Dynamic Metastability and Singular Perturbations

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Abstract. Certain singularly perturbed time-dependent partial differential equations exhibit a phenomenon known as dynamic metastability whereby the time-dependent solution approaches a steady-state solution only over an an asymptotically exponentially long time interval. This metastable behavior is directly related to the occurrence of an asymptotically exponentially small principal eigenvalue for the linearized equation. In this paper, we illustrate metastable behavior for various classes of perturbed problems and we show how this behavior can be analyzed asymptotically by supplementing the method of matched asymptotic expansions with certain spectral information associated with the linearized equation.

1. Introduction

The method of matched asymptotic expansions is a well-known and powerful method for systematically calculating asymptotic approximations to solutions of singularly perturbed problems. This method has been used successfully in a wide range of applications (cf. [**H**], [**HO**], [**KC**], [**LA**], [**O**], [**V**]), and its theoretical foundations are rather well-developed.

However, there are certain classes of steady-state singularly perturbed boundary value problems where a straightforward application of this method fails to determine the solution uniquely. In particular, for problems where asymptotically exponentially small terms need to be resolved, a failure to asymptotically resolve such terms typically leads to a matched asymptotic approximation with undetermined constants. Some examples where this indeterminacy occurs are for linear turning point problems associated with boundary layer resonance (cf. [**AO**]), for certain nonlinear autonomous boundary value problems with shock-type internal behavior (cf. [**L**]), and for a class of nonlinear elliptic equations with localized spike-layer solutions (cf. [**WN**], [**W95b**]). In many cases, this indeterminacy in the matched asymptotic approximation arises as a direct consequence of an exponential ill-conditioning of a certain linearization of the full perturbed problem. By exponential ill-conditioning we mean that the spectrum of the eigenvalue problem associated with the linearization contains exponentially small eigenvalues. As a result of this exponential ill-conditioning, the solution to the steady-state perturbed problem is typically very sensitive to exponentially small changes in the coefficients of the differential operator. Moreover, for the corresponding time-dependent problem, this exponential ill-conditioning can lead to the occurrence of a phenomenon

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known as dynamic metastability whereby the time-dependent solution approaches a steady-state solution only over an asymptotically exponentially long time interval. For example, this metastable behavior is known to occur for various phase separation models (cf. [**CP**], [**FH**], [**ABF**]) and for certain viscous shock problems (cf. [**KK**], [**LO94**], [**RW95a**]).

The goal of this paper is largely to illustrate and survey some results for metastable behavior and exponential ill-conditioning for various classes of linear and nonlinear singularly perturbed partial differential equations. In each case, we show how this behavior can be analyzed asymptotically by using an asymptotic projection method, which supplements the method of matched asymptotic expansions with certain spectral information associated with the linearized equation. This projection method exploits the existence of exponentially small eigenvalues by imposing limiting solvability conditions on the solution to the linearized equation.

The outline of this paper is as follows. In §2 and §3, the asymptotic projection method is used to study metastable behavior for various linear and nonlinear convection-diffusion equations. In §4, the projection method is used to construct localized spike-layer solutions for a class of steady-state problems in both one and two spatial dimensions. In §5 and §6, the projection method is used to study metastable behavior for various phase separation models.

2. Linear Convection-Diffusion Equations

In this section we study exponentially ill-conditioned linear convection-diffusion equations exhibiting metastable behavior in both one and two spatial dimensions.

2.1. One Spatial Dimension. We first consider the following convection-diffusion equation for $u = u(x,t)$:

$$u_t = \epsilon u_{xx} - \Psi'(x) u_x + \epsilon^{\nu} g(x) e^{-a/\epsilon} u \,, \quad -1 < x < 1 \,, \quad t > 0 \,, \tag{2.1}$$

$$u(-1,t) = u_l \,, \quad u(1,t) = u_r \,; \qquad u(x,0) = u_0(x) \,. \tag{2.2}$$

Here $a > 0$, ν, u_l and u_r are constants, $\epsilon \to 0^+$ and $g(x)$, $\Psi(x)$ and $u_0(x)$ are smooth. We assume that the potential $\Psi(x)$ has a global minimum on $[-1,1]$ at $x = 0$ with $\Psi(0) = 0$, $\Psi'(0) = 0$ and $\Psi''(0) > 0$. Thus, the spatial operator in (2.1) has a simple turning point at $x = 0$. We also assume that $\Psi'(x) \neq 0$ for $x \neq 0$ and thus $\Psi'(1) > 0$ and $\Psi'(-1) < 0$. Prototypical is $\Psi(x) = x^2/2$.

When $g(x) = 0$, the equilibrium problem corresponding to (2.1)-(2.2) and its associated eigenvalue problem arises in determining the exit time distribution for a Brownian particle confined by the potential well Ψ. Such problems, and the related multi-dimensional problems in §2.2 below, have been well-studied in [**LU**], [**MAS**], [**M80**] and [**MS**], among much additional literature.

The eigenvalue problem associated with (2.1)-(2.2) when $g(x) = 0$ is

$$L_\epsilon \phi \equiv \epsilon \phi_{xx} - \Psi'(x) \phi_x = -\lambda \phi \,, \quad -1 < x < 1 \,; \qquad \phi(\pm 1) = 0 \,, \tag{2.3}$$

$$(\phi, \phi)_w \equiv \int_{-1}^{1} \phi^2 w \, dx = 1 \,, \qquad w \equiv e^{-\Psi/\epsilon} \,. \tag{2.4}$$

The eigenvalues λ_j for $j \geq 0$ are real with $\lambda_j > 0$ and the orthogonality relations $(\phi_j, \phi_k)_w = \delta_{jk}$ for $j, k = 0, 1, .., $ hold. It is well-known (cf. [**D**], [**LU**], [**M80**], [**MS**], [**LW**]) that λ_0 is exponentially small as $\epsilon \to 0$ and has the asymptotic estimate

$$\lambda_0 \sim \left(\frac{\Psi''(0)}{2\pi\epsilon} \right)^{1/2} \left[\Psi'(1) b_r(\epsilon) e^{-\Psi(1)/\epsilon} - \Psi'(-1) b_l(\epsilon) e^{-\Psi(-1)/\epsilon} \right] . \tag{2.5}$$

Here $b_r(\epsilon)$ and $b_l(\epsilon)$ have asymptotic expansions in powers of ϵ with leading terms $b_r(0) = b_l(0) = 1$. To leading order, the corresponding eigenfunction ϕ_0 has the boundary layer form

$$\phi_0 \sim M_0 \left(1 - e^{-\Psi'(1)(1+x)/\epsilon} - e^{\Psi'(-1)(1-x)/\epsilon} \right) , \tag{2.6}$$

where M_0 is a normalization constant. Note that, as $\epsilon \to 0$, ϕ_0 is a constant away from the boundary layer regions.

For $\epsilon \to 0$, a leading order boundary layer analysis for the equilibrium solution $U(x;\epsilon)$ to (2.1)-(2.2) shows that $U(x;\epsilon) \sim \tilde{u}^\epsilon [x; A_{0e}]$, where

$$\tilde{u}^\epsilon [x; A_{0e}] \equiv A_{0e} + (u_r - A_{0e})\, e^{-\Psi'(1)(1-x)/\epsilon} + (u_l - A_{0e})\, e^{\Psi'(-1)(1+x)/\epsilon} , \tag{2.7}$$

for some undetermined constant A_{0e}. Singular perturbation problems of this type, where a conventional application of the method of matched asymptotic expansions fails to select certain constants uniquely, were first identified in [**AO**] and later studied extensively in [**D**], [**K**], [**LW**], [**M75**], [**M80**], [**S**], [**SR**], and [**W**] (and the references therein). The relationship between this apparent indeterminacy and the ill-conditioning of the underlying operator is emphasized in [**D**], [**K**] and [**LW**]. More specifically, since (2.3) has an exponentially small eigenvalue and $L_\epsilon \tilde{u}^\epsilon$ is exponentially small away from the boundary layer regions near $x = \pm 1$ for any choice of A_{0e}, it follows that the correct value of A_{0e} can only be determined by incorporating the effect of exponentially small terms into the asymptotic analysis. Methods for calculating A_{0e} are given in [**GM**], [**LW**], [**MS**] and [**SR**]. Since the equilibrium solution is exponentially ill-conditioned, it is natural to expect that it will be extremely sensitive to the exponentially small term in (2.1). This aspect has been studied in [**LW**], [**S**], [**SR**] and [**W**].

These previous studies have focused mainly on the equilibrium problem for (2.1)-(2.2). Since $\lambda_0 > 0$, the equilibrium solution is stable when a is sufficiently large. However, since λ_0 is exponentially small, the evolution of an arbitrary initial condition $u_0(x)$ to the equilibrium solution is exponentially slow. This metastable motion is studied in [**OW**] using the projection method.

We now outline this method and some of the results obtained from it. Following [**OW**], we seek a solution to (2.1)-(2.2) in the form

$$u(x,t) = \tilde{u}^\epsilon [x; A_0(t)] + v(x,t) , \tag{2.8}$$

where $\tilde{u}^\epsilon$ is defined in (2.7). Substituting (2.8) into (2.1)-(2.2), we obtain that $v(x,t)$ satisfies

$$v_t = L_\epsilon v - \tilde{u}^\epsilon_t + L_\epsilon \tilde{u}^\epsilon + \epsilon^\nu g(x) e^{-a/\epsilon} (\tilde{u}^\epsilon + v) , \quad -1 < x < 1 , \quad t > 0 , \tag{2.9}$$

$$v(-1,t) = u_l - \tilde{u}^\epsilon [-1; A_0(t)] , \qquad v(1,t) = u_r - \tilde{u}^\epsilon [1; A_0(t)] , \tag{2.10}$$

together with the initial condition $v(x,0) = u_0(x) - \tilde{u}^\epsilon [x; A_0(0)]$. We then expand $v(x,t)$ in terms of the eigenfunctions ϕ_j of (2.3)-(2.4) as

$$v(x,t) = \sum_{j=0}^{\infty} c_j(t) \phi_j(x) . \tag{2.11}$$

Using orthogonality, we find that $c_j(t)$ satisfies the differential equation

(2.12)
$$c_j^{'} + \lambda_j c_j = \left(\phi_j, L_\epsilon \tilde{u}^\epsilon\right)_w - \epsilon w v \phi_j^{'} \Big|_{-1}^{1} - \left(\phi_j, \tilde{u}_t^\epsilon\right)_w + \epsilon^\nu e^{-a/\epsilon} \left(g\phi_j, \tilde{u}^\epsilon + v\right)_w ,$$

and has the initial value

$$c_j(0) = \int_{-1}^{1} \left(u_0(x) - \tilde{u}^\epsilon[x; A_0(0)]\right) \phi_j w \, dx . \tag{2.13}$$

Since $\lambda_0 > 0$ and is exponentially small, it is necessary that $c_0(t) \equiv 0$ in order to ensure that $v \ll \tilde{u}^\epsilon$ over exponentially long time intervals . Therefore, the right sides of (2.12) and (2.13) must vanish when $j = 0$. Then, upon using $v \ll \tilde{u}^\epsilon$ to simplify the last term on the right of (2.12), we obtain

$$\left(\phi_0, \tilde{u}_t^\epsilon\right)_w \sim \left(\phi_0, L_\epsilon \tilde{u}^\epsilon\right)_w - \epsilon w v \phi_0^{'} \Big|_{-1}^{1} + \epsilon^\nu e^{-a/\epsilon} \left(g\phi_0, \tilde{u}^\epsilon\right)_w , \tag{2.14}$$

together with

$$\int_{-1}^{1} \tilde{u}^\epsilon[x; A_0(0)] \phi_0 w \, dx = \int_{-1}^{1} u_0(x) \phi_0 w \, dx . \tag{2.15}$$

Equation (2.14) is a differential equation for $A_0(t)$ and (2.15) determines the initial value $A_0(0)$.

To obtain an explicit differential equation for A_0 we use the form for ϕ_0 given in (2.6) to evaluate the various terms in (2.14)-(2.15) asymptotically for $\epsilon \to 0$ (see [**OW**] for details). To leading order as $\epsilon \to 0$, the analysis in [**OW**] shows that $A_0(t)$ satisfies the limiting differential equation

(2.16)
$$A_0^{'} \sim -\left(\lambda_0 - \epsilon^\nu g(0) e^{-a/\epsilon}\right) A_0 + \beta \left[u_r \Psi^{'}(1) e^{-\Psi(1)/\epsilon} - u_l \Psi^{'}(-1) e^{-\Psi(-1)/\epsilon}\right] ,$$

with the initial condition $A_0(0) \sim u_0(0)$. Here $\beta \equiv \left[\Psi^{''}(0)/(2\pi\epsilon)\right]^{1/2}$, and λ_0 is given in (2.5). The metastable dynamics for $u(x,t)$ is then given by $u(x,t) \sim \tilde{u}^\epsilon[x; A_0(t)]$, where $\tilde{u}^\epsilon$ is defined in (2.7). A higher order asymptotic differential equation for $A_0(t)$ than (2.16) is given in [**OW**]. In [**OW**] the asymptotic theory is also favorably compared with full numerical results for some specific examples. We remark that, when $g(x) \equiv 0$, the result (2.16) can also be obtained by an extension of the method used in [**MS**] (cf. [**M95**]).

If a is sufficiently large, it follows that $A_0(t) \to A_{0e}$ as $t \to \infty$, where A_{0e} is obtained by setting $A_0^{'} = 0$ in (2.16). This equilibrium result, which shows the extreme sensitivity of the equilibrium solution to the exponentially negligible term in (2.1), is obtained in [**W**] (see [**LW**] for a higher order asymptotic result).

