

# THE EFFECTIVE REACTIVITY FOR CAPTURING BROWNIAN MOTION BY PARTIALLY REACTIVE PATCHES ON A SPHERICAL SURFACE

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**Abstract.** We analyze the trapping of diffusing ligands, modeled as Brownian particles, by a sphere that has  $N$  partially reactive boundary patches, each of small area, on an otherwise reflecting boundary. For such a structured target, the partial reactivity of each boundary patch is characterized by a Robin boundary condition, with a local boundary reactivity  $\kappa_i$  for  $i = 1, \dots, N$ . For any spatial arrangement of well-separated patches on the surface of the sphere, the method of matched asymptotic expansions is used to derive explicit results for the capacitance  $C_T$  of the structured target, which is valid for any  $\kappa_i > 0$ . This target capacitance  $C_T$  is defined in terms of a Green's matrix, which depends on the spatial configuration of patches, the local reactive capacitance  $C_i(\kappa_i)$  of each patch and another coefficient that depends on the local geometry near a patch. The analytical dependence of  $C_i(\kappa_i)$  on  $\kappa_i$  is uncovered via a spectral expansion over Steklov eigenfunctions. For circular patches, the latter are readily computed numerically and provide an accurate fully explicit sigmoidal approximation for  $C_i(\kappa_i)$ . In the homogenization limit of  $N \gg 1$  identical uniformly-spaced patches with  $\kappa_i = \kappa$ , we derive an explicit scaling law for the effective capacitance and the effective reactivity of the structured target that is valid in the limit of small patch area fraction. From a comparison with numerical simulations, we show that this scaling law provides a highly accurate approximation over the full range  $\kappa > 0$ , even when there is only a moderately large number of reactive patches.

**1. Introduction.** Diffusive search for multiple targets is a key process in physics, chemistry, and biology [34, 43, 46, 49, 42, 33, 39, 21, 13]. For instance, signal transduction between neurons involves several diffusive search processes [1, 6, 32, 47]: (i) calcium ions that diffuse in the extracellular space and search for ionic channels on the plasma membrane to respond to an electrical signal; (ii) after entering the synaptic bouton, calcium ions search for calcium-sensing proteins on the vesicles filled with neurotransmitters to initiate their release into the inter-neuronal space; (iii) once released, the neurotransmitters diffusively search for suitable binding sites on the plasma membrane of a neighboring neuron. Likewise, small metabolites and various proteins such as transcription factors or histons search for nuclear pores on the plasmic membrane of the cell nucleus [1, 38]. From a chemical perspective, heterogeneous catalysis often involves porous catalysts with heterogeneous distribution of active sites on an otherwise passive (inert) boundary [34, 12, 19, 45]. In both applications, one generally deals with diffusive search for small targets or reactive patches that are distributed on an otherwise reflecting surface.

With this biophysical motivation, we consider the canonical problem of the trapping of ligands, modeled as diffusing Brownian particles, by the boundary  $\partial\mathcal{B}$  of a 3-D simply-connected and bounded domain  $\mathcal{B}$  that has many small partially reactive sites. The boundary is assumed to be smooth and consists of the union  $\partial\mathcal{B}_a$  of  $N$  small reactive patches  $\partial\mathcal{B}_i$ , on an otherwise reflecting boundary  $\partial\mathcal{B}_r$ . The steady-state concentration  $\mathcal{U}$  of diffusing ligands satisfies the mixed Neumann-Robin boundary value

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problem (BVP)

$$\begin{aligned}
(1.1a) \quad & \Delta \mathcal{U} = 0, \quad \mathbf{X} \in \mathbb{R}^3 \setminus \mathcal{B}, \\
(1.1b) \quad & D \partial_n \mathcal{U} + \mathcal{K}_i \mathcal{U} = 0, \quad \mathbf{X} \in \partial \mathcal{B}_i, \quad i = 1, \dots, N, \\
(1.1c) \quad & \partial_n \mathcal{U} = 0, \quad \mathbf{X} \in \partial \mathcal{B}_r, \\
(1.1d) \quad & \mathcal{U} \sim \mathcal{U}_{\text{inf}} \left( 1 - \frac{\mathcal{C}_T}{|\mathbf{X}|} + \mathcal{O}(|\mathbf{X}|^{-2}) \right), \quad \text{as } |\mathbf{X}| \rightarrow \infty,
\end{aligned}$$

where  $\partial \mathcal{B}_a = \cup_{i=1}^N \partial \mathcal{B}_i$ ,  $D > 0$  is a constant diffusivity,  $\mathcal{U}_{\text{inf}}$  is a constant concentration imposed at infinity,  $\Delta$  is the Laplacian in the dimensional coordinate  $\mathbf{X}$ , and  $\partial_n$  is the outward normal derivative to  $\partial \mathcal{B}$ , directed into  $\mathcal{B}$ . Each reactive boundary patch  $\partial \mathcal{B}_i$  of diameter  $2L_i$ , with reactivity  $\mathcal{K}_i > 0$ , is assumed to be simply-connected with a smooth boundary, but with an otherwise arbitrary shape. From the divergence theorem, the coefficient  $\mathcal{C}_T$  in the far-field (1.1d) is related to  $\mathcal{U}$  by the identity

$$(1.2) \quad \mathcal{C}_T = -\frac{1}{4\pi\mathcal{U}_{\text{inf}}} \int_{\partial \mathcal{B}} \partial_n \mathcal{U} ds = \frac{J}{4\pi D \mathcal{U}_{\text{inf}}},$$

where  $J$  is the total flux of particles reacted onto patches. This coefficient  $\mathcal{C}_T$  is interpreted as the capacitance of the heterogeneous partially reactive boundary  $\partial \mathcal{B}$ .

Our focus will be the spherical domain  $\mathcal{B} = \{\mathbf{X} \in \mathbb{R}^3 \mid |\mathbf{X}| \leq R\}$ , for which we will derive an analytical formula for the capacitance  $\mathcal{C}_T$  in the limit of small patch radius  $\varepsilon = L/R \ll 1$ , where  $L = \max_i \{L_i\}$ . For a homogeneous reactivity  $\mathcal{K}$  (i.e., a single patch covering the whole spherical boundary), the PDE (1.1) is radially symmetric and, from its readily obtained explicit solution, an exact formula for the flux  $J$  is [11]

$$(1.3) \quad J = \int_{\partial \mathcal{B}} (-D \partial_n \mathcal{U}) ds = \frac{J_{\text{Smol}}}{1 + D/(\mathcal{K}R)},$$

where  $J_{\text{Smol}} = 4\pi D R \mathcal{U}_{\text{inf}}$  is the Smoluchowski's diffusive flux onto a perfectly absorbing sphere [52]. In order to quantify the effect of a heterogeneous distribution of reactive patches, it is convenient to determine the effective reactivity  $\mathcal{K}_{\text{eff}}$  of an equivalent homogeneous sphere that produces the flux  $J$  from (1.3). In this way, by equating the fluxes in (1.2) and (1.3), we identify  $\mathcal{K}_{\text{eff}}$  in terms of  $\mathcal{C}_T$  as

$$(1.4) \quad k_{\text{eff}} = \frac{R}{D} \mathcal{K}_{\text{eff}} = \frac{1}{J_{\text{Smol}}/J - 1} = \frac{1}{R/\mathcal{C}_T - 1}.$$

Berg and Purcell [3] pioneered the study of the flux onto a spherical target that contains  $N$  identical small circular perfectly reactive boundary patches of radius  $\varepsilon R$ . Based on physical insight, they derived the following approximation for the flux:

$$(1.5) \quad J_{\text{BP}} = \frac{J_{\text{Smol}}}{1 + \pi/(\varepsilon N)}.$$

With this approximation, the effective reactivity of small patches becomes  $R\mathcal{K}_{\text{BP}}/D = \varepsilon N/\pi$ , which can also be written in terms of the patch area coverage fraction  $f$  on the surface of the sphere as (see also [50])

$$(1.6) \quad \mathcal{K}_{\text{BP}} = \frac{4fD}{\pi\varepsilon R}, \quad \text{where} \quad f \equiv \frac{N\pi(\varepsilon R)^2}{4\pi R^2} = \frac{\varepsilon^2 N}{4}.$$

Qualitatively, this result is equivalent to the reactivity of  $N$  one-sided disks of radius  $\varepsilon R$  in  $\mathbb{R}^3$ , as if they were trapping diffusing particles independently of each other (in fact, as  $2\varepsilon R/\pi$  is the capacitance of a two-sided disk, half of it corresponds to a one-sided disk). The seminal approximation (1.5) ignores the diffusion screening [18, 22] or diffusive interactions [53] between patches that compete for the diffusing particles. This competition is expected to reduce the fluxes to individual disks and can thus significantly diminish the effective reactivity. In addition, the approximation (1.6) ignores the curvature of the spherical boundary that may also be relevant when the patches are not too small. Despite these limitations, the seminal work by Berg and Purcell [3] stimulated the development of asymptotic and homogenization methods in both the mathematical and physical literature (see an overview in [28]).

From a mathematical perspective, the method of matched asymptotic expansions was used in [41] to provide a *first principles* derivation of the effective reactivity for  $N$  small circular and perfectly reactive patches on the boundary of a sphere, in the small patch area fraction limit  $f \ll 1$ . This analysis accounted for diffusive interactions between patches, as well the curvature of the sphere. For uniformly distributed identical patches, the leading-order term in the scaling law derived in [41] was found to agree with the classical Berg-Purcell result (1.5). This asymptotic scaling law for the effective reactivity was validated by comparing results with those computed from a numerical PDE-based approach for the Dirichlet-Neumann BVP, as developed in [41, 40], which effectively resolved the edge singularity near each patch (see also [16]). Moreover, a fast solver relying on an integral equation re-formulation of the mixed Dirichlet-Neumann BVP for a sphere with locally circular patches was developed in [36] to solve the PDE with up to 100,000 perfectly reacting patches. Other analytical and numerical studies with perfectly reactive patches include [40] and [5] for the boundary homogenization of periodic patterns of patches on a semi-infinite plane (see also the references therein). More recently, a hybrid analytical and numerical approach, based on a Kinetic Monte Carlo algorithm, has been developed to study time-dependent diffusive capture to an infinite plane and to a sphere [4].

In contrast, there have been much fewer analytical or numerical studies for the more realistic situation in (1.1) of partially reactive patches. In [59, 2], a heuristic interpolation formula was postulated, but without any mathematical justification, for the effective reactivity of a generic smooth surface that has identical small circular patches with a common reactivity  $\mathcal{K}$ :

$$(1.7) \quad \mathcal{K}_{\text{heur}} = \frac{f\mathcal{K}}{f\mathcal{K} + \mathcal{K}_{\text{BP}}} \mathcal{K}_{\text{BP}},$$

where  $\mathcal{K}_{\text{BP}}$  is given in (1.6). For identical patches of radii  $\varepsilon R$  on the surface of sphere of radius  $R$ , the heuristic approximation (1.7) can be written equivalently, by using (1.6) for  $\mathcal{K}_{\text{BP}}$ , as

$$(1.8) \quad \frac{R}{D} \mathcal{K}_{\text{heur}} = \frac{2f}{\varepsilon} \mathcal{A} \left( \frac{\mathcal{K}\varepsilon R}{D} \right), \quad \text{where} \quad \mathcal{A}(\mu) \equiv \frac{2\mu/\pi}{\mu + 4/\pi}.$$

In [44] a *first principles* leading-order asymptotic theory was developed to derive the effective reactivity for small circular patches of radius  $a$  on an infinite plane in  $\mathbb{R}^3$  in the limit of low, moderate, and large patch reactivity. With  $a = \varepsilon R$ , their leading-order analysis, resulting in Eq. (7.3) of [44], predicted that the effective reactivity can be well-approximated, uniformly in  $\mathcal{K}$ , by (1.8) with the term  $4/\pi$  in  $\mathcal{A}(\mu)$  replaced by  $2/(\pi K_w)$ . The value  $K_w \approx 0.5854$  was estimated in [44] by using Monte Carlo

simulations to calculate the numerical solution to a certain local PDE problem defined near a patch. This numerical value has now been corrected<sup>1</sup> to  $K_w \approx 0.5$ , so that the results in [44] do in fact provide the first theoretical justification of the empirical approximation (1.7) of [59, 2].

Related asymptotic studies for the mean first-passage time and splitting probabilities for the narrow capture of a Brownian particle in a bounded 3-D domain with small surface patches of finite reactivity include [31, 8], and our companion paper [30]. The analogous 2-D problem has been analyzed in [29]. Narrow capture in a 3-D bounded domain with reflecting walls but with a collection of small spherical inclusions with semipermeable interfaces has been analyzed in [7].

For the sphere, our main goal is to extend the previous analysis in [41], which was restricted to perfectly reactive patches, and the leading-order analysis in [44], to the more general case of partially reactive patches; in fact, we aim at developing a three-term asymptotic formula for  $C_T$  in (1.1) for patches of arbitrary shape and any reactivity. With this three-term asymptotic analysis we will incorporate inter-patch interactions, as well as the effect of the curvature of the sphere. By homogenizing our result for  $C_T$ , we will derive a new scaling law for the effective reactivity of the boundary of a sphere for identical partially reactive patches. Results from the asymptotic theory will be confirmed through simulations from a new Monte Carlo algorithm. We remark that the leading-order term in our asymptotic results is valid for an arbitrary bounded domain  $\mathcal{B}$  with a smooth boundary  $\partial\mathcal{B}$ . For this extension,  $\varepsilon = L/R$ , where  $2R$  is the diameter of  $\mathcal{B}$ , defined as the maximum Euclidean distance between any two points on  $\partial\mathcal{B}$ .

As convenient for our analysis, we will first non-dimensionalize (1.1) in a sphere, by introducing the dimensionless variables defined by

$$(1.9) \quad \mathbf{x} = \frac{\mathbf{X}}{R}, \quad \Omega = \frac{\mathcal{B}}{R}, \quad \varepsilon = \frac{L}{R}, \quad a_i = \frac{L_i}{L}, \quad \kappa_i = \frac{L\kappa_i}{D}, \quad u = \frac{\mathcal{U}}{\mathcal{U}_{\text{inf}}}, \quad C_T = \frac{C_T}{R}.$$

In the region exterior to the unit sphere  $\Omega$ , we obtain that (1.1) transforms to

$$(1.10a) \quad \Delta_{\mathbf{x}} u = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega,$$

$$(1.10b) \quad \varepsilon \partial_n u + \kappa_i u = 0, \quad \mathbf{x} \in \partial\Omega_i^\varepsilon, \quad i = 1, \dots, N,$$

$$(1.10c) \quad \partial_n u = 0, \quad \mathbf{x} \in \partial\Omega_r = \partial\Omega \setminus \partial\Omega_a,$$

$$(1.10d) \quad u \sim 1 - \frac{C_T}{|\mathbf{x}|} + \mathcal{O}(|\mathbf{x}|^{-2}), \quad \text{as } |\mathbf{x}| \rightarrow \infty,$$

where  $\Delta_{\mathbf{x}}$  is the Laplacian in  $\mathbf{x}$ , and  $\partial_n$  is the outward normal derivative to  $\partial\Omega$ , which points into the unit sphere. In (1.10), each rescaled patch  $\partial\Omega_i^\varepsilon = R^{-1}\partial\mathcal{B}_i$ , of small diameter  $\mathcal{O}(\varepsilon)$ , is assumed to satisfy  $\partial\Omega_i^\varepsilon \rightarrow \mathbf{x}_i \in \partial\Omega$  as  $\varepsilon \rightarrow 0$ . The patches are also assumed to be well-separated in the sense that  $|\mathbf{x}_i - \mathbf{x}_j| = \mathcal{O}(1)$  for all  $i \neq j$ .

In the limit  $\varepsilon \rightarrow 0$  of small patches, in §3 we will use strong localized perturbation theory, originating from [56], to derive a three-term asymptotic expansion for the capacitance  $C_T$ , valid for arbitrary  $\kappa_i > 0$ , in which only the third-order term depends on the spatial arrangement  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  of the centers of the patches. The main result is summarized in Proposition 1 of §3. For the special case where  $\kappa_i = \infty$  and when the patches are circular disks, such an asymptotic analysis has been performed in [41]

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<sup>1</sup>As communicated to MJW by S. Lawley, the Monte Carlo approximation should be  $K_w \approx 0.5$  and not  $K_w \approx 0.5854$ .

by using the traditional spherical coordinates for a sphere. Our new framework in §3 uses an alternative approach than in [41], which is based on the geodesic normal coordinates introduced in §2. These coordinates have been used previously in [55] and [20] to analyze localized pattern formation problems in reaction-diffusion systems. With this new, more efficient, coordinate system, we are readily able to extend the previous analysis of [41] to the case of finite  $\kappa_i$  and to arbitrary patch shapes.

