Traps, Patches, Spots, and Stripes: Localized Solutions to Diffusive and Reaction-Diffusion Systems

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BIRS Meeting; Multi-Scale Analysis of Self-Organization in Biology

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Lecture III: Localized Stripe Patterns for Reaction-Diffusion Systems

Singularly Perturbed RD Models

Consider the two-component system in $\Omega \in R^2$:

$$v_t = \varepsilon^2 \triangle v + g(u,v)\,; \quad \tau u_t = D\triangle u + f(u,v)\,, \quad \partial_n u = \partial_n v = 0\,, \quad \boldsymbol{x} \in \partial\Omega\,.$$

Since $\varepsilon \ll 1$, v can be localized in space on a curve of either zero or finite width, i.e. stripe. We will assume that both ends of the stripe connect, or intersect with the boundary (**no stripe fragments**).

- **Semi-Strong Interaction Regime:** $D = O(1)$ so that u is global.
- **Weak Interaction Regime:** $D = O(\varepsilon^2)$ so that u is also localized on a stripe
- **Terminology:** Homoclinic Stripes and Mesa Stripes. Stripe Instabilities: Breakup, Zigzag, and Self-Replication

Different Kinetics: (There is No Variational Structure)

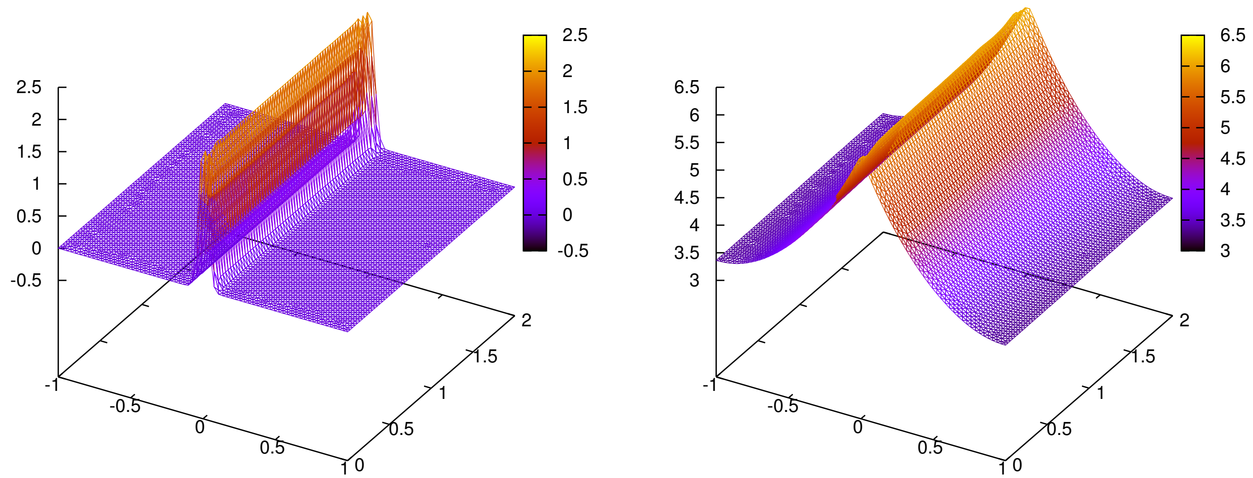
$$g(u,v) = -v + v^2/u \qquad f(u,v) = -u + v^2\,, \quad \text{GM model}$$

$$g(u,v) = -v + \frac{v^2}{u(1+\kappa v^2)}\,, \quad f(u,v) = -u + v^2\,. \quad \text{GMS model}$$

Homoclinic Stripes

Homoclinic Stripe: occurs when v localizes on a planar curve $\mathcal{C}$ of zero width. The stripe cross-section for v is approximated by a homoclinic orbit of the 1-D system in the direction perpendicular to the stripe.

- **Semi-strong regime:** For $D = O(1)$ only v is localized and not u. In this regime a stripe has a strong interaction (mediated by the global variable u) with either the boundary or with other stripes.

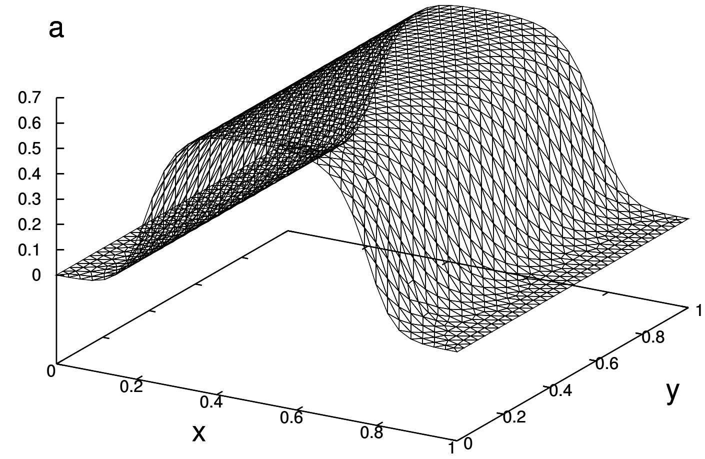


(a) v (localized)　　(b) u (global)

- **Weak-interaction regime:** Occurs for $D = O(\varepsilon^2)$ with $D = D_0\varepsilon^2$. Hence, both u and v are localized. In this regime the interaction of a stripe with either the boundary or with neighbouring stripes is exponentially weak.

Mesa Stripe (Mesa is Table in Spanish)

A **Mesa Stripe** has a flat plateau with edges determined by heteroclinic orbits of the 1-D system in the perpedicular direction.



- Such a stripe occurs in the GMS model under saturation effects:

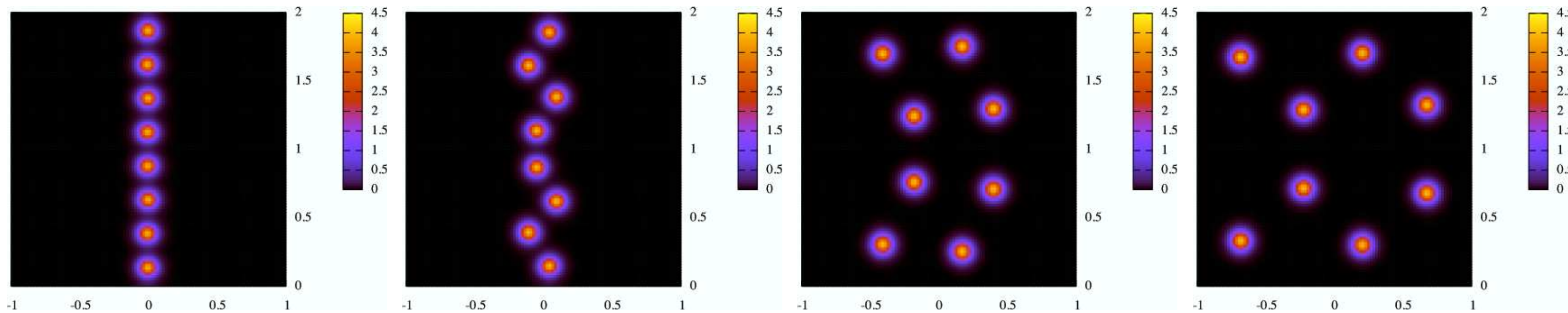
$$v_t = \varepsilon^2 \Delta v - v + \frac{v^2}{u(1+\kappa v^2)}\,, \qquad \tau u_t = D\Delta u - u + v^2\,.$$

"κ has a deep impact on the final pattern" (Koch and Meinhardt 1994).

- Mesa stripes also occur under bistable nonlinearities such as for Fitzhugh-Nagumo type models (Tanaguchi, Nishiura 1994, 1996).
- A classification of plateau (mesa) or spike (homoclinic) pulses is given by Hillen, SIAM Review (2006).

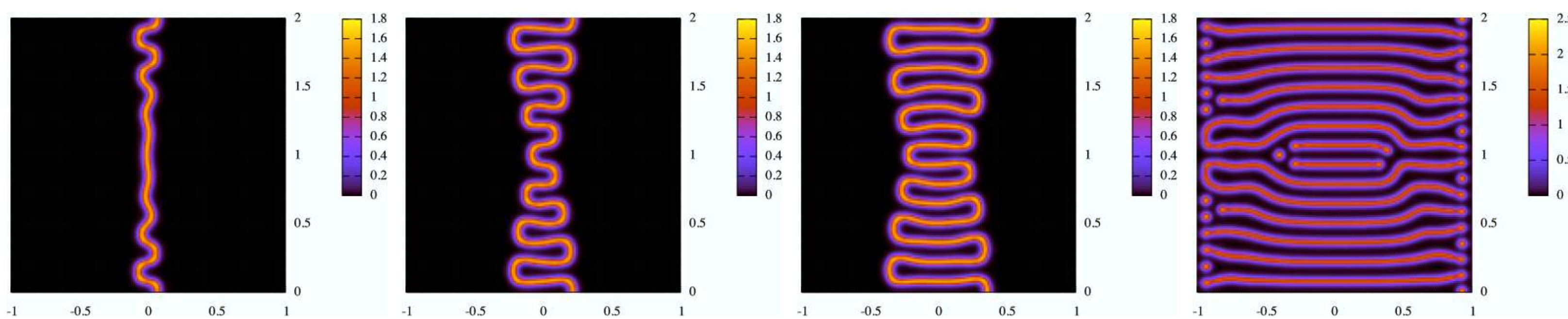
Breakup and Zigzag Instabilities

GM: Weak Interaction Regime: $D = 15\varepsilon^2$ and $\varepsilon_0 = 0.025$. Breakup Instability



Left: $t = 500$ Left Middle: $t = 800$ Right Middle: $t = 3000$ Right: $t = 30000$.

