4.13}$$

Next, we expand the solution v to (4.6)-(4.7) in terms of the normalized eigenfunctions of (4.8)-(4.9) as

$$v = \sum_{j=0}^{\infty} \frac{c_j}{\lambda_j} \phi_j \,. \tag{4.14}$$

Using orthogonality, we obtain the coefficients

$$c_j = -\epsilon^2 \phi_j v_x \Big|_{-1}^{1} \,. \tag{4.15}$$

Since $\lambda_0 \to 0$ exponentially as $\epsilon \to 0$, we require that the limiting solvability condition $c_0 \to 0$ be satisfied. This condition yields

$$\phi_0(1)\tilde{u}_x^{\epsilon}[1; x_0] \sim \phi_0(-1)\tilde{u}_x^{\epsilon}[-1; x_0] \,. \tag{4.16}$$

Substituting (4.10) and (4.4) into (4.16) we get that x_0 satisfies

$$e^{-4\epsilon^{-1}(1-x_0)} \sim e^{-4\epsilon^{-1}(1+x_0)} \,. \tag{4.17}$$

Therefore, $x_0 = 0$ and the equilibrium solution is $u(x) \sim \tanh(\epsilon^{-1}x)$.

Now consider the corresponding time-dependent problem associated with (4.1)-(4.2)

$$u_t = \epsilon^2 u_{xx} + 2(u - u^3) \,, \qquad -1 < x < 1 \,, \quad t > 0 \,, \tag{4.18}$$

$$u_x(\pm 1, t) = 0 \,; \qquad u(x, 0) = u_0(x) \,, \tag{4.19}$$

This is a special case of the Allen-Cahn equation. For a certain class of initial data, this problem leads to the formation of a one-shock layer solution of the form

$$u(x,t) \sim \tilde{u}^{\epsilon}[x; x_0(t)] \equiv \tanh\left[\epsilon^{-1}(x - x_0(t))\right] \,. \tag{4.20}$$

The motion of $x_0(t)$ is metastable since the principal eigenvalue λ_0 of (4.8)-(4.9), as estimated in (4.13), is exponentially small. To derive an equation of motion for $x_0(t)$ we linearize (4.18)-(4.19) about $\tilde{u}^{\epsilon}$ by writing

$$u(x,t) = \tilde{u}^{\epsilon}[x; x_0(t)] + v(x,t) \,, \tag{4.21}$$

where we assume that $v \ll \tilde{u}^{\epsilon}$ and $v_t \ll \partial_t \tilde{u}^{\epsilon}$. We then obtain that v satisfies (4.6)-(4.7) where the right side of (4.6) is replaced by $\partial_t \tilde{u}^{\epsilon}$. We then expand v as in (4.14) in terms of coefficients $c_j(t)$. Imposing the limiting solvability condition $c_0 \to 0$ as $\epsilon \to 0$, we obtain the following differential equation for $x_0(t)$:

$$\left(\partial_t \tilde{u}^{\epsilon}, \phi_0\right) \sim -\epsilon^2 \phi_0 \tilde{u}_x^{\epsilon} \Big|_{-1}^{1} \,. \tag{4.22}$$

To evaluate the terms in (4.22) we use (4.10) and (4.12) to get the following nonlinear asymptotic differential equation for $x_0(t)$:

$$x_0^{'} \sim 24 \left(e^{-4\epsilon^{-1}(1-x_0)} - e^{-4\epsilon^{-1}(1+x_0)}\right) \,. \tag{4.23}$$

Notice that $x_0(t)$ will collapse against the right wall (left wall) at $x = 1$ ($x = -1$) when the initial data $x_0(0)$ satisfies $x_0(0) > 0$ ($x_0(0) < 0$).

The simple calculations illustrated above can be extended to construct an asymptotic solution to (4.1)-(4.2) with n internal layers located at $x = x_i$ for

$i = 0, .., n-1$, where $|x_i| < 1$. Here $x_i < x_{i+1}$ for $i = 0, .., n-1$. In place of (4.4), $\tilde{u}^\epsilon$ now has the form $u(x) \sim \tilde{u}^\epsilon[x; x_0, .., x_{n-1}]$, where

(4.24)

$$\tilde{u}^\epsilon[x; x_0, .., x_{n-1}] \equiv \tanh\left[\epsilon^{-1}(x - x_0)\right] + \sum_{i=0}^{n-1} \left(\tanh\left[(-1)^i \epsilon^{-1}(x - x_i)\right] - (-1)^{i-1}\right) .$$

Here the x_i for $i = 0, .., n-1$ are to be determined. When the x_i are widely separated in the sense that $x_{i+1} - x_i \gg O(\epsilon)$ for $i = 0, .., n-1$, the first n eigenvalues of the corresponding linearized eigenvalue problem are exponentially small as $\epsilon \to 0$. By imposing n limiting solvability conditions we then obtain a set of algebraic equations for the x_i as in [**51**]. In a similar way, by letting $x_i = x_i(t)$ we can quantify the metastable dynamics of a collection of internal layers for the corresponding time-dependent problem (4.18)-(4.19). The analysis is given in [**52**].

4.2. The Viscous Cahn-Hilliard Equation. The viscous Cahn-Hilliard equation, introduced in [**37**], is a model of slow phase separation in binary alloys accounting for viscoelastic effects. In dimensionless form, this model is

$$(4.25) \qquad (1-\alpha)u_t = -\left(\epsilon^2 u_{xx} + Q(u) - \alpha\kappa u_t\right)_{xx} , \qquad -1 < x < 1 , \quad t > 0 ,$$

$$(4.26) \qquad u_x(\pm 1, t) = u_{xxx}(\pm 1, t) = 0 ; \qquad u(x, 0) = u_0(x) ,$$

where $u(x,t)$ is the concentration of one of the two components in the alloy. Here $\kappa > 0$ is the viscoelastic parameter, $\epsilon \to 0^+$ is the interfacial energy parameter, α with $0 \le \alpha < 1$ is a homotopy parameter, and $Q(u) = -V'(u)$ where $V(u)$ is a double-well potential with wells of equal depth. More specifically, we assume that $Q(u)$ has exactly three zeros at $u = s_- < 0$, $u = s_+ > 0$ and $u = 0$, with

$$(4.27) \qquad Q'(s_\pm) < 0 , \qquad Q'(0) > 0 , \qquad V(s_\pm) = 0 .$$

Prototypical is $Q(u) = 2(u - u^3)$, for which $s_\pm = \pm 1$ and $V(u) = (1-u^2)^2/2$. Since $Q(u)$ is non-monotone, the reduced equation $(1-\alpha)u_t = -\left[Q(u)\right]_{xx}$ is a backward heat equation for some range of u and consequently is ill-posed. The terms $-\epsilon^2 u_{xxxx}$ and κu_{txx} in (4.25) represent a gradient energy regularization and a viscoelastic regularization, respectively, of this ill-posed reduced equation. Note that the mass $m = \int_{-1}^{1} u(x,t)\, dx$ is conserved for (4.25)-(4.26). We assume below that $u_0(x)$ is such that $2s_- < m < 2s_+$.