2.2. Two Spatial Dimensions. We now generalize (2.1)-(2.2) to the case of two spatial dimensions. To this end we consider the following convection-diffusion equation in a closed and bounded two-dimensional domain D with a smooth boundary ∂D:

$$u_t = \epsilon \triangle u - \nabla\Psi \cdot \nabla u + \epsilon^\nu g(x) e^{-a/\epsilon} u , \quad x \in D , \quad t > 0 , \tag{2.17}$$
$$u = u_b(x) , \quad x \in \partial D ; \qquad u(x, 0) = u_0(x) . \tag{2.18}$$

Here $u = u(x,t)$ with $x = (x_1, x_2)$, ν and $a > 0$ are constants, $\epsilon \to 0^+$, and $g(x)$, $u_b(x)$ and $u_0(x)$ are smooth. Let $\bar{D} = D \cup \partial D$. We assume that the potential

$\Psi(x)$ has a unique global minimum on $\bar{D}$ at some interior point $x_0 \in D$. At this minimum point we assume that

$$\Psi(x_0) = 0\,, \qquad \nabla\Psi(x_0) = 0\,, \qquad H\left[\Psi(x_0)\right] > 0\,, \tag{2.19}$$

where $H(\Psi) \equiv \Psi_{x_1x_1}\Psi_{x_2x_2} - \Psi^2_{x_1x_2}$ is the Hessian. We also assume that $\nabla\Psi(x) \neq 0$ for $x \neq x_0$ and that $\nabla\Psi \cdot \hat{n} > 0$ on ∂D, where $\hat{n}$ is the unit outward normal to ∂D.

In place of (2.3)-(2.4), the eigenvalue problem associated with (2.17)-(2.18) when $g(x) = 0$ is

$$L_\epsilon\phi \equiv \epsilon\Delta\phi - \nabla\Psi\cdot\nabla\phi = -\lambda\phi\,, \quad x \in D\,; \qquad \phi = 0\,, \quad x \in \partial D\,, \tag{2.20}$$

$$(\phi,\phi)_w \equiv \int_D \phi^2 w\, dx = 1\,, \qquad w \equiv e^{-\Psi/\epsilon}\,. \tag{2.21}$$

The eigenvalues λ_j for $j \geq 0$ are real with $\lambda_j > 0$ and $(\phi_j,\phi_k)_w = \delta_{jk}$ for $j,k = 0,1,\ldots$ Suppose that the minimum value of Ψ on ∂D is taken at N distinct points $y_j \in \partial D$ for $j = 1,..,N$, and that these minima are non-degenerate. Then, as is shown in [**MS**], the principal eigenvalue λ_0 of (2.20) is exponentially small as $\epsilon \to 0$ and has the asymptotic estimate

$$\lambda_0 \sim (2\pi\epsilon)^{-1/2}\left(H\left[\Psi(x_0)\right]\right)^{1/2} e^{-\Psi^*/\epsilon} \sum_{j=1}^{N} |\nabla\Psi(y_j)| r_j^{-1/2}\,. \tag{2.22}$$

Here $\Psi^* \equiv \Psi(y_j)$ for $j = 1,..,N$, and

$$r_j \equiv |\nabla\Psi|^{-2}\left[\Psi_{x_1x_1}\Psi^2_{x_2} - 2\Psi_{x_1x_2}\Psi_{x_1}\Psi_{x_2} + \Psi_{x_2x_2}\Psi^2_{x_1} + \kappa_j|\nabla\Psi|^3\right]\Big|_{x=y_j}\,, \tag{2.23}$$

where $\kappa_j < 0$ is the curvature of ∂D at y_j. In terms of a normalization constant M_0, the corresponding eigenfunction ϕ_0 has the boundary layer form

$$\phi_0 \sim M_0\left(1 - e^{\gamma\eta/\epsilon}\right)\,, \qquad \text{for} \qquad \gamma = \gamma(s) \equiv \nabla\Psi\cdot\hat{n}\,\Big|_{\partial D} > 0\,. \tag{2.24}$$

Here s denotes arclength along ∂D and $-\eta$ is the distance from $x \in D$ to ∂D. Since $\gamma > 0$, $\phi_0 \to M_0$ as $\eta/\epsilon \to -\infty$.

For $\epsilon \to 0$, a leading order boundary layer approximation for the equilibrium solution $U(x;\epsilon)$ to (2.17)-(2.18) is given by

$$U(x;\epsilon) \sim \tilde{u}^\epsilon\left[x; A_{0e}\right] \equiv A_{0e} + \left(u_b(s) - A_{0e}\right) e^{\gamma\eta/\epsilon}\,, \tag{2.25}$$

in an $O(\epsilon)$ neighborhood near ∂D, for some undetermined constant A_{0e}. Here we have written u_b in terms of the arclength s. The outer limit of $\tilde{u}^\epsilon$, valid for $\eta/\epsilon \to -\infty$, is A_{0e}. For similar reasons as given in §2.1, to determine the correct value of A_{0e} requires exponential precision. Methods to calculate A_{0e} when $g(x) = 0$ are given in [**GM**] and [**MS**]. The equilibrium solution for (2.17)-(2.18) is stable when a is sufficiently large, and is very sensitive to the exponentially small term in (2.17). As in §2.1, the approach to the equilibrium solution is exponentially slow when a is sufficiently large.

To study this metastable motion analytically we extend the projection method as outlined in §2.1 to a multi-dimensional setting. In (2.17)-(2.18) we set $u(x,t) = \tilde{u}^\epsilon\left[x; A_0(t)\right] + v(x,t)$ and obtain that $v(x,t)$ satisfies

$$v_t = L_\epsilon v - \tilde{u}^\epsilon_t + L_\epsilon\tilde{u}^\epsilon + \epsilon^\nu g(x) e^{-a/\epsilon}(\tilde{u}^\epsilon + v)\,, \quad x \in D\,, \quad t > 0\,, \tag{2.26}$$

$$v = u_b - \tilde{u}^\epsilon\,, \quad x \in \partial D\,; \quad v(x,0) = u_0(x) - \tilde{u}^\epsilon\left[x; A_0(0)\right]\,, \quad x \in D\,. \tag{2.27}$$

We then expand v in terms of the eigenfunctions ϕ_j of (2.20)-(2.21) as in (2.11). In place of (2.12), we obtain that the coefficient $c_j(t)$ in (2.11) satisfies

(2.28)
$$c_j^{'} + \lambda_j c_j = \left(\phi_j, L_\epsilon \tilde{u}^\epsilon\right)_w - \int_{\partial D} \epsilon w v \, \partial_n \phi_j \, ds - \left(\phi_j, \tilde{u}_t^\epsilon\right)_w + \epsilon^\nu e^{-a/\epsilon} \left(g\phi_j, \tilde{u}^\epsilon + v\right)_w ,$$

with the initial value

$$c_j(0) = \int_D \left(u_0(x) - \tilde{u}^\epsilon\left[x; A_0(0)\right]\right) \phi_j w \, dx . \tag{2.29}$$

Here $w \equiv e^{-\Psi/\epsilon}$, and, in (2.28), ∂_n denotes the outward normal derivative to ∂D.

As in §2.1, to ensure that $v \ll \tilde{u}^\epsilon$ over exponentially long time intervals it is necessary that $c_0(t) \equiv 0$. Thus, we require that the right sides of (2.28) and (2.29) vanish when $j = 0$. This leads to the following differential equation for $A_0(t)$

$$\left(\phi_0, \tilde{u}_t^\epsilon\right)_w \sim \left(\phi_0, L_\epsilon \tilde{u}^\epsilon\right)_w - \int_{\partial D} \epsilon w v \, \partial_n \phi_0 \, ds + \epsilon^\nu e^{-a/\epsilon} \left(g\phi_0, \tilde{u}^\epsilon\right)_w , \tag{2.30}$$

together with the initial condition

$$\int_D \tilde{u}^\epsilon\left[x; A_0(0)\right] \phi_0 w \, dx = \int_D u_0(x) \phi_0 w \, dx . \tag{2.31}$$

To obtain an explicit ODE for $A_0(t)$, the form (2.24) is used to evaluate the various terms in (2.30)-(2.31) asymptotically as $\epsilon \to 0$. To leading order, it is found in [**SW96b**] that $A_0(t)$ satisfies the limiting differential equation

$$A_0^{'} \sim -\left(\lambda_0 - \epsilon^\nu g(x_0) e^{-a/\epsilon}\right) A_0 + \beta e^{-\Psi^*/\epsilon} \sum_{j=1}^{N} u_b(y_j) |\nabla \Psi(y_j)| r_j^{-1/2} , \tag{2.32}$$

with the initial value $A_0(0) \sim u_0(x_0)$. Here, λ_0 is given in (2.22), x_0 is the global minimum of Ψ in $\bar{D}$, and $\beta \equiv \left(H[\Psi(x_0)]/(2\pi\epsilon)\right)^{1/2}$ where H is the Hessian. Moreover, $y_j \in \partial D$, for $j = 1, .., N$, are those points where Ψ is minimized on ∂D (with a non-degenerate minimum) with minimum value $\Psi^* \equiv \Psi(y_j)$ for $j = 1, .., N$. Also, r_j is defined in (2.23). This result for $A_0(t)$ is analogous to the result (2.16) for the one-dimensional case.

In summary, the metastable dynamics for (2.17)-(2.18), away from an initial time layer, is given by $u(x,t) \sim \tilde{u}^\epsilon\left[x; A_0(t)\right]$, where $\tilde{u}^\epsilon$ is defined in (2.25) and $A_0(t)$ satisfies (2.32). If a is sufficiently large relative to Ψ^*, we obtain from (2.32) that $A_0(t) \to A_{0e}$ as $t \to \infty$, where

$$A_{0e} = \sum_{j=1}^{N} u_b(y_j) \frac{|\nabla \Psi(y_j)|}{r_j^{1/2}} \left(\sum_{j=1}^{N} \frac{|\nabla \Psi(y_j)|}{r_j^{1/2}} - \frac{\epsilon^\nu g(x_0)}{\beta} e^{-(a-\Psi^*)/\epsilon} \right)^{-1} . \tag{2.33}$$

When $g(x) \equiv 0$, this result for A_{0e} is given in [**MS**] (see also [**GM**] for a specific example). From (2.33) it is clear that A_{0e} is very sensitive with respect to changes in the value of a for $a \approx \Psi^*$.

3. Nonlinear Convection-Diffusion Equations

In this section we consider two different classes of nonlinear convection-diffusion equations in a finite interval that exhibit metastable behavior. In §3.1 metastable viscous shock-layer solutions are considered. In §3.2 we give some metastability results for the Burger-type equation of [**RAS**] and [**MIS**] modeling flame-front propagation in a vertical channel. For each of these problems, the metastable behavior is directly related to the existence of an exponentially small principal eigenvalue for a linearized equation that is similar in form to the exit problem (2.1).

3.1. Viscous Shocks. We first consider the viscous shock problem

$$u_t + [f(u)]_x = \epsilon\, u_{xx}\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{3.1}$$

$$u(-1,t) = \alpha_- > 0\,, \quad u(1,t) = \alpha_+ < 0\,; \qquad u(x,0) = u_0(x)\,. \tag{3.2}$$

Here $\alpha_\pm$ are constants, $u_0(x)$ is monotone decreasing with $u_0(\pm 1) = \alpha_\pm$, $\epsilon \to 0^+$, and $f(u)$ is a smooth convex function that satisfies $f(\alpha_+) = f(\alpha_-)$, $f(0) = f'(0) = 0$ and $uf'(u) > 0$ for $u \neq 0$. Two examples are $f(u) = u^2/2$, which yields the well-known Burgers equation (cf. [**WH**]), and $f(u) = (u-1)+1/(u+1)$, which arises in the study of one-dimensional transonic gas flow in a straight channel (cf. [**HOW**]).

The key condition $f(\alpha_+) = f(\alpha_-)$ ensures that (3.1) has a stationary wave solution $u_c(x/\epsilon)$ on the infinite line connecting α_+ and α_-. Here $u_c(z)$, called the viscous shock profile, is the unique solution to

$$u_c'(z) = f\,[u_c(z)] - f(\alpha_+)\,, \quad -\infty < z < \infty\,; \quad u_c(\pm\infty) = \alpha_\pm\,, \quad u_c(0) = 0\,, \tag{3.3}$$

with $u_c' < 0$. This solution has the far-field behavior

$$u_c(z) \sim \alpha_\pm \pm a_\pm e^{\mp \nu_\pm z}\,, \quad \text{as } z \to \pm\infty\,; \qquad \nu_\pm \equiv \mp f'(\alpha_\pm) > 0\,, \tag{3.4}$$

for some positive constants $a_\pm > 0$ (cf. [**RW95a**]). In particular, for Burgers equation with $\alpha_\pm = \mp 1$, we have $u_c(z) = -\tanh(z/2)$, $\nu_\pm = 1$ and $a_\pm = 2$.

A matched asymptotic expansion analysis shows that (3.1)-(3.2) has an equilibrium shock-layer solution $U(x;\epsilon)$ of the form $U(x;\epsilon) \sim u_c\,[(x - x_{0e})/\epsilon]$, for some undetermined $x_{0e} \in (-1,1)$. Since $u_c(z)$ decays exponentially as $z \to \pm\infty$, it follows that $u_c\,[(x - x_{0e})/\epsilon]$ satisfies the boundary conditions at $x = \pm 1$ to within exponentially small terms as $\epsilon \to 0$ for any shock-layer location $x_{0e} \in (-1,1)$. This suggests that the problem of determining x_{0e} is exponentially ill-conditioned. When $f(u)$ is even, it is clear by symmetry that $x_{0e} = 0$. For more general $f(u)$, the correct value $x_{0e} = (\nu_+ - \nu_-)/(\nu_+ + \nu_-) + O(\epsilon)$ can be obtained analytically by using either an integral identity (cf. [**HOW**]), a spectral projection method (cf. [**RW95a**]), or an extension of the method of matched asymptotic expansions (cf. [**LO94**]).