Generalized GM Model Weak Interaction Regime: $\varepsilon_0 = 0.01$, and $D_0 = 14.0$. Only zigzag instability leads to a curve of increasing length.



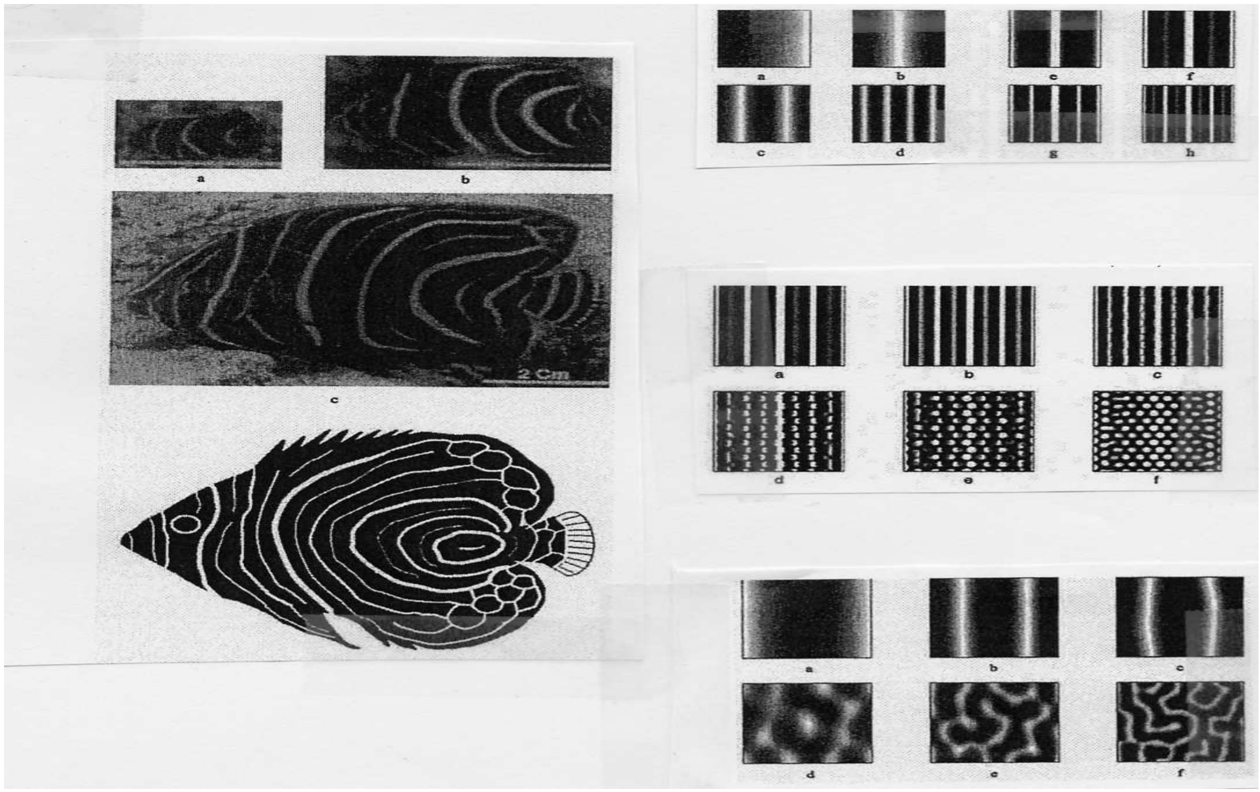
Left: $t = 1400$ Left Middle: $t = 1600$ Right Middle: $t = 1800$ Right: $t = 4300$.

Other Models: Hybrid RD–Chemotaxis I

Coloration changes in the development of spatial patterns on a young angelfish (K. Painter et al. Proc. Natl. Acad. Sci. 1999) modeled by a hybrid RD-chemotaxis system on a growing domain $L = L(t)$:

$$n_t = \frac{D_n}{L^2} \triangle n - \frac{1}{L^2} \nabla \cdot (\chi(u) \nabla n) \ ,$$

$$u_t = \frac{D_u}{L^2} \triangle u + f(u,v) \, , \quad v_t = \frac{D_v}{L^2} \triangle u + g(u,v) \, ,$$



Specific Questions

Kinetics: Either GM or GMS; Semi-Strong or Weak Interactions with GM

1. Need a theory to predict occurrence of zigzag and breakup instabilities of homoclinic stripes and to determine instability bands
   - Analytical theory in the **semi-strong regime**
   - 1-D numerical eigenvalue computations in **weak interaction regime**
2. Do mesa and homoclinic stripes have very different stability properties with regards to zigzags and breakup? Can we theoretically predict instability bands for mesa stripes?
3. What are the stability properties of fat homoclinics, which occur near a homoclinic bifurcation. This is an "intermediate state" between homoclinic and mesa stripes.
4. Self-replicating homoclinic or mesa stripes?
5. Can stripes undergo self-replication without zigzag or breakup instabilities? Important implications for patterns on growing domains.

Homoclinic Stripes for GM: S.S. Regime I

Consider the GM model in a rectangle $\Omega \equiv \{|x_1| < 1\,,\, 0 < x_2 < d_0\}$.

$$v_t = \varepsilon_0^2 \Delta v - v + \frac{v^2}{u}\,, \quad \tau u_t = D\Delta u - u + \frac{v^2}{\varepsilon_0} \quad x = (x_1, x_2) \in \Omega\,,$$

By re-scaling v and u, and from $X = x/l$ where $l = 1/\sqrt{D}$, we can set $D = 1$ above and replace Ω by Ω_l where

$$\Omega_l \equiv \{|x_1| < l\,,\; 0 < x_2 < d\}\,, \quad d \equiv d_0 l\,; \quad \varepsilon \equiv \varepsilon_0 l\,; \quad l \equiv 1/\sqrt{D}\,.$$

We examine the existence and stability of a stripe centered on the line $x_1 = 0$ in the semi-strong regime where $D = O(1)$ and $\varepsilon_0 \ll 1$.

For $\varepsilon \to 0$, the equilibrium stripe solution $v(x_1)$ satisfies

$$v(x_1) \sim u_c w\left(\varepsilon_0^{-1} x_1\right)\,,$$

for some explicit u_c, where $w(y)$ is the homoclinic sech^2 solution of

$$w'' - w + w^2 = 0\,, \quad w \to 0 \quad \text{as} \quad |y| \to \infty\,; \quad w'(0) = 0\,, \quad w(0) > 0\,,$$

Homoclinic Stripes for GM: S.S. Regime II

We introduce the perturbation

$$v = v_e + e^{\lambda t + imx_2}\phi\,, \quad u = u_e + e^{\lambda t + imx_2}\eta\,, \quad m = \frac{k\pi}{d}\,,$$

where d is the width of the rectangle.

Spectral Analysis: The $O(1)$ NLEP eigenvalues, which govern breakup instabilities, satisfy

$$\phi(x_1) \sim \Phi\left(\varepsilon^{-1}x_1\right)\,, \quad \int_{-\infty}^{\infty} w(y)\Phi(y)\,dy \neq 0\,.$$

The $O(\varepsilon^2)$ eigenvalues, which govern zigzag instabilities, satisfy

$$\phi(x_1) \sim w^{'}\left(\varepsilon^{-1}x_1\right) + \varepsilon\phi_1\left(\varepsilon^{-1}x_1\right) + \cdots\,.$$

Thus, in this regime, zigzags are dominated by breakup instabilities.

Homoclinic Stripes for GM: S.S. Regime III

A singular perturbation analysis leads to the NLEP problem for breakup instabilities:

$$L_0\Phi - \chi_m w^2 \frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w^2\, dy} = (\lambda + \varepsilon^2 m^2)\Phi\,, \quad \Phi \to 0 \text{ as } |y| \to \infty$$

$$\frac{1}{\chi_m(\lambda)} \equiv \frac{\theta_\lambda \tanh(\theta_\lambda l)}{2\tanh l}\,, \quad \theta_\lambda \equiv \sqrt{1 + m^2 + \tau\lambda}\,,$$

$$L_0\Phi \equiv \Phi'' - \Phi + 2w\Phi\,.$$

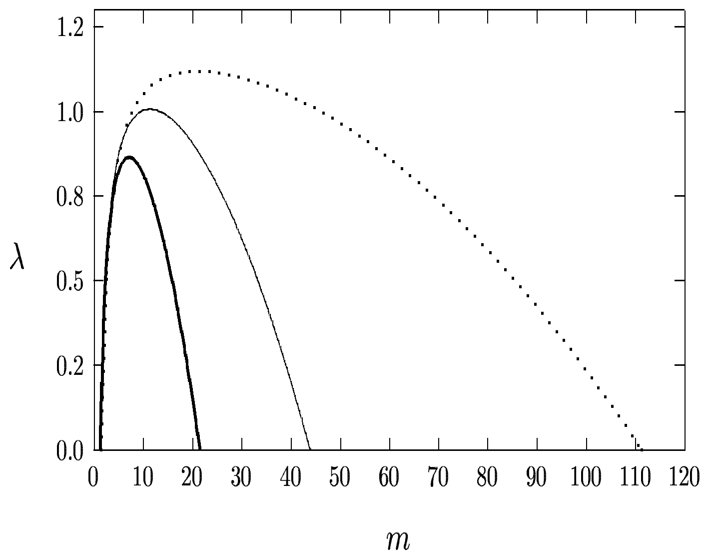
- A rigorous analysis (KSWW, SIADS 2006) proves stability Re$(\lambda) < 0$ for $m = 0$ and $m \gg 1$ when $\tau < \tau_H$. For any $\tau > 0$, there is an unstable breakup band of the form