Some related phase separation models are obtained by letting α take on limiting values in (4.25). The well-known Cahn-Hilliard model corresponds to $\alpha = 0$. If $\alpha = 1$, we can integrate the right side of (4.25) twice, explicitly impose a mass constraint, and re-scale t to obtain the constrained Allen-Cahn equation introduced in [**45**]

$$(4.28) \qquad u_t = \epsilon^2 u_{xx} + Q(u) - \sigma , \quad -1 < x < 1 , \quad t > 0 ,$$

$$(4.29) \qquad u_x(\pm 1, t) = 0 ; \qquad u(x,0) = u_0(x) ; \qquad \int_{-1}^{1} u(x,t)\, dx = m .$$

Here $\sigma = \sigma(t)$ is determined by the constant mass m. The well-studied unconstrained Allen-Cahn equation is obtained by setting $\sigma \equiv 0$ in (4.28) and disregarding the mass constraint. For an overview of mathematical problems and results for phase separation models see [**12**].

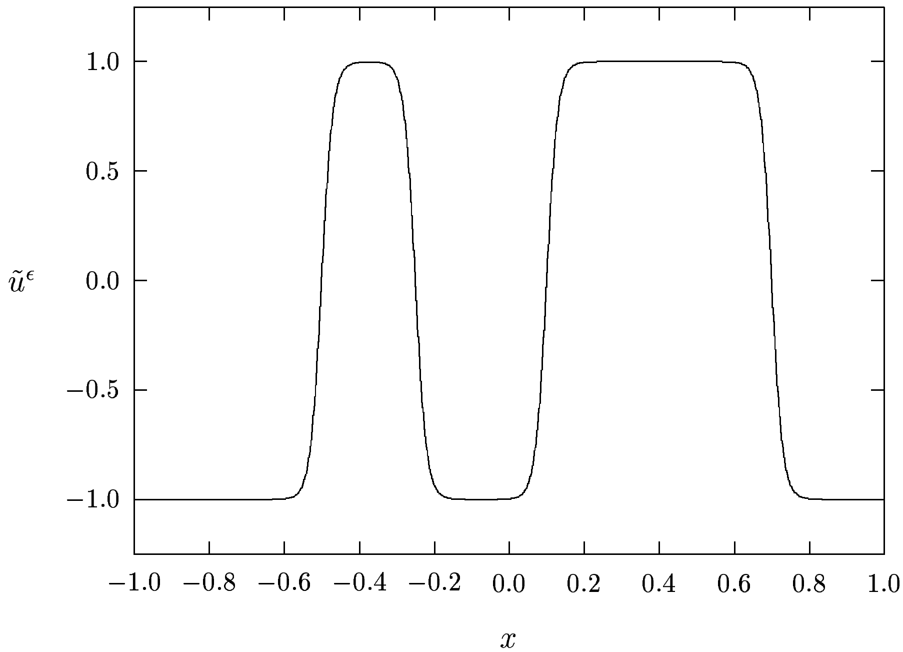

FIGURE 3. Schematic plot of a metastable pattern of internal layers.

There has been much recent work analyzing the dynamics associated with (4.25)-(4.26) and related models. These studies have revealed that the dynamics proceeds in two stages when ϵ is small. The first stage, occurring on an $O(1)$ time interval, involves the transient formation of a pattern of internal layers from initial data. The layers have width $O(\epsilon)$ and separate the two phases s_+ and s_- (see Fig. 3 for a schematic plot of a four layer pattern). This transient process is very intricate for (4.25)-(4.26), but is significantly less complex for the unconstrained Allen-Cahn equation. During the next stage of the dynamics, known as the coarsening process, the internal layers move exponentially slowly in time until, typically, they collapse together in pairs. For the unconstrained Allen-Cahn equation this process terminates when no layers remain. However, for models where mass is conserved, this process terminates when a pattern with only one layer, which is consistent with the mass, is attained.

The coarsening process for the unconstrained Allen-Cahn equation has been well-studied in [**10**], [**13**], [**52**]. The existence of metastable internal layer motion has been proved in [**3**], [**6**], [**8**], and [**15**] for the Cahn-Hilliard equation and in [**24**] for a system very similar in form to the constrained Allen-Cahn equation. An explicit characterization of metastability for the Cahn-Hilliard equation is given in [**6**] using a dynamical systems approach. In [**28**], [**41**], and [**43**] an asymptotic projection method is used to obtain similar results for the viscous Cahn-Hilliard equation (4.25)-(4.26) and for the constrained Allen-Cahn equation (4.28)-(4.29), respectively.

We now outline the metastability analysis in [**43**]. To begin, it is convenient to re-write (4.25)-(4.26) as a coupled system for $u(x,t)$ and $\sigma(x,t)$

$$\alpha\kappa u_t = \epsilon^2 u_{xx} + Q(u) - \sigma\,, \qquad u_x(\pm 1, t) = 0\,, \tag{4.30}$$

$$(1-\alpha)u_t = -\sigma_{xx}\,, \qquad \sigma_x(\pm 1, t) = 0\,. \tag{4.31}$$

For the equilibrium problem, σ is a constant and is asymptotically exponentially small as $\epsilon \to 0$. In [**43**] it is assumed that $\sigma(x,t)$ is also asymptotically exponentially small as $\epsilon \to 0$ for a metastable pattern with widely separated internal layers. Therefore, each layer is closely approximated by the stationary wave solution of the generalized CP problem $\epsilon^2 u_{xx} + Q(u) = 0$ on the infinite line, which connects the two states $u = s_+$ and $u = s_-$. Thus, we generalize (4.3) and introduce the heteroclinic orbit $u_s(z)$, which is the unique solution to

$$u_s^{''} + Q(u_s) = 0\,, \quad -\infty < z < \infty\,; \qquad u_s(\pm\infty) = s_\pm\,, \quad u_s(0) = 0\,, \tag{4.32}$$

with $u_s^{'}(z) > 0$. This solution has the far-field asymptotic behavior

$$u_s(z) \sim s_\pm \mp a_\pm e^{\mp \nu_\pm z}\,, \quad \text{as} \quad z \to \pm\infty\,; \qquad \nu_\pm \equiv \left[-Q^{'}(s_\pm)\right]^{1/2}\,, \tag{4.33}$$

for some explicit constants $a_\pm > 0$. When $Q(u) = 2(u - u^3)$, we have $\nu_\pm = 2$ and $a_\pm = 2$.