Metastable behavior for the time-dependent problem (3.1)-(3.2) was first observed numerically in [**KK**] for the special case of Burgers equation. Their numerical computations showed that a thin shock-layer, which connects $u = \alpha_+$ and $u = \alpha_-$, is formed quickly in time from the initial data $u_0(x)$. This shock-layer, which is closely approximated by the viscous profile $\tanh\left[(x_0^0 - x)/2\epsilon\right]$ for some x_0^0 depending on $u_0(x)$, then translates exceedingly slowly towards the equilibrium shock-layer solution centered at $x = 0$. As a partial explanation of these results, it is shown in [**KK**] that the equilibrium solution is linearly stable, but that the principal eigenvalue λ_0 associated with the linearization around this solution is exponentially small of the order $\lambda_0 = O(e^{-c/\epsilon})$ as $\epsilon \to 0$ for some $c > 0$. Motivated by these results of [**KK**] for Burgers equation, a qualitatively similar metastable shock-layer motion for (3.1)-(3.2) is analyzed in [**RW95a**] using a spectral projection method, and in [**LO94**] using an extension of the method of matched asymptotic expansions. A detailed asymptotic study of metastability for Burgers equation is given in [**LO95**] and [**RW95a**].

We now outline the projection method used in [**RW95a**]. In order to make a clear analogy with the analysis in §2.1, the discussion below differs somewhat from that given in [**RW95a**]. The method in [**RW95a**] is based on a quasi-steady linearization of (3.1)-(3.2) around the viscous shock profile, where the shock-layer trajectory $x_0 = x_0(t)$ is to be determined. Thus, in (3.1)-(3.2), we set

$$u(x,t) = u_c\left([x - x_0(t)]/\epsilon\right) + v(x,t)\,, \tag{3.5}$$

where $v \ll u_c$ and $v_t \ll \partial_t u_c$. Linearizing (3.1)-(3.2) around u_c, and using (3.4), we obtain the following quasi-steady problem for $v(x,t)$:

$$L_\epsilon v \equiv \epsilon v_{xx} - \left[f^{'}(u_c)v\right]_x = -\epsilon^{-1} x_0^{'} u_c^{'}\,, \tag{3.6}$$

$$v(-1,t) \sim a_- e^{-\nu_-(1+x_0)/\epsilon}\,, \qquad v(1,t) \sim -a_+ e^{-\nu_+(1-x_0)/\epsilon}\,. \tag{3.7}$$

Let $x_0 \in (-1,1)$ be fixed and consider the corresponding eigenvalue problem

$$L_\epsilon \phi = -\lambda\phi \quad -1 < x < 1\,; \qquad \phi(\pm 1) = 0\,. \tag{3.8}$$

The eigenvalue problem for the adjoint operator $L_\epsilon^\dagger$ is readily seen to be

$$L_\epsilon^\dagger \Phi \equiv \epsilon\Phi_{xx} - \Psi^{'}\Phi_x = -\Lambda\Phi\,, \quad -1 < x < 1\,; \qquad \Phi(\pm 1) = 0\,, \tag{3.9}$$

$$(\Phi,\Phi)_w \equiv \int_{-1}^{1} \Phi^2 w\, dx = 1\,, \qquad w = e^{-\Psi/\epsilon}\,. \tag{3.10}$$

Here $\Psi^{'} = \Psi^{'}(x;\epsilon)$ and the weight $w = w(x;\epsilon) > 0$ are defined by

$$\Psi^{'}(x;\epsilon) = f^{'}\left(u_c\left[(x-x_0)/\epsilon\right]\right)\,, \qquad w(x;\epsilon) = -u_c^{'}\left[(x-x_0)/\epsilon\right]\,. \tag{3.11}$$

The eigenvalues Λ_j for $j \geq 0$ are real with $\Lambda_j > 0$ for $j \geq 0$ and $(\Phi_j,\Phi_k)_w = \delta_{jk}$ for $j,k = 0,1,\ldots$ Let (λ_j,ϕ_j) and (Λ_j,Φ_j) for $j \geq 0$ be the eigenpairs of (3.8) and (3.9), respectively. Then, it easy to show that we can relate these (un-normalized) eigenpairs by

$$\phi_j(x) = -u_c^{'}\left[(x-x_0)/\epsilon\right]\Phi_j(x)\,, \qquad \lambda_j = \Lambda_j\,, \quad j = 0,1,\ldots \tag{3.12}$$

We now outline a key property of the spectrums of (3.8) and (3.9). From the properties of $f(u)$ and $u_c(z)$, it follows that Ψ in (3.9) has a global minimum on $[-1,1]$ at the point $x = x_0$, where $\Psi^{'} = 0$ and $\Psi^{''} > 0$. Therefore, (3.9) is very similar in form to the eigenvalue problem (2.3) for the linear convection-diffusion equation considered in §2.1. Thus, with this analogy, we have that Λ_0 and, hence λ_0, are exponentially small as $\epsilon \to 0$. The corresponding eigenfunction Φ_0 has the boundary layer form given by the right side of (2.6). Since $\Psi^{'} \sim -f^{'}(\alpha_\pm) = \pm\nu_\pm$ at $x = \pm 1$, we find that

$$\Phi_0 \sim M_0\left(1 - e^{-\nu_-(x+1)/\epsilon} - e^{-\nu_+(1-x)/\epsilon}\right)\,, \tag{3.13}$$

as $\epsilon \to 0$, where M_0 is a normalization constant. Then, Λ_0 can be estimated as $\epsilon \to 0$ from the identity

$$\Lambda_0\left(\Phi_0, 1\right)_w = -\epsilon e^{-\Psi/\epsilon}\Phi_0^{'}\Big|_{-1}^{1}\,. \tag{3.14}$$

From (3.12) and (3.13), we observe that ϕ_0 is proportional to $\partial_{x_0} u_c\left[(x-x_0)/\epsilon\right]$ away from the boundary layer regions near $x=\pm 1$. Thus, away from these boundary layers, ϕ_0 is asymptotically close to the translation eigenfunction associated with (3.1) on the infinite line. The finite domain in (3.1) breaks the translation invariance and, together with the exponential decay behavior (3.4), leads to the exponentially small eigenvalue λ_0.

A subtle, but important, difference between the two eigenvalue problems (3.9) and (2.3) is that Ψ' in (3.9) satisfies $\Psi' = \pm\nu_\pm + O(e^{-c/\epsilon})$ as $\epsilon \to 0$ in $O(\epsilon)$ regions near $x=\pm 1$, where $c>0$ is some constant. This result follows from the exponential decay behavior of $u_c(z)$ as $z \to \pm\infty$ given in (3.4). Therefore, in contrast to (2.6), there are no higher order boundary layer correction terms for Φ_0 in powers of ϵ near $x=\pm 1$. This implies that the result $\Phi_0'(\pm 1) \sim \mp\nu_\pm/\epsilon$, which is used in (3.14) to estimate Λ_0, is accurate to within exponentially small terms as $\epsilon \to 0$. The following asymptotic estimate is then obtained from (3.13) and (3.14) (cf. [**RW95a**]):

$$\Lambda_0 = \lambda_0 \sim \frac{1}{\epsilon(\alpha_- - \alpha_+)}\left[a_+\nu_+^2\, e^{-\nu_+(1-x_0)/\epsilon} + a_-\nu_-^2\, e^{-\nu_-(1+x_0)/\epsilon}\right]. \tag{3.15}$$

In contrast to the result (2.5) where the pre-exponential factors have power series in ϵ, the pre-exponential factors written in (3.15) are correct to within exponentially small terms as $\epsilon \to 0$. Hence, (3.15) is highly accurate even when ϵ is only moderately small (cf. [**RW95a**]).

Next, we expand the solution $v(x,t)$ to (3.6)-(3.7) as

$$v(x,t) = \sum_{j=0}^{\infty} \frac{c_j(t)}{\lambda_j}\phi_j(x). \tag{3.16}$$

Using Green's identity and the orthogonality condition $(\phi_j, \Phi_k) = 0$ for $j \neq k$, we obtain

$$c_j = \epsilon^{-1} x_0' \left(\Phi_j, u_c'\right) - \epsilon \Phi_j' v \Big|_{-1}^{1}. \tag{3.17}$$

Since $\lambda_0 \to 0$ exponentially as $\epsilon \to 0$, we must impose the limiting solvability condition that $c_0 \to 0$ as $\epsilon \to 0$. In this way, the following ODE for $x_0(t)$ is obtained by substituting (3.7) and (3.13) into the condition $c_0 = 0$ (see [**RW95a**]):

$$x_0' \sim \frac{1}{(\alpha_- - \alpha_+)}\left[a_-\nu_-\, e^{-\nu_-(1+x_0)/\epsilon} - a_+\nu_+\, e^{-\nu_+(1-x_0)/\epsilon}\right]. \tag{3.18}$$

This result is also derived in [**LO94**] using a different method. The initial condition for $x_0(t)$ can be obtained by studying the transient period describing the formation of the shock-layer from initial data (cf. [**RW95a**], [**LO94**]). Clearly $x_0(t) \to x_{0e}$, where

$$x_{0e} = \frac{\nu_+ - \nu_-}{\nu_+ + \nu_-} + \frac{\epsilon}{\nu_+ + \nu_-}\log\left(\frac{a_-\nu_-}{a_+\nu_+}\right). \tag{3.19}$$

In summary, the metastable shock-layer dynamics for (3.1)-(3.2) is given by $u(x,t) \sim u_c\left(\left[x - x_0(t)\right]/\epsilon\right)$ for large t, where $x_0(t)$ satisfies (3.18).

We close this section with a few remarks. The metastability result (3.18) applied to Burgers equation, where $\nu_\pm = 1$ and $a_\pm = 2$, can be verified analytically by expanding a certain exact solution for $\epsilon \to 0$ (cf. [**LO95**], [**RW95a**]). Such an exact solution is obtained by using the Cole-Hopf transformation on (3.1)-(3.2) when

$f(u) = u^2/2$. For other forms of $f(u)$, the result (3.18) has been favorably compared with corresponding numerical results in [**RW95a**]. The extreme sensitivity of the metastable shock-layer motion and the corresponding equilibrium solution with respect to exponentially small changes in either the boundary conditions of (3.2) or the differential operator of (3.1) has been studied in [**LO94**], [**LO95**], [**RW95c**] and [**RW95a**]. Finally, we remark that other classes of metastable viscous shock problems are considered in [**WO**] and [**LW**].

3.2. Metastable Flame-Front Propagation in a Vertical Channel. As discussed in [**RAS**], under certain conditions flame-front interfaces for upward flame propagation in vertical channels can assume a parabolic-type shape. Under various physical assumptions, a model equation describing the evolution of such a parabolic-shaped interface was derived in [**RAS**] using a weakly nonlinear analysis. In dimensionless variables, this model for the flame-front interface $y = y(x,t)$ is given by

$$y_t - \frac{1}{2}y_x^2 = \epsilon y_{xx} + y - \int_0^1 y\,dx\,, \qquad 0 < x < 1\,, \quad t > 0\,, \tag{3.20}$$

$$y_x(0,t) = y_x(1,t) = 0\,; \qquad y(x,0) = y_0(x)\,, \tag{3.21}$$

where $0 < \epsilon \ll 1$ is a small parameter. The substitution $u(x,t) = -y_x(x,t)$ in (3.20)-(3.21) leads to the following problem for $u(x,t)$ (cf. [**MIS**]):

$$u_t + uu_x - u = \epsilon\, u_{xx}\,, \qquad 0 < x < 1\,, \quad t > 0\,, \tag{3.22}$$

$$u(0,t) = u(1,t) = 0\,; \qquad u(x,0) = u_0(x)\,. \tag{3.23}$$

As shown in [**BKS**], this problem has several equilibrium solutions. For $\epsilon \to 0$, the leading order boundary layer analysis of [**SW96a**] shows that the equilibrium solution $U(x;\epsilon)$ for (3.22)-(3.23) corresponding to a concave parabolic-shaped flame-front interface $Y(x;\epsilon)$ has the form

$$U(x;\epsilon) \sim \tilde{u}^\epsilon[x;x_{0e}] \equiv x - x_{0e} + u_l[x/\epsilon;x_{0e}] + u_r[(1-x)/\epsilon;x_{0e}]\,, \tag{3.24}$$

for some undetermined $x_{0e} \in (0,1)$. Here the boundary layer functions u_l and u_r are defined by

$$u_l(y;\alpha) \equiv \alpha\,(1 - \tanh(\alpha y/2))\,, \quad u_r(y;\alpha) \equiv (\alpha - 1)\,(1 - \tanh[(1-\alpha)y/2])\,. \tag{3.25}$$

In Fig. 1, we plot $\tilde{u}^\epsilon[x;x_{0e}]$ for a fixed $x_{0e} \in (0,1)$. Since $U(x_{0e};\epsilon) = O(e^{-c/\epsilon})$ for some $c > 0$ and $Y_x = -U$, it follows that x_{0e} asymptotically represents the tip, or nose, location for the parabolic-shaped equilibrium interface $Y = Y(x:\epsilon)$. By symmetry we expect that the correct value for x_{0e} is $x_{0e} = 1/2$. However, this value for x_{0e} still cannot be determined by the method of matched asymptotic expansions even after one performs a higher order boundary layer analysis near the endpoints $x = 0$ and $x = 1$. To explain this apparent indeterminacy in x_{0e}, we note that $\tilde{u}_x^\epsilon - 1 = O(e^{-c/\epsilon})$ in the outer region $O(\epsilon) \ll x \ll 1 - O(\epsilon)$. Thus, in this region, the equation $\epsilon u_{xx} - u(u_x - 1) = 0$ is satisfied by $\tilde{u}^\epsilon$ to within exponentially small terms as $\epsilon \to 0$ for any choice $x_{0e} \in (0,1)$. Hence, this problem is exponentially ill-conditioned and exponential precision is required to determine x_{0e}.

For the time-dependent problem, the computational results of [**MIS**] showed that a concave parabolic-shaped flame-front interface can develop for some initial data. Their results showed that the location $x_0 = x_0(t)$ of the tip, or nose, of this interface drifts very slowly in time towards one of the endpoints of the interval.