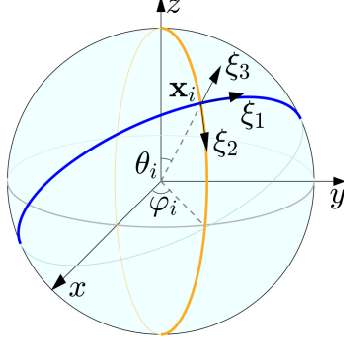
This extension involves the reactive capacitance  $C_i(\kappa_i)$  and an additional monopole coefficient  $E_i(\kappa_i)$ , which are determined in terms of the solutions to certain local (or inner) problems near each patch. From the analysis in the companion paper [30], we show that these key quantities can be expressed for all  $\kappa_i > 0$  via a Steklov eigenfunction expansion. For circular patches, an efficient numerical computation of the Steklov eigenfunctions is readily available [26]. Moreover, both  $C_i(\kappa_i)$  and  $E_i(\kappa_i)$  can be well-approximated over the full range  $0 < \kappa_i < \infty$  by some heuristic formulas that have been benchmarked to full numerical results. In this way, we obtain an explicit analytical approximation for  $C_T$  for any  $\kappa_i > 0$ . Preliminary results for  $C_i(\kappa_i)$ ,  $E_i(\kappa_i)$ , and the surface Neumann Green’s function, which are central to the analysis in §3, are summarized in §2. Although some of these results were previously derived in [30], they are summarized here for completeness and readability.

In §4 we consider the homogenization limit of  $N \gg 1$  identical patches that are uniformly distributed over the surface of the sphere, but in the limit of small patch area fraction. For this homogenization problem, we derive an explicit analytical scaling law for both the effective capacitance  $C_{\text{eff}}$  and the effective reactivity  $k_{\text{eff}}$  of the structured target. This scaling law can be applied over the full range  $\kappa > 0$  by using our empirical approximations for  $C(\kappa)$  and  $E(\kappa)$ . Substitution of the empirical approximation for  $C(\kappa)$  into the leading-order term of our asymptotic theory leads to the heuristic result (1.8) of [2]. In §5 we present a new efficient Monte Carlo method to numerically compute the capacitance  $C_T$  from the underlying PDE. From numerical results obtained by the Monte Carlo algorithm, we show in §6 that our explicit scaling laws for  $C_{\text{eff}}$  and  $k_{\text{eff}}$ , which are based on our three-term asymptotic theory, are still accurate even when there is only a moderately large number of patches, or when the reactive patch area fraction is not exceedingly small. Finally, in §7 we summarize our results and discuss a few additional problems that warrant further study.

**2. Preliminaries.** In this section we derive some preliminary results that are central for our asymptotic analysis in §3.

**2.1. Geodesic normal coordinates.** The inner problem near each patch is more readily analyzed in terms of geodesic normal coordinates, rather than the usual global spherical coordinates used in [41]. More specifically, for each patch  $\partial\Omega_i^\varepsilon$  on the unit sphere  $\Omega$ , we introduce the geodesic normal coordinates  $\xi = (\xi_1, \xi_2, \xi_3)^T \in (-\pi/2, \pi/2) \times (-\pi, \pi) \times [0, \infty]$  in  $\mathbb{R}^3 \setminus \Omega$  so that  $\xi = 0$  corresponds to  $\mathbf{x}_i \in \partial\Omega$ , while  $\xi_3 > 0$  corresponds to the exterior of  $\Omega$ . In these coordinates,  $\xi_2$  can be viewed as the polar angle of a spherical coordinate system centered at  $\mathbf{x}_i$  on the sphere, but defined on the range  $\xi_2 \in (-\pi/2, \pi/2)$  that avoids the usual coordinate singularity of spherical coordinates at the north pole. The curves obtained by setting  $\xi_3 = 0$  and fixing either  $\xi_1 = 0$  or  $\xi_2 = 0$  are geodesics on  $\partial\Omega$  that pass through  $\mathbf{x}_i$ .

From the global transformation  $\mathbf{x} = \mathbf{x}(\xi)$  between cartesian and geodesic coordinates, as given explicitly in (A.2) of Appendix A, we derive an exact expression for the Laplacian of a generic function  $\mathcal{V}(\xi) \equiv u(\mathbf{x}(\xi))$ . Then, in terms of the local (or



**Fig. 2.1:** Geodesic normal coordinates  $(\xi_1, \xi_2, \xi_3)^T$  at  $\mathbf{x}_i \in \partial\Omega$ , with the geodesics (orange and blue curves) indicated.

inner) variables,  $\mathbf{y} = (y_1, y_2, y_3)^T$ , defined by

$$(2.1) \quad \xi_1 = \varepsilon y_1, \quad \xi_2 = \varepsilon y_2, \quad \xi_3 = \varepsilon y_3,$$

in Appendix A we show, for  $\varepsilon \rightarrow 0$  and for  $V(\mathbf{y}) = \mathcal{V}(\varepsilon \mathbf{y})$ , that

$$(2.2) \quad \Delta_{\mathbf{x}} u = \varepsilon^{-2} \Delta_{\mathbf{y}} V + \varepsilon^{-1} [-2y_3 (V_{y_1 y_1} + V_{y_2 y_2}) + 2V_{y_3}] + \mathcal{O}(1),$$

with  $\Delta_{\mathbf{y}} V \equiv V_{y_1 y_1} + V_{y_2 y_2} + V_{y_3 y_3}$ . This result is essential to our local analysis in §3.

**2.2. Reactive capacitance.** Defined on the tangent plane to the sphere centered at  $\mathbf{x}_i$ , the leading-order term in our inner expansion near  $\mathbf{x} = \mathbf{x}_i$  will involve the solution  $w_i = w_i(\mathbf{y}; \kappa_i)$  to

$$(2.3a) \quad \Delta_{\mathbf{y}} w_i = 0, \quad \mathbf{y} \in \mathbb{R}_+^3,$$

$$(2.3b) \quad -\partial_{y_3} w_i + \kappa_i w_i = \kappa_i, \quad y_3 = 0, (y_1, y_2) \in \Gamma_i,$$

$$(2.3c) \quad \partial_{y_3} w_i = 0, \quad y_3 = 0, (y_1, y_2) \notin \Gamma_i,$$

$$(2.3d) \quad w_i \sim \frac{C_i(\kappa_i)}{|\mathbf{y}|} + \frac{\mathbf{p}_i(\kappa_i) \cdot \mathbf{y}}{|\mathbf{y}|^3} + \dots, \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

Here  $\mathbb{R}_+^3 \equiv \{\mathbf{y} = (y_1, y_2, y_3) \mid y_3 \geq 0, -\infty < y_1, y_2 < \infty\}$  is the upper half-space, and  $\Gamma_i \asymp \varepsilon^{-1} \partial\Omega_i^\varepsilon$  is the flattened Robin patch on the horizontal plane  $y_3 = 0$ . In (2.3d), the dipole vector  $\mathbf{p}_i = \mathbf{p}_i(\kappa_i)$  must have the form  $\mathbf{p}_i = (p_{i1}, p_{i2}, 0)^T$  to ensure that the far-field behavior (2.3d) satisfies (2.3c). When  $\Gamma_i$  is symmetric in  $y_1$  and  $y_2$ , such as is the case when  $\Gamma_i$  is a disk, it follows from symmetry that  $p_{i1} = p_{i2} = 0$ , so that the dipole term in the far-field (2.3d) is absent.

We refer to the monopole coefficient  $C_i(\kappa_i)$  in (2.3d) as the *reactive capacitance*. Although by applying the divergence theorem we can readily express  $C_i(\kappa_i)$  in terms of the *charge density*  $q_i$ ,

$$(2.4) \quad C_i(\kappa_i) = \frac{1}{\pi} \int_{\Gamma_i} q_i(y_1, y_2; \kappa_i) dy_1 dy_2, \quad \text{where} \quad q_i(y_1, y_2; \kappa_i) \equiv -\frac{1}{2} \partial_{y_3} w_i|_{y_3=0},$$

we emphasize that there is no *explicit* solution to (2.3) for arbitrary  $\kappa_i > 0$ . As a result,  $C_i(\kappa_i)$  must in general be computed numerically. In Appendix B we summarize

some results of Appendix D of [30] that determined an easily computable spectral representation for  $C_i(\kappa_i)$  in terms of a suitable Steklov eigenvalue problem. More specifically, this spectral representation is

$$(2.5) \quad C_i(\kappa_i) = \frac{\kappa_i}{2\pi} \sum_{k=0}^{\infty} \frac{\mu_{ki} d_{ki}^2}{\mu_{ki} + \kappa_i},$$

where the eigenvalues  $\mu_{ki} > 0$  and the spectral weights  $d_{ki} \neq 0$  of the Steklov problem (B.1) can be computed numerically for a specified patch shape  $\Gamma_i$ . Although the spectrum of (B.1) must be computed numerically, the functional form of  $C_i(\kappa_i)$  and its dependence on reactivity is universal. In the special case where the patches are all of the same shape, but with a variable size, we have the scaling law

$$(2.6) \quad C_i(\kappa_i) = a_i \mathcal{C}(\kappa_i a_i), \quad i = 1, \dots, N,$$

where  $\mathcal{C}(\mu)$  is the reactive capacitance of the rescaled common patch shape  $\Gamma_c \equiv \Gamma_i/a_i$ , which needs to be computed only once. In particular, this scaling result is applicable to the case where the patches are all circular disks of different radii. For a circular patch of unit radius, the first eight Steklov eigenvalues and weights, corresponding to eigenfunctions that are axially symmetric on the patch, are given in Table B.1.

**Circular patch.** We recall that when  $\Gamma_i$  is a disk of radius  $a_i$ , the limiting problem with  $\kappa_i = \infty$  is the classical problem for the capacitance of a flat disk (cf. [35]), whose solution  $w_i(\mathbf{y}; \infty)$  is (see page 38 of [17])

$$(2.7a) \quad w_i(\mathbf{y}; \infty) \equiv \frac{2}{\pi} \sin^{-1} \left( \frac{a_i}{B(y_3, \rho_0)} \right),$$

where  $\rho_0 \equiv (y_1^2 + y_2^2)^{1/2}$  and

$$(2.7b) \quad B(y_3, \rho_0) \equiv \frac{1}{2} \left( [(\rho_0 + a_i)^2 + y_3^2]^{1/2} + [(\rho_0 - a_i)^2 + y_3^2]^{1/2} \right).$$

The far-field behavior is  $w_i(\mathbf{y}; \infty) \sim C_i(\infty)/|\mathbf{y}| + \mathcal{O}(|\mathbf{y}|^{-3})$ , where  $C_i(\infty) = 2a_i/\pi$ . Moreover, from (2.7a), and by using the radial symmetry, the charge density is

$$(2.7c) \quad q_i(y_1, y_2; \infty) = q_i(\rho_0; \infty) \equiv -\frac{1}{2} \partial_{y_3} w_i(\mathbf{y}; \infty)|_{y_3=0} = \frac{1}{\pi \sqrt{a_i^2 - \rho_0^2}}, \quad 0 \leq \rho_0 \leq a_i.$$

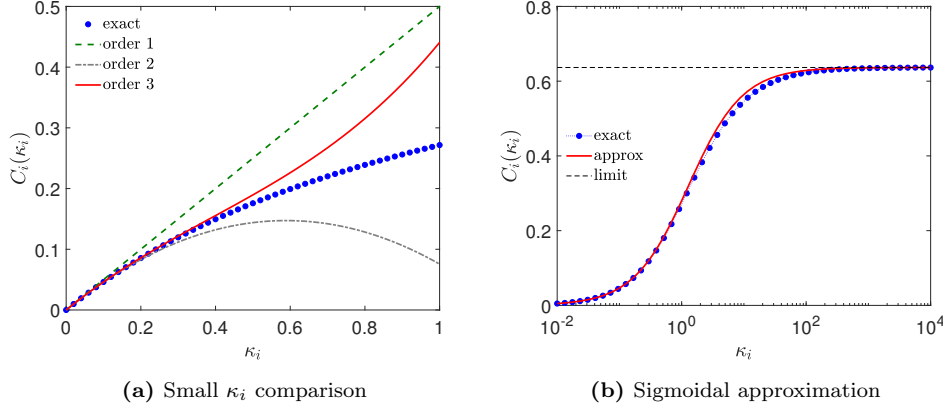
As a result of the edge singularity in (2.7c) at  $\rho = a_i$ , it can be shown from the analysis in [31] (see also Appendix D.3 of [30]) that the difference  $C_i(\kappa_i) - C_i(\infty)$  is not analytic for  $\kappa_i \gg 1$ . More specifically,  $C_i(\kappa_i)$  has the refined far-field behavior

$$(2.8) \quad C_i(\kappa_i) \sim \frac{2a_i}{\pi} - 2 \frac{[\log(a_i \kappa_i) + \log 2 + \gamma_e + 1]}{\pi^2 \kappa_i}, \quad \text{as } \kappa_i \rightarrow \infty,$$

where  $\gamma_e \approx 0.5772 \dots$  is Euler's constant.

In Fig. 2.2(a) we illustrate that a three-term Taylor expansion (B.5) of  $C_i(\kappa_i)$  at small  $\kappa_i$ , with the coefficients in (B.6), provides a rather close approximation to  $C_i(\kappa_i)$  on the range  $0 < \kappa_i < 0.45$ . Finally, in Fig. 2.2(b) we show that the heuristic sigmoidal approximation, given by

$$(2.9) \quad C_i(\kappa_i) \approx C_i^{\text{app}}(\kappa_i) = a_i \mathcal{C}^{\text{app}}(a_i \kappa_i), \quad \text{where} \quad \mathcal{C}^{\text{app}}(\mu) = \frac{2\mu/\pi}{\mu + 4/\pi},$$



**Fig. 2.2:** The reactive capacitance  $C_i(\kappa_i)$  for the circular patch  $\Gamma_i$  of unit radius ( $a_i = 1$ ). **(a):** A comparison of  $C_i(\kappa_i)$  numerically computed from (2.5), with the one-, two-, and three-term approximations obtained from (B.5) and (B.6), valid for  $\kappa_i \ll 1$ . **(b):** The sigmoidal approximation (2.9) provides a decent approximation of the numerical result for  $C_i(\kappa_i)$  on the full range  $\kappa_i > 0$ .

is within 4%, over the entire range  $\kappa_i > 0$ , of the spectral expansion result (2.5) when applied to a circular patch. These results are summarized in the following lemma of [30].

LEMMA 2.1. (*Lemma 2.1 of [30]*) When  $\Gamma_i$  is the disk  $y_1^2 + y_2^2 \leq a_i^2$ ,  $C_i(\kappa_i)$  has the limiting asymptotics

$$(2.10a) \quad C_i(\kappa_i) \sim C_i(\infty) + \mathcal{O}\left(\frac{\log \kappa_i}{\kappa_i}\right), \quad \text{as } \kappa_i \rightarrow \infty, \quad \text{with } C_i(\infty) = \frac{2a_i}{\pi},$$

$$(2.10b) \quad C_i(\kappa_i) \sim a_i \left[ c_{1i} \kappa_i a_i - c_{2i} (\kappa_i a_i)^2 + c_{3i} (\kappa_i a_i)^3 + \mathcal{O}((\kappa_i a_i)^4) \right], \quad \text{as } \kappa_i \rightarrow 0,$$

where  $c_{1i} = 0.5$ ,  $c_{2i} \approx 0.4241$  and  $c_{3i} \approx 0.3651$  (see (B.6) of Appendix B), are independent of the patch radius  $a_i$ . The sigmoidal approximation (2.9) is exactly consistent with only the leading-order coefficient  $c_{1i}$ . The other exact Taylor coefficients  $c_{2i}$  and  $c_{3i}$  are numerically comparable, but distinct, from the higher Taylor coefficients of (2.9).