As suggested in (4.24) above, an n-layer metastable pattern is represented as a superposition of translates of this heteroclinic connection. Let $x_i = x_i(t)$ for $i = 0, .., n-1$ be the internal layer locations for such a pattern and define $x_{-1} = -1$ and $x_n = 1$. We assume that the layers satisfy the ordering $x_{i+1} > x_i$, and that the layers are widely separated at time t in the sense that $x_{i+1} - x_i \gg O(\epsilon)$ for $i = 0, .., n-1$. Then, an n-layer metastable pattern for (4.25)-(4.26) is represented by

$$u(x,t) \sim \tilde{u}^\epsilon[x; x_0, .., x_{n-1}] \equiv u_s\left[\xi_0(x - x_0)/\epsilon\right] + \sum_{i=1}^{n-1} \left(u_s\left[\xi_i(x - x_i)/\epsilon\right] - s_i\right)\,, \tag{4.34}$$

where $x_i = x_i(t)$ for $i = 0, .., n-1$ are to be determined. Here $\xi_i = (-1)^i \xi_0$, where $\xi_0 = +1$ or $\xi_0 = -1$ is the orientation of the first layer. In addition, $s_i = s_+$ when $\xi_i = -1$ and $s_i = s_-$ when $\xi_i = +1$. For instance, in Fig. 3 we have $n = 4$, $\xi_0 = 1$ and $\xi_i = (-1)^i$ for $i = 1, 2, 3$. Notice that the trajectories $x_i = x_i(t)$ are asymptotically exponentially close to the zeroes of $u(x,t)$ as a function of x.

The projection method used in [**43**] provides an explicit differential-algebraic system of ODE's for the $x_i(t)$, $i = 0, .., n-1$, in (4.34). To derive this system, we first perform a quasi-steady linearization of (4.30)-(4.31) around $\tilde{u}^\epsilon$ by substituting

$$u(x,t) = \tilde{u}^\epsilon\left[x;\, x_0(t), .., x_{n-1}(t)\right] + v(x,t) \tag{4.35}$$

into (4.30)-(4.31), where $v \ll u^*$ and $v_t \ll \partial_t \tilde{u}^\epsilon$. From (4.31) we get

$$\sigma \sim (1-\alpha)\sum_{i=0}^{n-1} x_i^{'} M_i(x) + \sigma_c\,, \quad M_i(x) \equiv \int_{-1}^{x} \left(u_s\left[\xi_i(\eta - x_i)/\epsilon\right] - s_i\right)\, d\eta\,, \tag{4.36}$$

provided that the mass constraint $m = \int_{-1}^{1} \tilde{u}^\epsilon \, dx$ holds. In (4.36), $\sigma_c = \sigma_c(t)$ is to be determined. Then, from (4.30), we find that v satisfies

$$L_\epsilon v \equiv \epsilon^2 v_{xx} + Q^{'}(\tilde{u}^\epsilon) v = \sigma + E + \alpha\, \kappa\, \partial_t \tilde{u}^\epsilon \,, \tag{4.37}$$

$$v_x(-1,t) = -\tilde{u}_x^\epsilon\,(-1; x_0,..,x_{n-1})\ , \qquad v_x(1,t) = -\tilde{u}_x^\epsilon\,(1; x_0,..,x_{n-1})\ . \tag{4.38}$$

Here E represents the exponentially weak interactions between neighboring layers, and is defined by

$$E \equiv E(x;\, x_0,..,x_{n-1}) = \sum_{i=0}^{n-1} Q\left(u_s\left[\xi_i(x-x_i)/\epsilon\right]\right) - Q\left(\tilde{u}^\epsilon\right)\ . \tag{4.39}$$

Let x_i for $i = 0,..,n-1$ be fixed and consider the associated eigenvalue problem

$$L_\epsilon \phi = \lambda \phi\,, \quad -1 < x < 1\,; \qquad \phi^{'}(\pm 1) = 0\,; \qquad (\phi,\phi) \equiv \int_{-1}^{1} \phi^2\, dx = 1\,, \tag{4.40}$$

where L_ϵ is defined in (4.37).

This eigenvalue problem generalizes (4.8)-(4.9) to the case of n internal layers. The first n eigenvalues λ_i, for $i = 0,..,n-1$, of (4.40) are exponentially small as $\epsilon \to 0$ and the corresponding (un-normalized) eigenfunctions are given asymptotically by $\phi_i(x) \sim u_s^{'}\left[\xi_i(x-x_i)/\epsilon\right]$ for $i = 0,..,n-1$ (cf. [**10**], [**51**]). This key property arises as a result of the combined effects of $u_s^{'} > 0$, the decay behavior (4.33), and the near translation invariance of the system.

Next, we expand the solution to (4.37)-(4.38) in terms of the eigenfunctions of (4.40) as

$$v(x,t) = \sum_{i=0}^{\infty} \frac{c_i(t)}{\lambda_i} \phi_i(x)\,. \tag{4.41}$$

Using orthogonality, we obtain from (4.37)-(4.38) that the coefficients are given by

$$c_i = (\sigma,\phi_i) + (E,\phi_i) + \alpha\kappa\left(\partial_t \tilde{u}^\epsilon, \phi_i\right) - \epsilon^2 \phi_i v_x \Big|_{-1}^{1}\ . \tag{4.42}$$

Since λ_i for $i = 0,..,n-1$ are exponentially small, we must impose the n limiting solvability conditions that $c_i \to 0$ as $\epsilon \to 0$ for $i = 0,..,n-1$. By enforcing this projection condition and by substituting (4.36) into (4.42), we obtain a differential-algebraic (DAE) system of degree n for the n unknowns $\sigma_c(t)$ and $x_i(t)$, for $i = 0,..,n-1$ (cf. [**43**]). This system has the form

$$(1-\alpha)\sum_{k=0}^{n-1} x_k^{'}\ (M_k,\phi_i) + (\sigma_c,\phi_i) + \alpha\kappa\left(\partial_t \tilde{u}^\epsilon,\phi_i\right) \sim \epsilon^2 \phi_i v_x \Big|_{-1}^{1} - (E,\phi_i)\ , \tag{4.43}$$

for $i = 0,..,n-1$, together with the mass constraint

$$\int_{-1}^{1} \tilde{u}^\epsilon\left[x; x_0,..,x_{n-1}\right]\, dx = m\,. \tag{4.44}$$