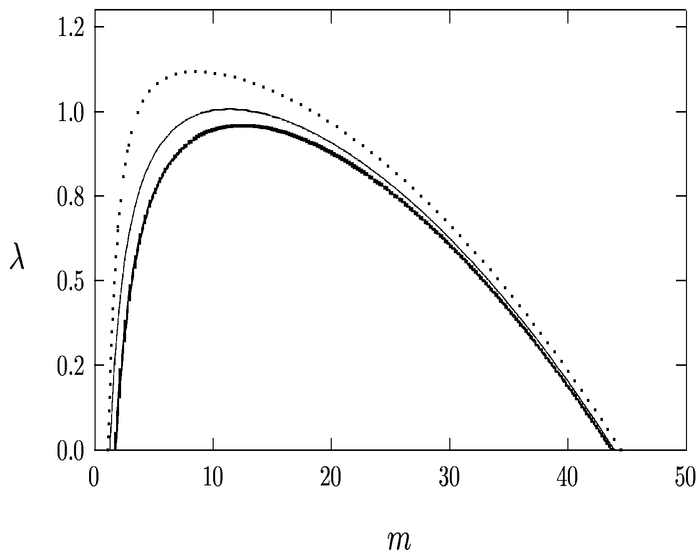
$$m_{b-} < m < m_{b+} = O(\varepsilon^{-1})\,, \quad m_{b+} \sim \frac{\sqrt{5}}{2\varepsilon} + O(\varepsilon)\,.$$

- Related analysis of Doelman, Van der Ploeg (SIADS 2002).

Homoclinic Stripes for GM: S.S. Regime IV

From numerical computations of the spectrum of the NLEP we can compute the theoretically predicted instability band.



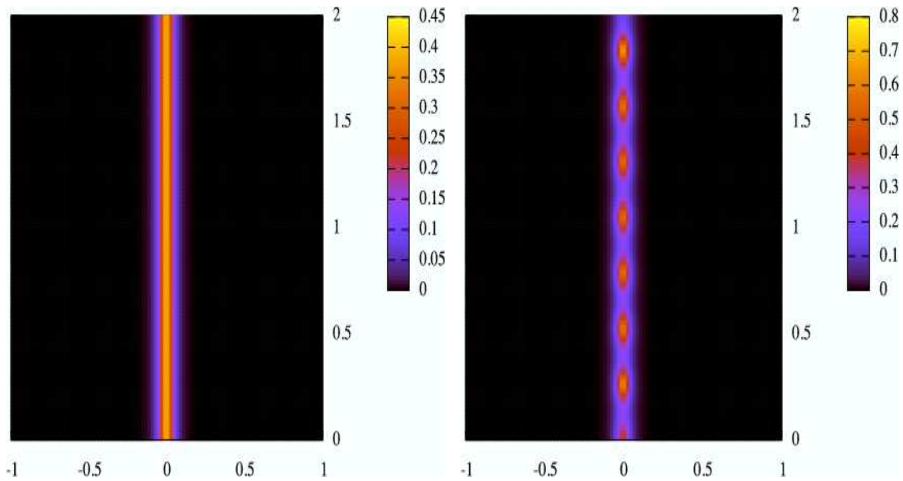


Caption: Left: Most unstable λ versus m for $D = 1$, $\tau = 0$, and for different ε. Note $\varepsilon = 0.025$ is the solid curve. **Right:** λ versus m for $\varepsilon = 0.025$ and $D = 0.1$ (heavy solid), $D = 1$ (solid), and $D = 10$ (dashed).

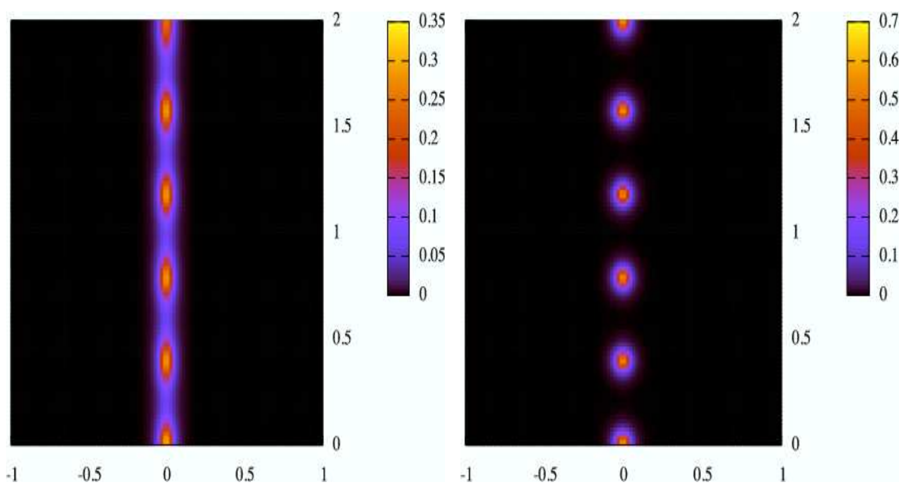
Key: Since $m_{b+} = O(\varepsilon^{-1})$, then unless the domain is asymptotically thin, a homoclinic stripe for the GM model will always break into spots.

Homoclinic Stripes for GM: S.S. Regime V

Experiment 1 and 2: Take $\varepsilon_0 = 0.025$, $\tau = 0.1$ and $\Omega = [-1,1] \times [0,2]$. The predicted number N of spots is the number of maxima of the eigenfunction $\cos(my)$ on $0 < y < d_0 = 2$, which is $N = md_0/(2\pi) = m/\pi$ where m is most unstable mode.



- For $D = 1.0$, we predict $N = 8$. Numerically, the initial stripe breaks up into seven spots on an $O(1)$ time-scale. There is then a competition instability.
 The movie.



- For $D = 0.1$ we predict $N = 4$. Numerically, the initial stripe breaks up into five spots on an $O(1)$ time-scale with no competition instability. The movie.

Fat Homoclinics for GM: Small Saturation I

In the semi-strong regime, we consider the GMS model in

$$v_t = \varepsilon^2 \Delta v - v + \frac{v^2}{u\,(1+kv^2)}\,, \quad \tau u_t = D\Delta u - u + \frac{v^2}{\varepsilon}\,,$$

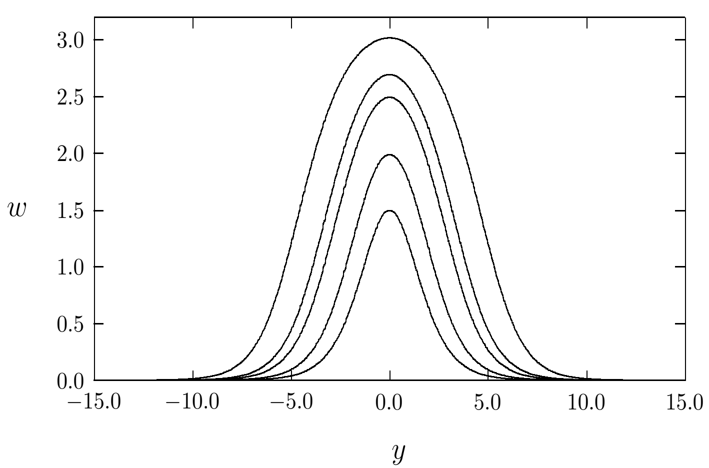
in $\Omega = [-1,1] \times [0, d_0]$. For $\varepsilon_0 \ll 1$, the homoclinic stripe solution is

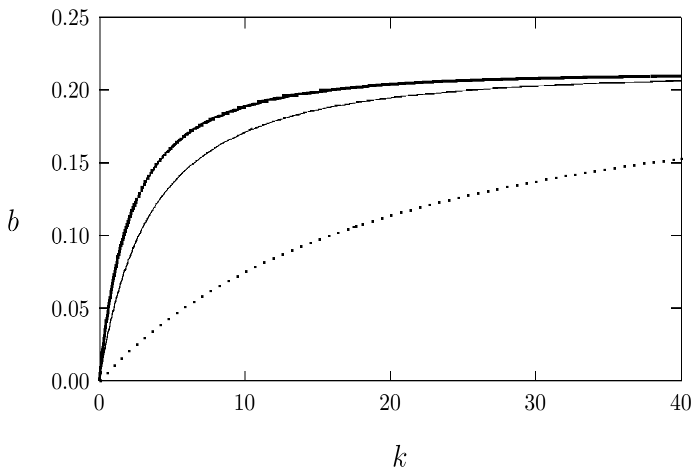
$$v \sim \mathcal{H} w\left[\varepsilon^{-1} x_1\right]\,, \quad u \sim \mathcal{H}\frac{\cosh\left[\theta_0(1-|x_1|)\right]}{\cosh\theta_0}\,, \quad \theta_0 \equiv \frac{1}{\sqrt{D}}\,,$$

$$\mathcal{H} \equiv \frac{2\tanh\theta_0}{\beta\theta_0}\,, \quad \beta \equiv \int_{-\infty}^{\infty} w^2\, dy\,, \quad b\beta^2 = 4kD\tanh^2(\theta_0)\,.$$