**2.3. Monopole From a Higher-Order Inner Solution.** In our asymptotic analysis of (1.10) in §3 we show in Appendix C that we have to calculate the monopole coefficient  $E_i = E_i(\kappa_i)$  defined by the solution  $\Phi_{2hi}$  to the inner problem

$$(2.11a) \quad \Delta_{\mathbf{y}} \Phi_{2hi} = 0, \quad \mathbf{y} \in \mathbb{R}_+^3,$$

$$(2.11b) \quad -\partial_{y_3} \Phi_{2hi} + \kappa_i \Phi_{2hi} = \kappa_i \mathcal{F}_i, \quad y_3 = 0, (y_1, y_2) \in \Gamma_i,$$

$$(2.11c) \quad \partial_{y_3} \Phi_{2hi} = 0, \quad y_3 = 0, (y_1, y_2) \notin \Gamma_i,$$

$$(2.11d) \quad \Phi_{2hi} \sim -\frac{E_i}{\rho}, \quad \text{as } \rho = |\mathbf{y}| \rightarrow \infty.$$



In (2.11b), the inhomogeneous term  $\mathcal{F}_i = \mathcal{F}_i(y_1, y_2; \kappa_i)$  is the unique solution, defined in terms of  $C_i = C_i(\kappa_i)$  and  $q_i(y_1, y_2; \kappa_i)$  as related by (2.4), to the 2-D problem

$$(2.12a) \quad \mathcal{F}_{i,y_1 y_1} + \mathcal{F}_{i,y_2 y_2} = q_i(y_1, y_2; \kappa_i) I_{\Gamma_i}(y_1, y_2), \quad I_{\Gamma_i} \equiv \begin{cases} 1, & (y_1, y_2) \in \Gamma_i \\ 0, & (y_1, y_2) \notin \Gamma_i \end{cases},$$

$$(2.12b) \quad \mathcal{F}_i \sim \frac{C_i}{2} \log \rho_0 + o(1), \quad \text{as } \rho_0 \equiv (y_1^2 + y_2^2)^{1/2} \rightarrow \infty.$$

It is the  $o(1)$  condition in the far-field (2.12b) which ensures that  $\mathcal{F}_i$  is unique.

For an arbitrary patch shape, in Appendix C we show in (C.3) that  $E_i(\kappa_i)$  can be determined up to a quadrature. The following result, established in [30], and discussed in Appendix C, more fully characterizes  $E_i$  when  $\Gamma_i$  is a disk:

LEMMA 2.2. (Lemma 2.2 of [30]) *When the Robin patch  $\Gamma_i$  is the disk  $y_1^2 + y_2^2 \leq a_i^2$ , we have*

$$(2.13) \quad E_i = E_i(\kappa_i) = -\frac{\log a_i}{2} [C_i(\kappa_i)]^2 + a_i^2 \mathcal{E}_i(\kappa_i a_i),$$

where  $C_i = C_i(\kappa)$  is the reactive capacitance for a disk given by

$$(2.14) \quad C_i = 2 \int_0^{a_i} q_i(\rho_0; \kappa_i) \rho_0 d\rho_0, \quad q_i(\rho_0; \kappa_i) = -\frac{1}{2} w_{i,y_3}|_{y_3=0},$$

and  $\mathcal{E}_i(\mu)$  is defined by

$$(2.15) \quad \mathcal{E}_i(\mu) \equiv 2 \int_0^1 \frac{1}{\rho_0} \left( \int_0^{\rho_0} a_i q_i(\eta a_i; \mu/a_i) \eta d\eta \right)^2 d\rho_0.$$

The limiting asymptotic behavior of  $E_i(\kappa_i)$  is

$$(2.16a) \quad E_i \sim E_i(\infty) \equiv -\frac{2a_i^2}{\pi^2} \left( \log a_i + \log 4 - \frac{3}{2} \right), \quad \text{as } \kappa_i \rightarrow \infty,$$

$$(2.16b) \quad E_i \sim \frac{\kappa_i^2 a_i^4}{8} \left( \frac{1}{4} - \log a_i \right), \quad \text{as } \kappa_i \rightarrow 0.$$

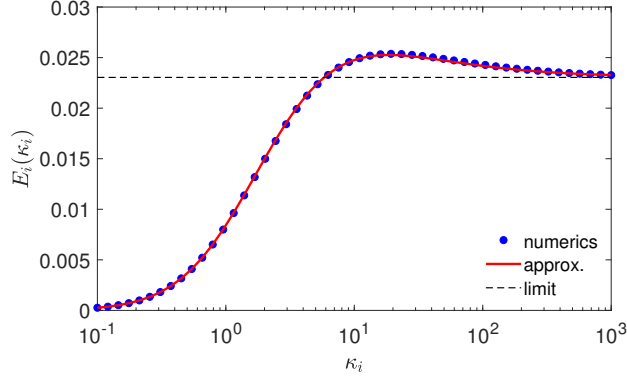
Since there is no analytical formula for  $E_i(\kappa_i)$  for arbitrary  $\kappa_i > 0$  when  $\Gamma_i$  is a disk, in Appendix D of [30] we showed how it can be computed numerically to high precision by expanding the charge density  $q_i$  in terms of Steklov eigenfunctions (see [30] for details). Moreover, labeling  $\mu \equiv a_i \kappa_i$  and with  $\mathcal{C}^{\text{app}}(\mu)$  as given by the sigmoidal approximation (2.9), in Appendix D of [30] we showed that the heuristic approximation

$$(2.17a) \quad E_i(\kappa_i) \approx E_i^{\text{app}} = -\frac{a_i^2 \log a_i}{2} [\mathcal{C}^{\text{app}}(\kappa_i a_i)]^2 + a_i^2 \mathcal{E}^{\text{app}}(\kappa_i a_i),$$

where

$$(2.17b) \quad \mathcal{E}^{\text{app}}(\mu) \equiv [\mathcal{C}^{\text{app}}(\mu)]^2 \left( \frac{3}{4} - \log 2 + \frac{1}{\frac{1}{\log 2 - 5/8} + 5.17 \mu^{0.81}} \right),$$

agrees with the corresponding numerical result, with a maximal relative error of 0.7% over the entire range of  $\mu > 0$  (see Fig. 2.3). Moreover, the established limits from (2.16) as  $\mu \rightarrow 0$  and  $\mu \rightarrow \infty$  are satisfied by (2.17).



**Fig. 2.3:** For a circular patch of radius  $a_i = 1$ , the heuristic approximation  $\mathcal{E}^{\text{app}}(\kappa_i)$  (solid curve) from (2.17b) is compared with  $E_i(\kappa_i) = \mathcal{E}_i(\kappa_i)$  (filled circles) given by (2.13) and computed via the numerical approach described in Appendix D of [30]. The dashed horizontal line is the asymptotic limiting value  $(3 - 4 \log 2)/\pi^2$  consistent with (2.16a).

**2.4. The Exterior Surface Neumann Green's function.** In our asymptotic analysis in §3, the asymptotic expansion of the outer solution is represented in terms of the *exterior* surface Neumann Green's function  $G_s(\mathbf{x}; \mathbf{x}_i)$  for the unit sphere  $\Omega$ , defined as the unique solution to

(2.18)

$$\Delta G_s = 0, \quad r > 1; \quad -\partial_r G_s = \delta(\mathbf{x} - \mathbf{x}_i), \quad r = 1; \quad G_s \sim \frac{1}{4\pi|\mathbf{x}|}, \quad \text{as } |\mathbf{x}| \rightarrow \infty,$$

where  $r = |\mathbf{x}|$  and  $|\mathbf{x}_i| = 1$ . The exact solution to (2.18) is (see [41])

$$(2.19) \quad G_s(\mathbf{x}; \mathbf{x}_i) = \frac{1}{2\pi|\mathbf{x} - \mathbf{x}_i|} - \frac{1}{4\pi} \log \left( 1 + \frac{2}{|\mathbf{x} - \mathbf{x}_i| + |\mathbf{x}| - 1} \right).$$

To determine the local behavior of  $G_s(\mathbf{x}; \mathbf{x}_i)$  as  $\mathbf{x} \rightarrow \mathbf{x}_i$  in terms of the local geodesic coordinates  $\mathbf{y}$ , we use (A.7) to estimate  $|\mathbf{x} - \mathbf{x}_i|$ . In this way, we obtain that

$$G_s \sim \frac{1}{2\pi\varepsilon|\mathbf{y}|} \left( 1 - \frac{\varepsilon y_3}{2|\mathbf{y}|^2} (y_1^2 + y_2^2) \right) - \frac{1}{4\pi} \log \left( 1 + \frac{2}{\varepsilon(|\mathbf{y}| + y_3)} \right),$$

which can be simplified asymptotically to

$$(2.20) \quad G_s \sim \frac{1}{2\pi\varepsilon|\mathbf{y}|} + \frac{1}{4\pi} \log \left( \frac{\varepsilon}{2} \right) - \frac{y_3(y_1^2 + y_2^2)}{4\pi|\mathbf{y}|^3} + \frac{1}{4\pi} \log(|\mathbf{y}| + y_3) + o(1).$$

**3. Analysis for the Capacitance  $C_T$ .** We now use the method of matched asymptotic expansions to construct solutions to (1.10) in the limit  $\varepsilon \rightarrow 0$ . For our analysis it is convenient to introduce  $U$  by  $u = -C_T U$ , so that from (1.10)  $U$  satisfies

$$(3.1a) \quad \Delta_{\mathbf{x}} U = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega,$$

$$(3.1b) \quad \varepsilon \partial_n U + \kappa_i U = 0, \quad \mathbf{x} \in \partial\Omega_i^\varepsilon, \quad i = 1, \dots, N,$$

$$(3.1c) \quad \partial_n U = 0, \quad \mathbf{x} \in \partial\Omega_r = \partial\Omega \setminus \partial\Omega_a,$$

$$(3.1d) \quad U \sim -\frac{1}{C_T} + \frac{1}{|\mathbf{x}|} + \frac{\mathbf{p} \cdot \mathbf{x}}{C_T |\mathbf{x}|^3} + \dots, \quad \text{as } |\mathbf{x}| \rightarrow \infty.$$

In this way, we can write the far-field condition (3.1d) as the following flux condition over the boundary  $\partial\Omega_R$  of a large sphere of radius  $R$  centered at  $\mathbf{x} = 0$ :

$$(3.1e) \quad \lim_{R \rightarrow \infty} \int_{\partial\Omega_R} \partial_r U|_{r=R} ds = -4\pi.$$

In the outer region away from the Robin patches we expand the outer solution as

$$(3.2) \quad U \sim \varepsilon^{-1} U_0 + U_1 + \varepsilon \log\left(\frac{\varepsilon}{2}\right) U_2 + \varepsilon U_3 + \dots,$$

where  $U_0$  is a constant to be determined, and where  $U_k$  for  $k \geq 1$  satisfies

$$(3.3) \quad \begin{aligned} \Delta_{\mathbf{x}} U_k &= 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega; \quad \partial_n U_k = 0, \quad \mathbf{x} \in \partial\Omega \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_N\}, \\ \lim_{R \rightarrow \infty} \int_{\partial\Omega_R} \partial_n U_k|_{r=R} ds &= -4\pi \delta_{k1}. \end{aligned}$$

Here  $\delta_{k1}$  is the Kronecker symbol. Our analysis will provide singularity behaviors for each  $U_k$  as  $\mathbf{x} \rightarrow \mathbf{x}_i$ , for  $i = 1, \dots, N$ .

In the inner region near the  $i$ -th Robin patch we introduce the local geodesic coordinates (2.1) and we expand each inner solution as

$$(3.4) \quad U \sim \varepsilon^{-1} V_{0i} + \log\left(\frac{\varepsilon}{2}\right) V_{1i} + V_{2i} + \dots$$

Upon substituting (3.4) into (2.2), we obtain that  $V_{ki}$  for  $k = 0, 1, 2$  satisfies

$$(3.5a) \quad \Delta_{\mathbf{y}} V_{ki} = \delta_{k2} (-2y_3 V_{0i, y_3 y_3} - 2V_{0i, y_3}), \quad \mathbf{y} \in \mathbb{R}_+^3,$$

$$(3.5b) \quad -\partial_{y_3} V_{ki} + \kappa_i V_{ki} = 0, \quad y_3 = 0, (y_1, y_2) \in \Gamma_i,$$

$$(3.5c) \quad \partial_{y_3} V_{ki} = 0, \quad y_3 = 0, (y_1, y_2) \notin \Gamma_i.$$

The leading-order matching condition is that  $V_{0i} \sim U_0$  as  $|\mathbf{y}| \rightarrow \infty$  for each  $i = 1, \dots, N$ . As a result, we write the leading-order inner solution as

$$(3.6) \quad V_{0i} = U_0 (1 - w_i),$$

where  $w_i = w_i(\mathbf{y}; \kappa_i)$  is the solution to (2.3). The asymptotic matching condition requires that the local behavior of the outer expansion (3.2) as  $\mathbf{x} \rightarrow \mathbf{x}_i$  must agree with the far-field behavior as  $|\mathbf{y}| \rightarrow \infty$  of the inner expansion (3.4), so that

$$(3.7) \quad \begin{aligned} \frac{U_0}{\varepsilon} + U_1 + \varepsilon \log\left(\frac{\varepsilon}{2}\right) U_2 + \varepsilon U_3 + \dots \\ \sim \frac{U_0}{\varepsilon} \left(1 - \frac{C_i}{|\mathbf{y}|} - \frac{\mathbf{p}_i \cdot \mathbf{y}}{|\mathbf{y}|^3}\right) + \log\left(\frac{\varepsilon}{2}\right) V_{1i} + V_{2i} + \dots, \end{aligned}$$

where  $C_i = C_i(\kappa_i)$  is the reactive capacitance of the  $i$ -th patch. Since  $|\mathbf{y}| \sim \varepsilon^{-1} |\mathbf{x} - \mathbf{x}_i|$  from (A.8) of Appendix A, we require that  $U_1$  must satisfy (3.3), with the singular behavior  $U_1 \sim -U_0 C_i / |\mathbf{x} - \mathbf{x}_i|$  as  $\mathbf{x} \rightarrow \mathbf{x}_i$  for  $i = 1, \dots, N$ , so that

$$(3.8a) \quad \Delta_{\mathbf{x}} U_1 = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega; \quad \partial_n U_1 = 0, \quad \mathbf{x} \in \partial\Omega \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_N\},$$

$$(3.8b) \quad U_1 \sim -\frac{U_0 C_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_i \in \partial\Omega, \quad i = 1, \dots, N,$$

$$(3.8c) \quad \lim_{R \rightarrow \infty} \int_{\partial\Omega_R} \partial_r U_1|_{r=R} ds = -4\pi.$$

From the divergence theorem, the solvability condition for (3.8) determines  $U_0$  as

$$(3.9) \quad U_0 = -\frac{2}{\overline{C}}, \quad \text{where} \quad \overline{C} \equiv \sum_{i=1}^N C_i(\kappa_i).$$

The solution to (3.8) is represented in terms of the Green's function of (2.19) as

$$(3.10) \quad U_1 = \overline{U}_1 - 2\pi U_0 \sum_{j=1}^N C_j G_s(\mathbf{x}; \mathbf{x}_j).$$

As in [41], the unknown constant  $\overline{U}_1$  in (3.10) has been expanded in terms of additional constants  $\overline{U}_{10}$  and  $\overline{U}_{11}$ , independent of  $\varepsilon$ , as

$$(3.11) \quad \overline{U}_1 = \overline{U}_{10} \log\left(\frac{\varepsilon}{2}\right) + \overline{U}_{11}.$$

The term  $\overline{U}_{10} \log(\varepsilon/2)$  in (3.11), which arises from the logarithmic gauge function in (2.20), is known as a “switchback term” (cf. [37]), as it effectively inserts a constant term between  $U_0/\varepsilon$  and  $U_1$  in the outer expansion (3.2).

