A Metastable Flame-Front Interface and Carrier's Problem

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Dedicated with admiration to Prof. Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

Metastable dynamics for a nonlocal PDE modeling the upwards propagation of a flame-front interface in a vertical channel is analyzed in the one-dimensional case where the channel cross-section is taken to be the slab $-1 < x < 1$. In a certain asymptotic limit, the interface assumes a roughly concave parabolic shape, and the tip of the parabola drifts asymptotically exponentially slowly towards the boundary of the domain. In contrast to previous analyses that studied this behavior by transforming the governing nonlocal PDE to a convection-diffusion equation, a novel nonlinear transformation is introduced that transforms the problem to a singularly perturbed quasilinear PDE. The steady-state problem for this transformed PDE, for which the parabolic interface shape maps onto a one-spike solution, is closely related to a class of two-point boundary value problems with seemingly spurious solutions studied initially by G. Carrier in 1968. Rigorous and formal asymptotic results for a one-spike solution to this transformed PDE are obtained together with a formal metastability analysis of certain time-dependent solutions.

Key words: Metastability, flame-front interface, Carrier's problem, exponentially small eigenvalue

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1 Introduction

We analyze the nonlinear evolution equation of (14) and (19) that models the location of a flame-front that propagates upwards in a vertical channel. In (19) and Appendix B of (6) a nonlocal PDE was derived for the location of the flame-front interface by taking into account the competing effects of buoyancy and gravity. In non-dimensional variables, and in a certain asymptotic limit, the flame-front interface $S = S(x,t)$ was found to satisfy

$$S_t - \frac{1}{2}S_x^2 = \varepsilon^2 S_{xx} + S - \langle S \rangle \,, \qquad -1 < x < 1 \,, \quad t > 0 \,, \tag{1.1a}$$

$$S_x(\pm 1, t) = 0 \,, \qquad S(x,0) = S_0(x) \,, \qquad \langle S \rangle \equiv \frac{1}{2}\int_{-1}^{1} S(x,t)\, dx \,. \tag{1.1b}$$

Here $\varepsilon > 0$ is a small parameter defined in terms of the gravitational acceleration, the speed of the flame, the width of the channel, the thermal expansion coefficient of the gas, and other parameters (see equation (1.4) of (6)).

Upward propagating flames often assume a characteristic paraboloidal shape with the tip of the paraboloid located somewhere near the centerline of the channel (cf. (19)). In the one-dimensional case, the interface $S(x,t)$ for (1.1) assumes a roughly concave parabolic shape where the tip of the parabola drifts slowly towards one of the walls of the channel. For a certain class of initial conditions, the numerical results of (14) suggested that the solution to (1.1) exhibits a phenomenon known as dynamic metastability when $\varepsilon \ll 1$. This phenomenon is characterized by an asymptotically exponentially slow drift of the tip of the parabolic flame-front towards the wall of the channel.

For $\varepsilon \ll 1$, it was proved in (5) and (6) that the speed of this slow drift of the tip of the parabola is asymptotically exponentially slow as $\varepsilon \to 0$. However, no law of motion for the tip of the interface was derived in (5) and (6). In (21) a formal asymptotic analysis was used to derive the following nonlinear ODE for the tip $x_0(t)$ of the flame-front interface for (1.1) in the limit $\varepsilon \to 0$

$$x_0' \sim \sqrt{\frac{2}{\pi\varepsilon^2}} \left[\left((1-x_0)^2 + O(\varepsilon^2) \right) e^{-(1-x_0)^2/2\varepsilon^2} - \left((1+x_0)^2 + O(\varepsilon^2) \right) e^{-(1+x_0)^2/2\varepsilon^2} \right] . \tag{1.2}$$

This result is a scaled version of that given in Corollary 2 of (21), where a different domain length and definition of ε was used. The ODE (1.2) shows that the parabolic tip will move exponentially slowly but monotonically either towards the wall at $x = 1$ or at $x = -1$, depending on whether $x_0(0) > 0$ or $x_0(0) < 0$, respectively. The analysis leading to (1.2) is, however, not valid when x_0 is $O(\varepsilon)$ close to either ± 1.