Here M_i and E are defined in (4.36) and (4.39), respectively. Finally, a lengthy calculation of asymptotically evaluating the various terms in (4.43)-(4.44) as $\epsilon \to 0$ leads to the following explicit result (cf. [**43**]):

Proposition (Metastable Motion): *For $\epsilon \to 0$, an n-layer metastable pattern for (4.25)-(4.26) with widely separated internal layers is represented by (4.34),* *where $x_i(t)$ for $i = 0, .., n-1$, and $\sigma_c(t)$ satisfy the explicit DAE system*

$$\alpha\kappa\beta\epsilon^{-1}\, x_i^{'} + (1-\alpha)\sum_{k=0}^{n-1} x_k^{'}\, b_{ik} \sim \sigma_c\, \xi_i(s_+ - s_-) + H_i\,, \quad i = 0, .., n-1\,, \tag{4.45}$$

$$\sum_{k=0}^{n} s_k\,(x_k - x_{k-1}) = m + O(\epsilon)\,. \tag{4.46}$$

In (4.45) the exponentially weak forces H_i for $i = 0, .., n-1$ and the coupling coefficient b_{ik} for $i, k = 0, .., n-1$ are defined by

$$H_i = 2\left(a_{i+1}^2 \nu_{i+1}^2\, e^{-(1+\delta_{i\, n-1})\nu_{i+1} d_{i+1}/\epsilon} - a_i^2 \nu_i^2\, e^{-(1+\delta_{i\,0})\nu_i d_i/\epsilon}\right)\,, \tag{4.47}$$

$$b_{ik} = \int_{-1}^{1} \left(u_s\left[\xi_k(x-x_k)/\epsilon\right] - s_k\right)\left(u_s\left[\xi_i(x-x_i)/\epsilon\right] - s_{i+1}\right)\, dx\,, \tag{4.48}$$

where δ_{ik} is the Kronecker symbol, $\beta = \int_{-\infty}^{\infty} [u_s^{'}(z)]^2\, dz$, $d_i = x_i - x_{i-1}$ for $i = 0, .., n$ are the inter-layer separations, and the triplet (s_i, a_i, ν_i) is defined by $(s_i, a_i, \nu_i) = (s_\pm, a_\pm, \nu_\pm)$ when $\xi_i = \mp 1$. Here $a_\pm$ and $\nu_\pm$ are defined in (4.33). To obtain a more explicit result, the coefficients b_{ik} in (4.48) can easily be evaluated asymptotically as $\epsilon \to 0$ (cf. [**43**]).

There are various special cases of (4.45)-(4.46). The constrained Allen-Cahn equation (4.28)-(4.29) corresponds to $\alpha = 1$ and $\kappa = 1$, and the Cahn-Hilliard equation corresponds to $\alpha = 0$. Note in (4.45) that there is a distinguished limit when $\alpha = O(\epsilon)$. The unconstrained Allen-Cahn equation is obtained by setting $\alpha = \kappa = 1$ and $\sigma_c = 0$ in (4.45) and disregarding the mass constraint (4.46). This leads to the well-known dynamics $x_i^{'} \sim \epsilon\beta^{-1} H_i$, for $i = 0, .., n-1$, which was proved in [**10**] and [**13**], and which generalizes the one-layer result (4.23). For the special case of a two-layer evolution (i. e. $n = 2$), the result (4.45)-(4.46) has been favorably compared in [**43**] with full numerical results for different values of α. Starting from initial conditions $x_i(0) = x_i^0$, with $d_i = x_i^0 - x_{i-1}^0 > 0$ for $i = 0, .., n$, the dynamics (4.45)-(4.46) is valid until the first time $t = t_c$ where $d_I(t_c) = O(\epsilon)$ for some $I \in \{0, , ., n\}$. Thus, the metastability result does not hold when a layer collapse event is initiated. A quantitative analysis of the coarsening process for (4.25)-(4.26), which describes both the metastable dynamics and the sudden collapse events, is given in [**48**].

Finally, we remark that the projection method described above is closely related to a similar method used in [**4**] and [**5**] to analyze weakly interacting pulse-type solutions for other classes of nonlinear evolution equations. In many of these other problems, the localized pulse solution is a translate of a homoclinic orbit, which has exponentially damped oscillations at infinity. For a train of well-separated pulses, this damped oscillatory far-field behavior can lead to chaotic dynamics between neighboring pulses (cf. [**5**]). A very interesting survey of results for this class of problems is given in [**4**].

5. An Activator-Inhibitor Problem

In 1957 Turing [**49**] proposed a mathematical model for morphogenesis that describes the development of complex organisms from a single cell. He speculated that localized peaks, or spikes, in the concentration of a chemical substance, known as an activator, could be responsible for a group of cells developing differently from the surrounding cells. From a simplified linear analysis he showed how a coupled system of reaction-diffusion equations could generate isolated peaks in the activator concentration. To model morphogenesis, a general and highly nonlinear activator-inhibitor system, known as the Gierer Meinhardt model of morphogenesis, was developed in [**14**]. The numerical finite difference computations of [**14**], and more recently those of [**19**], showed that such a system can indeed support spike solutions for the full nonlinear problem.

In the limit for which the inhibitor diffuses much more quickly than the activator, the dimensionless activator concentration $a(x,t)$ and the inhibitor field $h(t)$ were found in [**20**] to satisfy the following nonlocal reaction-diffusion model in one spatial dimension:

$$a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{h^q}, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{5.1}$$

$$h = \left(\frac{1}{2\epsilon\mu} \int_{-1}^{1} a^m \, dx \right)^{\frac{1}{s+1}}, \tag{5.2}$$

$$a_x(\pm 1, t) = 0 \quad . \tag{5.3}$$

Here the exponents (p,q,m,s) satisfy

$$p > 1, \qquad q > 0, \qquad m > 0, \qquad s \geq 0, \qquad 0 < \frac{p-1}{q} < \frac{m}{s+1}\,. \tag{5.4}$$

In [**20**], a one-spike solution to (5.1)-(5.3) was shown to be metastable in the limit $\epsilon \to 0$. There are several key steps in the analysis. Firstly, a quasi-equilibrium solution to the steady-state problem corresponding to (5.1)-(5.3) was constructed. The center of the spike for this quasi-equilibrium solution is at an arbitrary location $x_0 \in (-1,1)$. Secondly, the stability of this solution was determined by analyzing the spectrum of the linearization of (5.1)-(5.3) about this quasi-equilibrium solution. The eigenvalue problem associated with this linearization is a nonlocal Sturm-Liouville problem, and it has one exponentially small eigenvalue. It was shown in [**20**] that the nonlocal term is essential for ensuring that this exponentially small eigenvalue is the principal eigenvalue associated with the linearization. This exponentially small principal eigenvalue leads to a dynamic metastability for the spike solution. To characterize the metastable spike dynamics, the projection method was then used to provide a limiting ODE for the location of the center of the spike. This ODE shows that the spike drifts exponentially slowly towards the endpoint of the domain that is closest to the initial location of the spike.