To determine  $\overline{U}_{10}$  and  $\overline{U}_{11}$  we should proceed to higher order. To do so, we expand  $U_1$  in (3.10) as  $\mathbf{x} \rightarrow \mathbf{x}_i$  by using the local behavior (2.20) of  $G_s$  near the  $i$ -th patch. The matching condition (3.7) becomes

$$(3.12) \quad \begin{aligned} & \frac{U_0}{\varepsilon} \left(1 - \frac{C_i}{|\mathbf{y}|}\right) + \left(-\frac{U_0 C_i}{2} + \overline{U}_{10}\right) \log\left(\frac{\varepsilon}{2}\right) + \frac{U_0 C_i}{2} \left(\frac{y_3(y_1^2 + y_2^2)}{|\mathbf{y}|^3} - \log(y_3 + |\mathbf{y}|)\right) \\ & \quad + U_0 \beta_i + \overline{U}_{11} + \varepsilon \log\left(\frac{\varepsilon}{2}\right) U_2 + \varepsilon U_3 + \dots \\ & \sim \frac{U_0}{\varepsilon} \left(1 - \frac{C_i}{|\mathbf{y}|} - \frac{\mathbf{p}_i \cdot \mathbf{y}}{|\mathbf{y}|^3}\right) + \log\left(\frac{\varepsilon}{2}\right) V_{1i} + V_{2i} + \dots \end{aligned}$$

In (3.12), the constant  $\beta_i$  is defined by the  $i$ -th component of the matrix-vector product

$$(3.13) \quad \beta_i = -2\pi (\mathcal{G}_s \mathbf{C})_i,$$

where  $\mathbf{C} \equiv (C_1, \dots, C_N)^T$ , with  $C_i = C_i(\kappa_i)$ , and  $\mathcal{G}_s$  is the symmetric Green's matrix:

$$(3.14a) \quad \mathcal{G}_s \equiv \begin{pmatrix} 0 & G_{12} & \cdots & G_{1N} \\ G_{21} & 0 & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & \cdots & G_{N,N-1} & 0 \end{pmatrix},$$

with

$$(3.14b) \quad G_{ij} \equiv G_s(\mathbf{x}_i; \mathbf{x}_j) = \frac{1}{2\pi|\mathbf{x}_i - \mathbf{x}_j|} - \frac{1}{4\pi} \log\left(1 + \frac{2}{|\mathbf{x}_i - \mathbf{x}_j|}\right).$$

Upon comparing the  $\mathcal{O}(\log \varepsilon)$  terms on both sides of (3.12) we conclude that the inner correction  $V_{1i}$ , satisfying (3.5) with  $k = 1$ , must have the far-field behavior  $V_{1i} \sim \bar{U}_{10} - U_0 C_i / 2$  as  $|\mathbf{y}| \rightarrow \infty$ . As a result, the solution for  $V_{1i}$  is

$$(3.15) \quad V_{1i} = \left( \bar{U}_{10} - \frac{U_0 C_i}{2} \right) (1 - w_i),$$

where  $w_i = w_i(\mathbf{y}; \kappa_i)$  satisfies (2.3). Upon using the far-field behavior (2.3d) for  $w_i$ , we obtain that

$$(3.16) \quad V_{1i} \sim \left( \bar{U}_{10} - \frac{U_0 C_i}{2} \right) \left( 1 - \frac{C_i}{|\mathbf{y}|} + \dots \right), \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

By substituting (3.16) into the matching condition (3.12), we conclude that  $U_2$  must satisfy (3.3) with the singular behavior  $U_2 \sim -(\bar{U}_{10} - \frac{1}{2} U_0 C_i) C_i / |\mathbf{x} - \mathbf{x}_i|$  as  $\mathbf{x} \rightarrow \mathbf{x}_i$ . As a result, we find that  $U_2$  satisfies

$$(3.17a) \quad \Delta_{\mathbf{x}} U_2 = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega; \quad \partial_n U_2 = 0, \quad \mathbf{x} \in \partial\Omega \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_N\},$$

$$(3.17b) \quad U_2 \sim -\left( \bar{U}_{10} - \frac{U_0 C_i}{2} \right) \frac{C_i}{|\mathbf{x} - \mathbf{x}_i|}, \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_i \in \partial\Omega, \quad i = 1, \dots, N,$$

$$(3.17c) \quad \lim_{R \rightarrow \infty} \int_{\partial\Omega_R} \partial_r U_2|_{r=R} ds = 0.$$

By using the divergence theorem, we readily find that (3.17) is solvable only when  $\sum_{j=1}^N C_j [\bar{U}_{10} - U_0 C_j / 2] = 0$ , which determines  $\bar{U}_{10}$  as

$$(3.18) \quad \frac{\bar{U}_{10}}{U_0} = \frac{1}{2\bar{C}} \sum_{j=1}^N C_j^2 \rightarrow \bar{U}_{10} = -\frac{\sum_{j=1}^N C_j^2}{(\bar{C})^2},$$

where  $\bar{C}$  was defined by (3.9). With  $\bar{U}_{10}$  determined in this way, the solution to (3.17) is given in terms of the Green's function in (2.19) and an additional unknown constant  $\bar{U}_2$  as

$$(3.19) \quad U_2 = \bar{U}_2 + 2\pi \sum_{j=1}^N C_j \left( \frac{U_0 C_j}{2} - \bar{U}_{10} \right) G_s(\mathbf{x}; \mathbf{x}_j).$$

To determine  $\bar{U}_{11}$ , we must match the  $\mathcal{O}(1)$  terms in (3.12). We obtain that  $V_{2i}$  satisfies (3.5) with  $k = 2$  subject to the far-field behavior

$$(3.20) \quad V_{2i} \sim \beta_i U_0 + \bar{U}_{11} + \frac{U_0 C_i}{2} \left( \frac{y_3(y_1^2 + y_2^2)}{|\mathbf{y}|^3} - \log(y_3 + |\mathbf{y}|) \right), \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

Since  $V_{0i} = U_0(1 - w_i)$  from (3.6), we decompose  $V_{2i}$  as

$$(3.21) \quad V_{2i} = U_0 \left( \Phi_{2i} + \left( \beta_i + \frac{\bar{U}_{11}}{U_0} \right) (1 - w_i) \right),$$

and obtain from (3.5) and (3.20) that  $\Phi_{2i}$  satisfies

$$(3.22a) \quad \Delta_{\mathbf{y}} \Phi_{2i} = (2y_3 w_{i,y_3 y_3} + 2w_{i,y_3}), \quad \mathbf{y} \in \mathbb{R}_+^3,$$

$$(3.22b) \quad -\partial_{y_3} \Phi_{2i} + \kappa_i \Phi_{2i} = 0, \quad y_3 = 0, (y_1, y_2) \in \Gamma_i,$$

$$(3.22c) \quad \partial_{y_3} \Phi_{2i} = 0, \quad y_3 = 0, (y_1, y_2) \notin \Gamma_i,$$

$$(3.22d) \quad \Phi_{2i} \sim \frac{C_i}{2} \left( \frac{y_3(y_1^2 + y_2^2)}{|\mathbf{y}|^3} - \log(y_3 + |\mathbf{y}|) \right), \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

From the analysis of the solution to (3.22) given in Appendix C, we identify the monopole coefficient  $E_i = E_i(\kappa_i)$  from the following refined far-field behavior:

$$(3.23) \quad \Phi_{2i} - \frac{C_i}{2} \left( \frac{y_3(y_1^2 + y_2^2)}{|\mathbf{y}|^3} - \log(y_3 + |\mathbf{y}|) \right) \sim -\frac{E_i}{|\mathbf{y}|}, \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

For an arbitrary patch shape  $\Gamma_i$ , the determination of  $E_i(\kappa_i)$  is reduced to quadrature in (C.3) of Appendix C. We recall that some properties of  $E_i(\kappa_i)$  for circular patches were summarized in Lemma 2.2 of §2.3, with a highly accurate but heuristic formula for  $E_i = E_i(\kappa_i)$  given in (2.17).

To determine  $\bar{U}_{11}$ , we will impose a solvability condition for the problem for the outer correction  $U_3$  in (3.2). To derive the problem for  $U_3$ , we substitute (3.23) into (3.21) and use  $w_i \sim C_i/|\mathbf{y}|$  as  $|\mathbf{y}| \rightarrow \infty$ . We conclude that  $V_{2i}$  satisfies the refined far-field behavior

$$(3.24) \quad V_{2i} \sim \beta_i U_0 + \bar{U}_{11} + \frac{U_0 C_i}{2} \left( \frac{y_3(y_1^2 + y_2^2)}{|\mathbf{y}|^3} - \log(y_3 + |\mathbf{y}|) \right) - (E_i U_0 + (\beta_i U_0 + \bar{U}_{11}) C_i) \frac{1}{|\mathbf{y}|}, \quad \text{as } |\mathbf{y}| \rightarrow \infty.$$

The monopole term in (3.24), given by the coefficient of  $1/|\mathbf{y}|$ , is one of the two terms that needs to be accounted for by  $U_3$  in the matching condition (3.12). The second term is the dipole term in (3.12), which arises from (2.3d). This term is written in terms of outer variables using (A.8) of Appendix A.

In this way, we conclude from (3.3), (3.12) and (3.24) that  $U_3$  must satisfy

$$(3.25a) \quad \Delta_{\mathbf{x}} U_3 = 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Omega; \quad \partial_n U_3 = 0, \quad \mathbf{x} \in \partial\Omega \setminus \{\mathbf{x}_1, \dots, \mathbf{x}_N\},$$

$$(3.25b) \quad U_3 \sim -\frac{[U_0 E_i + (\beta_i U_0 + \bar{U}_{11}) C_i]}{|\mathbf{x} - \mathbf{x}_i|} - U_0 \frac{\mathbf{p}_i \cdot \mathcal{Q}_i^T(\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3} \quad \text{as } \mathbf{x} \rightarrow \mathbf{x}_i \in \partial\Omega, \quad i = 1, \dots, N,$$

$$(3.25c) \quad \lim_{R \rightarrow \infty} \int_{\partial\Omega_R} \partial_r U_3|_{r=R} ds = 0.$$

Here  $\mathbf{p}_i$  is the dipole vector in (2.3d), while the orthogonal matrix  $\mathcal{Q}_i$  is defined in (A.8) in terms of the basis vectors of the geodesic coordinate system.

By applying the divergence theorem, we find that (3.25) is solvable only when

$$(3.26) \quad \frac{\bar{U}_{11}}{U_0} \bar{C} + \sum_{j=1}^N \beta_j C_j + \bar{E} = 0, \quad \text{where} \quad \bar{E} \equiv \sum_{j=1}^N E_j.$$

We remark that the contribution to the solvability condition from the dipole term vanishes identically by symmetry owing to the fact that  $\mathbf{p}_i$  has the form  $\mathbf{p}_i = (p_{1i}, p_{2i}, 0)^T$ . Upon recalling (3.13) for  $\beta_i$ , we solve (3.26) for  $\bar{U}_{11}$  to obtain

$$(3.27) \quad \frac{\bar{U}_{11}}{U_0} = \frac{2\pi}{\bar{C}} \mathbf{C}^T \mathcal{G}_s \mathbf{C} - \frac{\bar{E}}{\bar{C}} \rightarrow \bar{U}_{11} = \frac{2}{(\bar{C})^2} (\bar{E} - 2\pi \mathbf{C}^T \mathcal{G}_s \mathbf{C}).$$

Finally, to determine an expression for the capacitance  $C_T$  in (3.1d), we must take the limit as  $|\mathbf{x}| \rightarrow \infty$  of our outer asymptotic expansion (3.2) and compare it with

the required limiting behavior (3.1d). By comparing the  $\mathcal{O}(1)$  terms in the resulting expression we obtain a three-term asymptotic expansion for  $C_T$ . We summarize our main result for  $C_T$  as follows.

PROPOSITION 1. *For  $\varepsilon \rightarrow 0$ , the capacitance  $C_T$  for the dimensionless problem (1.10) outside the unit sphere in the presence of  $N$  well-separated partially reactive Robin patches, centered at  $\mathbf{x}_i$  for  $i = 1, \dots, N$  and with local reactivities  $\kappa_i$  for  $i = 1, \dots, N$ , has the three-term asymptotics*

$$(3.28a) \quad \frac{1}{C_T} \sim \frac{|U_0|}{\varepsilon} \left[ 1 + \varepsilon \frac{\bar{U}_{10}}{U_0} \log\left(\frac{\varepsilon}{2}\right) + \varepsilon \frac{\bar{U}_{11}}{U_0} + \mathcal{O}\left(\varepsilon^2 [\log(\varepsilon/2)]^2\right) \right],$$

where

$$(3.28b) \quad |U_0| = \frac{2}{\bar{C}}, \quad \frac{\bar{U}_{10}}{U_0} = \frac{1}{2\bar{C}} \sum_{i=1}^N C_i^2, \quad \frac{\bar{U}_{11}}{U_0} = \frac{2\pi}{\bar{C}} \mathbf{C}^T \mathcal{G}_s \mathbf{C} - \frac{\bar{E}}{\bar{C}}.$$

Here  $\bar{C} = \sum_{i=1}^N C_i$  and  $\bar{E} = \sum_{i=1}^N E_i$ . Properties of the reactive capacitance  $C_i = C_i(\kappa_i)$  and the monopole coefficient  $E_i = E_i(\kappa_i)$ , as defined by the far-field behaviors in (2.3d) and (2.11d), respectively, were summarized in Lemmas 2.1 and 2.2. The Green's matrix  $\mathcal{G}_s$  in (3.28b) is given by (3.14). In terms of the dimensional capacitance  $C_T$ , defined in (1.1d), we use (1.9) to obtain for a sphere of radius  $R$  and with a collection of partially reactive patches of maximum diameter  $L$  that

$$(3.29) \quad \mathcal{C}_T = R C_T.$$

Here in calculating  $C_T$  from (3.28) we use  $C_i = C_i(L\kappa_i/D)$  and  $E_i = E_i(L\kappa_i/D)$  and we evaluate the Green's matrix  $\mathcal{G}_s$  at  $\mathbf{x}_i = \mathbf{X}_i/R$ .

We remark that for the special case of  $N$  locally circular patches with perfect reactivity ( $\kappa_i = \infty$ ), for which  $C_i = 2a_i/\pi$  and  $E_i = -2a_i^2(\log a_i + \log 4 - 3/2)/\pi^2$ , our three-term asymptotic result (3.28) for  $1/C_T$  agrees with that in Principal Result 3.1 of [41]. We emphasize that (3.28) extends the previous result of [41] to arbitrary-shaped patches and to finite reactivity  $\kappa_i > 0$ . Moreover, we remark that our three-term asymptotic result remains well-ordered in  $\varepsilon$  over the full range of reactivities. As a result, we can explore the limits  $\kappa_i \ll 1$  and  $\kappa_i \gg 1$  directly from (3.28).

There are a few additional key features of our main result for  $C_T$ . Firstly, since  $\bar{U}_{11}$  depends on the Green's matrix  $\mathcal{G}_s$ , we conclude that only the third-order term in the asymptotic expansion of  $C_T$  depends on the spatial configuration  $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$  of the centers of the reactive patches on the surface of the unit sphere. It is this term that incorporates spatial effects from the diffusive interactions between patches. Secondly, since the leading-order term in (3.28) does not depend on the details of the Green's function, it is valid for an arbitrary bounded domain  $\mathcal{B}$  with smooth boundary. By retaining only the leading-order terms in our three-term main result (3.28), we obtain

$$(3.30) \quad \mathcal{C}_T \approx \sum_{i=1}^N \frac{L}{2} C_i(L\kappa_i/D).$$

This shows that the capacitance of the structured target is, to leading order, the sum of reactive capacitances  $LC_i/2$  of individual patches, in analogy to a parallel connection of capacitors in electrostatics. The factor  $1/2$  accounts for the fact that only one side of the patch located on the boundary is accessible to Brownian particles.

As in the Berg-Purcell result, this leading-order term does not account for diffusive interactions between the patches (which appears only in the third-order term), nor for the curvature of the boundary. Thirdly, to numerically evaluate the asymptotic result (3.28), we need only compute  $C_i(\kappa_i)$  from the Steklov eigenfunction expansion (2.5) and  $E_i(\kappa_i)$  from the quadrature in (C.3) of Appendix C. When the Robin patches are all locally circular, by using the heuristic approximations given in (2.9) and (2.17) for  $C_i$  and  $E_i$ , respectively, we obtain an explicit expression for  $C_T$  valid for all  $\kappa_i > 0$ .

**4. The Effective Capacitance and Effective Reactivity.** We now derive a new scaling law for the effective capacitance  $C_{\text{eff}}$  and the effective reactivity  $k_{\text{eff}}$  for (1.10) applicable to a large number of uniformly distributed identical circular patches of a common radius  $\varepsilon$  and reactivity  $\kappa$ . Our result applies to the low patch area fraction limit where  $f \equiv N\pi\varepsilon^2/|\partial\Omega| = N\varepsilon^2/4 \ll 1$ . A similar result was given in [41] for perfectly reactive patches, and in [10] for the mean first-passage time (MFPT) narrow capture problem within a sphere with small surface patches of finite reactivity.