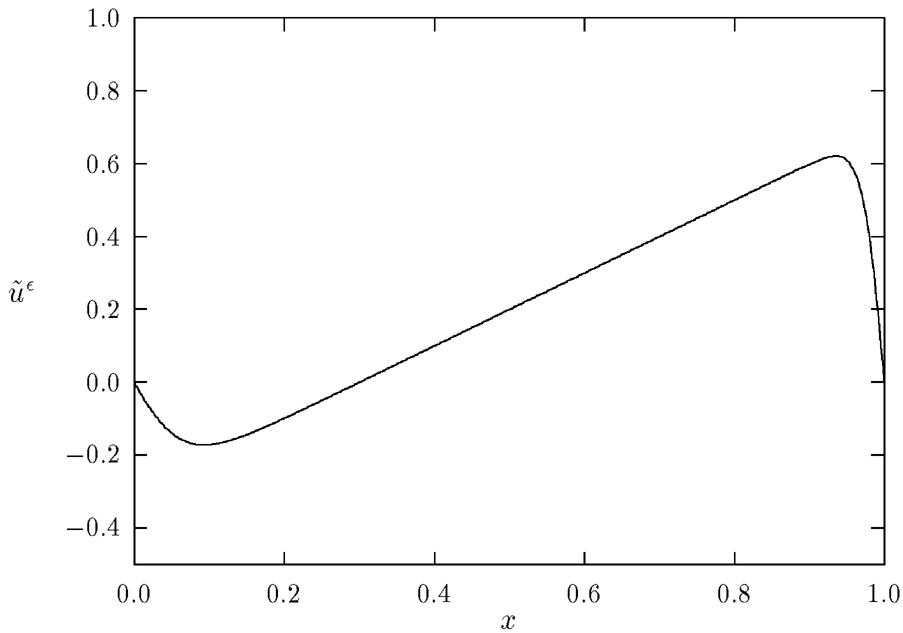

FIGURE 1. Plot of $\tilde{u}^\epsilon$ versus x when $x_{0e} = 0.3$ and $\epsilon = 0.01$.

The analysis of [**BKS**] proved that this interface is dynamically metastable in the sense that the tip location $x_0(t)$ remains in a small neighborhood of its initial value for an asymptotically exponentially long time interval as $\epsilon \to 0$. In [**SW96a**], this metastable behavior is analyzed asymptotically using the projection method, and an explicit ODE for $x_0(t)$ analogous to (3.18) is derived.

We now very briefly outline the analysis and the results given in [**SW96a**]. The metastable dynamics for (3.22)-(3.23) is represented by

$$u(x,t) = \tilde{u}^\epsilon[x; x_0(t)] + v(x,t)\,, \tag{3.26}$$

where $\tilde{u}^\epsilon$ is defined in (3.24) and $v \ll \tilde{u}^\epsilon$ and $v_t \ll \partial_t \tilde{u}^\epsilon$. The quasi-steady linearization of (3.22)-(3.23) around $\tilde{u}^\epsilon$ yields

$$L_\epsilon v \equiv \epsilon v_{xx} - \left(\tilde{u}^\epsilon v\right)_x + v = -R_\epsilon(x; x_0) + x_0^{'} \partial_{x_0} \tilde{u}^\epsilon\,, \qquad 0 < x < 1\,, \tag{3.27}$$

$$v(0,t) = -\tilde{u}^\epsilon[0; x_0]\,, \qquad v(1,t) = -\tilde{u}^\epsilon[1; x_0]\,, \tag{3.28}$$

where the residual R_ϵ is defined by

$$R_\epsilon(x; x_0) \equiv \epsilon \partial_{xx} \tilde{u}^\epsilon - \tilde{u}^\epsilon \left(\partial_x \tilde{u}^\epsilon - 1\right)\,. \tag{3.29}$$

Let $x_0 \in (0,1)$ be fixed, and consider the associated eigenvalue problem

$$L_\epsilon \phi = -\lambda \phi\,, \quad 0 < x < 1\,; \qquad \phi(0) = \phi(1) = 0\,, \tag{3.30}$$

$$(\phi, \phi)_w = \int_0^1 \phi^2 w \, dx = 1\,, \qquad w \equiv \exp\left(-\epsilon^{-1} \int_{x_0}^x \tilde{u}^\epsilon[z; x_0]\, dz\right)\,. \tag{3.31}$$

The eigenvalues λ_j for $j \geq 0$ are real and the orthogonality relations $(\phi_j, \phi_k)_w = \delta_{jk}$ for $j, k = 0, 1, .., $ hold. Note that $L_\epsilon(\partial_{x_0}\tilde{u}^\epsilon)$ is uniformly small on $[0,1]$ and $\partial_{x_0}\tilde{u}^\epsilon$ is of one sign. Thus, we have that $\phi_0 \sim M_0 \partial_{x_0}\tilde{u}^\epsilon$, where M_0 is a normalization constant (cf. [**SW96a**]). In [**SW96a**] it is shown that the corresponding eigenvalue λ_0 is exponentially small as $\epsilon \to 0$ and has the estimate

(3.32)
$$\lambda_0 \sim -\frac{1}{\epsilon}\left[x_0\left(x_0 - \epsilon^{1/2}c\right)e^{-x_0^2/2\epsilon} + (1-x_0)\left((1-x_0) - \epsilon^{1/2}c\right)e^{-(1-x_0)^2/2\epsilon}\right],$$

where $c = \sqrt{8/\pi}$.

To derive an equation of motion for $x_0(t)$, we expand $v(x,t)$ in terms of the eigenfunctions ϕ_j of (3.30)-(3.31) as in (3.16). In place of (3.17), the coefficient c_j is now given by

$$c_j = -x_0^{'}\left(\phi_j, \partial_{x_0}\tilde{u}^\epsilon\right)_w + \left(\phi_j, R_\epsilon\right)_w - \epsilon w v \phi_j^{'}\Big|_0^1, \tag{3.33}$$

where w and the inner product are defined in (3.31). Since $\lambda_0 \to 0$ exponentially as $\epsilon \to 0$, we impose the limiting solvability condition $c_0 \to 0$ as $\epsilon \to 0$. This condition yields a differential equation for $x_0(t)$. After a lengthy calculation of evaluating the inner product $(\phi_0, R_\epsilon)_w$ as $\epsilon \to 0$, the following explicit ODE for the tip location $x_0(t)$ is obtained in [**SW96a**]:

$$x_0^{'} \sim \left(\frac{2}{\pi\epsilon}\right)^{1/2}\left[\left((1-x_0)^2 + \frac{\pi^2\epsilon}{3}\right)e^{-(1-x_0)^2/2\epsilon} - \left(x_0^2 + \frac{\pi^2\epsilon}{3}\right)e^{-x_0^2/2\epsilon}\right]. \tag{3.34}$$

The initial condition $x_0(0)$ for (3.34) is found from a transient analysis describing the formation of the interface from initial data. The ODE (3.34) differs significantly in form from the corresponding result (3.18) for the viscous shock problem. From (3.34), the (unstable) equilibrium value for x_0 is $x_{0e} = 1/2$. Also note that $x_0(t)$ will collapse against the endpoint $x = 1$ ($x = 0$) on an exponentially long time interval when $x_0(0) > 1/2$ ($x_0(0) < 1/2$). This qualitative behavior is observed in the computational results of [**MIS**]. In [**SW96a**], the asymptotic results (3.32) and (3.34) are favorably compared with full numerical results.

4. Spike-Layer Solutions for Reaction-Diffusion Equations

In this section we use a projection method to construct equilibrium spike-layer solutions for a class of reaction-diffusion equations in both one and two dimensions. The characteristic feature of these solutions is that they are localized in an $O(\epsilon)$ region near some peak location x_0 and that they tend to zero away from the peak. Although these solutions are exponentially ill-conditioned, they are not metastable for the corresponding time-dependent problem.

4.1. One Spatial Dimension. We first construct a spike-layer solution for the following nonlinear boundary value problem in one spatial dimension:

$$\epsilon^2 u^{''} + Q(u) = 0, \quad -1 < x < 1; \qquad u^{'}(\pm 1) = 0. \tag{4.1}$$

Here $Q(u)$ is smooth and has exactly two roots at $u = 0$ and $u = u_+ > 0$ with $Q^{'}(0) < 0$ and $Q^{'}(u_+) > 0$. In addition, we assume that there exists a value u_m with $u_m > u_+$ such that $\int_0^{u_m} Q(u)\,du = 0$. Prototypical is $Q(u) = -u + u^2/2$.

Under these assumptions on $Q(u)$ there exists a unique solution $u_s(\rho)$ to

$$u_s'' + Q(u_s) = 0 \,, \quad -\infty < \rho < \infty \,; \qquad u_s(\pm\infty) = 0 \,, \quad u_s(0) = u_m \,. \tag{4.2}$$

This solution, which is a homoclinic orbit, satisfies $\rho u_s' < 0$ for $\rho \neq 0$ and has the far-field asymptotic behavior

$$u_s(\rho) \sim a e^{-\nu|\rho|} \,, \quad \text{as} \;\; |\rho| \to \infty \,; \qquad \nu \equiv \left[-Q'(0)\right]^{1/2} \,, \tag{4.3}$$

for some $a > 0$. In particular, if $Q(u) = -u + u^2/2$, then $u_s(\rho) = 3\text{sech}^2(\rho/2)$, $\nu = 1$ and $a = 12$.

We now look for a solution to (4.1) with exactly one interior spike. Such a solution is given asymptotically by $u(x) \sim u_s\left[(x - x_0)/\epsilon\right]$, where $x_0 \in (-1, 1)$ is to be determined. However, since $u_s(\rho)$ decays exponentially as $\rho \to \pm\infty$, the problem of determining x_0 using asymptotic methods requires exponential precision. This ill-conditioning results from the fact that $u_s\left[(x - x_0)/\epsilon\right]$ satisfies the differential equation in (4.1) exactly and satisfies the boundary conditions in (4.1) to within exponentially small terms as $\epsilon \to 0$ for any choice of $x_0 \in (-1, 1)$. There have been several asymptotic methods used to calculate the correct value $x_0 = 0$; explicit matching of exponentially small terms (cf. [**L**]); a variational method (cf. [**KKM**]); a spectrally-based projection method (cf. [**W92**]).

There are three key steps in the projection method of [**W92**]. First, we linearize (4.1) around $u_s\left[(x - x_0)/\epsilon\right]$. Then we show that the eigenvalue problem associated with this linearization has an exponentially small eigenvalue. The final step is to enforce a limiting solvability condition on the solution to the linearized problem. This condition, which ensures that the solution to the linearized problem is orthogonal to the eigenfunction associated with the exponentially small eigenvalue, yields an algebraic equation for x_0. This last step relies heavily on constructing this critical eigenfunction using a boundary layer analysis.

We now give a few details of the analysis. We first linearize (4.1) around $u_s\left[(x - x_0)/\epsilon\right]$ by substituting

$$u(x) = u_s\left[(x - x_0)/\epsilon\right] + v(x) \,, \tag{4.4}$$

into (4.1), where $v \ll u_s$. Using (4.3), we obtain that v satisfies the linearized problem

$$L_\epsilon v \equiv \epsilon^2 v'' + Q'(u_s) v = 0 \,, \qquad -1 < x < 1 \,, \tag{4.5}$$

$$v'(1) \sim a\nu e^{-\nu(1-x_0)/\epsilon} \,, \qquad v'(-1) \sim -a\nu e^{-\nu(1+x_0)/\epsilon} \,. \tag{4.6}$$

Next, we study the following eigenvalue problem associated with (4.5)-(4.6):

$$L_\epsilon \phi = -\lambda\phi \,, \quad -1 < x < 1 \,; \qquad \phi'(\pm 1) = 0 \,; \qquad (\phi, \phi) \equiv \int_{-1}^{1} \phi^2 \, dx = 1 \,. \tag{4.7}$$

The eigenvalues λ_j for $j \geq 0$ are real and $(\phi_j, \phi_k) = \delta_{jk}$ for $j, k = 0, 1, \ldots$

To study (4.7) it is instructive to compare it with the related eigenvalue problem $L_\epsilon \bar{\phi} = -\bar{\lambda}\bar{\phi}$ defined on the infinite line $-\infty < x < \infty$ with $\bar{\phi} \to 0$ sufficiently rapidly as $|x| \to \infty$. Let $(\bar{\lambda}_j, \bar{\phi}_j)$ for $j = 0, .., J$ be the discrete eigenpairs of this related problem. The parameter ϵ in this related problem can be eliminated by scaling the x axis. Thus, we conclude that $\bar{\lambda}_j$ is independent of ϵ. Next, by translation invariance, it is clear that $L_\epsilon \hat{\phi} = 0$, where $\hat{\phi}(x) \equiv u_s'\left[(x - x_0)/\epsilon\right]$. Note that $\hat{\phi} \to 0$ exponentially as $|x| \to \infty$ and $\hat{\phi}$ has exactly one zero-crossing located at $x = x_0$.

Therefore, it follows that the *second* eigenpair of the related eigenvalue problem is given by $\bar{\phi}_1 = \hat{\phi}$ and $\bar{\lambda}_1 = 0$. Since $\bar{\phi}_1$ is localized near $x = x_0$ and $x_0 \in (-1, 1)$, it is clear that the second eigenfunction ϕ_1 of (4.7) is approximated closely by $\hat{\phi}$ in the interior of the interval $(-1, 1)$. However, boundary layer correction terms of exponentially small amplitude must be added to $\hat{\phi}$ near the endpoints $x = \pm 1$ in order to satisfy the boundary conditions $\phi_1(\pm 1) = 0$. This slight break in the translation invariance, together with the exponential decay behavior (4.3), leads to an exponentially small second eigenvalue λ_1 for (4.7). For $\epsilon \to 0$, the boundary layer analysis of [**W92**] shows that

$$\phi_1 \sim M_1 \left(u_s^{'} \left[(x - x_0)/\epsilon \right] + a_l e^{-\nu(1+x)/\epsilon} + a_r e^{-\nu(1-x)/\epsilon} \right) . \tag{4.8}$$

Here $a_l = a\nu e^{-\nu(1+x_0)/\epsilon}$, $a_r = -a\nu e^{-\nu(1-x_0)/\epsilon}$, and M_1 is a normalization constant. Moreover, λ_1 has the asymptotic estimate

$$\lambda_1 \sim -2a^2\nu^3\beta^{-1} \left[e^{-2\nu(1+x_0)/\epsilon} + e^{-2\nu(1-x_0)/\epsilon} \right] , \qquad \beta \equiv \int_{-\infty}^{\infty} [u_s^{'}(\rho)]^2 \, d\rho \, . \tag{4.9}$$

The results (4.8) and (4.9), which quantify the exponential ill-conditioning of (4.5)-(4.6), are analogous to the results (2.6) and (2.5), respectively, obtained for the linear convection-diffusion equation.