We now outline some of the details of the analysis in [**20**]. We first construct a one-spike quasi-equilibrium solution a_E for (5.1)-(5.3) in the form

$$a = a_E(x;x_0) \equiv h^\gamma u_c[\epsilon^{-1}(x-x_0)]\,, \qquad \gamma = q/(p-1)\,. \tag{5.5}$$

Here x_0, with $|x_0| < 1$, is the center of the spike. The function $u_c(y)$, called the canonical spike solution, satisfies

$$u_c'' - u_c + u_c^p = 0, \qquad 0 < y < \infty\,, \tag{5.6}$$

$$u_c'(0) = 0\,; \qquad u_c(y) \sim a e^{-y}\,, \qquad \text{as} \quad y \to \infty\,, \tag{5.7}$$

where $a > 0$ is a constant. It is easily seen from phase plane considerations that such a solution $u_c(y)$ exists. In terms of this solution, $h = h_E$, where

$$h_E \sim \left(\frac{\beta}{\mu}\right)^{\frac{p-1}{(s+1)(p-1)-qm}}, \qquad \beta \equiv \int_0^\infty [u_c(y)]^m \, dy \,. \tag{5.8}$$

For the special case $p = 2$, we have

$$u_c(y) = \frac{3}{2}\text{sech}^2\left(\frac{y}{2}\right), \qquad a = 6\,. \tag{5.9}$$

For other values of p, the function $u_c(y)$ and the constant a are readily computed numerically.

We note that, for any x_0 with $|x_0| < 1$, the solution $a_E(x;x_0)$ will satisfy the steady-state problem corresponding to (5.1), but will fail to satisfy the boundary conditions in (5.3) by only exponentially small terms as $\epsilon \to 0$. From the viewpoint of the method of matched asymptotic expansions, this indeterminacy in selecting the correct value of x_0 results from a need of exponential precision in the asymptotic analysis to resolve the exponentially weak interactions between the tail of the spike and the boundaries at $x = \pm 1$. From the viewpoint of ill-conditioning, this indeterminacy results from a near translation invariance of the underlying problem. More specifically, this means that that the spectrum of the eigenvalue problem associated with the linearization about a_E will contain an exponentially small eigenvalue.

5.1. A Nonlocal Eigenvalue Problem. Let $x_0 \in (-1,1)$ be fixed and linearize (5.1)-(5.3) around a_E, h_E, by introducing $\phi \ll 1$ and $\eta \ll 1$ by

$$a(x,t) = a_E(x;x_0) + e^{\lambda t}\phi \tag{5.10}$$

$$h(t) = h_E + e^{\lambda t}\eta\,. \tag{5.11}$$

Substituting (5.10)-(5.11) into (5.1)-(5.3) we obtain the following nonlocal eigenvalue problem of Sturm-Liouville type on $[-1,1]$:

$$L_\epsilon\phi \equiv \epsilon^2\phi_{xx} + (-1 + pu_c^{p-1})\phi - \frac{mq\epsilon^{-1}u_c^p}{2\beta(s+1)}\int_{-1}^1 u_c^{m-1}\phi\,dx = \lambda\phi, \tag{5.12}$$

$$\phi_x(\pm 1) = 0\,. \tag{5.13}$$

Since u_c is localized near x_0, we will only seek eigenfunctions that are localized near $x = x_0$. These eigenfunctions have the form

$$\tilde{\phi}(y) = \phi(x_0 + \epsilon y)\,, \qquad y = \epsilon^{-1}(x - x_0)\,. \tag{5.14}$$

Therefore, we can replace the finite interval by an infinite interval in the integral in (5.12) and impose a decay condition for $\tilde{\phi}(y)$ as $y \to \pm\infty$. This gives us the nonlocal eigenvalue problem for the infinite domain $-\infty < y < \infty$:

$$\tilde{L}_\epsilon\tilde{\phi} \equiv \tilde{\phi}_{yy} + (-1 + pu_c^{p-1})\tilde{\phi} - \frac{mqu_c^p}{2\beta(s+1)}\int_{-\infty}^\infty u_c^{m-1}\tilde{\phi}\,dy = \lambda\tilde{\phi}, \tag{5.15}$$

$$\tilde{\phi} \to 0 \quad \text{as} \quad y \to \pm\infty\,. \tag{5.16}$$

For the special case where $p = 2$, the coefficients in this eigenvalue problem are known explicitly using (5.9). For this case, we again get an eigenvalue problem that has a sech^2 potential.

We now outline the key properties of the nonlocal eigenvalue problem as determined in [**20**]. The first property is that if we know an eigenpair of (5.15)-(5.16) for which the corresponding eigenfunction decays exponentially as $|y| \to \infty$, then there is an eigenpair of the (5.12)-(5.13) that is exponentially close to this pair. This observation is used to show that (5.12)-(5.13) has an exponentially small eigenvalue. To see this we note that $\tilde{L}_\epsilon u_c^{'} = 0$. This follows from translation invariance and from the fact that the integrand in (5.15) is odd. Hence, (5.15) has a zero eigenvalue. It then follows that (5.12)-(5.13) has an eigenfunction and eigenvalue exponentially close to $u_c^{'}$ and zero, respectively. Label this exponentially small eigenvalue by λ_1. To estimate λ_1 we proceed as in [**20**]. We construct an eigenfunction ϕ_1 for L_ϵ that is exponentially close to $u_c^{'}$, but that has boundary layers near $x = \pm 1$. These boundary layers are needed to satisfy the boundary conditions at $x = \pm 1$ in (5.13). This eigenfunction ϕ_1 has the form

(5.17)
$$\phi_1(x) \sim \left(u_c' \left[\epsilon^{-1}(x - x_0) \right] + a e^{-\epsilon^{-1}(1+x_0)} e^{-\epsilon^{-1}(1+x)} - a e^{-\epsilon^{-1}(1-x_0)} e^{-\epsilon^{-1}(1-x)} \right) .$$

To estimate λ_1 we first derive Lagrange's identity for $(u, L_\epsilon v)$, where $(u,v) \equiv \int_{-1}^{1} uv \, dx$. Upon integration by parts we derive

$$(v, L_\epsilon u) = \epsilon^2 \left(u_x v - v_x u \right) |_{-1}^{1} + (u, L_\epsilon^* v) \, , \tag{5.18}$$

where L_ϵ^* is the adjoint operator

$$L_\epsilon^* v \equiv \epsilon^2 v_{xx} + (-1 + p u_c^{p-1}) v - \frac{m q \epsilon^{-1} u_c^{m-1}}{2\beta(s+1)} \int_{-1}^{1} u_c^p v \, dx \, . \tag{5.19}$$