For $0 < b < b_0 \approx 0.21138$, w is the fat homoclinic solution to

$$w'' - w + w^2/(1+bw^2) = 0\,, \quad w(\pm\infty) = 0\,.$$





Fat Homoclinics for GM: Small Saturation II

A singular perturbation analysis leads to the NLEP governing breakup instabilities:

$$L_0\Phi - \frac{\chi_m w^2}{1+bw^2}\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w^2\, dy} = (\lambda + \varepsilon^2 m^2)\Phi\,, \quad \Phi \to 0\,, \text{ as } |y| \to \infty$$

$$\chi_m(\lambda) \equiv \frac{2\theta_0 \tanh\theta_0}{\theta_\lambda \tanh\theta_\lambda}\,, \quad \theta_\lambda \equiv \sqrt{m^2 + \frac{(1+\tau\lambda)}{D}}\,, \quad \theta_0 \equiv \frac{1}{\sqrt{D}}\,,$$

$$L_0\Phi \equiv \Phi^{''} - \Phi + \frac{2w}{(1+bw^2)^2}\Phi\,.$$

Since the pulse is stable when $m = 0$, and that $\chi_m \to 0$ as $m \to \infty$, the breakup instability band has the form

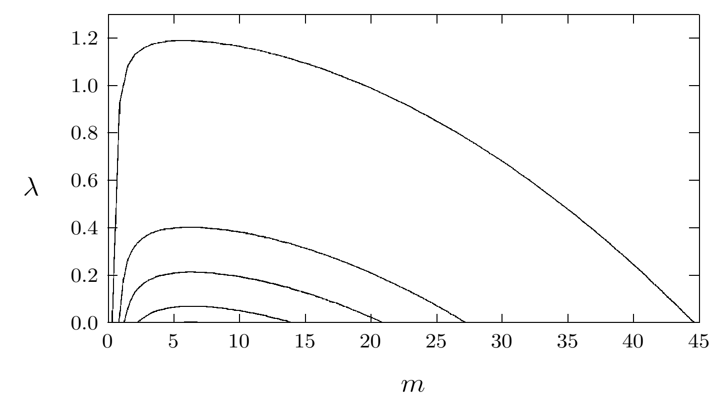
$$m_{b-} < m < \sqrt{\nu_0}/\varepsilon\,.$$

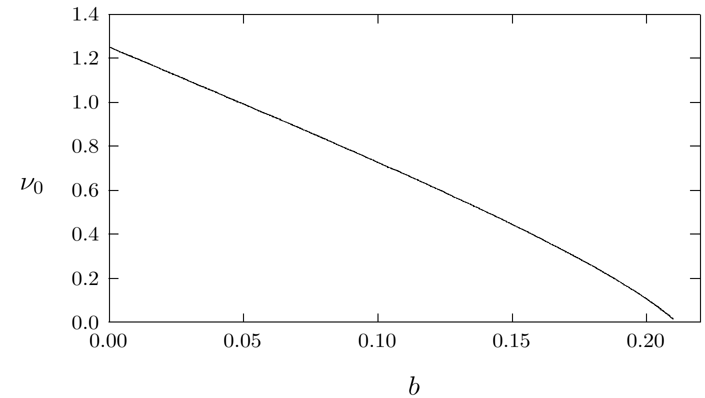
Here $\nu_0 = \nu_0(b)$ is the positive eigenvalue of the local operator L_0.

Since $\nu_0 \to 0$ as $b \to b_0 = 0.211$, where the homoclinic bifurcation occurs, it suggests that the breakup band disappears for some b slightly below b_0. Ghost effect of homoclinic bifurcation. This is precisely what is observed numerically! (next slide)

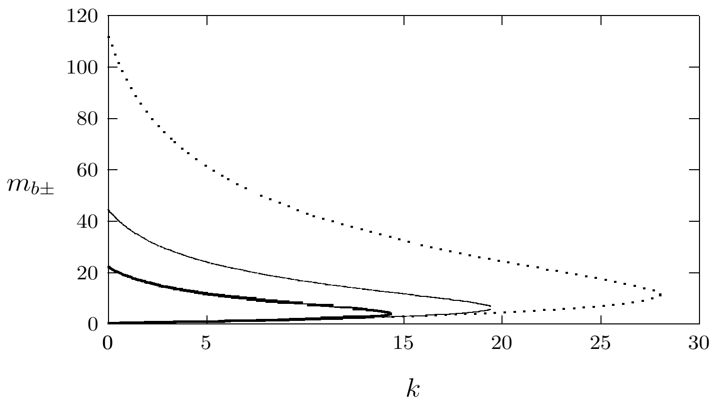
Fat Homoclinics for GM: Small Saturation III

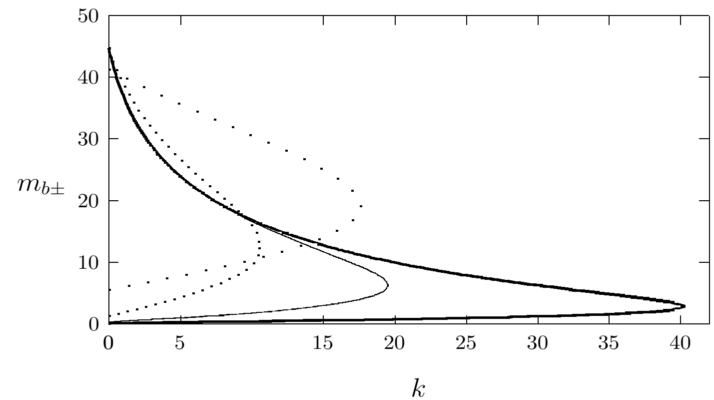
From numerical computations of the spectrum of the NLEP we get:





Left: Most unstable λ versus m for $D = 10.0$, $\varepsilon = 0.025$ with $k = 0$ (top), $k = 3.6$, $k = 6.8$, and $k = 12.5$. The band disappears near $k = 12.5$. **Right:** positive eigenvalue ν_0 of L_0 vesus b.





Left: $m_{b\pm}$ versus k for $D = 10$ and different ε. **Right:** $m_{b\pm}$ versus k for $\varepsilon = 0.025$ and different D.

Mesa Stripes for GM: Large Saturation

On a rectangular domain $0 < x_1 < 1, 0 < x_2 < d_0$, we consider

$$v_t = \varepsilon_0^2 \Delta v - v + \frac{v^2}{u\left(1 + \kappa v^2\right)}\,, \qquad \tau u_t = D\Delta u - u + u^2\,.$$

- If we set $\kappa = \varepsilon_0^2 k$ and enlarge a and h by a factor of $1/\varepsilon_0$ we recover the small saturation formulation.
- The constant $\kappa > 0$ is the saturation parameter. For $\kappa > 0$, with $\kappa = O(1)$, the numerical simulations of Koch, Meinhardt (1994), Meinhardt (1995) suggest that a stripe will not disintegrate into spots. This model seems to robustly support stable stripe patterns. Theoretical Analysis?
- For $\kappa = O(1) > 0$ we will construct a mesa stripe centered about the mid-line $x_1 = 0$ when $D = O(\varepsilon_0^{-1})$. We write

$$D = \frac{\mathcal{D}}{\varepsilon_0}\,.$$

- Key result: There are no $\lambda = O(1)$ breakup instabilities into spots in this regime, only weak zigzag and breather instabilities with $\lambda = O(\varepsilon^2)$.