Since the principal eigenvalue $\bar{\lambda}_0$ of the related eigenvalue problem is non-degenerate and $\bar{\lambda}_1 = 0$, it is clear that $\bar{\lambda}_0 < 0$ and is independent of ϵ. In particular, if $Q(u) = -u + u^2/2$, it is easy to show that $\bar{\lambda}_0 = -5/4$ and that $\bar{\phi}_0$ is proportional to $\text{sech}^3 \left[(x - x_0)/2\epsilon \right]$. Since the corresponding eigenfunction $\bar{\phi}_0$ is localized near $x = x_0$, it follows that the principal eigenvalue λ_0 of (4.7) is exponentially close to $\bar{\lambda}_0$. Thus, as $\epsilon \to 0$, we have the strict inequality that $\lambda_0 < 0$ as $\epsilon \to 0$. A consequence of this result is that spike-layer solutions for the corresponding parabolic time-dependent problem are dynamically unstable and, thus, do not exhibit metastable behavior.

The final step in the projection method is to expand the solution to (4.5)-(4.6) in terms of the eigenfunctions of (4.7) as

$$v(x) = \sum_{j=0}^{\infty} \frac{c_j}{\lambda_j} \phi_j(x) \, . \tag{4.10}$$

The coefficients c_j for $j \geq 0$ are given by

$$c_j = \epsilon^2 \left[\phi_j(1) v^{'}(1) - \phi_j(-1) v^{'}(-1) \right] \, . \tag{4.11}$$

Since $\lambda_1 \to 0$ exponentially as $\epsilon \to 0$, it is necessary that the limiting solvability condition $c_1 \to 0$ as $\epsilon \to 0$ be satisfied. Setting $c_1 = 0$, and using both (4.6) and the asymptotic form for ϕ_1 given in (4.8), we derive the following algebraic equation for x_0:

$$e^{-2\nu(1-x_0)/\epsilon} \sim e^{-2\nu(1+x_0)/\epsilon} \, . \tag{4.12}$$

This yields the correct value $x_0 = 0$. Geometrically, this shows that the spike is located at that point in $(-1, 1)$ which is furthest from the endpoints at $x = \pm 1$. We remark that similar ideas can be used to construct multi-spike solutions to (4.1) and to study the extreme sensitivity of such solutions to exponentially small perturbations (see [**W92**]).

4.2. Two Spatial Dimensions. We now construct spike-layer solutions for the following multi-dimensional analogue of (4.1):

$$\epsilon^2 \triangle u + Q(u) = 0\,, \qquad x \in D\,, \tag{4.13}$$

$$\epsilon \partial_n u + bu = 0\,, \qquad x \in \partial D\,. \tag{4.14}$$

Here $\epsilon \to 0^+$, ∂_n denotes the outward normal derivative, b is a constant, and $Q(u)$ satisfies the conditions given following (4.1) above. For simplicity, we let D be a closed, bounded and convex two-dimensional domain.

Spike-layer solutions to (4.13)-(4.14) where the spikes are located on ∂D have been constructed in [**GU**], [**NT91**] and [**NT93**] for the case $b = 0$ in (4.14). In [**W95b**] the projection method is used to construct a spike-layer solution for (4.13)-(4.14) where the spike location is not on ∂D but, instead, is strictly contained within D. For the case $b = \infty$, some related work for an interior spike is given in [**WN**] using very different methods.

A spike-layer solution to (4.13)-(4.14) with an interior spike has the form $u(x) \sim u_c\left[|x - x_0|/\epsilon\right]$, where the spike location x_0 satisfies $\text{dist}(x_0, \partial D) = O(1)$ as $\epsilon \to 0$. Here $u_c(\rho)$ is the unique, positive, radially symmetric solution to (4.13) in all of R^2, which satisfies

$$u_c^{''} + \frac{1}{\rho} u_c^{'} + Q(u_c) = 0\,, \qquad \rho \geq 0\,; \qquad u_c^{'}(0) = 0\,, \tag{4.15}$$

$$u_c(\rho) \sim a\rho^{-1/2} e^{-\nu\rho}\,, \quad \text{as} \quad \rho \to \infty\,; \qquad \nu \equiv \left[-Q^{'}(0)\right]^{1/2}\,, \tag{4.16}$$

for some constant $a > 0$. The exponential decay behavior (4.16) is analogous to (4.3), except that there is a geometrical spreading factor of $\rho^{-1/2}$ in R^2, which is not present for the one-dimensional problem. Since $u_c(\rho)$ decays exponentially as $\rho \to \infty$, it follows that $u_c\left[|x - x_0|/\epsilon\right]$ fails to satisfy (4.14) by only exponentially small terms for any $x_0 \in D$. Thus, once again, the problem of determining x_0 is exponentially ill-conditioned.

We now outline the projection method used in [**W95b**] to determine x_0. We first linearize (4.13)-(4.14) around $u_c\left[|x - x_0|/\epsilon\right]$ by substituting

$$u(x) = u_c\left[|x - x_0|/\epsilon\right] + v(x)\,, \tag{4.17}$$

into (4.13), where $v \ll u_c$. In place of (4.5)-(4.6), the linearized problem for v is

$$L_\epsilon v \equiv \epsilon^2 \triangle v + Q^{'}(u_c) v = 0\,, \qquad x \in D\,, \tag{4.18}$$

$$\epsilon\, \partial_n v + bv = -\left(\epsilon\, \partial_n + b\right) u_c\,, \qquad x \in \partial D\,. \tag{4.19}$$

The eigenvalue problem associated with (4.18)-(4.19) is

$$L_\epsilon \phi = -\lambda \phi\,, \qquad x \in D\,, \tag{4.20}$$

$$\epsilon\, \partial_n \phi + b\phi = 0 \qquad x \in \partial D\,; \qquad (\phi, \phi) \equiv \int_D \phi^2\, dx = 1\,. \tag{4.21}$$

The eigenvalues λ_j for $j \geq 0$ are real and $(\phi_j, \phi_k) = \delta_{jk}$ for $j, k = 0, 1, \ldots$

Next, we study the spectral properties of (4.20)-(4.21). These properties are, in a sense, rather similar to those of (4.7). Define $\hat{\phi}_j$ by $\hat{\phi}_j \equiv \partial_{x_j} u_c(r/\epsilon)$ for $j = 1, 2$, where $r \equiv |x - x_0|$. By translation invariance, it follows that $L_\epsilon \hat{\phi}_j = 0$ for $j = 1, 2$. Thus, in all of R^2, there are two zero eigenvalues of (4.20) with corresponding eigenfunctions $\hat{\phi}_j$ for $j = 1, 2$. Since $u_c(\rho)$ decays exponentially as $\rho \to \infty$, $\hat{\phi}_j$ fails to satisfy the boundary condition (4.21) by exponentially small terms as $\epsilon \to 0$.

This slight break in the translation invariance as a result of the finite domain, together with the exponential decay behavior (4.16), leads to the existence of two exponentially small eigenvalues λ_j, for $j = 1, 2$. The corresponding normalized eigenfunctions ϕ_j have the boundary layer form $\phi_j \sim M_j \left[\partial_{x_j} u_c(r/\epsilon) + \phi_{Lj}\right]$ for $j = 1, 2$. Here ϕ_{Lj} is a boundary layer function, localized near ∂D, which allows the boundary condition in (4.21) to be satisfied. This boundary layer structure is analogous to the corresponding form (4.8) for the one-dimensional problem. Since $\hat{\phi}_j$ for $j = 1$ and $j = 2$ has exactly one nodal line, it follows, for similar reasons as outlined in the one-dimensional case, that the principal eigenvalue λ_0 for (4.20)-(4.21) satisfies the strict inequality $\lambda_0 < 0$ as $\epsilon \to 0$.

A boundary layer analysis and Green's identity leads to the following asymptotic estimates for ϕ_j on ∂D and for λ_j, for $j = 1, 2$ (see [**W95b**]):

$$\phi_j \sim -(\epsilon\beta\pi)^{-1/2} \frac{a\nu^2}{\nu + b}(x_j - x_{0j})r^{-3/2}e^{-\nu r/\epsilon}\left[1 + \hat{r}\cdot\hat{n}\right], \qquad x \in \partial D, \tag{4.22}$$

$$\lambda_j \sim \frac{a^2\nu^3}{\beta\pi}\int_{\partial D}\left(\frac{x_j - x_{0j}}{r}\right)^2 r^{-1}e^{-2\nu r/\epsilon}\left(\frac{b - \nu\hat{r}\cdot\hat{n}}{\nu + b}\right)\left[1 + \hat{r}\cdot\hat{n}\right] ds\,. \tag{4.23}$$

Here $\beta = \int_0^\infty \rho\,[u_c'(\rho)]^2\,d\rho$, a and ν are given in (4.16), $r = |x(s) - x_0|$ where $x(s) \in \partial D$, $\hat{r} = (x(s) - x_0)r^{-1}$, $\hat{n} = \hat{n}(s)$ is the outward normal to ∂D, x_j is the j^{th} component of x, and s is arclength along ∂D. A more precise estimate for λ_j can be obtained by evaluating the integral in (4.23) using Laplace's method. The resulting asymptotic formula for λ_j is similar to (2.22).

The final step in the projection method is to expand the solution to (4.18)-(4.19) in terms of the eigenfunctions of (4.20)-(4.21) as in (4.10). In place of (4.11), the coefficient c_j in (4.10) is given by

$$c_j = -\epsilon\int_{\partial D}\phi_j\left[\epsilon\partial_n + b\right]u_c(r/\epsilon)\,ds\,. \tag{4.24}$$

Since $\lambda_j \to 0$ exponentially as $\epsilon \to 0$ for $j = 1, 2$, it is necessary that the limiting solvability conditions $c_j \to 0$ as $\epsilon \to 0$ for $j = 1, 2$ be satisfied. Setting $c_j = 0$ for $j = 1, 2$ and using (4.16) and (4.22), we obtain that x_0 satisfies

$$I(x_0) = 0\,, \quad \text{for} \quad I(x_0) \equiv \int_{\partial D} r^{-1}\,e^{-2\nu r/\epsilon}\left[1 + \hat{r}\cdot\hat{n}\right]\left(\frac{b - \nu\hat{r}\cdot\hat{n}}{\nu + b}\right)\hat{r}\,ds\,. \tag{4.25}$$

This equation for x_0 is analogous to (4.12).

Suppose that there exists a unique largest inscribed circle B for D, such as is the case when D is strictly convex. Let r_{in} and x_{in} denote the radius and center of B. Then, as shown in [**W95b**], there is a root to $I(x_0) = 0$ satisfying $|x_0 - x_{in}| = O(\epsilon)$ as $\epsilon \to 0$. This result, which is geometrically analogous to the corresponding one-dimensional result, is obtained by using Laplace's method on (4.25). A similar conclusion is obtained in [**WN**] for the case $b = \infty$. The following more precise result appears in [**W95b**]:

Proposition (Spike-Layer Location): *Assume that B is uniquely defined and that B makes exactly three-point contact with ∂D at $x(s_i) \in \partial D$ for $i = 1, 2, 3$. Suppose that $b \neq \nu$ and that $\kappa_i r_{in} > -1$ for $i = 1, 2, 3$, where $\kappa_i \leq 0$ is the curvature of ∂D at $x(s_i)$. Then, the spike-layer location $x_0 = x_0(\epsilon)$ satisfies*

$$x_0(\epsilon) = x_{in} + \epsilon x_0^1 + O(\epsilon^2)\,, \tag{4.26}$$

where x_0^1 is the solution to the linear system

$$(\hat{n}_3 - \hat{n}_1) \cdot x_0^1 = (2\nu)^{-1} \left\{ \log(A_1/A_3) + \log\left(-\hat{n}_1 \cdot \hat{t}_2 / \hat{n}_3 \cdot \hat{t}_2\right) \right\} , \tag{4.27}$$

$$(\hat{n}_1 - \hat{n}_2) \cdot x_0^1 = (2\nu)^{-1} \left\{ \log(A_2/A_1) + \log\left(-\hat{n}_2 \cdot \hat{t}_3 / \hat{n}_1 \cdot \hat{t}_3\right) \right\} . \tag{4.28}$$

Here $A_i = (1 + \kappa_i r_{in})^{-1/2}$. Also, $\hat{n}_i$ and $\hat{t}_i$ are the unit outward normal vector and the unit tangent vector at $x(s_i) \in \partial D$, respectively (oriented in the counter-clockwise sense). A similar result is given in [**W95b**] for the case where B makes exactly two-point contact with ∂D. Further work is needed to construct spike-layer solutions to (4.13)-(4.14) having several interior peaks and to investigate the extreme sensitivity of these solutions to small perturbations.