For the case of  $N$  identical circular patches, we set  $a_i = 1$  for  $i = 1, \dots, N$  in (3.28) to obtain that our three-term expansion becomes

$$(4.1) \quad \frac{1}{C_T} \sim \frac{2}{NC\varepsilon} \left[ 1 + \frac{\varepsilon C}{2} \log\left(\frac{\varepsilon}{2}\right) + \varepsilon C \left( \frac{2}{N} \mathcal{H}(\mathbf{x}_1, \dots, \mathbf{x}_N) - \frac{E}{C^2} \right) \right].$$

Here  $C = C(\kappa)$  is the common reactive capacitance of each patch, defined by the far-field behavior of the solution  $w = w(\mathbf{y}; \kappa)$  to (2.3), where  $\Gamma$  is the unit disk and where we omit the subscript  $i$ . In addition, the monopole coefficient  $E = E(\kappa)$  in (4.1) is given by (2.13) in which we set  $a_i = 1$  and omit the subscript  $i$ . In (4.1), the discrete energy  $\mathcal{H}$ , representing inter-patch interactions from the Green's matrix, is

$$(4.2) \quad \mathcal{H}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{i=1}^N \sum_{j=i+1}^N \left( \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} - \frac{1}{2} \log \left( 1 + \frac{2}{|\mathbf{x}_i - \mathbf{x}_j|} \right) \right).$$

As discussed in [41] (see also analogous results in Appendix of [10] and [9] for the MFPT narrow capture problem), for a large collection of uniformly distributed patches on the surface of the sphere with centers at  $\mathbf{x}_i$  for  $i = 1, \dots, N$  we have

$$(4.3) \quad \mathcal{H} = \frac{N^2}{4} - d_1 N^{3/2} + \frac{1}{8} N \log N + d_2 N + d_3 N^{1/2} + \mathcal{O}(\log N), \quad \text{for } N \gg 1,$$

with  $d_1 \approx 0.55230$ ,  $d_2 = 1/8$  and  $d_3 = 1/4$ . Defining  $\beta = \beta(\kappa) \equiv e^{-2E/C^2} e^{4d_2}$ , the effective capacitance  $C_{\text{eff}}$  is obtained by substituting (4.3) into (4.1), which yields

$$(4.4) \quad \frac{1}{C_{\text{eff}}} \sim \frac{2}{NC\varepsilon} + 1 + \frac{1}{N} \log \left( \frac{\varepsilon\sqrt{N}}{2} \beta \right) - \frac{4d_1}{N^{1/2}} + \frac{4d_3}{N^{3/2}} + \mathcal{O} \left( \frac{\log N}{N} \right).$$

Next, we write (4.4) in terms of the patch area fraction  $f \equiv N\varepsilon^2/4$ . Since for the well-ordering of our asymptotic expansion the first term in (4.4) must be the dominant term, we require that  $N \ll \mathcal{O}(\varepsilon^{-1})$ . This enforces that our result will apply to the small patch area fraction limit  $f \ll 1$ . Upon eliminating  $N$  in (4.4) using  $N = 4f/\varepsilon^2$ , we obtain our main result for the dimensionless effective capacitance:

$$(4.5a) \quad \frac{1}{C_{\text{eff}}} \sim 1 + \frac{\varepsilon}{2Cf} \left[ 1 - 4d_1 C \sqrt{f} + \frac{\varepsilon C}{2} \log(\beta \sqrt{f}) + \frac{d_3 C \varepsilon^2}{\sqrt{f}} \right],$$



where

$$(4.5b) \quad f = \frac{\varepsilon^2 N}{4} \ll 1, \quad \beta = e^{-2E/C^2} e^{4d_2}, \quad d_1 \approx 0.5523, \quad d_2 = \frac{1}{8}, \quad d_3 = \frac{1}{4}.$$

The dependence of  $C_{\text{eff}}$  on the reactivity  $\kappa$  is inherited through  $C(\kappa)$  and  $\beta(\kappa)$ .

To determine the effective reactivity, we use the dimensionless form of (1.4) given by  $1/k_{\text{eff}} = -1 + 1/C_{\text{eff}}$ . In this way, by using (4.5a) for  $C_{\text{eff}}$  in this result, we obtain a scaling law for the dimensionless effective reactivity given by

$$(4.6) \quad k_{\text{eff}} \sim \frac{2Cf}{\varepsilon} \left[ 1 - 4d_1 C \sqrt{f} + \frac{\varepsilon C}{2} \log(\beta \sqrt{f}) + \frac{d_3 C \varepsilon^2}{\sqrt{f}} \right]^{-1}.$$

Setting  $d_3 = 1/4$ , we can also rewrite this expression in an alternative form as

$$(4.7) \quad k_{\text{eff}} \sim \frac{N \varepsilon C(\kappa)}{2} \left[ 1 - \sqrt{N} \varepsilon C(\kappa) \left( 2d_1 - \frac{1}{2N} - \frac{1}{2\sqrt{N}} \log(\beta(\kappa) \varepsilon \sqrt{N}/2) \right) \right]^{-1}.$$

Even though (4.6) or (4.7) can be formally re-expanded by using the binomial approximation, the resulting expressions turn out to be less accurate than (4.6) or (4.7) when  $\varepsilon$  is not too small, as witnessed by comparison with Monte Carlo simulations. For this reason, we avoid using such re-expansions and retain (4.6) or (4.7) in the following numerical illustrations.

Our main results for  $C_{\text{eff}}$  and  $k_{\text{eff}}$  are determined in terms of both  $C(\kappa)$  and  $E(\kappa)$ . For  $\kappa \ll 1$ , we set  $a_i = 1$  in (2.10b) and (2.16b) to obtain

$$(4.8) \quad C(\kappa) \sim \frac{\kappa}{2} - \frac{4}{3\pi} \kappa^2 + 0.3651 \kappa^3, \quad E(\kappa) \sim \frac{\kappa^2}{32} \quad \text{as } \kappa \rightarrow 0.$$

To the leading order in  $\kappa$ , this yields  $E/C^2 \sim 1/8$  so that  $\beta \sim e^{1/4} \approx 1.2840$  as  $\kappa \rightarrow 0$ . In contrast, in the limit  $\kappa \rightarrow \infty$  of large reactivity, we have from setting  $a_i = 1$  in (2.10a) and (2.16a) that

$$(4.9) \quad C \sim 2/\pi, \quad E \sim (3 - 4 \log 2)/\pi^2 \quad \text{as } \kappa \rightarrow \infty.$$

This gives  $E/C^2 \sim (3 - 4 \log 2)/4$ , which yields that  $\beta \sim 4/e \approx 1.4715$  as  $\kappa \rightarrow \infty$  in (4.6). In this way, we recover the scaling law derived in §4 of [41] for the effective reactivity for the case of perfectly reactive patches.

In summary, the results (4.5), (4.6) and (4.7) determine the dimensionless effective capacitance and effective reactivity over the full range  $\kappa > 0$  of reactivity of a large collection of identical uniformly distributed circular patches of a common radius  $\varepsilon$ . By setting  $a_i = 1$  in the heuristic approximations for  $C$  and  $E$  in (2.9) and (2.17), we obtain explicit analytical results for these two effective parameters for all  $\kappa > 0$ .

Next, we deduce the corresponding homogenized result for the dimensional problem (1.1) in a sphere of radius  $R$  and with circular patches of a common radius  $L \ll R$  and common dimensional reactivity  $\mathcal{K}_i = \mathcal{K}$  for  $i = 1, \dots, N$ . Upon recalling the scaling relations (1.9), we have from (4.6) that

$$(4.10) \quad \mathcal{K}_{\text{eff}} = \frac{D}{L} k_{\text{eff}} = \frac{D}{\varepsilon R} k_{\text{eff}},$$

where  $C = C(LK/D)$  and  $E = E(LK/D)$ . The dimension of  $\mathcal{K}_{\text{eff}}$  is length/time.

Finally, in order to compare with the heuristic approximation (1.8) of [2], we consider our leading-order theory from (4.6) where we use  $k_{\text{eff}} \sim 2C/f$  together with  $L = \varepsilon R$  for the patch radius and (2.9) to approximate  $C(\kappa)$ . This yields that

$$(4.11) \quad \mathcal{K}_{\text{eff}} = \frac{1}{\varepsilon} \left( \frac{2fD}{\varepsilon R} \right) \mathcal{C} \left( \frac{\mathcal{K}\varepsilon}{D} \right), \quad \text{where} \quad \mathcal{C}(\mu) \equiv \frac{2\mu/\pi}{\mu + 4/\pi},$$

with the extra pre-factor of  $1/\varepsilon$  resulting from the fact that we homogenized  $\varepsilon \partial_n U + \kappa_i U = 0$  in (3.1c) rather than  $\partial_n U + \kappa_i U = 0$ . We conclude that our leading-order theory exactly reproduces the heuristic formula (1.8) of [2], which was first derived theoretically in [44] as was discussed in §1.

**5. Monte Carlo simulations.** To assess the accuracy and the range of validity of our asymptotic results, we resort to a numerical calculation of the capacitance  $\mathcal{C}_T$ . For this purpose, one can generate random trajectories of diffusing particles and determine the flux  $J$ , from which the effective reactivity  $\mathcal{K}_{\text{eff}}$  and the capacitance  $\mathcal{C}_T$  follow from (1.4). In the steady-state regime, any trajectory terminates either by reaction on a patch, or by escape to infinity (due to the transient character of diffusion in three dimensions), and the fraction of reacted particles is proportional to the flux (see below). Different numerical schemes were proposed to undertake such Monte Carlo simulations, including variable-jump techniques based on the walk-on-spheres algorithm [58, 48, 57]. However, for all of these approaches, modeling reflections on an inert boundary is the most difficult and time-consuming part of the computation. We now propose an alternative, highly efficient Monte Carlo algorithm, which relies on the rotational symmetry of the spherical domain.

**5.1. Efficient algorithm for a sphere.** Let us introduce a spherical surface of radius  $R + a$ ,  $\partial \mathcal{B}^a = \{\mathbf{X} \in \mathbb{R}^3 \mid |\mathbf{X}| = R + a\}$ , and define the sequence of stopping times for a random trajectory  $\mathbf{X}_t$  of a diffusing particle as

$$(5.1) \quad \tau_{2j+1} = \inf\{t > \tau_{2j} : \mathbf{X}_t \in \partial \mathcal{B}\}, \quad \tau_{2j} = \inf\{t > \tau_{2j-1} : \mathbf{X}_t \in \partial \mathcal{B}^a\},$$

with  $j = 1, 2, \dots$ , and  $\tau_0 \equiv 0$ . In other words,  $\tau_1$  is the first-passage time to the sphere  $\partial \mathcal{B}$ ,  $\tau_2$  is the first instance when  $\mathbf{X}_t$  crosses  $\partial \mathcal{B}^a$  after  $\tau_1$ ,  $\tau_3$  is the first instance of the return to the sphere after  $\tau_2$ , etc. The stopping times  $\tau_j$  partition the trajectory into consecutive *independent* paths:  $\mathbf{X}_0 \rightsquigarrow \mathbf{X}_{\tau_1}$ ,  $\mathbf{X}_{\tau_1} \rightsquigarrow \mathbf{X}_{\tau_2}$ , etc. If the particle escapes to infinity (and thus never returns to  $\partial \mathcal{B}$ ) or reacts on  $\partial \mathcal{B}$  during the  $j$ -th path, all stopping times  $\tau_i$  with  $i \geq j$  are set to  $+\infty$ .

(i) The first path consists in the first arrival onto the sphere from a given starting point  $\mathbf{X}_0$ . Instead of simulating the random trajectory over  $t \in (0, \tau_1)$ , one can generate the first arrival point  $\mathbf{X}_{\tau_1} = \hat{\mathbf{X}}_1$  on the sphere according to the harmonic measure density, which is given explicitly for the exterior of a sphere of radius  $R$  by

$$(5.2) \quad \omega_{\mathbf{X}_0}(\mathbf{X}_1) = \frac{|\mathbf{X}_0|^2 - R^2}{4\pi R |\mathbf{X}_1 - \mathbf{X}_0|^3}.$$

Let us introduce the local spherical coordinates with the north pole oriented along  $\mathbf{X}_0$  such that  $\mathbf{X}'_0 = (r_0, 0, 0)$ , while  $\mathbf{X}'_1 = (R, \theta, \phi)$  is determined by the angles  $\theta$  and  $\phi$ , where the prime indicate the local coordinates. By symmetry, the azimuthal angle  $\phi$  is uniformly distributed over  $(0, 2\pi)$ . In turn, the probability density of the polar angle  $\theta \in (0, \pi)$  follows from (5.2) as

$$(5.3) \quad \omega_\rho(\theta) = \frac{\rho^2 - 1}{2(1 - 2\rho \cos \theta + \rho^2)^{3/2}} \sin \theta,$$

where  $\rho = r_0/R > 1$ . Here we multiplied (5.2) by  $2\pi \sin \theta$  to account for the Jacobian and for the already generated azimuthal angle. The integral of this density yields the cumulative distribution function

$$(5.4) \quad F_\rho(\theta) = \int_0^\theta \omega_\rho(\theta') d\theta' = \frac{\rho+1}{2\rho} \left( 1 - \frac{\rho-1}{\sqrt{1-2\rho \cos \theta + \rho^2}} \right).$$

One easily checks that  $F_\rho(0) = 0$  and  $F_\rho(\pi) = 1/\rho < 1$ . As expected, the probability densities in (5.2) and (5.3) are not normalized to unity owing to the transient character of diffusion in three dimensions. In fact, the particle can escape to infinity with probability  $1 - 1/\rho = 1 - R/r_0$ . To account for this possibility, one generates a random variable  $\eta$  with uniform distribution on  $(0, 1)$ . If  $\eta > 1/\rho$ , the simulation stops due to escape to infinity. Otherwise, one sets  $\eta = F_\rho(\theta)$  and inverts the relation (5.4) to generate the angle  $\theta$  of the arrival point as

$$(5.5) \quad \theta = \arccos \left( 1 - \frac{(\rho-1)^2}{2\rho} \left( \frac{1}{(1 - \frac{2\rho}{1+\rho}\eta)^2} - 1 \right) \right).$$

In this way, one generates the random point  $\hat{\mathbf{X}}'_1 = (R, \theta, \phi)$  on the sphere in the local spherical coordinates. To complete this step, it remains to transform these local coordinates into the global coordinates to get  $\mathbf{X}_{\tau_1} = \hat{\mathbf{X}}_1$ . We emphasize that in this step a long simulation of the random path  $\mathbf{X}_0 \rightsquigarrow \mathbf{X}_{\tau_1}$  is replaced by the generation of two random variables  $\phi$  and  $\theta$  with explicitly known distributions.

(ii) The next path is diffusion near a partially reactive sphere until hitting the surface  $\partial\mathcal{B}^a$  (or reacting on  $\partial\mathcal{B}$ ). This is the most time-consuming and delicate step for common Monte Carlo techniques, which use a very small timestep to approximate the random path  $\mathbf{X}_{\tau_1} \rightsquigarrow \mathbf{X}_{\tau_2}$  by a sequence of small random jumps, with eventual reflections/reactions on the boundary. Following the concept of the *escape-from-a-layer* approach [57], we aim at replacing such a detailed and time-consuming simulation by a single escape event. Let us first consider the case when the reactivity is homogeneous. The extension to the heterogeneous case is discussed below. With a homogeneous reactivity, the splitting probability  $\Pi(r)$  of the escape event can be easily found by solving the Laplace equation in a spherical layer  $\Omega^a = \{\mathbf{X} \in \mathbb{R}^3 \mid R < |\mathbf{X}| < R+a\}$ , with Dirichlet boundary condition at  $r = R+a$  and Robin boundary condition at  $r = R$ , yielding

$$(5.6) \quad \Pi(r) = 1 - \frac{\mathcal{K}R/D}{1 + \frac{\mathcal{K}a/D}{1+a/R}} \left( \frac{R}{r} - \frac{R}{R+a} \right) \quad \text{for } R \leq r \leq R+a.$$

For the starting point on the reactive sphere, one has

$$(5.7) \quad \Pi(R) = \left( 1 + \frac{a\mathcal{K}/D}{1+a/R} \right)^{-1}.$$

In other words, one can generate a uniform random variable  $\eta \in (0, 1)$  to choose between two possible outcomes: (a) if  $\eta > \Pi(R)$ , the particle reacts on  $\partial\mathcal{B}$  before escaping to  $\partial\mathcal{B}^a$ , and the simulation stops; or (b) if  $\eta < \Pi(R)$ , the particle hits  $\partial\mathcal{B}^a$  before reacting on  $\partial\mathcal{B}$ . To complete this step, one needs to generate the random escape position  $\mathbf{X}_{\tau_2} = \hat{\mathbf{X}}_2$  on  $\partial\mathcal{B}^a$ . Even though the conditional distribution of  $\hat{\mathbf{X}}_2$  can be derived and implemented, we resort to a much simpler approximation, which

consists in replacing the random variable  $\hat{\mathbf{X}}_2$  by its mean value, which by symmetry is simply  $\hat{\mathbf{X}}_1(R+a)/R$  (i.e., the previous hitting point  $\hat{\mathbf{X}}_1$  on the sphere  $\partial\mathcal{B}$  is lifted to  $\partial\mathcal{B}^a$  by rescaling). In fact, it is unlikely for a particle to diffuse far away from the point  $\hat{\mathbf{X}}_1$  in a thin reactive layer  $\Omega^a$  so that the distribution of  $\hat{\mathbf{X}}_2$  is peaked around  $\hat{\mathbf{X}}_1(R+a)/a$  when  $a/R \ll 1$ . We conclude that the most time-consuming simulation of the path  $\mathbf{X}_{\tau_1} \rightsquigarrow \mathbf{X}_{\tau_2}$  is replaced by the generation of the uniform random variable  $\eta$ .