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The Equilibrium Mesa Stripe

For $\varepsilon_0 \ll 1$, the cross-section of the mesa stripe is given by

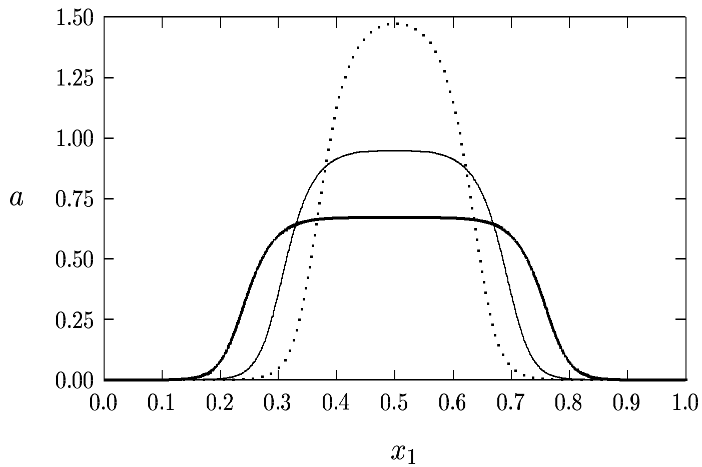
$$v \sim \mathcal{H}\left[w_l + w_r - w_+\right], \quad u \sim \mathcal{H} = \frac{1}{w_+^2 L},$$

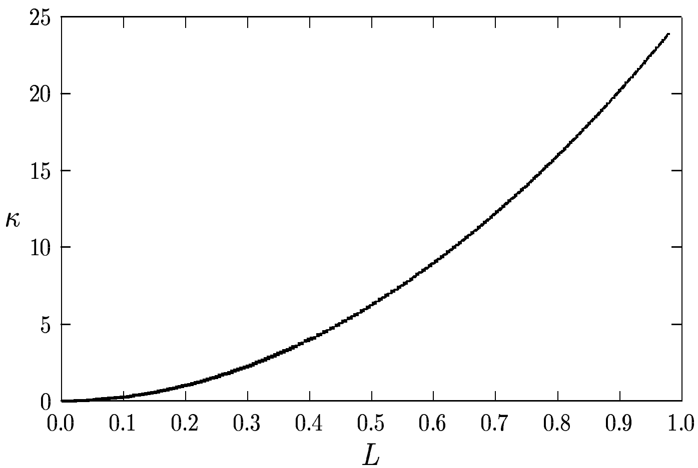
$$w_l(y_l) \equiv w\left[\varepsilon_0^{-1}(x_1 - \xi_l)\right], \quad w_r(y_r) \equiv w\left[\varepsilon_0^{-1}(\xi_r - x_1)\right],$$

$$L \equiv \xi_r - \xi_l \sim \frac{\sqrt{\kappa}}{\sqrt{b_0 w_+^2}} < 1\,.$$

Here L is the length of the mesa plateau. With $b \equiv \kappa \mathcal{H}^2$, there is a heteroclinic solution $w(y)$ for $b_0 \approx 0.21138$ to

$$w'' - w + \frac{w^2}{1 + b_0 w^2} = 0\,, \quad w(\infty) = w_+ \approx 3.295\,, \quad w(-\infty) = 0\,.$$





The Stability of the Mesa Stripe I

Let v_e, u_e be the equilibrium mesa stripe. We linearize as

$$v = v_e + e^{\lambda t + imx_2}\phi \,, \quad u = u_e + e^{\lambda t + imx_2}\psi \,, \quad m = \frac{k\pi}{d_0} \,, \quad k = 1, 2, \ldots \,,$$

where $\phi = \phi(x_1) \ll 1$ and $\psi = \psi(x_1) \ll 1$. We assume that $\tau = O(1)$.
For $\varepsilon_0 \ll 1$ the eigenfunction for ϕ has the form

$$\phi \sim \begin{cases} c_l \left(w^{'}(y_l) + O(\varepsilon_0) + \cdots \right) \,, & y_l \equiv \varepsilon_0^{-1}(x_1 - \xi_l) = 0(1) \,, \\ \phi_i \equiv \mu\psi \,, & \xi_l < x_1 < \xi_r \,, \\ c_r \left(w^{'}(y_r) + O(\varepsilon_0) + \cdots \right) \,, & y_r \equiv \varepsilon_0^{-1}(\xi_r - x_1) = 0(1) \,, \end{cases}$$

with $\mu \equiv \frac{w_+^2}{2 - w_+}$.
The eigenfunction for ψ on $0 < x_1 < 1$ has the form

$$\psi_{x_1 x_1} - \theta^2 \psi = -\frac{\varepsilon_0^2 \mathcal{H} w_+^2}{\mathcal{D}} \left[c_r \delta(x_1 - \xi_r) + c_l \delta(x_1 - \xi_l) \right] \,,$$
$$\psi_{x_1}(0) = \psi_{x_1}(1) = 0 \,.$$

The Stability of the Mesa Stripe II

Assume that $\tau = O(1)$. In terms of the plateau length L, a SLEP-type method (Nishiura) yields

$$\lambda_{\pm} \sim \frac{\varepsilon_0^2}{\alpha}\left[-\alpha m^2 + \frac{L}{2}(1-L) - \sigma_{\pm}\right], \quad \boldsymbol{c}_{\pm} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

Note that $\boldsymbol{c}_+$ is a breather mode while $\boldsymbol{c}_-$ is a zigzag mode. Here

$$\sigma_+ = \left[\theta_+ \tanh\left(\frac{\theta_+ L}{2}\right) + \theta_- \tanh\left(\frac{\theta_-(1-L)}{2}\right)\right]^{-1},$$

$$\sigma_- = \left[\theta_+ \coth\left(\frac{\theta_+ L}{2}\right) + \theta_- \tanh\left(\frac{\theta-(1-L)}{2}\right)\right]^{-1}.$$

$$\theta \equiv \begin{cases} \theta_- \equiv \left[m^2 + \frac{\varepsilon_0}{\mathcal{D}}\right]^{1/2}, & 0 < x_1 < \xi_l\,, \quad \xi_r < x_1 < 1\,, \\ \theta_+ \equiv \left[m^2 + \frac{\varepsilon_0}{\mathcal{D}}\left(1 + \frac{2w_+}{L(w_+-2)}\right)\right]^{1/2}, & \xi_l < x_1 < \xi_r\,. \end{cases}$$

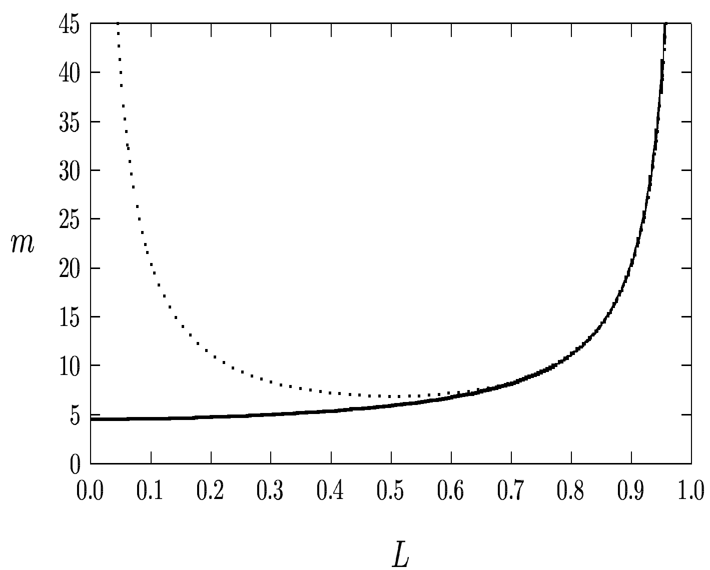
The Stability of the Mesa Stripe

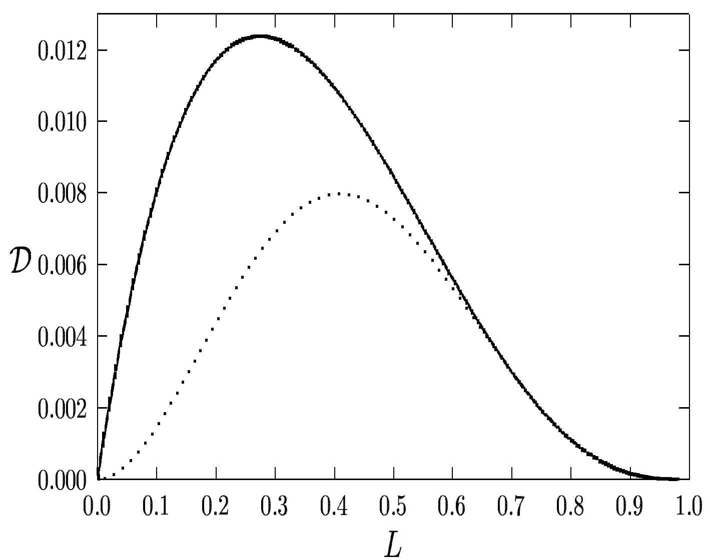
The critical values $\mathcal{D}_z$ and $\mathcal{D}_b$ where a zigzag and breather instability emerge at some $m = m_z$ and $m = m_b$ are roots of

$$\frac{m}{2}\frac{d\sigma_-}{dm} = \sigma_- - \frac{L}{2}(1-L)\,, \quad \mathcal{D}_z = \frac{w_+^2}{2\beta L m_z^2}\left[\frac{L}{2}(1-L) - \sigma_-\right].$$

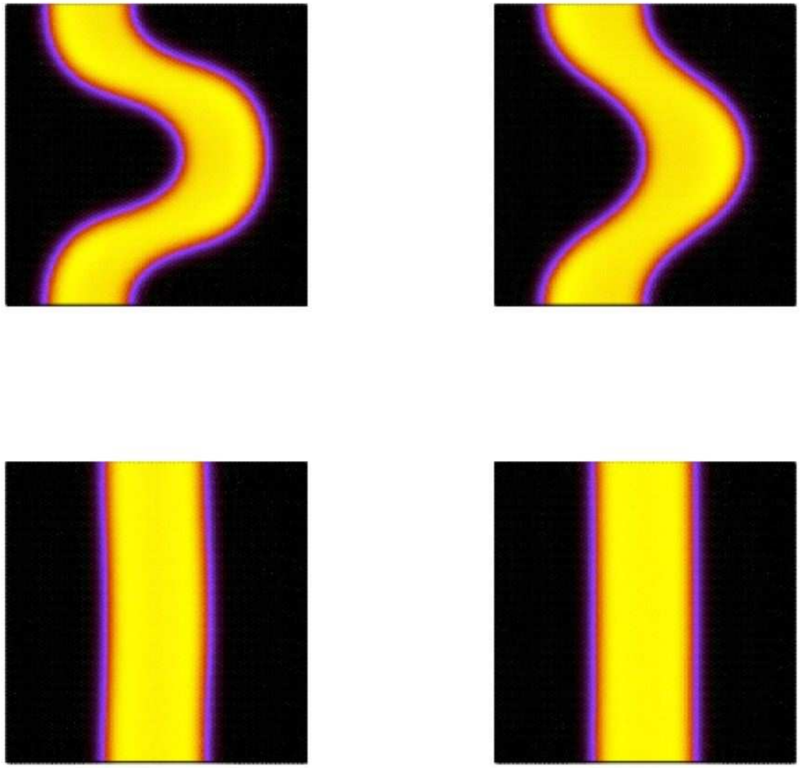
$$\frac{m}{2}\frac{d\sigma_+}{dm} = \sigma_+ - \frac{L}{2}(1-L)\,, \qquad \mathcal{D}_b = \frac{w_+^2}{2\beta L m_z^2}\left[\frac{L}{2}(1-L) - \sigma_+\right].$$

Since $\mathcal{D}_z > \mathcal{D}_b$ a zigzag instability always occurs before a breather instability as $\mathcal{D}$ is decreased. Dotted curve is breather mode and solid curve is zigzag.