Applying this identity to ϕ_1 and $u_c^{'}[\epsilon^{-1}(x - x_0)]$, we get

$$\lambda_1 (u_c', \phi_1) = -\epsilon \phi_1 u_c'' |_{-1}^{1} + (\phi_1, L_\epsilon^* u_c') \, . \tag{5.20}$$

The various terms in (5.20) can be estimated asymptotically as $\epsilon \to 0$ as in [**20**] to get the following key estimate:

Proposition: (Exponentially Small Eigenvalue) *For $\epsilon \to 0$, the exponentially small eigenvalue of (5.12)-(5.13) satisfies*

$$\lambda_1 \sim 2a^2 \hat{\beta}^{-1} \left(e^{-2(1+x_0)/\epsilon} + e^{-2(1-x_0)/\epsilon} \right) . \tag{5.21}$$

Here a is given in (5.7) and $\hat{\beta}$ is defined by

$$\hat{\beta} = \int_{-\infty}^{\infty} \left[u_c'(y) \right]^2 \, dy \, . \tag{5.22}$$

The estimate (5.21) holds for p, q, m and s satisfying (5.4). Since $\lambda_1 > 0$ the quasi-equilibrium solution is unstable. However, since it is exponentially small, the spike is metastable and will persist for extremely long times. For the special case when $p = 2$ where u_c is known, we can calculate $\hat{\beta} = 6/5$ and $a = 6$. This gives us an explicit estimate for λ_1 in this case.

The second key property of the spectrum of (5.12)-(5.13) is that the exponentially small eigenvalue λ_1 is the principal eigenvalue for (5.12)-(5.13). To show this, a homotopy method was employed in [**20**] to calculate the principal eigenvalue of (5.12)-(5.13). Let δ be a continuation parameter and consider the infinite domain problem

$$\tilde{L}_\delta \tilde{\phi} \equiv \tilde{\phi}_{yy} + (-1 + p u_c^{p-1}) \tilde{\phi} - \frac{\delta m q u_c^p}{2\beta(s+1)} \int_{-\infty}^{\infty} u_c^{m-1} \tilde{\phi} \, dy = \lambda \tilde{\phi}, \tag{5.23}$$

$$\tilde{\phi} \to 0 \quad \text{as} \quad y \to \pm\infty \, . \tag{5.24}$$

When $\delta = 0$, we have a local eigenvalue problem with principal eigenpair $\lambda = \lambda_0^{(0)}$ and $\tilde{\phi} = \tilde{\phi}_0^{(0)}$. When $\delta = 1$, we have our operator $\tilde{L}_\epsilon$. It is easy to show that $\lambda_0^{(0)} > 0$, and hence in the absence of the nonlocal term the spike would be unstable on an $O(1)$ time interval. In particular, the special case $p = 2$ yields the well-known eigenvalue problem associated with the scattering by a sech^2 potential well, and we have

$$\lambda_0^{(0)} = 5/4\,, \qquad \tilde{\phi}_0^{(0)} = \text{sech}^3(y/2)\,. \tag{5.25}$$

Since $\tilde{L}_\delta$ has a positive eigenvalue when $\delta = 0$, we must consider what happens to this eigenvalue as δ ranges from 0 to 1. If this eigenvalue remains positive then, since the eigenvalues of $\tilde{L}_\delta$ at $\delta = 1$ and L_ϵ will differ by only exponentially small amounts as $\epsilon \to 0$, we can conclude that the one-spike quasi-equilibrium solution is unstable. However, as shown in the numerical computations of [**20**], this eigenvalue crosses through zero at some finite value $\delta_0 < 1$, it then become complex at some δ_c with $\delta_0 < \delta_c < 1$, and remains in the left half-plane $\text{Re}(\lambda) < 0$ when $\delta = 1$. Hence, the nonlocal term in (5.23) has pushed the positive eigenvalue $\lambda_0^{(0)}$, which exists when $\delta = 0$, into the left half-plane when $\delta = 1$. This ensures that the principal eigenvalue of L_ϵ is exponentially small and that the one-spike solution is metastable.

5.2. Spike Motion. The quasi-equilibrium solution fails to satisfy the steady-state problem corresponding to (5.1)-(5.3) by only exponentially small terms for any value of x_0 in $|x_0| < 1$. Moreover, the linearization about this solution admits a principal eigenvalue that is exponentially small. Therefore, we expect that the one-spike quasi-equilibrium solution evolves on an exponentially slow time-scale.

To derive an equation of motion for the center of the spike corresponding to the quasi-equilibrium solution, we linearize around a moving spike by introducing

$$a(x,t) = a_E(x; x_0(t)) + w(x,t)\,, \tag{5.26}$$

where a_E is defined in (5.5) and $x_0 = x_0(t)$ is the trajectory of the spike. Since (5.12)-(5.13) does not have an $O(1)$ positive eigenvalue, we may assume that $w \ll a_E$ and $w_t \ll \partial_t a_E$. Substituting (5.26) into (5.1)-(5.3), we get

$$L_\epsilon w = \partial_t a_E\,, \qquad -1 < x < 1\,, \quad t \geq 0 \tag{5.27}$$

$$w_x(\pm 1, t) = -\partial_x a_E(\pm 1; x_0)\,. \tag{5.28}$$

Next, we expand w in terms of the eigenfunctions ϕ_i of L_ϵ as

$$w = \sum_{i=1}^{\infty} D_i(t)\phi_i\,. \tag{5.29}$$

The solvability condition for w is that w is orthogonal to the eigenspace of L_ϵ^* associated with the exponentially small eigenvalue. Let ϕ_i^* be the i^{th} eigenfunction of L_ϵ^*. Then, since $(\phi_i, \phi_j^*) = \delta_{ij}$, we integrate by parts to show that

$$D_i(t) = (w, \phi_i^*) = \frac{1}{\lambda_i^*}\left[(L_\epsilon w, \phi_i^*) - \epsilon^2 w_x \phi_i^* |_{-1}^{1}\right]\,, \tag{5.30}$$

where $L_\epsilon^* \phi_i^* = \lambda_i^* \phi_i^*$. Using (5.27)-(5.28), we have

$$D_i(t) = \frac{1}{\lambda_i^*}\left[(\partial_t a_E, \phi_i^*) + \epsilon^2 \phi_i^* \partial_x a_E |_{-1}^{1}\right]\,. \tag{5.31}$$

When $\epsilon \ll 1$, it was shown in [**20**] that the nonlocal term in the eigenvalue problem (5.12)-(5.13) is insignificant in the asymptotic estimation of the eigenspace associated with the exponentially small eigenvalue of L_ϵ. Therefore, we can replace ϕ_1^* and λ_1^* by ϕ_1 and λ_1 in (5.31), where ϕ_1 and λ_1 are given in (5.17) and (5.21), respectively.