5. Phase Transition Models in One Spatial Dimension

The viscous Cahn-Hilliard equation, introduced in [**NO**], is a model of slow phase separation in binary alloys accounting for viscoelastic effects. In dimensionless form, this model is

$$(1-\alpha)u_t = -\left(\epsilon^2 u_{xx} + Q(u) - \alpha\kappa u_t\right)_{xx} , \qquad -1 < x < 1 , \quad t > 0 , \tag{5.1}$$

$$u_x(\pm 1, t) = u_{xxx}(\pm 1, t) = 0 ; \qquad u(x, 0) = u_0(x) , \tag{5.2}$$

where $u(x,t)$ is the concentration of one of the two components in the alloy. Here $\kappa > 0$ is the viscoelastic parameter, $\epsilon \to 0^+$ is the interfacial energy parameter, α with $0 \leq \alpha < 1$ is a homotopy parameter, and $Q(u) = -V'(u)$ where $V(u)$ is a double-well potential with wells of equal depth. More specifically, we assume that $Q(u)$ has exactly three zeros at $u = s_- < 0$, $u = s_+ > 0$ and $u = 0$, with

$$Q'(s_\pm) < 0 , \qquad Q'(0) > 0 , \qquad V(s_\pm) = 0 . \tag{5.3}$$

Prototypical is $Q(u) = u - u^3$, for which $s_\pm = \pm 1$ and $V(u) = (1-u^2)^2/4$. Since $Q(u)$ is non-monotone, the reduced equation $(1-\alpha)u_t = -\left[Q(u)\right]_{xx}$ is ill-posed for some range of u. The terms $-\epsilon^2 u_{xxxx}$ and κu_{txx} in (5.1) represent a gradient energy regularization and a viscoelastic regularization, respectively, of this ill-posed reduced equation. Note that the mass $m = \int_{-1}^{1} u(x,t)\, dx$ is conserved for (5.1)-(5.2). We assume below that $u_0(x)$ is such that $2s_- < m < 2s_+$.

Some related phase separation models are obtained by letting α take on limiting values in (5.1). The well-known Cahn-Hilliard model corresponds to $\alpha = 0$. If $\alpha = 1$, we can integrate the right side of (5.1) twice, explicitly impose a mass constraint, and re-scale t to obtain the constrained Allen-Cahn equation introduced in [**RS**]

$$u_t = \epsilon^2 u_{xx} + Q(u) - \sigma , \quad -1 < x < 1 , \quad t > 0 , \tag{5.4}$$

$$u_x(\pm 1, t) = 0 ; \qquad u(x, 0) = u_0(x) ; \qquad \int_{-1}^{1} u(x,t)\, dx = m . \tag{5.5}$$

Here $\sigma = \sigma(t)$ is determined by the constant mass m. The well-studied unconstrained Allen-Cahn equation is obtained by setting $\sigma \equiv 0$ in (5.4) and disregarding the mass constraint. For an overview of mathematical problems and results for phase separation models see [**F**].

There has been much recent work analyzing the dynamics associated with (5.1)-(5.2) and related models. These studies have revealed that the dynamics proceeds in two stages when ϵ is small. The first stage, occurring on an $O(1)$ time interval, involves the transient formation of a pattern of internal layers from initial data.

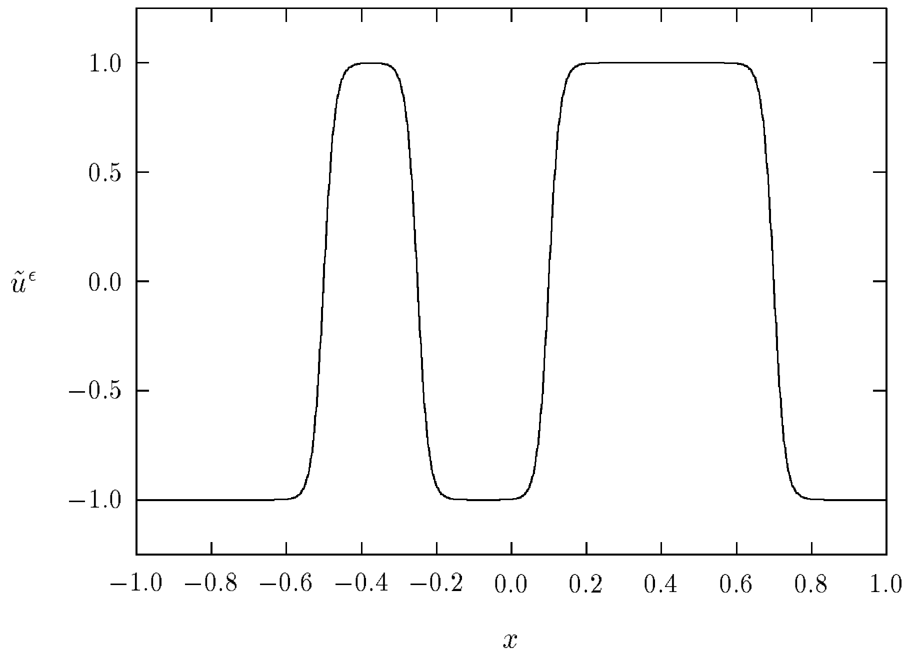

FIGURE 2. Schematic plot of a metastable pattern of internal layers.

The layers have width $O(\epsilon)$ and separate the two phases s_+ and s_- (see Fig. 2 for a schematic plot of a four layer pattern). This transient process is very intricate for (5.1)-(5.2), but is significantly less complex for the unconstrained Allen-Cahn equation. During the next stage of the dynamics, known as the coarsening process, the internal layers move exponentially slowly in time until, typically, they collapse together in pairs. For the unconstrained Allen-Cahn equation this process terminates when no layers remain. However, for models where mass is conserved, this process terminates when a pattern with only one layer, which is consistent with the mass, is attained.

The coarsening process for the unconstrained Allen-Cahn equation has been well-studied in [**CP**], [**FH**], [**N**] and [**W94**]. The existence of metastable internal layer motion has been proved in [**ABF**], [**BH**], [**BX**], and [**G**] for the Cahn-Hilliard equation and in [**KEM**] for a system very similar in form to the constrained Allen-Cahn equation. An explicit characterization of metastability for the Cahn-Hilliard equation is given in [**BX**] using a dynamical systems approach. In [**RW95b**] and [**RW94**] an asymptotic projection method is used to obtain similar results for the viscous Cahn-Hilliard equation (5.1)-(5.2) and for the constrained Allen-Cahn equation (5.4)-(5.5), respectively.

We now outline the metastability analysis in [**RW95b**]. To begin, it is convenient to re-write (5.1)-(5.2) as a coupled system for $u(x,t)$ and $\sigma(x,t)$

$$\alpha\kappa u_t = \epsilon^2 u_{xx} + Q(u) - \sigma\,, \qquad u_x(\pm 1, t) = 0\,, \tag{5.6}$$

$$(1-\alpha) u_t = -\sigma_{xx}\,, \qquad \sigma_x(\pm 1, t) = 0\,. \tag{5.7}$$

For the equilibrium problem, σ is a constant and is asymptotically exponentially small as $\epsilon \to 0$ (cf. [**CGS**]). In [**RW95b**] it is assumed that $\sigma(x,t)$ is also asymptotically exponentially small as $\epsilon \to 0$ for a metastable pattern with widely separated internal layers. Therefore, each layer is closely approximated by the stationary wave solution of $\epsilon^2 u_{xx} + Q(u) = 0$ on the infinite line, which connects the two states $u = s_+$ and $u = s_-$. Thus, we introduce the heteroclinic orbit $u_s(z)$, which is the unique solution to

$$u_s^{''} + Q(u_s) = 0\,, \quad -\infty < z < \infty\,; \qquad u_s(\pm\infty) = s_\pm\,, \quad u_s(0) = 0\,, \tag{5.8}$$

with $u_s^{'}(z) > 0$. This solution has the far-field asymptotic behavior

$$u_s(z) \sim s_\pm \mp a_\pm e^{\mp \nu_\pm z}\,, \quad \text{as} \quad z \to \pm\infty\,; \qquad \nu_\pm \equiv \left[-Q^{'}(s_\pm)\right]^{1/2}\,, \tag{5.9}$$

for some explicit constants $a_\pm > 0$.

An n-layer metastable pattern is represented as a superposition of translates of this heteroclinic connection. Let $x_i = x_i(t)$ for $i = 0,..,n-1$ be the internal layer locations for such a pattern, satisfying the ordering $x_{i+1} > x_i$. Let $x_{-1} = -1$ and $x_n = 1$ be the endpoint locations. Assume that the layers are widely separated at time t, so that $x_{i+1} - x_i \gg O(\epsilon)$ for $i = 0,..,n-1$. Then, an n-layer metastable pattern for (5.1)-(5.2) is represented by

(5.10)
$$u(x,t) \sim \tilde{u}^\epsilon[x; x_0,..,x_{n-1}] \equiv u_s\left[\xi_0(x-x_0)/\epsilon\right] + \sum_{i=1}^{n-1} \left(u_s\left[\xi_i(x-x_i)/\epsilon\right] - s_i\right)\,,$$

where $x_i = x_i(t)$ for $i = 0,..,n-1$ are to be determined. Here $\xi_i = (-1)^i \xi_0$, where $\xi_0 = +1$ or $\xi_0 = -1$ is the orientation of the first layer. In addition, $s_i = s_+$ when $\xi_i = -1$ and $s_i = s_-$ when $\xi_i = +1$. For instance, in Fig.2 we have $n = 4$, $\xi_0 = 1$ and $\xi_i = (-1)^i$ for $i = 1,2,3$.

The projection method used in [**RW95b**] provides an explicit differential-algebraic system of ODE's for the $x_i(t)$, $i = 0,..,n-1$, in (5.10). To derive this system, we first perform a quasi-steady linearization of (5.6)-(5.7) around $\tilde{u}^\epsilon$ by substituting

$$u(x,t) = \tilde{u}^\epsilon\left[x;\, x_0(t),..,x_{n-1}(t)\right] + v(x,t) \tag{5.11}$$

into (5.6)-(5.7), where $v \ll u^*$ and $v_t \ll \partial_t \tilde{u}^\epsilon$. From (5.7) we get

$$\sigma \sim (1-\alpha) \sum_{i=0}^{n-1} x_i^{'} M_i(x) + \sigma_c\,, \quad M_i(x) \equiv \int_{-1}^{x} \left(u_s\left[\xi_i(\eta - x_i)/\epsilon\right] - s_i\right)\, d\eta\,, \tag{5.12}$$

provided that the mass constraint $m = \int_{-1}^{1} \tilde{u}^\epsilon\, dx$ holds. In (5.12), $\sigma_c = \sigma_c(t)$ is to be determined. Then, from (5.6), we find that v satisfies

$$L_\epsilon v \equiv \epsilon^2 v_{xx} + Q^{'}(\tilde{u}^\epsilon) v = \sigma + E + \alpha\,\kappa\,\partial_t \tilde{u}^\epsilon\,, \tag{5.13}$$

$$v_x(-1,t) = -\tilde{u}_x^\epsilon\left(-1; x_0,..,x_{n-1}\right)\,, \qquad v_x(1,t) = -\tilde{u}_x^\epsilon\left(1; x_0,..,x_{n-1}\right)\,. \tag{5.14}$$

Here E represents the exponentially weak interactions between neighboring layers, and is defined by

$$E \equiv E(x;\, x_0, .., x_{n-1}) = \sum_{i=0}^{n-1} Q\left(u_s\left[\xi_i(x-x_i)/\epsilon\right]\right) - Q\left(\tilde{u}^{\epsilon}\right) \,. \tag{5.15}$$

Let x_i for $i = 0, .., n-1$ be fixed and consider the associated eigenvalue problem

$$L_{\epsilon}\phi = -\lambda\phi \,, \quad -1 < x < 1 \,; \qquad \phi^{'}(\pm 1) = 0 \,; \qquad (\phi, \phi) \equiv \int_{-1}^{1} \phi^2 \, dx = 1 \,, \tag{5.16}$$

where L_ϵ is defined in (5.13). The first n eigenvalues λ_i, for $i = 0, .., n-1$, of (5.16) are exponentially small as $\epsilon \to 0$ and the corresponding (un-normalized) eigenfunctions are given asymptotically by $\phi_i(x) \sim u_s^{'}\left[\xi_i(x-x_i)/\epsilon\right]$ for $i = 0, .., n-1$ (cf. [**CP**], [**W92**]). This key property arises as a result of the combined effects of $u_s^{'} > 0$, the decay behavior (5.9), and the near translation invariance of the system.

Next, we expand the solution to (5.13)-(5.14) in terms of the eigenfunctions of (5.16) as in (3.16). Since λ_i for $i = 0, .., n-1$ are exponentially small, we must impose n limiting solvability conditions. This projection step, together with the mass constraint, provides a differential-algebraic (DAE) system of degree n for the n unknowns $\sigma_c(t)$ and $x_i(t)$, for $i = 0, .., n-1$ (cf. [**RW95b**]). This system has the form

$$(1-\alpha)\sum_{k=0}^{n-1} x_k^{'}\,(M_k, \phi_i) + (\sigma_c, \phi_i) + \alpha\kappa\left(\partial_t \tilde{u}^{\epsilon}, \phi_i\right) \sim \epsilon^2 \phi_i v_x \Big|_{-1}^{1} - (E, \phi_i) \,, \tag{5.17}$$

for $i = 0, .., n-1$, together with,

$$\int_{-1}^{1} \tilde{u}^{\epsilon}\left[x; x_0, .., x_{n-1}\right] dx = m \,. \tag{5.18}$$

Here M_i and E are defined in (5.12) and (5.15), respectively. Finally, a lengthy calculation of asymptotically evaluating the various terms in (5.17)-(5.18) as $\epsilon \to 0$ leads to the following explicit result (cf. [**RW95b**]):

Proposition (Metastable Motion): *For $\epsilon \to 0$, an n-layer metastable pattern for (5.1)-(5.2) with widely separated internal layers is represented by (5.10), where $x_i(t)$ for $i = 0, .., n-1$, and $\sigma_c(t)$ satisfy the explicit DAE system*

$$\alpha\kappa\beta\epsilon^{-1}\, x_i^{'} + (1-\alpha)\sum_{k=0}^{n-1} x_k^{'}\, b_{ik} \sim \sigma_c\, \xi_i(s_+ - s_-) + H_i \,, \quad i = 0, .., n-1 \,, \tag{5.19}$$

$$\sum_{k=0}^{n} s_k\left(x_k - x_{k-1}\right) = m + O(\epsilon) \,. \tag{5.20}$$

In (5.19) the exponentially weak forces H_i for $i = 0, .., n-1$ and the coupling coefficient b_{ik} for $i, k = 0, .., n-1$ are defined by

$$H_i = 2\left(a_{i+1}^2 \nu_{i+1}^2\, e^{-(1+\delta_{i\,n-1})\nu_{i+1}d_{i+1}/\epsilon} - a_i^2\nu_i^2\, e^{-(1+\delta_{i\,0})\nu_i d_i/\epsilon}\right) \,, \tag{5.21}$$

$$b_{ik} = \int_{-1}^{1}\left(u_s\left[\xi_k(x-x_k)/\epsilon\right] - s_k\right)\left(u_s\left[\xi_i(x-x_i)/\epsilon\right] - s_{i+1}\right) dx \,, \tag{5.22}$$

where δ_{ik} is the Kronecker symbol, $\beta = \int_{-\infty}^{\infty}[u_s^{'}(z)]^2\, dz$, $d_i = x_i - x_{i-1}$ for $i = 0, .., n$ are the inter-layer separations, and the triplet (s_i, a_i, ν_i) is defined by $(s_i, a_i, \nu_i) = (s_\pm, a_\pm, \nu_\pm)$ when $\xi_i = \mp 1$. Here $a_\pm$ and $\nu_\pm$ are defined in (5.9). To obtain a more explicit result, the coefficients b_{ik} in (5.22) can easily be evaluated asymptotically as $\epsilon \to 0$ (cf. [**RW95b**]).