(iii) From the point  $\hat{\mathbf{X}}_2$ , the particle resumes its diffusion until its arrival onto  $\partial\mathcal{B}$ , or escape to infinity. The simulation of the next path  $\mathbf{X}_{\tau_2} \rightsquigarrow \mathbf{X}_{\tau_3}$  can thus be substituted by the generation of the arrival point  $\mathbf{X}_{\tau_3} = \hat{\mathbf{X}}_3$  according to the harmonic measure density, as described in step (i). After the arrival onto the boundary  $\partial\mathcal{B}$ , the particle needs to escape from a thin layer of width  $a$ , as described in step (ii), and so on. As a consequence, the detailed simulation of the random trajectory  $\mathbf{X}_t$  is replaced by a succession of steps (i) and (ii), that sample a sequence of random points  $\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \hat{\mathbf{X}}_3, \dots$  of that trajectory. The simulation of the trajectory is stopped when either the particle escapes to infinity, corresponding to a step with some odd index  $2j-1$ , or reacts on the sphere during a step with some even index  $2j$ . Repeating such a simulation  $M$  times, one can approximate the probability of reaction on the sphere as  $P_{\text{react}}(\mathbf{X}_0) \approx M_{\text{react}}/M$ , where  $M_{\text{react}}$  is the number of simulated trajectories that terminated by reaction. Finally, the latter probability determines the flux of particles started from  $\mathbf{X}_0$  as  $J(\mathbf{X}_0) = J_{\text{Smol}} P_{\text{react}}(\mathbf{X}_0)$ .

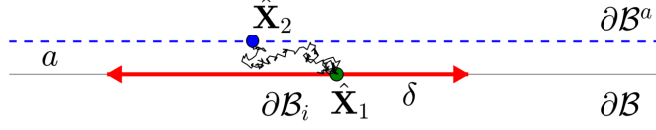
Since the steady-state flux  $J$  from (1.3) corresponds to a source at infinity, in the sense that  $\mathcal{U}(\mathbf{X}) \rightarrow \mathcal{U}_{\text{inf}}$  as  $|\mathbf{X}| \rightarrow \infty$ , one would formally need to start simulations with  $\mathbf{X}_0$  at infinity. However, the rotational symmetry of the sphere implies that the first arrival point  $\hat{\mathbf{X}}_1$  is distributed uniformly on the sphere so that one can simply replace the very first step (the path from  $\mathbf{X}_0$  to  $\hat{\mathbf{X}}_1$ ) by the generation of the uniform point  $\hat{\mathbf{X}}_1$  on  $\partial\mathcal{B}$ . Keeping the remaining steps unchanged, one can thus determine the probability of reaction  $P_{\text{react}}(\circ)$  with the uniform starting point, from which  $J = J_{\text{Smol}} P_{\text{react}}(\circ)$ . Substituting this expression into (1.4), one can directly access the effective reactivity:

$$(5.8) \quad \frac{R}{D} \mathcal{K}_{\text{eff}} = \left( \frac{1}{P_{\text{react}}(\circ)} - 1 \right)^{-1}.$$

In the present form, the proposed algorithm is formally limited to the case of homogeneous reactivity, for which the solution is already known. In the heterogeneous case, the reactivity is a piecewise constant function on the spherical boundary, which can be written in the form

$$(5.9) \quad \mathcal{K}(\mathbf{X}) = \sum_{i=1}^N \mathcal{K}_i \mathcal{I}_{\partial\mathcal{B}_i}(\mathbf{X}) \quad \text{for } \mathbf{X} \in \partial\mathcal{B},$$

where  $\mathcal{I}_{\partial\mathcal{B}_i}(\mathbf{X})$  is the indicator function of the  $i$ -th patch  $\partial\mathcal{B}_i$  with reactivity  $\mathcal{K}_i$ . As a consequence, the escape probability from a thin layer depends on the spatial arrangement of patches and on their reactivities, and (5.7) is in general not applicable. However, if the layer width  $a$  is much smaller than the patch sizes, one can still employ (5.7) by setting  $\mathcal{K}$  to be the reactivity at the arrival point  $\hat{\mathbf{X}}_1 \in \partial\mathcal{B}$  (or at  $\hat{\mathbf{X}}_{2j-1} \in \partial\mathcal{B}$  for other paths). For instance, let us assume that  $\hat{\mathbf{X}}_1 \in \partial\mathcal{B}_i$ , and we denote  $\delta \equiv |\hat{\mathbf{X}}_1 - \partial_e\mathcal{B}_i|$  to be the distance between  $\hat{\mathbf{X}}_1$  and the boundary  $\partial_e\mathcal{B}_i$  of the patch  $\partial\mathcal{B}_i$  (see Fig. 5.1). If  $\delta \gg a$ , the probability of diffusing inside the reactive



**Fig. 5.1:** Schematic two-dimensional illustration of a thin layer of width  $a$  near the boundary  $\partial B$  with a reactive patch  $\partial B_i$  (thick red interval). A random path  $\hat{X}_1 = \mathbf{X}_{\tau_1} \rightsquigarrow \mathbf{X}_{\tau_2} = \hat{X}_2$  from the first arrival point  $\hat{X}_1$  to the escape point  $\hat{X}_2$  on the surface  $\partial B^a$  is shown. Two arrows indicate the boundary  $\partial_e B_i$  of the patch  $\partial B_i$  (in three dimensions,  $\partial_e B_i$  is a curve but here it is reduced to two endpoints of the shown interval).

thin layer  $B_a$  from  $\hat{X}_1$  to points farther than the distance  $\delta$  is exponentially small, of the order  $\mathcal{O}(e^{-\delta/a})$ . In other words, since it is unlikely that the particle may be exposed to a reactivity other than  $\mathcal{K}_i$ , the formula (5.7) can be used, and the algorithm described above applies. Similarly, if  $\hat{X}_1$  does not belong to any reactive patch, the particle diffuses near an inert reflecting boundary with reactivity  $\mathcal{K} = 0$ , and the escape probability is equal to 1. The only difficult case is the situation when the arrival point  $\hat{X}_1$  occurs in the very close vicinity of the boundary  $\partial_e B_i$  of a reactive patch. In this case, (5.7) is not valid because the particle may encounter the boundary on either side of  $\partial_e B_n$  that would alter  $\Pi(R)$ . However, since the relative surface area of the regions near the boundary of the patches, estimated as  $2a(|\partial_e B_1| + \dots + |\partial_e B_N|)/(4\pi R^2) \propto a\varepsilon N/R^2$ , is small, the contribution of inaccurate estimations of the escape probability via (5.7) is expected to be small as well. This is actually confirmed by our numerical simulations, as discussed below. In summary, we still employ (5.7) for the heterogeneous case by setting  $\mathcal{K} = \mathcal{K}(\hat{X}_{2j-1})$ .

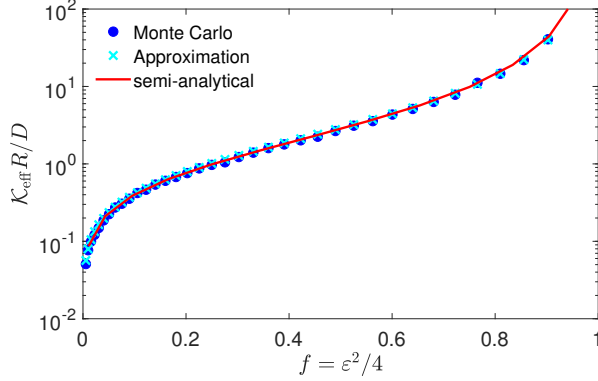
**5.2. Validation.** For most of our computations below and in §6 we set  $a = 10^{-2}$  and used  $M = 10^5$  realizations. We verified that a further decrease of  $a$  did not affect the numerical results. Although an increase of  $M$  would result in smaller statistical errors, given that the relative error is of the order of  $1/\sqrt{M}$ , our choice of  $M = 10^5$  was sufficient to provide good accuracy.

Our Monte Carlo algorithm has been validated in several settings, as described in (i)–(iii) below, such as when the solution is either known analytically or could be obtained numerically by other methods.

(i) When the reactivity is homogeneous, the probability of reaction  $P_{\text{react}}(\mathbf{X}_0)$  is known explicitly as  $P_{\text{react}}(\mathbf{X}_0) = \frac{R}{|\mathbf{X}_0|} \frac{1}{1+D/(\mathcal{K}R)}$ . This relation was used to validate the algorithm (results not shown). In addition, one can quantify the distribution of the reaction events on the sphere via the spread harmonic measure [23, 24]. In the spherical coordinates oriented such that the starting point is  $\mathbf{X}_0 = (r_0, 0, 0)$ , the random position  $\mathbf{X}_r = (R, \theta, \phi)$  of the reaction event is characterized by the uniformly distributed azimuthal angle  $\phi$  and the polar angle  $\theta$  obeying the following probability density in terms of the Legendre polynomials  $P_n(z)$ :

$$(5.10) \quad \omega_{r_0, \mathcal{K}}(\theta) = \sin \theta \sum_{n=0}^{\infty} P_n(\cos \theta) (R/r_0)^{n+1} \frac{n+1/2}{1+(n+1)D/(\mathcal{K}R)}.$$

In the limit  $\mathcal{K} = \infty$ , this spread harmonic measure density reduces to (5.3). The comparison of (5.10) to an empirical probability density obtained from Monte Carlo simulations served for validation purposes (results not shown).



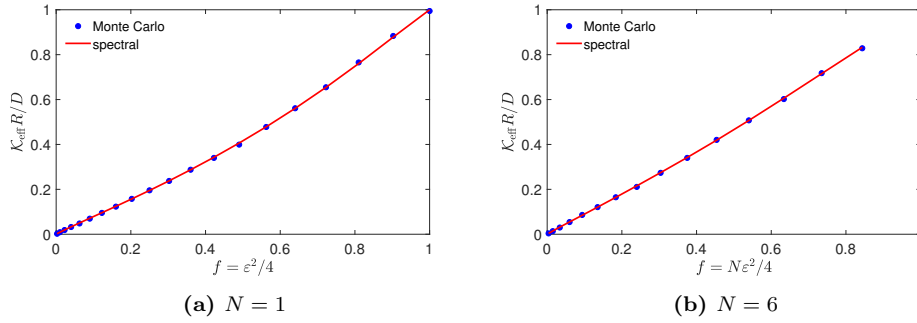
**Fig. 5.2:** Rescaled effective reactivity  $\mathcal{K}_{\text{eff}} R/D$  as a function of the patch surface fraction  $f = \varepsilon^2/4$  for a single absorbing patch of radius  $\varepsilon$  on the unit sphere (with  $\mathcal{K} = \infty$ ). Comparison between Monte Carlo simulations (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ), the empirical approximation (5.11), and semi-analytical solution from [54].

(ii) For a single perfectly reactive circular patch of radius  $\varepsilon R$  and infinite reactivity  $\mathcal{K} = \infty$ , the mixed BVP (1.1) can be reduced to dual infinite series relations that offers an efficient semi-analytical solution. In this way, Traytak [54] calculated the probability of reaction  $P_T$ , as given in the last column of Table 1 from [54], and compared its values for different patch sizes  $\varepsilon$  with those obtained by other numerical methods. We used his semi-analytical values to validate our Monte Carlo algorithm for perfectly reactive patches. Figure 5.2 shows an excellent agreement between our estimation of the effective reactivity  $\mathcal{K}_{\text{eff}}$  and that derived from Traytak’s solution, given by  $\frac{R}{D} \mathcal{K}_T = (1/P_T - 1)^{-1}$ , over the whole range  $0 < f < 1$  of patch area fraction  $f = \pi \varepsilon^2 R^2 / (4\pi R^2) = \varepsilon^2/4$ . Moreover, we also compare our results to an empirical approximation for the effective reactivity given in [27] by

$$(5.11) \quad \frac{R}{D} \mathcal{K}_G = \frac{2}{\pi} \sqrt{f} \frac{1 + 2.32\sqrt{f} - 1.47f^2}{(1-f)^{3/2}}.$$

(note that the original approximation from [14] failed to describe the reactivity in the limit  $f \rightarrow 1$ ). We observe from Fig. 5.2 that our Monte Carlo numerical results agree very closely with both Traytak’s semi-analytical solution and the empirical formula (5.11) over the entire range of  $f$ .

(iii) The case of partially reactive patches is least documented. For the validation of our algorithm, we compared our Monte Carlo results with those computed using the spectral approach presented in [25]. This spectral method allows one to accurately compute the flux  $J$  onto a sphere with a finite number of circular patches as  $J = J_{\text{Smol}} h_{00}^{(0)}$  (see Eq. (52) of [25]), where  $h_{00}^{(0)} = [(\mathbf{M} + \mathbf{K})^{-1} \mathbf{K}]_{00,00}$  is precisely the probability of reaction. Here  $\mathbf{M}$  and  $\mathbf{K}$  are two infinite-dimensional matrices that represent the Dirichlet-to-Neumann operator and the reactivity distribution, respectively. This representation is exact and valid for any bounded distribution of reactivity. A practical implementation of this representation requires truncating the matrices  $\mathbf{M}$  and  $\mathbf{K}$ , whose efficient construction for a sphere was provided in [25]. Note that higher truncation orders are needed to deal with larger reactivities and/or smaller patches. We used this numerical approach to validate our Monte Carlo simulations for  $N$  cir-

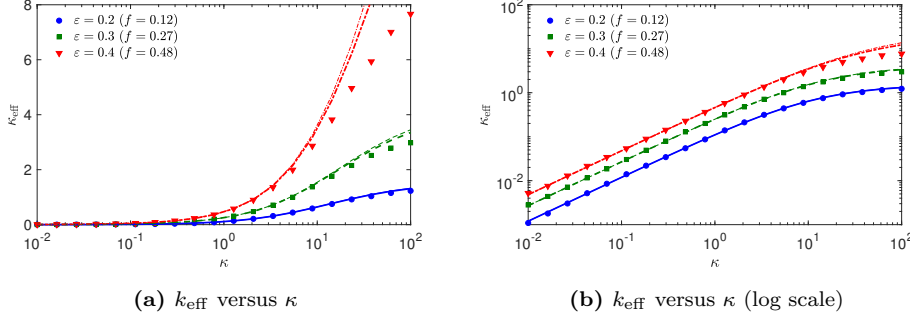


**Fig. 5.3:** Rescaled effective reactivity  $\mathcal{K}_{\text{eff}}R/D$  as a function of the patch surface fraction  $f = N\varepsilon^2/4$  for  $N$  identical partially reactive patches of radius  $\varepsilon$  on the unit sphere (with  $\mathcal{K}R/D = 1$ ). Comparison between Monte Carlo simulations (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ) and the spectral solution from [25], in which the matrices  $\mathbf{M}$  and  $\mathbf{K}$  were truncated to the size  $121 \times 121$ . (a)  $N = 1$ ,  $\varepsilon$  ranges from 0 to 2; (b)  $N = 6$ , with patches centered at the vertices of an octahedron, and  $\varepsilon$  ranges from 0 to 0.75.