Since $\lambda_1 \to 0$ exponentially as $\epsilon \to 0$, we must impose the limiting solvability condition that $D_1 = 0$. This projection step yields the following implicit differential equation for $x_0(t)$:

$$(\partial_t a_E, \phi_1) = -\epsilon^2 \phi_1 \partial_x a_E|_{-1}^{1} \,. \tag{5.32}$$

The dominant contribution to the left side of (5.32) arises from the region near x_0. For $\epsilon \to 0$, we calculate from Laplace's method that

$$(\partial_t a_E, \phi_1) \sim -h_E^{\gamma} x_0^{'} \hat{\beta} \epsilon^{-1} \,, \tag{5.33}$$

where $x_0^{'} \equiv dx_0/dt$. Finally, we can evaluate the right side of (5.32) using our estimates for $\phi_1(\pm 1)$ in (5.17) and for $u_c(y)$ as $|y| \to \infty$. This yields the metastability result of [**20**]:

Proposition: (Metastability) *A metastable spike solution for (5.1)-(5.3), is represented by $a(x,t) = a_E(x; x_0(t))$, where a_E is defined in (5.5) and $x_0(t)$ satisfies*

$$x_0^{'} \sim 2a^2 \hat{\beta}^{-1} \epsilon \left[e^{-2(1-x_0)/\epsilon} - e^{-2(1+x_0)/\epsilon} \right] . \tag{5.34}$$

Here a and $\hat{\beta}$ are defined in (5.7) and (5.22), respectively.

For a given initial condition $x_0(0) \in (-1,1)$, this ODE shows that the spike drifts towards the endpoint that is closest to the initial location $x_0(0)$. However, it takes an exponentially long time for the spike to collapse against the wall at $x = 1$ or $x = -1$. For the special case where $p = 2$, we have $a = 6$ and $\hat{\beta} = 6/5$. Thus, we get the explicit result

$$x_0^{'} \sim 60\epsilon \left[e^{-2(1-x_0)/\epsilon} - e^{-2(1+x_0)/\epsilon} \right] . \tag{5.35}$$

This equation can be integrated explicitly to give

$$x_0(t) = \frac{\epsilon}{2} \log \left(\frac{1 + f^0 e^{t/t_s}}{1 - f^0 e^{t/t_s}} \right) , \tag{5.36}$$

where

$$f^0 \equiv \tanh(x_0(0)/\epsilon) \,, \qquad t_s \equiv \frac{1}{240} e^{2/\epsilon} \,. \tag{5.37}$$

Acknowledgements

It is my pleasure to acknowledge the contributions of my collaborator Prof. R. O'Malley and my graduate students David Iron and Xiaodi Sun to some of the work described herein. The support of NSERC grant 5-81541 is also gratefully acknowledged.

References

[1] R. Ackerberg, R. O'Malley, *Boundary Layer Problems Exhibiting Resonance*, Stud. Appl. Math. **49**, (1970), pp. 277–295.

[2] S. Adjerid, M. Aiffa, J. E. Flaherty, *Computational Methods for Singularly Perturbed Systems*, (1998), these proceedings.

[3] N. Alikakos, P.W. Bates, G. Fusco, *Slow Motion for the Cahn-Hilliard Equation in One Space Dimension*, J. Diff. Equat. **90**, (1991), pp. 81–135.

[4] N.J. Balmforth, *Solitary Waves amd Homoclinic Orbits*, Annual Review of Fluid Mechanics, **27**, (1995), pp. 335–374.

[5] N.J. Balmforth, G.R. Ierley, E.A. Spiegel, *Chaotic Pulse Trains*, SIAM J. Appl. Math. **54**, (1993), pp. 1291–1334.

[6] P.W. Bates, J. Xun, *Metastable Patterns for the Cahn-Hilliard Equation: Parts 1 and 2*, J. Diff. Equat. **111**, (1994), pp. 421–457; J. Diff. Equat. **117**, (1995), pp. 165–216.

[7] H. Berestycki, S. Kamin, G. Sivashinsky, *Nonlinear Dynamics and Metastability in a Burgers Type Equation*, Compte Rendus Acad. Sci., Paris t. **321**, Serié 1, (1995), pp. 185-190.

[8] L. Bronsard, D. Hilhorst, *On the Slow Dynamics for the Cahn-Hilliard Equation in One Space Dimension*, Proc. Roy. Soc. London A, **439**, (1992), pp. 669–682.

[9] G. Carrier, C. Pearson, *Ordinary Differential Equations*, Blaisdell, Waltham, Mass., (1968).

[10] J. Carr, R. Pego, *Metastable Patterns in Solutions of* $u_t = \epsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. **42**, (1989), pp. 523–576.

[11] P.P.N. De Groen, *The Nature of Resonance in a Singular Perturbation Problem of Turning Point Type*, SIAM J. Math. Anal. **11**, (1980), pp. 1–22.

[12] P. Fife, *Models for Phase Separation and Their Mathematics*, in 'Nonlinear Partial Differential Equations and Applications' (M. Mimura, Y. Nishiura editors), Taniguyia Publications, (1992).

[13] G. Fusco, J.K. Hale, *Slow Motion Manifolds, Dormant Instability and Singular Perturbations*, J. Dyn. Diff. Equat. **1**, (1989), pp. 75–94.

[14] A. Gierer, H. Meinhardt, *A Theory of Biological Pattern Formation*, Kybernetik, **12**, (1972), pp. 30-39.

[15] C. Grant, *Slow Motion in One-Dimensional Cahn-Morral Systems*, SIAM J. Math. Anal. **26**, (1995), pp. 21–34.

[16] J. Grasman, B.J. Matkowsky, *A Variational Approach to Singularly Perturbed Boundary Value Problems for Ordinary and Partial Differential Equations with Turning Points*, SIAM J. Appl. Math. **32**, (1977), pp. 588–597.

[17] E.J. Hinch, *Perturbation Methods*, Cambridge Texts in Applied Math., Cambridge University Press, Cambridge, (1991).

[18] M. Holmes, *Introduction to Perturbation Methods*, Texts in Appl. Math. Vol. **20**, Springer-Verlag, New York, (1995).