Numerical Realization of Zigzag Instability



- Top left: $D = 0.6$ at $t = 10,000$.
- Top right: $D = 0.8$ at $t = 10,000$.
- Bottom left: $D = 1.0$ at $t = 20,000$.
- Bottom right: $D = 1.4$ at $t = 20,000$.

Experiment: The numerical solution to the saturated GM model in $[0,1] \times [0,1]$ for $\varepsilon_0 = 0.01$ and $\kappa = 1.92$.

A zigzag instability occurs in each case, except when $D = 1.4$ which is above the zigzag instability threshold of $D = 1.24$.

Summary: Semi-Strong Regime

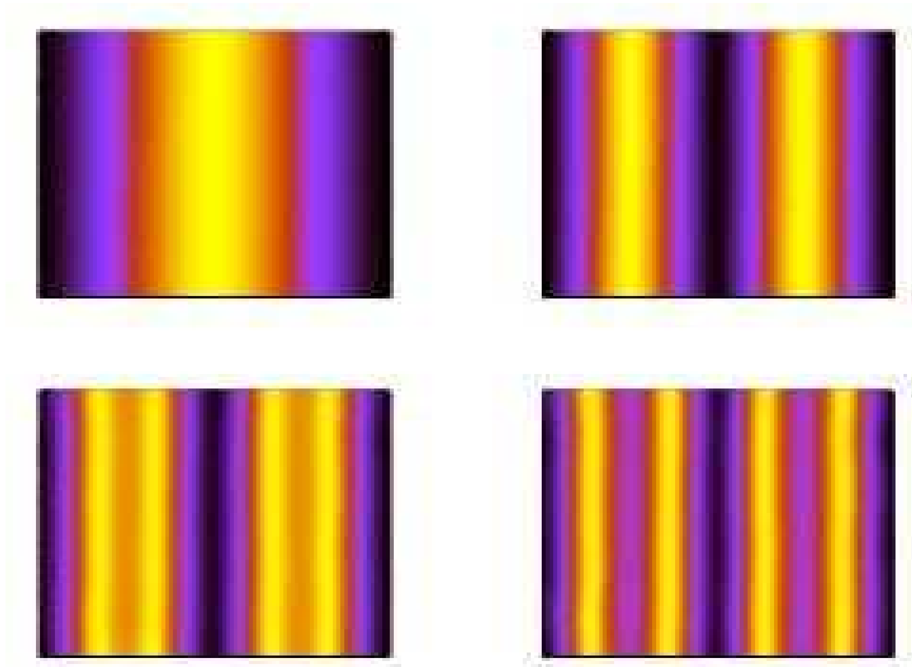
GM and GMS Models:

- With no saturation, a stripe for the GM model will disintegrate into spots (breakup instability) unless the domain width is asymptotically thin. Rigorous theory based on analysis of NLEP.
- With no saturation zigzag instabilities are dominated by the faster breakup instabilities.
- With a small saturation the breakup instability band for a ''fat homoclinic" stripe can disappear close to a homoclinic bifurcation.
- With a large saturation, a mesa stripe exists that is formed from front-back heteroclinic connections. For $D = O(\varepsilon_0^{-1}) \gg 1$, the dominant instability of this stripe is zigzag instabilities.

Mesa Stripe-Splitting: Growing Domains

Pattern formation on growing domains: Kondo, Asai (Nature, 1995); Crampin, Maini (J. Math Bio. 2000-2002); Painter et al. (PNAS, 2001).

- Previous theory for $D \gg 1$. For the GMS model we can get new behavior: mesa-splitting for some $D = O(1)$ depending on κ. **Theory for Mesa-Splitting:** KWW, Physica D Vol. 236 No. 2, (2007), pp. 104–122.
- Decreasing D and ε^2 exponentially slowly in time on a fixed domain is equivalent to fixing ε and D on a slowly exponentially growing domain.
- From a single stripe, further stripes are robustly obtained through mesa-split events without triggering zigzag instabilities



Experiment: Take $\kappa = 1.9$, $D(t) = 0.2e^{-0.0002t}$ and $\varepsilon(t) = 0.15 * D(t)$. First mesa-split at $t = 350$ when $D < 0.17$. Further splits occur and slow zigzag instabilities do not have time to develop. The movie.

Homoclinic Stripes: Weak Interaction Regime

The GM model in a rectangle in the weak interaction regime is

$$a_t = \varepsilon_0^2 \Delta a - a + \frac{a^p}{h^q}\,, \quad \tau h_t = \varepsilon_0^2 D_0 \Delta h - h + \frac{a^r}{h^s}\,, \quad x = (x_1, x_2) \in \Omega\,.$$

Here $D = D_0 \varepsilon^2$ and $\Omega = [-1, 1] \times [0, d_0]$.

Examine the existence and stability of a stripe centered along the midline $x_1 = 0$ of Ω. The effect of sidewalls is insignificant in this regime.

Questions for a Pulse and a Stripe:

- What is the bifurcation behavior of equilibrium homoclinic pulses in terms of D_0 (Doelman, Van der Ploeg (2002), KWW (2004))
- What are the stability properties for a pulse in 1-D?
- For a stripe (composed of a pulse trivially extended in the second direction) what are the zigzag and breakup instability bands as a function of D_0. Can the breakup band disappear, leaving only the zigzag band? If so, we predict the possibility of labyrinthian patterns

Homoclinic Pulse: Pulse-Splitting Criteria

Consider the general system:

$$u_t = u_{xx} + F(u;\sigma)\,, \quad x \in \mathbb{R}^1\,, \quad u \in \mathbb{R}^2$$

The following criteria for pulse-splitting are due to Nishiura and Ueyama (Physica D (1998)), and Ei et al. (JJIAM (2001)):

- Bifurcation branches of k-pulse equilibria have a saddle-node structure for $\sigma > \sigma_c$ with the fold point occuring at $\sigma = \sigma_c$ for each branch. This is the **Lining-up property**. If a one-pulse solution has a fold-point structure, this property is always satisfied in the weak interaction regime.
- Each upper branch is stable, and each lower branch is unstable.
- There is a **dimple-shaped, even eigenfunction**, at the fold point location for one component of u (one negative lobe, and two positive lobes)
- Pulse-splitting should occur for $\sigma < \sigma_c$, with $\sigma_c - \sigma$ small, due to the **ghost** of the dimple eigenfunction.

In KWW (Studies in Applied Math 2004) it was shown that the profile of a component of u has a dimple near the fold point; i.e. fat homoclinics exist near fold point.

Homoclinic Pulse: Equilibrium Solution I

We look for solutions of the form

$$a_{e\pm}(x_1) \sim u\left[\varepsilon_0^{-1} x_1\right] , \quad h_{e\pm}(x) \sim v\left[\varepsilon_0^{-1} x_1\right] .$$

The functions $u(y)$ and $v(y)$ are homoclinic solutions on $0 < y < \infty$ to

$$u^{''} - u + u^p/v^q = 0 , \quad D_0 v^{''} - v + u^r/v^s = 0 ,$$

with

$$u(0) = \alpha , \;\; u^{'}(0) = v^{'}(0) = 0 , \;\; u(\infty) = v(\infty) .$$

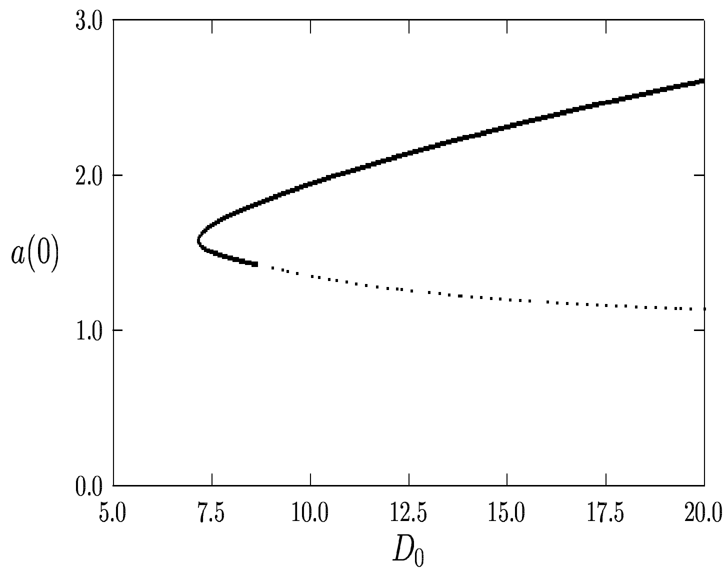
- Solution by AUTO for various (p, q, r, s).
- Some existence results by geometric theory of singular perturbations (Doelman and Van der Ploeg, SIADS 2002)
- Numerical computations also by Nishiura (AMS Monograph Translations 2002)
- Bifurcation and stability behavior KWW (2004).