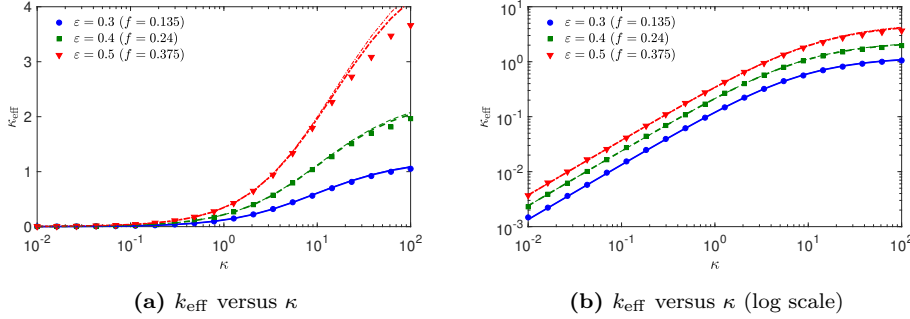
cular patches of size  $\varepsilon$  with reactivity  $R\mathcal{K}/D = 1$ . Fig. 5.3(a) presents the effective reactivity  $\mathcal{K}_{\text{eff}}$  of a single patch as a function of its surface fraction  $f$ . As expected, the effective reactivity tends to 1 as  $f \rightarrow 1$ , i.e., when the whole surface is covered by the reactive patch. In turn, Fig. 5.3(b) illustrates the effective reactivity of six identical partially reactive patches centered at the vertices of an octahedron. To avoid overlapping between patches, their common radius  $\varepsilon$  is limited to  $2 \sin(\pi/8) \approx 0.7654$  that yields the maximal covered fraction 0.88. In both cases, we observe an excellent agreement between Monte Carlo simulations and the spectral method, which provides a further benchmark for the accuracy of our Monte Carlo algorithm.

**6. Numerical Comparison.** We now employ the Monte Carlo algorithm of §5 to assess the accuracy and range of validity of our asymptotic results. We recall that the homogenized formula (4.7) (or equivalently (4.6)) for the effective reactivity was derived under the assumption of a large  $N \gg 1$  number of small, uniformly distributed, circular patches of radius  $\varepsilon$ , in the limit of small patch area fraction:  $f = \varepsilon^2 N/4 \ll 1$ . Therefore, our numerical experiments will seek to explore the following issues: (i) whether (4.7) remains accurate for moderately large  $\varepsilon$  and moderately small  $N$ ; (ii) does (4.7) hold on the full range  $\kappa > 0$  of reactivity, and (iii) how relevant is the assumption of equally spaced patches?

In Fig. 6.1 we plot the effective reactivity  $\kappa_{\text{eff}}$  as a function of  $\kappa$  for 12 patches centered at the vertices of an icosahedron on the unit sphere, and with three choices of the patch radius  $\varepsilon$ , that give different patch coverage fractions  $f$ . We compare Monte Carlo results (shown by symbols) to two asymptotic formulas: the general formula (1.4), in which the capacitance  $C_T$  is given in (4.1), and the homogenized formula (4.7). For patches of moderate size  $\varepsilon = 0.2$  that cover one-eighth of the spherical surface, both asymptotic formulas are in remarkable agreement with Monte Carlo simulations over the whole considered range of reactivities from  $10^{-2}$  to  $10^2$ . When  $\varepsilon = 0.3$ , corresponding to a one-quarter coverage, the agreement is still excellent for small and moderate reactivities, but small deviations appear at high reactivities. These deviations at large  $\kappa$  are further enhanced when  $\varepsilon = 0.4$ , which corresponds to a



**Fig. 6.1:** (a): Dimensionless effective reactivity  $k_{\text{eff}}$  versus  $\kappa = \varepsilon \mathcal{K}R/D$  for  $N = 12$  identical circular patches of radius  $\varepsilon$  centered at the vertices of an icosahedron. Symbols are the Monte Carlo results (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ), thick lines show (1.4) with  $C_T$  from the asymptotic formula (4.1), and thin lines are the homogenized asymptotic formula (4.7). (b): Same plot but with a logarithmic scale on the vertical axis to better show the small  $\kappa$  comparison.

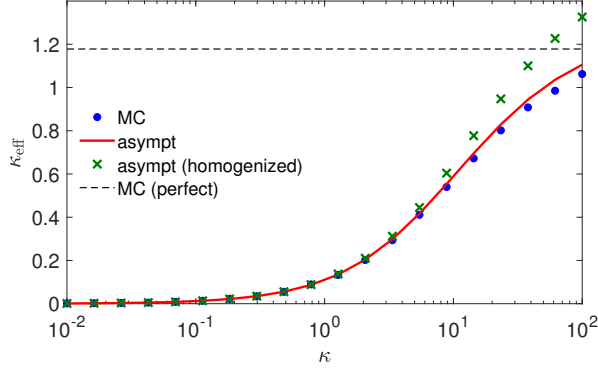


**Fig. 6.2:** (a): Dimensionless effective reactivity  $k_{\text{eff}}$  versus  $\kappa = \varepsilon \mathcal{K}R/D$  for  $N = 6$  identical circular patches of radius  $\varepsilon$  centered at the vertices of an octahedron. Symbols are the Monte Carlo results (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ), thick lines show (1.4) with  $C_T$  from the asymptotic formula (4.1), and thin lines are the homogenized asymptotic formula (4.7). (b): Same plot but with a logarithmic scale on the vertical axis to better show the small  $\kappa$  comparison.

one-half coverage of the sphere by patches. Despite these deviations, the asymptotic formula is still applicable for small and moderate reactivities. The origin of this remarkable agreement is revealed by the structure of the asymptotic formula (4.7), in which the patch size  $\varepsilon$  is multiplied by the reactive capacitance  $C(\kappa)$  (except for the logarithmic term). In other words, the actual small parameter is  $\varepsilon C(\kappa)$ , which vanishes in the limit  $\kappa \rightarrow 0$  since  $C(\kappa) \sim \kappa/2$  for  $\kappa \ll 1$ . As shown in Fig. 6.2, similar results are obtained for  $N = 6$  patches that are centered at the vertices of an octahedron.

In Fig. 6.3 we illustrate the effect of random locations of twelve identical circular patches of radius  $\varepsilon = 0.2$ . The general asymptotic formula (1.4) with (4.1) yields



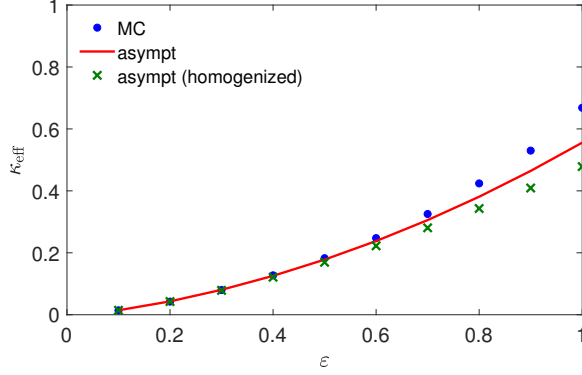


**Fig. 6.3:** Dimensionless effective reactivity  $k_{\text{eff}}$  versus  $\kappa = \varepsilon \mathcal{K} R / D$  for  $N = 12$  identical circular patches of radius  $\varepsilon = 0.2$ , that are centered at independent random points uniformly distributed on the sphere. Round symbols are the Monte Carlo results (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ), the thick line shows (1.4) with  $C_T$  from the asymptotic formula (4.1), and crosses are the homogenized asymptotic formula (4.7). The thin horizontal line is the Monte Carlo computed value for perfect patch reactivity.

accurate results over the whole considered range of reactivities. As the patches are not equally spaced, the homogenized asymptotic formula (4.7) is expectedly less accurate at high reactivities, but still provides a reasonable approximation at smaller reactivities.

Finally, in Fig. 6.4 we consider the extreme case  $N = 1$  of a single reactive patch and we plot the effective reactivity as a function of the patch radius  $\varepsilon$ , for a fixed moderately large reactivity  $R\mathcal{K}/D = 10$ . Even for large patches for which  $\varepsilon = 1$ , corresponding to a one-quarter coverage of the sphere, the general asymptotic formula (1.4) with (4.1), as well as the homogenized asymptotic formula (4.7), remain reasonably accurate. Their accuracy is further improved when  $\mathcal{K}$  is decreased.

**7. Discussion.** By using an asymptotic approach based on strong localized perturbation theory [56] we have derived a three-term asymptotic formula for the capacitance of a spherical target that has many small surface patches of finite reactivity on an otherwise reflecting boundary. This problem has broad applications in biophysical modeling, where a common theme in diverse applications is the study of how diffusing ligands can bind to surface receptors that have finite reactivity. Our analysis, relying on a non-traditional geodesic normal coordinate system, has extended the previous analysis in [41] for perfectly reactive circular patches on a sphere and the leading-order analysis in [44] for partially reactive circular patches on an infinite plane. Our three-term asymptotic result for the capacitance in the small patch radius limit incorporates the effect of inter-patch interactions, accounts for the curvature of the boundary of the sphere, and remains well-ordered over the full range of reactivities. By homogenizing our result for the capacitance we have derived a scaling law for the effective reactivity of the structured target that has identical, uniformly distributed, partially reactive small patches. The two coefficients, depending on the local reactivity and the shape of the patches, that appear in our scaling law can be calculated from Steklov eigenfunction expansions, while for circular patches they are well-approximated by heuristic approximations. Finally, our asymptotic results have



**Fig. 6.4:** Dimensionless effective reactivity  $k_{\text{eff}}$  versus  $\varepsilon$  for one circular patch of radius  $\varepsilon$  with reactivity  $RK/D = 10$ . Bullet symbols are Monte Carlo results (with  $M = 10^5$  realizations and  $a = 10^{-2}$ ), the thick line shows (1.4) with  $C_T$  from the asymptotic formula (4.1), and crosses are the homogenized asymptotic formula (4.7). The patch area fraction  $f = \varepsilon^2/4$  is 25% when  $\varepsilon = 1$ .

been validated for certain patch configurations by numerical results obtained from a new Monte Carlo algorithm. This comparison has suggested that the asymptotic theory can still be reliably used beyond its expected range of validity (small patches).

We remark that it is readily possible to extend the leading-order analysis in [44] by deriving a three-term expansion for the capacitance of an infinite plane that has small circular partially reactive patches on an otherwise reflecting boundary. For this simpler scenario, where boundary curvature plays no role, there is no monopole coefficient  $E$  and no subdominant logarithmic singularity of the surface Green's function. A similar analysis for circular patches of infinite reactivity, with either random or periodic patch locations on the infinite plane, was undertaken in [40] and [5], respectively. More specifically, consider a collection of  $N$  well-separated disks centered at  $\mathbf{x}_i$ , for  $i = 1, \dots, N$ , on the infinite plane, each with radius  $\varepsilon a_i$  and reactivity  $\kappa_i$ . Then, by a simple adaptation of the analysis in §2 of [40], which dealt with the case of infinite reactivity, we obtain that the asymptotic expansion for the dimensionless capacitance  $C_T$  of this planar structured target is

$$(7.1) \quad \frac{1}{C_T} \sim \frac{1}{\varepsilon \bar{C}} \left[ 1 + \varepsilon \gamma + \varepsilon^2 \left( \gamma^2 - \frac{1}{\bar{C}} \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq j}}^N \sum_{\substack{k=1 \\ k \neq i}}^N \frac{C_i C_j C_k}{|\mathbf{x}_k - \mathbf{x}_j| |\mathbf{x}_j - \mathbf{x}_i|} \right) \right],$$

where  $\gamma \equiv \mathbf{C}^T \mathcal{G}_s \mathbf{C} / \bar{C}$ ,  $\mathbf{C} \equiv (C_1, \dots, C_N)^T$ ,  $\bar{C} \equiv \sum_{j=1}^N C_j$ , and  $C_j = a_j \mathcal{C}(a_j \kappa_j)$  is the reactive capacitance defined by the inner problem (2.3). Here the  $N \times N$  symmetric Green's matrix  $\mathcal{G}_s$  needed for  $\gamma$  has matrix entries  $(\mathcal{G}_s)_{ii} = 0$  and  $(\mathcal{G}_s)_{ij} = |\mathbf{x}_i - \mathbf{x}_j|^{-1}$  for  $i \neq j$ . In this way, by using the heuristic approximation (2.9), (7.1) is readily evaluated over the full range of reactivities of the patches. Moreover, the analysis in [5] that determined an effective capacitance for periodic patterns of identical circular patches centered on a Bravais lattice on the infinite plane with unit area of the fundamental Wigner-Seitz cell is easily extended to the case of partial reactivity.

Finally, we highlight two problems that warrant further study. Firstly by exploit-

ing a local tangential-normal coordinate system together with a careful resolution of the subdominant logarithmic singularity of the surface Green's function (see [15, 51]), it should be possible to derive a similar three-term result for the capacitance of a generic bounded domain with a smooth boundary covered by small surface patches of finite reactivity. Such a result would also depend on the mean curvature of the boundary at the center of each patch as well as the regular (i.e., non-singular) part of the surface Green's function at each patch location, the latter of which must be computed numerically. With these modifications, the overall analysis should be rather similar to that done for the structured sphere in §3 and the Appendices. Secondly, from a computational viewpoint, it would be worthwhile to devise a numerical PDE approach to accurately solve (3.1) for a large collection of partially reactive patches. A key challenge in the numerics is to carefully resolve the behavior at the boundary of each patch. For perfectly reactive patches, corresponding to the Dirichlet-Neumann problem, such a scheme has previously been developed (cf. [41, 40, 36]).

**Acknowledgements.** D.S.G. acknowledges the Simons Foundation for supporting his sabbatical sojourn in 2024 at the CRM (CNRS – University of Montréal, Canada), and the Alexander von Humboldt Foundation for support within a Bessel Prize award. M.J.W. was supported by the NSERC Discovery grant program. We are grateful to Prof. Sean Lawley of the University of Utah for discussions related to [44] and to the sigmoidal approximation in (2.9).

#### Appendix A. Geodesic Normal Coordinates to the Unit Sphere $\Omega$ .

We define geodesic normal coordinates  $\xi = (\xi_1, \xi_2, \xi_3)^T \in (-\pi/2, \pi/2) \times (-\pi, \pi) \times [0, \infty]$  in the exterior  $|\mathbf{x}| \geq 1$  of the unit sphere  $\Omega$ . In these coordinates,  $\xi = 0$  corresponds to  $\mathbf{x}_i \in \partial\Omega$ , while  $\xi_3 > 0$  holds in the exterior of  $\Omega$ . In terms of the spherical angles  $\theta_i \in (0, \pi)$  and  $\varphi_i \in [0, 2\pi)$ , and for  $|\mathbf{x}_i| = 1$ , we first define the orthonormal vectors  $\mathbf{x}_i$ ,  $\mathbf{v}_{2i}$  and  $\mathbf{v}_{3i}$  by

$$(A.1) \quad \mathbf{x}_i \equiv \begin{bmatrix} \cos \varphi_i \sin \theta_i \\ \sin \varphi_i \sin \theta_i \\ \cos \theta_i \end{bmatrix}, \quad \mathbf{v}_{2i} = \partial_\theta \mathbf{x}_i \equiv \begin{bmatrix} \cos \varphi_i \cos \theta_i \\ \sin \varphi_i \cos \theta_i \\ -\sin \theta_i \end{bmatrix}, \quad \mathbf{v}_{3i} = \mathbf{x}_i \times \partial_\theta \mathbf{x}_i \equiv \begin{bmatrix} -\sin \varphi_i \\ \cos \varphi_i \\ 0 \end{bmatrix},$$

where  $\mathbf{v}_{2i}$  and  $\mathbf{v}_{3i}$  provide a basis for the tangent plane to the sphere at  $\mathbf{x} = \mathbf{x}_i$ . In terms of these vectors, the geodesic coordinates  $\xi = (\xi_1, \xi_2, \xi_3)^T$  are defined by the global transformation

$$(A.2) \quad \mathbf{x}(\xi) = (1 + \xi_3) (\cos \xi_1 \cos \xi_2 \mathbf{x}_i + \cos \xi_1 \sin \xi_2 \mathbf{v}_{2i} + \sin \xi_1 \mathbf{v}_{3i}).$$

The geodesic coordinate curves obtained by setting  $\xi_3 = 0$ , and fixing either  $\xi_2 = 0$  or  $\xi_1 = 0$  are, respectively,  $\mathbf{x}(\xi_1, 0, 0) = \cos \xi_1 \mathbf{x}_i + \sin \xi_1 \mathbf{v}_{3i}$  or  $\mathbf{x}(0, \xi_2, 0) = \cos \xi_2 \mathbf{x}_i + \sin \xi_2 \mathbf{v}_{2i}$ . These circles correspond to intersections of  $\partial\Omega$  with planes spanned by  $\{\mathbf{x}_i, \mathbf{v}_{3i}\}$  or  $\{\mathbf{x}_i, \mathbf{v}_{2i}\}$ , respectively.