There are various special cases of (5.19)-(5.20). The constrained Allen-Cahn equation corresponds to $\alpha = 1$ and $\kappa = 1$, and the Cahn-Hilliard equation corresponds to $\alpha = 0$. Note in (5.19) that there is a distinguished limit when $\alpha = O(\epsilon)$. The unconstrained Allen-Cahn equation is obtained by setting $\alpha = \kappa = 1$ and $\sigma_c = 0$ in (5.19) and disregarding the mass constraint (5.20). This leads to the well-known dynamics $x_i' \sim \epsilon\beta^{-1}H_i$, for $i = 0, .., n-1$, which was proved in [**CP**] and [**FH**]. For the special case of a two-layer evolution (i. e. $n = 2$), the result (5.19)-(5.20) has been favorably compared in [**RW95b**] with full numerical results for different values of α. Starting from initial conditions $x_i(0) = x_i^0$, with $d_i = x_i^0 - x_{i-1}^0 > 0$ for $i = 0, .., n$, the dynamics (5.19)-(5.20) is valid until the first time $t = t_c$ where $d_I(t_c) = O(\epsilon)$ for some $I \in \{0, , ., n\}$. Thus, the metastability result does not hold when a layer collapse event is initiated. Further work is needed to characterize these events, and, hence, to obtain a complete description of the coarsening process for (5.1)-(5.2).

Finally, we remark that the projection method described above is closely related to a similar method used in [**B**], [**BIS**] and [**EMS**] to analyze weakly interacting pulse-type solutions for other classes of nonlinear evolution equations. In many of these other problems, the localized pulse solution is a translate of a homoclinic orbit, which has exponentially damped oscillations at infinity. For a train of well-separated pulses, this damped oscillatory far-field behavior can lead to chaotic dynamics between neighboring pulses (cf. [**BIS**], [**EMS**]). A very interesting survey of results for this class of problems is given in [**B**].

6. A Phase Transition Problem in a Multi-Dimensional Domain

In a multi-dimensional setting, dynamic metastability can occur for phase separation models that conserve mass. For such models, the motion of radially symmetric internal layer solutions, referred to in [**AF94a**] as bubble solutions, exhibit metastable behavior (cf. [**AF94a**], [**AF94b**], [**W95a**]). These solutions are characterized by an exponentially slow drift of the center of the bubble towards the boundary of the domain. The bubble maintains its spherical shape during this evolution. For the Cahn-Hilliard equation, the existence and some qualitative properties of this metastable bubble motion have been studied in [**AF94a**] and [**AF94b**]. Results concerning the spectrum of the linearization around the bubble solution for the Cahn-Hilliard equation are given in [**AFS**].

In [**W95a**] the projection method is used to give an explicit asymptotic description of metastable bubble motion for the constrained Allen-Cahn equation introduced in [**RS**]

$$u_t = \epsilon^2 \triangle u + Q(u) - \sigma\,, \qquad x \in D \subset R^N\,, \quad N = 2, 3\,, \tag{6.1}$$

$$\partial_n u = 0\,, \qquad x \in \partial D\,; \qquad \int_D u(x,t)\, dx = m\,. \tag{6.2}$$

Here $u = u(x,t)$, $\sigma = \sigma(t)$, $\epsilon \to 0^+$, D is a bounded convex domain in two or three dimensions, and the mass m is constant. As in §5, we assume that $Q(u) = -V'(u)$, where $V(u)$ is a double-well potential with equal minima at $u = s_\pm$ where $V(s_\pm) = 0$. In Fig. 3 we give a sketch of the geometry for this problem. A thin internal layer of width $O(\epsilon)$ separates the phases $u \sim s_+$ and $u \sim s_-$ that are outside and inside

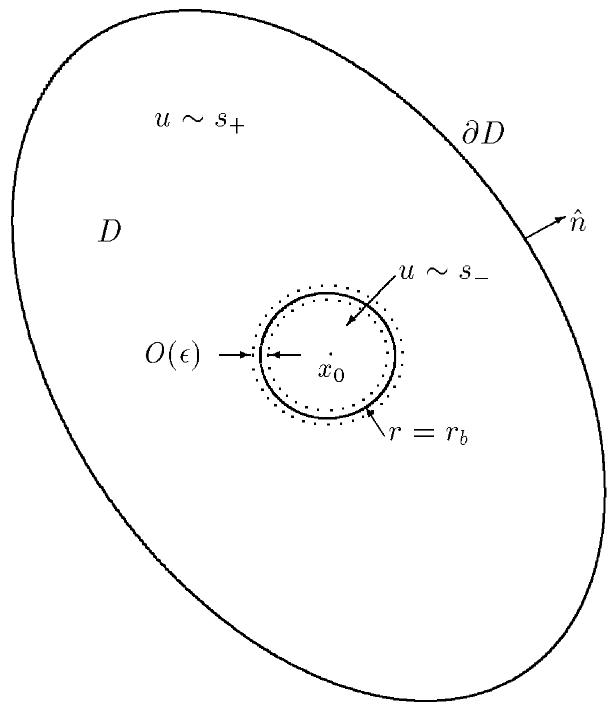

FIGURE 3. Schematic plot of a bubble of radius r_b in a two-dimensional convex domain.

the bubble, respectively. The radius r_b of the bubble is determined by the mass m, and is assumed to be sufficiently small so that the bubble is contained within D at $t = 0$. In [**W95a**] the bubble is shown to drift exponentially slowly across the domain, without change of shape, towards the closest point on ∂D. Also, an explicit ODE for the motion of the center of the bubble is derived. The geometric criteria determining the (unstable) equilibrium location for the center of the bubble is found to be very similar to that of the spike-layer solution for (4.13)-(4.14).

There are three basic steps in the projection method used in [**W95a**]. The first step is to construct a radially symmetric equilibrium bubble solution of radius r_b in all of R^2 and then to linearize (6.1)-(6.2) around this solution. This step is significantly more difficult than for the spike-layer problem in §4.1 in that the bubble must be constructed asymptotically using the method of matched asymptotic expansions. The second step is to analyze the spectrum associated with this linearization. As a result of the slight break in translation invariance, this spectrum contains exponentially small eigenvalues. Asymptotic estimates for the corresponding eigenfunctions on the boundary of the domain are derived using a boundary layer analysis. The final step is to ensure that mass is conserved and that the solution to the quasi-steady linearized problem is orthogonal to the eigenspace associated with the exponentially small eigenvalues. This projection step yields an ODE for the center $x_0 = x_0(t)$ of the bubble. We now outline some of the details of this analysis and give the main result obtained in [**W95a**].

The equilibrium bubble solution $u = U_b(r;\epsilon)$, $\sigma = \sigma_b(\epsilon)$ satisfies

$$\epsilon^2\left(U_b^{''} + \frac{N-1}{r}U_b^{'}\right) + Q(u_b) = \sigma_b\,, \qquad 0 < r < \infty\,, \tag{6.3}$$

$$U_b(r;\epsilon) \to S_\pm(\epsilon)\,, \quad \text{as } \epsilon^{-1}(r-r_b) \to \pm\infty\,; \qquad U_b(r_b;\epsilon) = 0\,. \tag{6.4}$$

Here $S_\pm(\epsilon)$ are the roots of $Q\left[S_\pm(\epsilon)\right] = \sigma_b(\epsilon)$ for which $S_\pm(\epsilon) \to s_\pm$ and $\sigma_b(\epsilon) \to 0$ as $\epsilon \to 0$. The function $U_b(r;\epsilon)$ varies rapidly in an $O(\epsilon)$ neighborhood near $r = r_b$ and it varies slowly away from this region. A matched asymptotic expansion analysis for the region $r \geq r_b$ shows that (cf. [**W95a**])

$$U_b(r;\epsilon) = \begin{cases} S_+(\epsilon) - a_+\left(r/r_b\right)^{(1-N)/2}\, e^{-\nu_+^\epsilon (r-r_b)/\epsilon} & \text{for } r > r_b\,, \\ u_s\left[(r-r_b)/\epsilon\right] + O(\epsilon) & \text{for } r - r_b = O(\epsilon)\,, \end{cases} \tag{6.5}$$

and $\sigma_b(\epsilon) = \epsilon\sigma_1 + O(\epsilon^2)$. Here,

$$S_+(\epsilon) = s_+ - \epsilon\sigma_1\nu_+^{-2} + O(\epsilon^2)\,, \qquad \nu_+^\epsilon = \nu_+ + \frac{\epsilon}{2}\sigma_1\nu_+^{-3}Q^{''}(s_+) + O(\epsilon^2)\,, \tag{6.6}$$

$$\sigma_1 = \frac{\beta(N-1)}{(s_+ - s_-)r_b}\,, \qquad \beta = \int_{-\infty}^{\infty}\left[u_s^{'}(\rho)\right]^2\,d\rho\,, \tag{6.7}$$

and a_+, ν_+ are defined in (5.9) in terms of the one-dimensional profile $u_s(\rho)$, which satisfies (5.8). The exponential decay behavior of U_b for $r > r_b$ is similar to that for the spike-layer problem given in (4.16).

The quasi-steady linearization of (6.1)-(6.2) is obtained by substituting

$$u(x,t) = U_b\left[|x - x_0(t)|;\epsilon\right] + v(x,t)\,, \qquad \sigma(t) = \sigma_b(\epsilon) + \mu(t)\,, \tag{6.8}$$

into (6.1)-(6.2), where $v \ll U_b$, $v_t \ll \partial_t U_b$ and $\mu \ll \sigma_b$. Here $x_0 = x_0(t)$ is the unknown trajectory of the center of the bubble. This leads to the following problem for v and μ:

$$L_\epsilon v \equiv \epsilon^2\triangle v + Q^{'}(U_b)v = \partial_t U_b + \mu\,, \qquad x \in D\,, \tag{6.9}$$

$$\partial_n v = -\partial_n U_b\,, \qquad x \in \partial D\,; \qquad \int_D v\,dx = 0\,. \tag{6.10}$$

Let (λ_j, ϕ_j) for $j \geq 0$ be the eigenpairs of the associated eigenvalue problem

$$L_\epsilon\phi = -\lambda\phi\,, \quad x \in D\,; \quad \partial_n\phi = 0\,, \quad x \in \partial D\,; \quad (\phi,\phi) \equiv \int_D \phi^2\,dx = 1\,. \tag{6.11}$$

The principal eigenvalue satisfies $\lambda_0 < 0$ with $\lambda_0 = O(\epsilon^2)$ as $\epsilon \to 0$, and the corresponding eigenfunction ϕ_0 has the form $\phi_0 \sim M_0\left(U_b^{'} + \phi_{L0}\right)$. Here M_0 is a normalization constant and ϕ_{L0} is a boundary layer function of exponentially small amplitude, localized near ∂D, which allows $\partial_n\phi_0 = 0$ on ∂D to be satisfied. In addition, as a result of the near translation invariance and the exponential decay behavior (6.5), there are N exponentially small eigenvalues λ_j for $j = 1,..,N$. The corresponding eigenfunctions are given asymptotically by $\phi_j \sim M_j\left(\partial_{x_j}U_b + \phi_{Lj}\right)$ for some boundary layer functions ϕ_{Lj}, $j = 1,..,N$. A boundary layer analysis determines ϕ_{Lj} for $j = 0,..,N$ and hence we can obtain explicit asymptotic formulas for ϕ_j on ∂D for $j = 0,..,N$ (see [**W95a**]).

Next, we expand the solution $v(x,t)$ to (6.9)-(6.10) in terms of the eigenfunctions ϕ_j of (6.11) as in (3.16). The coefficients $c_j = c_j(t)$ for $j \geq 0$ in (3.16) are found to be

$$c_j = -(\partial_t U_b, \phi_j) - \mu(1,\phi_j) - \epsilon^2 \int_{\partial D} \phi_j \partial_n U_b \, dS \,, \tag{6.12}$$

where $(f,g) \equiv \int_D fg\,dx$ denotes the inner product. Since U_b and ϕ_j, for $j = 0,..,N$, are known when $\epsilon \ll 1$, we can calculate the inner products and the surface integral in (6.12) asymptotically to determine c_j for $j = 0,..,N$.