[19] D. Holloway, *Reaction-Diffusion Theory of Localized Structures with Application to Vertebrate Organogenesis*, Ph.D Thesis (Chemistry), University of British Columbia, Vancouver, (1995).

[20] D. Iron, M. J. Ward, *A Metastable Spike Solution for a Non-Local Reaction-Diffusion Model*, submitted SIAM J. Appl. Math. (1998).

[21] J. Kevorkian, J.Cole, *Multiple Scale and Singular Perturbation Problems*, Appl. Math. Sciences, Vol. **1**14, Springer-Verlag, New York, (1996).

[22] G. Kreiss, H. Kreiss, *Convergence to Steady State of Solutions of Burgers Equation*, Appl. Numerical Math. **2**, (1986), pp. 161–179.

[23] H. Kreiss, *Resonance for Singular Perturbation Problems*, SIAM J. Appl. Math. **41**, (1981), pp. 331–344.

[24] M. Kuwamura, S.I. Ei, M. Mimura, *Very Slow Dynamics for Some Reaction-Diffusion Systems of the Activator-Inhibitor Type*, Japan J. Indust. Appl. Math. **9**, (1992), pp. 35–77.

[25] J. Laforgue, R.E. O'Malley, *On the Motion of Viscous Shocks and the Super-Sensitivity of their Steady-State Limits*, Methods and Appl. of Anal. **1**, (1994), pp. 465–487.

[26] J. Laforgue, R.E. O'Malley, *Shock layer movement for Burgers equation*, SIAM J. Appl. Math. **55**, (1995), pp. 332–348.

[27] J. Laforgue, R. O'Malley, *Exponential Asymptotics, the Viscid Burgers Equation, and Standing Wave Solutions for a Reaction-Advection-Diffusion Model*, to appear, Studies in Appl. Math. (1998).

[28] J. Laforgue, R. O'Malley, M. J. Ward, *Metastable Travelling Wave Solutions of Singularly Perturbed Reaction-Diffusion Equations*, to appear, European J. Appl. Math.

[29] P.A. Lagerstrom, *Matched Asymptotic Expansions*, Appl. Math. Sciences, Vol. **76**, Springer-Verlag, New York, (1988).

[30] E. von Lavante, R.A. Strehlow, *The Mechanism of Lean Limit Flame Extinction* , Combustion and Flame, **49**, (1983), pp. 123-140.
[31] J.Y. Lee, M.J. Ward, *On the Asymptotic and Numerical Analysis of Exponentially Ill-Conditioned Singularly Perturbed Boundary Value Problems*, Stud. Appl. Math. **94**, (1995), pp. 271–326.
[32] D. Ludwig, *Persistence of Dynamical Systems Under Random Perturbations*, SIAM Review **17**, (1975), pp. 605–640.
[33] R.S. Maier, D.L. Stein, *Limiting Exit Location Distributions in the Stochastic Exit Problem*, SIAM J. Appl. Math. **57**, (1997), pp. 752-790.
[34] B.J. Matkowsky, *On Boundary Layer Problems Exhibiting Resonance*, SIAM Rev. **17**, (1975), pp. 82–100.
[35] B.J. Matkowsky, Z. Schuss, *The Exit Problem for Randomly Perturbed Dynamical Systems*, SIAM J. Appl. Math. **33**, (1977), pp. 365–382.
[36] A. Mikishev, G. Sivashinsky, *Quasi-equilibrium in Upward Propagating Flames*, Physics Letters A **175**, (1993), pp. 409-414.
[37] A. Novick-Cohen, *On the Viscous Cahn-Hilliard Equation*, in 'Material Instabilities in Continuum Mechanics and Related Mathematical Problems' (J. Ball, editor), Oxford Science Publications, Clarendon Press, (1988), pp. 329–342.
[38] R.E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag, New York, (1991).
[39] R.E. O'Malley, M.J. Ward, *Exponential Asymptotics, Boundary Layer Resonance, and Dynamic Metastability*, in 'Mathematics is for Solving Problems', (L. P. Cook et al. editors), SIAM, (1996), pp. 189-203.
[40] Z. Rakib, G.I. Sivashinsky, *Instabilities in Upward Propagating Flames*, Combust. Sci. and Tech. **54**, (1987), pp. 69–84.
[41] L.G. Reyna, M.J. Ward, *Resolving Weak Internal Layer Interactions for the Ginzburg-Landau Equation*, European J. Appl. Math. **5**, (1994), pp. 495–523.
[42] L.G. Reyna, M.J. Ward, *On the Exponentially Slow Motion of a Viscous Shock*, Comm. Pure Appl. Math. **48**, (1995), pp. 79–120.
[43] L.G. Reyna, M.J. Ward, *Metastable Internal Layer Dynamics for the Viscous Cahn-Hilliard Equation*, Methods and Appl. of Anal. **2** No. 3, (1995), pp. 285–306.
[44] L.G. Reyna, M.J. Ward, *On Exponential Ill-Conditioning and Internal Layer Behavior*, J. Num. Func. Analysis and Optimization Vol. **16** No. 4, (1995), pp. 475–500.
[45] J. Rubinstein, P. Sternberg, *Nonlocal Reaction-Diffusion Equations and Nucleation*, IMA J. Appl. Math. **48**, (1992), pp. 249–264.
[46] S. Yu. Slavyanov, *Asymptotic Solutions of the One-Dimensional Schrodinger Equation*, AMS Translations of Mathematical Monographs, Vol. **1**51, AMS, (1996).
[47] X. Sun, M. J. Ward, *Metastability for a Generalized Burgers Equation with Applications to Propagating Flame-Fronts*, to appear, European J. Appl. Math. (1998).
[48] X. Sun, M. J. Ward, *The Analysis and Computation of Metastable Dynamics for the Viscous Cahn-Hilliard Equation in One-Spatial Dimension*, in preparation.
[49] A. Turing, *The Chemical Basis of Morphogenesis*, Phil. Trans. Roy. Soc. B, **237**, (1952), pp. 37-72.
[50] M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford, (1975).
[51] M.J. Ward, *Eliminating Indeterminacy in Singularly Perturbed Boundary Value Problems with Translation Invariant Potentials*, Stud. Appl. Math **87**, (1992), pp. 95–135.
[52] M.J. Ward, *Metastable Patterns, Layer Collapses, and Coarsening for a One-Dimensional Ginzburg-Landau Equation*, Stud. Appl. Math. **91**, (1994), pp. 51–93.
[53] G. Wolansky, *On the Slow Evolution of Quasi-Stationary Shock Waves*, J. Dyn. Diff. Equat. **6**, (1994), pp. 247–276.

Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z2

E-mail address: `ward@math.ubc.ca`