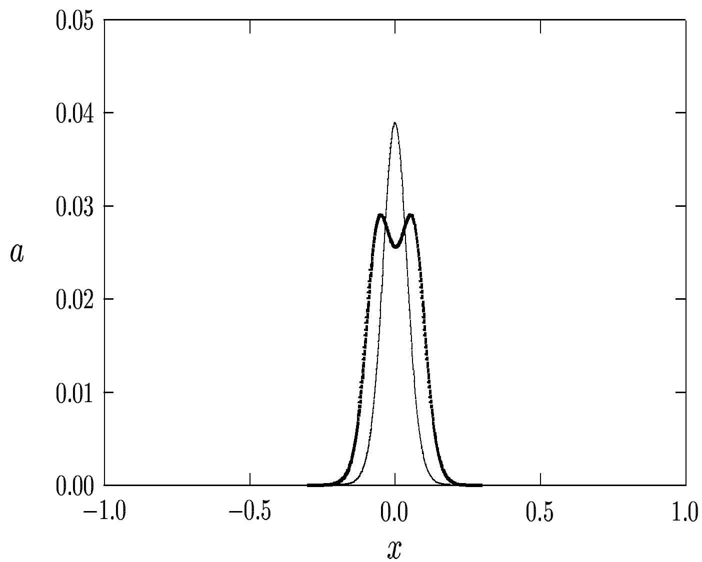
Homoclinic Pulse: Equilibrium Solution II

For each exponent set (p, q, r, s) tested, we find qualitatively that:

- There are two solutions when $D_0 > D_{c0}$ and none when $0 < D_0 < D_{c0}$. Therefore, we have a fold bifurcation at $D = D_{c0} > 0$.
- On the dashed portion of the lower branch the homoclinic profile for u has a double-bump structure so that the maximum of u occurs on either side of $y = 0$. We refer to this as the multi-bump transition condition. (KWW Studies in Appl. Math. 2004).

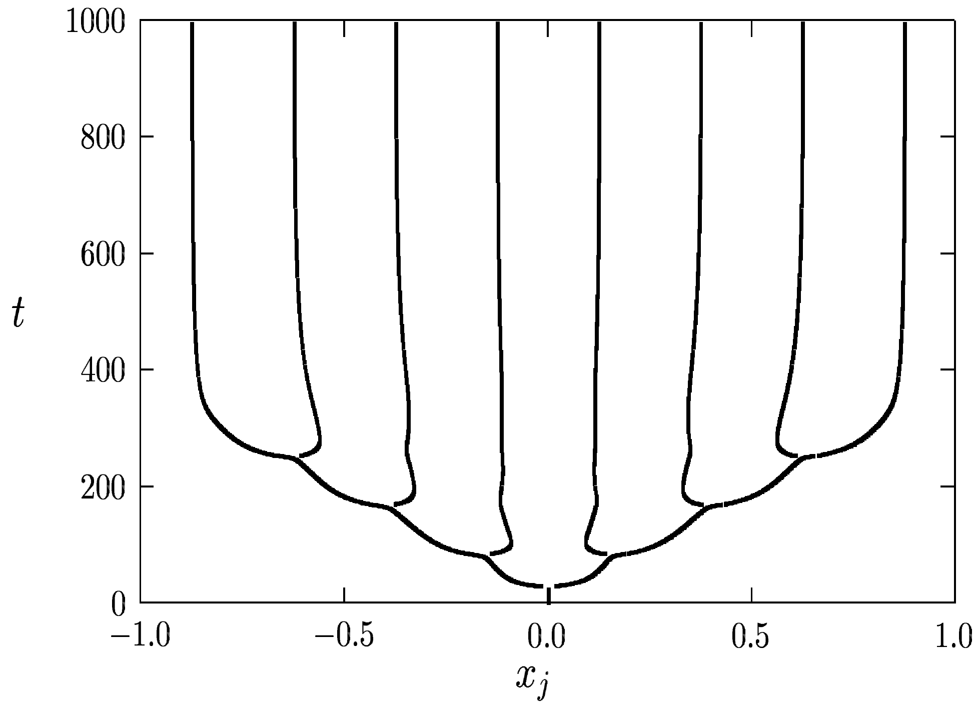
Left figure: $a(0)$ versus D_0 for $(2, 1, 2, 0)$. The dotted part is where a_e has a double-bump structure. Right figure: a_{e+} (solid curve) for $D_0 = 9.83$ and a_{e-} (heavy solid curve) for $D_0 = 10.22$.





Homoclinic Pulse: Pulse-Splitting

Experiment: Let $(p, q, r, s) = (2, 1, 2, 0)$, $\varepsilon = 0.02$, and $D_0 = 6 < D_{c0} = 7.17$. We plot the trajectories $x_j(t)$ of the local maxima of a. Notice that there is an edge-splitting behavior and not simultaneous splitting.

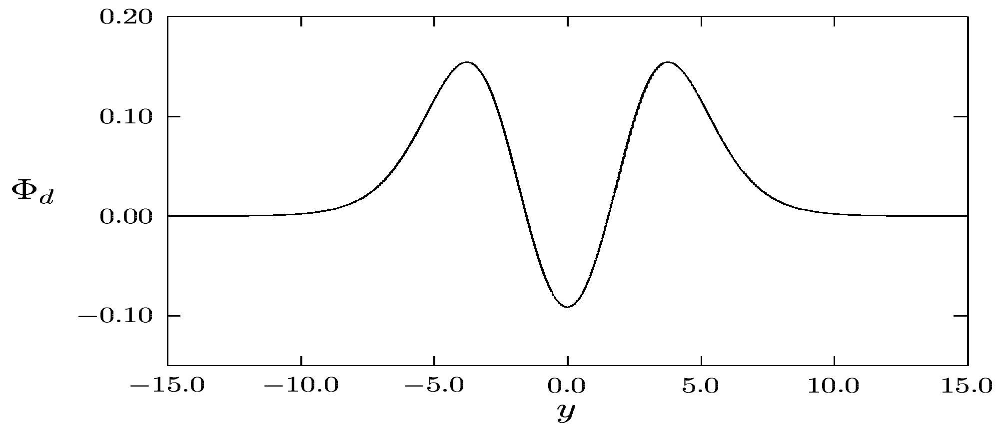


Homoclinic Pulse: Stability Properties

- Numerically, the upper branch is stable for $\tau < \tau_H(D_0)$. For $\tau > \tau_H(D_0)$, the pulse on the upper branch undergoes a Hopf bifurcation. The lower branch is unstable for any $\tau \geq 0$.
- The translation mode $\Phi = u^{'}$, $N = v^{'}$, is always a neutral mode.
- At the fold point $D_0 = D_{c0}$, there is an even dimple-shaped neutral eigenmode $\Phi = \Phi_d(y)$ satisfying

$$\Phi_d(0) < 0 \, ; \quad \Phi_d(y) > 0 \, , \;\; y > y_0 \, ; \;\; \Phi_d(y) < 0 \, , \;\; 0 < y < y_0 \, .$$

- Hence, the key global bifurcation conditions of Ei and Nishiura (JJIAM 2001) for edge-splitting pulse-replication behavior are satisfied. The extra condition of a multi-bump transition structure is also necessary (KWW, Studies in Appl. Math., 2004).



Homoclinic Stripe: Stability Properties I

Extend the pulse trivially in x_2 direction to obtain a straight stripe. To study stripe stability we let $y = \varepsilon_0^{-1} x$ and

$$a = u(y) + \Phi\,(y)\, e^{\lambda t} \cos(m x_2)\,, \quad h = v(y) + N\,(y)\, e^{\lambda t} \cos(m x_2)\,.$$

Defining $\mu \equiv \varepsilon_0^2 m^2$, we get an eigenvalue problem on $0 < y < \infty$:

$$\Phi_{yy} - (1+\mu)\Phi + \frac{p u^{p-1}}{v^q}\Phi - \frac{q u^p}{v^{q+1}} N = \lambda \Phi\,,$$

$$D_0 N_{yy} - (1 + D_0 \mu)\, N + \frac{r u^{r-1}}{v^s}\Phi - \frac{s u^r}{v^{s+1}} N = \tau \lambda N\,.$$

- **Breakup instabilities:** Even eigenfunctions with $\Phi_y(0) = N_y(0) = 0$.
- **Zigzag instabilities:** Odd eigenfunctions with $\Phi(0) = N(0) = 0$.
- **Computations:** Auto, finite differences, Lapack, and quasi-Newton to find boundaries of instability band in μ

Homoclinic Stripes: Non-Zero Curvature I

For $\varepsilon \to 0$, the existence and stability theory for a straight stripe in weak interaction regime also applies to a curved stripe $\mathcal{C}$ that is either a closed curve or that has both ends intersecting the boundary. (for the curvature κ of $\mathcal{C}$ to be a secondary effect we need $\kappa \ll O(\varepsilon_0^{-1})$.)