The conditions to determine $\mu(t)$ and $x_0(t)$ are as follows. First, we must ensure that $\int_D w\,dx = 0$ in order to conserve mass. For $\epsilon \to 0$, this condition requires that $c_0(t) \equiv 0$, which then determines μ (see [**W95a**]). Thus, in contrast to the spike-layer problem considered in §4, the existence of the negative eigenvalue λ_0 for (6.11) does not lead to an instability of the bubble solution. We remark that if mass was not conserved, the bubble would shrink to a point under a mean curvature flow on a time scale $|\lambda_0^{-1}| = O(\epsilon^{-2})$. Next, since λ_j for $j = 1,..,N$ are exponentially small as $\epsilon \to 0$, we must also require that the limiting solvability conditions $c_j \to 0$ as $\epsilon \to 0$ for $j = 1,..,N$ be satisfied. These conditions yield a differential equation for $x_0(t)$ that governs the metastable bubble motion. In this way, the following main result is obtained in [**W95a**]:

Proposition (Slow Bubble Motion): *When the bubble is strictly contained inside D, its center location $x_0 = x_0(t)$ satisfies the asymptotic ODE*

$$x_0' \sim \frac{\epsilon N a_+^2 \nu_+^2}{\Omega_N \beta} \int_{\partial D} r^{1-N} e^{-2\nu_+^\epsilon (r-r_b)/\epsilon}\, \hat{r}\, [1 + \hat{r}\cdot\hat{n}]\, \hat{r}\cdot\hat{n}\, dS \,. \tag{6.13}$$

Here $r = |x(\xi) - x_0(t)|$, $\hat{r} = [x(\xi) - x_0(t)]/r$, $\hat{n} = \hat{n}(\xi)$ is the unit outward normal to ∂D, Ω_N is the surface area of the unit N-ball and $\xi = (\xi_1,..,\xi_{N-1})$ parameterizes ∂D. Also, β and ν_+^ϵ are defined in (6.6)-(6.7) and a_+ and ν_+ are given in (5.9).

The (unstable) equilibrium location x_{0e} for the bubble center is obtained by setting $x_0' = 0$ in (6.13). Then, by comparing (6.13) and (4.25), it is clear that a result very similar to (4.26) also holds for x_{0e} when $N = 2$. Next, an asymptotic evaluation of the surface integral in (6.13) yields the following explicit result:

Corollary (Explicit Motion): *Assume that at $t = 0$, $x(\xi_0)$ is the unique point on ∂D that is closest to the initial center location $x_0(0) \equiv x_0^0$. Then, for $t > 0$ and $\epsilon \to 0$, the motion of the center of the bubble is in the direction of $x(\xi_0) - x_0^0$ and the distance $r_m(t) = |x(\xi_0) - x_0(t)|$ satisfies the asymptotic ODE*

$$r_m' \sim -\zeta r_m \left(\frac{\epsilon}{r_m}\right)^{(N+1)/2} H(r_m)\, e^{-2\nu_+^\epsilon (r_m - r_b)/\epsilon} \,, \tag{6.14}$$

together with the initial condition $r_m(0) = |x(\xi_0) - x_0^0| > r_b$. Here ζ and $H(r_m)$ are defined by

$$\zeta = \frac{2N a_+^2 \nu_+^2}{\Omega_N \beta} \left(\frac{\pi}{\nu_+^\epsilon}\right)^{(N-1)/2} \,, \qquad H(r_m) = \Pi_{i=1}^{N-1} (1 - r_m/R_i)^{-1/2} \,, \tag{6.15}$$

where $R_i \geq 0$ for $i = 1,..,N-1$, are the principal radii of curvature of ∂D at $x(\xi_0) \in \partial D$. The result (6.14), which is asymptotically valid only when $r_m(t) > r_b$, shows that the bubble will collapse against ∂D on an exponentially long time scale.

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References

[AO] R. Ackerberg, R. O'Malley, *Boundary layer problems exhibiting resonance*, Stud. Appl. Math. **49**, (1970), pp. 277–295.

[ABF] N. Alikakos, P.W. Bates, G. Fusco, *Slow motion for the Cahn-Hilliard equation in one space dimension*, J. Diff. Equat. **90**, (1991), pp. 81–135.

[AF94a] N. Alikakos, G. Fusco, *Slow dynamics for the Cahn-Hilliard equation in higher spatial dimensions, part 1: spectral estimates*, Comm. Part. Diff. Equat. **19**, (1994), pp. 1397–1447.

[AF94b] N. Alikakos, G. Fusco, *Slow dynamics for the Cahn-Hilliard equation in higher spatial dimensions, part 2: the motion of bubbles*, preprint, (1994).

[AFS] N. Alikakos, G. Fusco, V. Stefanopoulis, *Critical spectrum and stability of interfaces for a class of reaction-diffusion equations*, J. Diff. Equat., to appear.

[B] N.J. Balmforth, *Solitary waves amd homoclinic orbits*, Annual Review of Fluid Mechanics, **27**, (1995), pp. 335–374.

[BIS] N.J. Balmforth, G.R. Ierley, E.A. Spiegel, *Chaotic pulse trains*, SIAM J. Appl. Math. **54**, (1993), pp. 1291–1334.

[BX] P.W. Bates, J. Xun, *Metastable patterns for the Cahn-Hilliard equation: parts 1 and 2*, J. Diff. Equat. **111**, (1994), pp. 421–457; J. Diff. Equat. **117**, (1995), pp. 165–216.

[BKS] H. Berestycki, S. Kamin, G. Sivashinsky, *Nonlinear dynamics and metastability in a Burgers type equation*, Compte Rendus Acad. Sci., Paris t. **321**, Serié 1, (1995), pp. 185-190.

[BH] L. Bronsard, D. Hilhorst, *On the slow dynamics for the Cahn-Hilliard Equation in one space dimension*, Proc. Roy. Soc. London A, **439**, (1992), pp. 669–682.

[CGS] J. Carr, M. Gurtin, M. Slemrod, *Structural phase transitions on a finite interval*, Archive for Rat. Mechanics and Analysis, **86**, (1984), pp. 317–351.

[CP] J. Carr, R. Pego, *Metastable patterns in solutions of* $u_t = \epsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. **42**, (1989), pp. 523–576.

[D] P.P.N. De Groen, *The nature of resonance in a singular perturbation problem of turning point type*, SIAM J. Math. Anal. **11**, (1980), pp. 1–22.

[EMS] C. Elphick, E. Meron, E.A. Spiegel, *Patterns of propagating pulses*, SIAM J. Appl. Math. **50**, (1990), pp. 490–503.

[F] P. Fife, *Models for phase separation and their mathematics*, in 'Nonlinear Partial Differential Equations and Applications' (M. Mimura, Y. Nishiura editors), Taniguyia Publications, (1992).

[FH] G. Fusco, J.K. Hale, *Slow motion manifolds, dormant instability and singular perturbations*, J. Dyn. Diff. Equat. **1**, (1989), pp. 75–94.

[G] C. Grant, *Slow motion in one-dimensional Cahn-Morral systems*, SIAM J. Math. Anal. **26**, (1995), pp. 21–34.

[GM] J. Grasman, B.J. Matkowsky, *A variational approach to singularly perturbed boundary value problems for ordinary and partial differential equations with turning points*, SIAM J. Appl. Math. **32**, (1977), pp. 588–597.

[GU] C. Gui, *Multi-peak solutions for a semilinear Neumann problem*, to appear.

[H] E.J. Hinch, *Perturbation methods*, Cambridge Texts in Applied Math., Cambridge University Press, Cambridge, (1991).

[HO] M. Holmes, *Introduction to perturbation methods*, Texts in Appl. Math. Vol. **20**, Springer-Verlag, New York, (1995).

[HOW] F. Howes, *An analytical treatment of the formation of one-dimensional steady shock waves in uniform and diverging ducts*, J. Comput. and Appl. Math, **10**, (1984), pp. 195–201.

[KKM] W.L. Kath, C. Knessl, B.J. Matkowsky, *A variational approach to nonlinear singularly perturbed boundary value problems*, Stud. Appl. Math. **77**, (1987), pp. 61–88.

[KC] J. Kevorkian, J.Cole, *Perturbation methods in applied mathematics*, Springer-Verlag, New York, 1981.

[KK] G. Kreiss, H. Kreiss, *Convergence to steady state of solutions of Burgers equation*, Appl. Numerical Math. **2**, (1986), pp. 161–179.

[K] H. Kreiss, *Resonance for singular perturbation problems*, SIAM J. Appl. Math. **41**, (1981), pp. 331–344.

[KEM] M. Kuwamura, S.I. Ei, M. Mimura, *Very slow dynamics for some reaction-diffusion systems of the activator-inhibitor type,* Japan J. Indust. Appl. Math. **9**, (1992), pp. 35–77.

[LO94] J. Laforgue, R.E. O'Malley, *On the motion of viscous shocks and the super-sensitivity of their steady-state limits*, Methods and Appl. of Anal. **1**, (1994), pp. 465–487.

[LO95] J. Laforgue, R.E. O'Malley, *Shock layer movement for Burgers equation*, SIAM J. Appl. Math. **55**, (1995), pp. 332–348.

[LA] P.A. Lagerstrom, *Matched asymptotic expansions*, Appl. Math. Sciences, Vol. **76**, Springer-Verlag, New York, (1988).

[L] C. Lange, *On spurious solutions of singular perturbation problems*, Stud. Appl. Math. **68**, (1983), pp. 227–257.

[LW] J.Y. Lee, M.J. Ward, *On the asymptotic and numerical analysis of exponentially ill-conditioned singularly perturbed boundary value problems*, Stud. Appl. Math. **94**, (1995), pp. 271–326.

[LU] D. Ludwig, *Persistence of dynamical systems under random perturbations*, SIAM Review **17**, (1975), pp. 605–640.

[MAS] R.S. Maier, D.L. Stein, *Asymptotic exit location distributions in the stochastic exit problem*, to appear.

[M75] B.J. Matkowsky, *On boundary layer problems exhibiting resonance*, SIAM Rev. **17**, (1975), pp. 82–100.

[M80] B.J. Matkowsky, *Singular perturbations, stochastic differential equations, and applications,* in 'Singular Perturbations and Asymptotics' (R.E. Meyer and S.V. Parter, editors), Academic Press, New York, (1980), pp. 109–147.

[M95] B.J. Matkowsky, *personal communication*, (1995).

[MS] B.J. Matkowsky, Z. Schuss, *The exit problem for randomly perturbed dynamical systems*, SIAM J. Appl. Math. **33**, (1977), pp. 365–382.

[MIS] A. Mikishev, G. Sivashinsky, *Quasi-equilibrium in upward propagating flames*, Physics Letters A **175**, (1993), pp. 409-414.

[N] J. Neu, *Unpublished notes.*

[NT91] W.M. Ni, I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44**, (1991), pp. 819–851.

[NT93] W.M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70**, (1993), pp. 247–281.

[NO] A. Novick-Cohen, *On the viscous Cahn-Hilliard equation*, in 'Material Instabilities in Continuum Mechanics and Related Mathematical Problems' (J. Ball, editor), Oxford Science Publications, Clarendon Press, (1988), pp. 329–342.

[O] R.E. O'Malley, *Singular perturbation methods for ordinary differential equations*, Springer-Verlag, New York, (1991).

[OW] R.E. O'Malley, M.J. Ward, *Exponential asymptotics, boundary layer resonance, and dynamic metastability*, submitted for proceedings of conference honoring Prof. J.D. Cole.

[RAS] Z. Rakib, G.I. Sivashinsky, *Instabilities in upward propagating flames*, Combust. Sci. and Tech. **54**, (1987), pp. 69–84.

[RW94] L.G. Reyna, M.J. Ward, *Resolving weak internal layer interactions for the Ginzburg-Landau equation*, European J. Appl. Math. **5**, (1994), pp. 495–523.

[RW95a] L.G. Reyna, M.J. Ward, *On the exponentially slow motion of a viscous shock*, Comm. Pure Appl. Math. **48**, (1995), pp. 79–120.

[RW95b] L.G. Reyna, M.J. Ward, *Metastable internal layer dynamics for the viscous Cahn-Hilliard equation*, Methods and Appl. of Anal. **2** No. 3, (1995), pp. 285–306.

[RW95c] L.G. Reyna, M.J. Ward, *On exponential ill-conditioning and internal layer behavior*, J. Num. Func. Analysis and Optimization Vol. **16** No. 4, (1995), pp. 475–500.

[RS] J. Rubinstein, P. Sternberg, *Nonlocal reaction-diffusion equations and nucleation*, IMA J. Appl. Math. **48**, (1992), pp. 249–264.

[S] L.A. Skinner, *Uniform solutions of boundary value problems exhibiting resonance*, SIAM J. Appl. Math. **47**, (1987), pp. 225–231.

[SR] R. Srinivasan, *A variational principle for the Ackerberg-O'Malley resonance problem*, Stud. Appl. Math. **79**, (1988), pp. 271–289.

[SW96a] X. Sun, M.J. Ward, *Metastability for a generalized Burgers equation with applications to propagating flame fronts*, in preparation.

[SW96b] X. Sun, M.J. Ward, *Exponentially ill-conditioned convection-diffusion equations in multi-dimensional domains*, in preparation.

[V] M. Van Dyke, *Perturbation methods in fluid mechanics*, Parabolic Press, Stanford, (1975).

[W92] M.J. Ward, *Eliminating indeterminacy in singularly perturbed boundary value problems with translation invariant potentials*, Stud. Appl. Math **87**, (1992), pp. 95–135.

[W94] M.J. Ward, *Metastable patterns, layer collapses, and coarsening for a one-dimensional Ginzburg-Landau equation*, Stud. Appl. Math. **91**, (1994), pp. 51–93.

[W95a] M.J. Ward, *Metastable bubble solutions for the Allen-Cahn equation with mass conservation*, to appear, SIAM J. Appl. Math.

[W95b] M.J. Ward, *An asymptotic analysis of localized solutions for some reaction-diffusion models in multi-dimensional domains*, to appear, Stud. Appl. Math.

[WN] J. Wei, W.M. Ni, *On the location and profile of spike-solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. **48**, (1995), pp. 731–768.

[WH] G. Whitham, *Linear and nonlinear waves*, Wiley-Interscience, New York, (1974).

[W] M. Williams, *Another look at Ackerberg-O'Malley resonance*, SIAM J. Appl. Math. **41**, (1981), pp. 288–293.

[WO] G. Wolansky, *On the slow evolution of quasi-stationary shock waves*, J. Dyn. Diff. Equat. **6**, (1994), pp. 247–276.

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