The scale factors  $h_{\xi_j} \equiv |\partial \mathbf{x} / \partial \xi_j|$  for  $j = 1, 2, 3$  are readily calculated as

$$(A.3) \quad h_{\xi_1} = (1 + \xi_3), \quad h_{\xi_2} = (1 + \xi_3) \cos \xi_1, \quad h_{\xi_3} = 1.$$

For a generic function  $\mathcal{V}(\xi) \equiv u(\mathbf{x}(\xi))$ , we calculate, as similar to that done in Appendix A of [30], that the Laplacian transforms according to

$$(A.4) \quad \Delta_{\mathbf{x}} u = \mathcal{V}_{\xi_3 \xi_3} + \frac{2}{1 + \xi_3} \mathcal{V}_{\xi_3} + \frac{1}{(1 + \xi_3)^2 \cos^2 \xi_1} \mathcal{V}_{\xi_2 \xi_2} + \frac{1}{(1 + \xi_3)^2 \cos \xi_1} \frac{\partial}{\partial \xi_1} (\cos \xi_1 \mathcal{V}_{\xi_1}).$$

Then, upon introducing the local (or inner) variables,  $\mathbf{y} = (y_1, y_2, y_3)^T$ , defined by

$$(A.5) \quad \xi_1 = \varepsilon y_1, \quad \xi_2 = \varepsilon y_2, \quad \xi_3 = \varepsilon y_3,$$

we use the Taylor series approximations  $(1 + \xi_3)^{-1} \sim 1 - \varepsilon y_3$ ,  $(1 + \xi_3)^{-2} \sim 1 - 2\varepsilon y_3$ ,  $\cos^2 \xi_1 = 1 + \mathcal{O}(\varepsilon^2)$  and  $\sin \xi_1 \sim \varepsilon y_1$ , to show that (A.4) reduces to (2.2).

To determine a two-term approximation for the Euclidian distance  $|\mathbf{x} - \mathbf{x}_i|$  near the patch, we proceed in a similar way as in Appendix A of [30]. By substituting (A.5) in (A.2), we obtain from a Taylor series approximation that

$$(A.6a) \quad \mathbf{x} - \mathbf{x}_i = \varepsilon \mathbf{b}_0 - \varepsilon^2 \mathbf{b}_1 + \mathcal{O}(\varepsilon^3), \quad |\mathbf{x} - \mathbf{x}_i|^2 \sim \varepsilon^2 (\mathbf{b}_0^T \mathbf{b}_0 - 2\varepsilon \mathbf{b}_0^T \mathbf{b}_1),$$

where  $\mathbf{b}_0$  and  $\mathbf{b}_1$  are defined by

$$(A.6b) \quad \mathbf{b}_0 = y_3 \mathbf{x}_i + y_2 \mathbf{v}_{2i} + y_1 \mathbf{v}_{3i}, \quad \mathbf{b}_1 = \frac{1}{2} (y_1^2 + y_2^2) \mathbf{x}_i - y_3 y_2 \mathbf{v}_{2i} - y_3 y_1 \mathbf{v}_{3i}.$$

In this way, and by labeling  $\rho = |\mathbf{y}|$ , with  $\mathbf{y} = (y_1, y_2, y_3)^T$ , we conclude that

$$(A.7) \quad |\mathbf{x} - \mathbf{x}_i| \sim \varepsilon \rho + \frac{\varepsilon^2 y_3}{2\rho} (y_1^2 + y_2^2) + \mathcal{O}(\varepsilon^3), \quad \frac{1}{|\mathbf{x} - \mathbf{x}_i|} \sim \frac{1}{\varepsilon \rho} \left( 1 - \frac{\varepsilon y_3}{2\rho^2} (y_1^2 + y_2^2) + \mathcal{O}(\varepsilon^2) \right).$$

In matrix form, and to the leading order in  $\varepsilon$ , we can write (A.6) in terms of  $\mathbf{y} = (y_1, y_2, y_3)^T$  and an orthogonal matrix  $\mathbf{Q}_i$  as

$$(A.8) \quad \mathbf{y} \sim \varepsilon^{-1} \mathbf{Q}_i^T (\mathbf{x} - \mathbf{x}_i), \quad \text{where} \quad \mathbf{Q}_i \equiv \begin{bmatrix} | & | & | \\ \mathbf{v}_{3i} & \mathbf{v}_{2i} & \mathbf{x}_i \\ | & | & | \end{bmatrix} \rightarrow |\mathbf{y}| \sim \varepsilon^{-1} |\mathbf{x} - \mathbf{x}_i|.$$

Finally, since  $|\mathbf{x} - \mathbf{x}_i| = \varepsilon \rho + \mathcal{O}(\varepsilon^3)$  when  $y_3 = 0$  from (A.7), and by recalling the scale factor  $h_{\xi_3} = 1$  from (A.3), we conclude that a Robin boundary condition on a locally circular patch is well-approximated in the local geodesic coordinates by

$$(A.9) \quad -\partial_{y_3} U + \kappa U = 0, \quad \text{for} \quad y_3 = 0, \quad (y_1^2 + y_2^2)^{1/2} \leq a + \mathcal{O}(\varepsilon^2).$$

To the order of our asymptotic expansion we can neglect the  $\mathcal{O}(\varepsilon^2)$  term in (A.9).

## Appendix B. Computing the Reactive Capacitance.

In this Appendix, we summarize some of the results from Appendix D of [30] that determined a Steklov eigenfunction expansion for the reactive capacitance  $C_i(\kappa_i)$  for an arbitrary patch shape  $\Gamma_i$ , and for the special case of a circular patch.

Written in terms of the local geodesic coordinates, the following Steklov eigenvalue problem for eigenpairs  $\Psi_{ki}, \mu_{ki}$  in a half-space  $\mathbb{R}_+^3$  plays a central role in determining  $C_i(\kappa_i)$ :

$$(B.1a) \quad \Delta \Psi_{ki} = 0, \quad \mathbf{y} \in \mathbb{R}_+^3,$$

$$(B.1b) \quad \partial_n \Psi_{ki} = \mu_{ki} \Psi_{ki}, \quad y_3 = 0, \quad (y_1, y_2) \in \Gamma_i,$$

$$(B.1c) \quad \partial_n \Psi_{ki} = 0, \quad y_3 = 0, \quad (y_1, y_2) \notin \Gamma_i,$$

$$(B.1d) \quad \Psi_{ki}(\mathbf{y}) = \mathcal{O}(1/|\mathbf{y}|), \quad \text{as} \quad |\mathbf{y}| \rightarrow \infty.$$

As discussed in Appendix D of [30], the Steklov eigenvalues, enumerated by the index  $k = 0, 1, 2, \dots$ , can be ordered as

$$(B.2) \quad 0 < \mu_{0i} < \mu_{1i} \leq \dots \nearrow +\infty,$$

$k$	0	1	2	3	4	5	6	7
$\mu_{ki}$	1.1578	4.3168	7.4602	10.602	13.744	16.886	20.028	23.169
$d_{ki}$	1.7524	0.2298	0.1000	0.0587	0.0397	0.0291	0.0225	0.0180

**Table B.1:** The first eight Steklov eigenvalues  $\mu_{ki}$  and weights  $d_{ki}$  for the unit disk  $\Gamma_i$  in the upper half-space that correspond to axially symmetric eigenfunctions on the patch, for which  $d_{ki} \neq 0$ , as computed numerically in [26].

where the principal eigenvalue  $\mu_{0i}$  is simple and strictly positive. The corresponding eigenfunctions, when restricted to the patch  $\Gamma_i$ , as labeled by  $\Psi_{ki}(\mathbf{y})|_{\Gamma_i}$ , form a complete orthonormal basis in  $L^2(\Gamma_i)$ , in the sense that  $\int_{\Gamma_i} \Psi_{ki} \Psi_{k'i} d\mathbf{y} = \delta_{k,k'}$ . By applying the divergence theorem to (B.1), the far-field behavior of  $\Psi_{ki}(\mathbf{y})$  has the form

$$(B.3) \quad \Psi_{ki}(\mathbf{y}) \sim \frac{\mu_{ki} d_{ki}}{2\pi|\mathbf{y}|} + \dots, \quad \text{as } |\mathbf{y}| \rightarrow \infty, \quad \text{where } d_{ki} = \int_{\Gamma_i} \Psi_{ki} d\mathbf{y}.$$

In terms of these Steklov eigenpairs and weights  $d_{ki}$ , it was shown in Appendix D of [30] that

$$(B.4) \quad C_i(\kappa_i) = \frac{\kappa_i}{2\pi} \sum_{k=0}^{\infty} \frac{\mu_{ki} d_{ki}^2}{\mu_{ki} + \kappa_i}.$$

From this Steklov eigenfunction expansion, we conclude that  $C_i(\kappa_i)$  is monotonically increasing on  $\kappa_i > 0$ , so that  $C_i(\infty)$  is an upper bound for  $C_i(\kappa_i)$  on  $\kappa_i > 0$ . Moreover, for  $\kappa_i \rightarrow 0^+$ , a Taylor series approximation yields that

$$(B.5) \quad C_i(\kappa_i) = -a_i \sum_{n=1}^{\infty} c_{ni} (-\kappa_i a_i)^n, \quad \text{with } c_{ni} = \frac{1}{2\pi a_i^{n+1}} \sum_{k=0}^{\infty} \frac{d_{ki}^2}{\mu_{ki}^{n-1}},$$

where  $c_{1i} = |\Gamma_i|/(2\pi)$  (see Appendix D of [30]).

**B.1. Circular Patch.** For a circular patch  $\Gamma_i$  of unit radius, the first eight axially symmetric eigenpairs for (B.1) were computed numerically in [26] (see also Appendix D.2 of [30]). The eigenvalues and corresponding weights are given in Table B.1.

To characterize the limiting asymptotics of  $C_i(\kappa_i)$  for  $\kappa_i \ll 1$ , in Appendix B of [30] it was shown analytically that the first three Taylor coefficients in (B.5) are

$$(B.6) \quad c_{1i} = \frac{1}{2}, \quad c_{2i} = \frac{4}{3\pi} \approx 0.4244, \quad c_{3i} = \frac{4}{\pi^2} \int_0^1 r [E(r)]^2 dr \approx 0.3651,$$

where  $E(r)$  is the complete elliptic integral of the second kind. Moreover, Appendix D.2 of [30] established that all of the Taylor coefficients  $c_{ni}$  in (B.5) are well-approximated by

$$(B.7) \quad c_{ni} \approx \frac{0.4888}{(1.1578)^{n-1}} + \frac{0.0084}{(4.3168)^{n-1}}, \quad \text{for } n \geq 2.$$

In contrast, in the limit  $\kappa_i \rightarrow +\infty$ , it was shown in [31] that the difference  $C_i(\kappa_i) - C_i(\infty)$  is not analytic in  $\kappa_i$  for  $\kappa_i \gg 1$ . In particular, the results in [31] (see also Appendix D.3 of [30]) yield the refined asymptotic behavior given in (2.8).

### Appendix C. Inner Problem Beyond Tangent Plane Approximation.

In this Appendix we construct the solution to (3.22) and show how to determine the monopole coefficient  $E_i$  in the limiting behavior (3.23). Since this Appendix is similar to Appendix C of [30] for the narrow capture MFPT problem *inside* the unit sphere, we only briefly outline the analysis. The key distinction from the analysis in [30] is that for our *exterior* problem one must account for the different algebraic sign of the curvature of the sphere, as viewed from the exterior of the sphere.

The central issue in solving (3.22) for  $\Phi_{2i}$  is to find an explicit particular solution  $\Phi_{2pi}$  that accounts for the inhomogeneous term in the PDE (3.22a) for  $\Phi_{2i}$ . This inhomogeneous term is directly responsible for the non-monopole behavior in the far-field (3.22d). However, a second issue is that we must also account for the fact that this particular solution  $\Phi_{2pi}$  does not satisfy the Robin boundary condition (3.22b) on the patch. As a result, in our decomposition of  $\Phi_{2i}$  we need to introduce an auxiliary function  $\Phi_{2hi}$ , which satisfies the homogeneous part of the PDE (3.22a), but that allows the homogeneous Robin condition (3.22b) for the full solution  $\Phi_{2i}$  to hold. The far-field behavior of this auxiliary function yields the monopole coefficient  $E_i$ . Our decomposition is summarized as follows:

LEMMA C.1. *The solution to (3.22) can be decomposed as*

$$(C.1) \quad \Phi_{2i} = \Phi_{2pi} + \Phi_{2hi},$$

where

$$(C.2) \quad \Phi_{2pi} = \frac{y_3^2}{2} w_{i,y_3} + \frac{y_3}{2} w_i - \frac{1}{2} \int_0^{y_3} w_i(y_1, y_2, \eta; \kappa_i) d\eta - \mathcal{F}_i(y_1, y_2; \kappa_i),$$

with  $w_i$  being the solution to (2.3). Here  $\mathcal{F}_i(y_1, y_2; \kappa_i)$ , with  $\Delta_S \mathcal{F}_i \equiv \mathcal{F}_{i,y_1 y_1} + \mathcal{F}_{i,y_2 y_2}$ , is the unique solution to (2.12), while  $\Phi_{2hi}$  is the unique solution to (2.11). For an arbitrary patch shape  $\Gamma_i$ , the monopole coefficient  $E_i = E_i(\kappa_i)$  in (2.11d) is given by

$$(C.3a) \quad E_i = -\frac{1}{\pi} \int_{\Gamma_i} q_i(y_1, y_2; \kappa_i) \mathcal{F}_i(y_1, y_2; \kappa_i) dy_1 dy_2,$$

where, in terms of a double integral over the patch, we have

$$(C.3b) \quad \mathcal{F}_i(y_1, y_2; \kappa_i) = \frac{1}{4\pi} \int_{\Gamma_i} q_i(y'_1, y'_2; \kappa_i) \log \left( (y_1 - y'_1)^2 + (y_2 - y'_2)^2 \right) dy'_1 dy'_2.$$

The proof of this result is analogous to that done in Appendix C of [30] for the narrow capture MFPT problem *interior* to a sphere, and is omitted.

As shown in Appendix C of [30], when  $\Gamma_i$  is a disk of radius  $a_i$  we can determine  $E_i$  in (C.3) up to a quadrature. For a locally circular patch, both the charge density  $q_i$  and the solution to (2.12) are radially symmetric in  $\rho_0 = (y_1^2 + y_2^2)^{1/2}$ , and we obtain

$$(C.4) \quad E_i = - \int_0^{a_i} 2\rho_0 q_i(\rho_0; \kappa_i) \mathcal{F}_i(\rho_0; \kappa_i) d\rho_0; \quad \mathcal{F}_{i,\rho_0} = \frac{1}{\rho_0} \int_0^{\rho_0} \eta q_i(\eta; \kappa_i) d\eta, \quad 0 \leq \rho_0 \leq a_i,$$

with  $\mathcal{F}_i = (C_i/2) \log a_i$  at  $\rho = a_i$ . Upon integrating this result by parts we obtain (2.13) in Lemma 2.2.

To determine the limiting asymptotics in (2.16) of Lemma 2.2 when  $\Gamma_i$  is a disk, we proceed as in Appendix C of [30]. When  $\kappa_i = \infty$ , we use (2.7c) for  $q_i(\rho_0; \infty)$  together with  $C_i = C_i(\infty) = 2a_i/\pi$  in (2.13). Upon evaluating the resulting integrals

analytically we obtain the expression for  $E_i(\infty)$  given in (2.16) of Lemma 2.2. Finally, to approximate  $E_i$  for  $\kappa_i \ll 1$ , we observe from (2.3) that in this limit  $-\partial_{y_3} w_i \sim \kappa$  on the patch  $y_3 = 0$ ,  $(y_1, y_2) \in \Gamma_i$ . As a result, we identify that  $q_i(\rho_0; \kappa_i) \sim \kappa_i/2$  for  $0 \leq \rho_0 \leq a_i$ . With this approximation for  $q_i$ , we can evaluate the integrals in (2.13), while using  $C_i \sim \kappa_i a_i^2/2$  for  $\kappa_i \ll 1$  from (2.10b). In this way, we readily derive the limiting asymptotics for  $E_i$  for  $\kappa_i \ll 1$  as given in (2.16) of Lemma 2.2.

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