The profile of the stripe is $v = v_e(\eta)$ and $u = u_e(\eta)$ where $\eta =$ is the local in ε perpendicular distance to the stripe and v_e, u_e is the 1-D solution.

For stability let $v = v_e\,(\eta) + e^{\lambda t + im\sigma}\Phi\,(\eta)$, etc.. where σ is arclength along $\mathcal{C}$. Let L be the length of $\mathcal{C}$. Then, $m = 2\pi k/L$, and the most unstable zigzag or breakup mode is

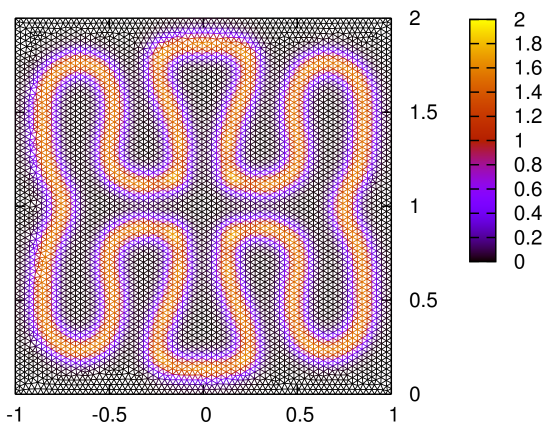
$$k = \frac{(\varepsilon_0 m)L}{2\pi\varepsilon_0}\,.$$

Here $\sqrt{\mu} = \varepsilon_0 m$ is computed from the eigenvalue problem for a straight stripe done earlier

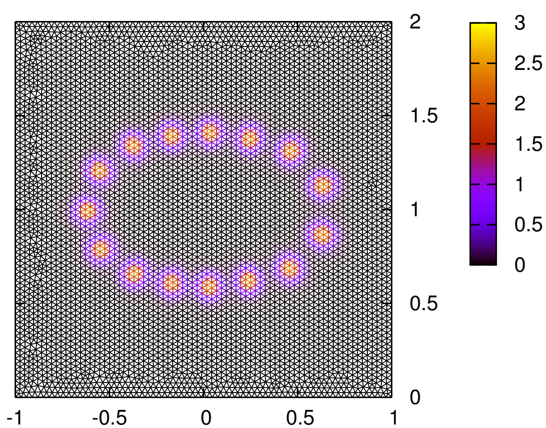
Conjecture: A zigzag instability in the absence of a breakup instability can lead to some type of space-filling labyrinthine pattern.

Homoclinic Stripes: Non-Zero Curvature II

Experiment: Consider the GM model with exponent set $(p, q, r, s) = (2, 1, 3, 0)$, $\tau = 0.1$ and $\varepsilon_0 = 0.02$ in $\Omega = [-1, 1] \times [0, 2]$. Suppose that the initial data concentrates on the ellipse $\mathcal{C} : x_1^2 + 2(x_2 - 1)^2 = 1/4$ in Ω. Then, since $L = 2.71$ for $\mathcal{C}$, we use the eigenvalue computation for a straight stripe to predict the following:



- For $D_0 = 14$ we predict only zigzags with most unstable mode $\varepsilon_0 m \approx 0.227$ and $\lambda = 0.0131$. Thus, we predict $k = 5$ zigzag crests. Plot at $t = 625$. The movie.



- For $D_0 = 20$ we predict a breakup with with most unstable mode $\varepsilon_0 m \approx 0.682$ and $\lambda = 0.0145$. Thus, we predict $k = 15$ spots. Plot at $t = 187$. The movie.

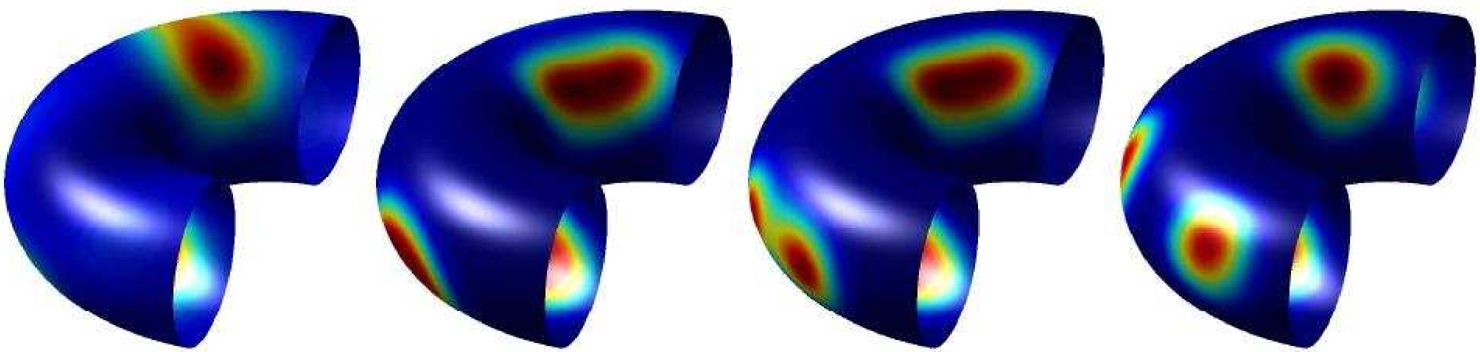
Open Issues and Further Directions I

- **Universality:** Apply framework to RD systems with classes of kinetics, to derive general principles for breakup, zigzag, and self-replication instabilities of either homoclinic stripes or mesas.
- **Fat Homoclinics**: Study the spectrum of the linearization around a fat homoclinic. Prove rigorously that breakup bands can disappear, leaving only zigzag band, due to ghost effect of a homoclinic bifurcation.
- **Growing Domains:** Study delayed bifurcation effects and self-replication of mesa stripes on growing domains for RD systems. This typically gives stable stripe self-replication.
- **Other Systems:** study stripe self-replication in slowly growing domains for other systems, such as the hybrid chemotaxis-RD systems of Othmer and Painter (reproducing Kondo and Asai's angelfish).
- **Stripe Fragments:** study wormlike stripe fragments for RD systems; derive laws of motion for the tip of the filament.

Open Issues and Further Directions II

- **Patterns on Manifolds:** Pattern formation on manifolds, **where the geometry of the manifold interfaces with localization**; equilibrium stripes on geodesics? dynamics of spots induced by Gaussian curvature? Spot replication on slowly evolving manifolds etc..Require properties of Green's functions on manifolds.

  Schnakenburg model on a Manifold: S. Ruuth (JCP, 2008)



  Key: New PDE numerical approaches "Closest Point Algorithms to Compute PDE's on Surfaces", by S. Ruuth (SFU) , C. McDonald (Oxford), allow for "routine" full numerical simulations to test any asymptotic theories.

Closing I: Quotes on Localization

With regards to the intricate patterns computed by him for the Gray-Scott model (*Complex Patterns in a Simple System*, Science 1993), John Pearson (Los Alamos) remarked:

Most work in this field has focused on pattern formation from a spatially uniform state that is near the transition from linear stability to linear instability. With this restriction, standard bifurcation-theoretic tools such as amplitude equations have been used with considerable success (ref: Cross and Hohenburg (Rev. Mod. Physics 1993)). It is unclear whether the patterns presented here will yield to these standard technologies.

In his survey book chapter on pattern formation in *Nonlinear Dynamics; Where Do We Fo From Here?, Editor: A. Champneys, 2004*, Edgar Knobloch (Berkeley) remarks that: *The question of the stability of finite amplitude structures, be they periodic or localized, and their bifurcation properties is a major topic that requires new insights.*

Closing II: Long-Term Goals

Long-term goal for **nonlinear RD systmems with localized patterns**:

Theory Building: Develop ultimately a comprehensive theory for the dynamics and stability of quasi-equilibrium localized solutions for classes of singularly perturbed reaction diffusion systems.

- Can we classify all the instability mechanisms (oscillatory, competition, self-replication, zigzag, breakup, etc..) for various **generic** objects such as **spots**, **homoclinic stripes**, **mesa stripes** etc..
- Such a classfication will depend on the parameter regime; **weak versus semi-strong interactions.**
- Aim to have as clear a picture as exists with conventional Turing-type linearizations, and with weakly nonlinear theory (Eckhaus instabilities, Busse balloons etc..).
- Investigate specific new RD models with clear scientific applications through this **theoretical lens.**

Even in the realm of **linear diffusive systems**, there are many applications of localization that warrant careful and rigorous study; diffusion with traps, narrow escape time and the MFPT, persistence thresholds in highly patchy environments etc..

References

Available at: http://www.math.ubc.ca/ ward/prepr.html

Lecture III:

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- T. Kolokolnikov, W. Sun, M. J. Ward, J. Wei, *The Stability of a Stripe for the Gierer-Meinhardt Model and the Effect of Saturation*, SIAM J. Appl. Dyn. Sys., Vol. 5, No. 2, (2006), pp. 313-363.
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