The analyses of (5), (6), and (21), are based on introducing the transformation $y = -S_x$ in (1.1) to eliminate the nonlocal term. The resulting PDE problem for $y = y(x,t)$ on $-1 < x < 1$ is the following nonlinear convection-diffusion problem, known as the Burgers-Sivashinsky equation:

$$y_t + yy_x - y = \varepsilon^2 y_{xx}\,, \qquad y(\pm 1, t) = 0\,, \quad y(x,0) = -S_{0x}\,. \tag{1.3}$$

The tip $x_0 = x_0(t)$ of the interface is then given by $y_x[x_0(t), t] = 0$. The metastability analysis of (21) based on (1.3), and leading to the ODE (1.2), is then similar, but significantly more intricate, than the analysis of metastability of viscous shocks for Burgers equation given in (12) and (20). A survey of metastable behavior for other problems in one-dimensional domains is given in (23).

The goal of this paper is to study metastable flame-front motion for (1.1) by using a different transformation that, in contrast to the change of variables $y = -S_x$, extends readily to the two-dimensional case of flame-front propagation in a channel of arbitrary cross-section. To this end, we introduce the nonlinear change of variables

$$S = 2\varepsilon^2 \log v\,. \tag{1.4}$$

In terms of $v(x,t)$, (1.1) transforms exactly to

$$v_t = \varepsilon^2 v_{xx} + v\log v - v\langle \log v\rangle\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.5a}$$

$$v_x(\pm 1, t) = 0\,, \qquad v(x,0) = e^{S_0(x)/(2\varepsilon^2)}\,, \qquad \langle \log v\rangle \equiv \frac{1}{2}\int_{-1}^{1} \log v(x,t)\, dx\,. \tag{1.5b}$$

A further time-dependent transformation can be made to eliminate the nonlocal term in (1.5a). To do so, we introduce $u(x,t)$, defined by

$$v(x,t) = f(t)u(x,t)\,, \qquad f(t) \equiv \exp\left(- \int_0^t \langle \log u(x,\tau)\rangle\, d\tau \right)\,. \tag{1.6}$$

Then, a simple calculation shows that (1.5) transforms to the quasilinear problem

$$u_t = \varepsilon^2 u_{xx} + u\log u\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1.7a}$$

$$u_x(\pm 1, t) = 0\,, \qquad u(x,0) = e^{S_0(x)/(2\varepsilon^2)}\,. \tag{1.7b}$$

In terms of u, the flame-front interface S, obtained from (1.4) and (1.6), is

$$S(x,t) = 2\varepsilon^2 \log u - 2\varepsilon^2 \int_0^t \langle \log u(x,\tau)\rangle\, d\tau\,. \tag{1.8}$$

In §2 we relate (1.7) to Carrier's problem (cf. pages 202-205 of (7)) where spike-layer solutions are known to occur. It is shown that a one-spike equilibrium solution to (1.7) with the spike located at $x = 0$, corresponds to a steady-state concave flame-front interface where the tip is at the centerline of the channel. Rigorous and formal asymptotic results for this steady-state solution are obtained.

In §3 we study the spectral problem associated with (1.5) and (1.7). For $\varepsilon \ll 1$, it is shown that there is an asymptotically exponentially small eigenvalue in the spectrum of the linearization.

This eigenvalue is responsible for the metastable flame-front behavior. In §3.2 we also analyze metastability in terms of the transformed problem, and we re-derive the result (1.2). The advantage of the transformation (1.4) and suggestions for further work are given in §4.

2 The Equilibrium Problem

In this section we study the steady-state problem for (1.7) given by

$$\varepsilon^2 U_{xx} + Q(U) = 0\,, \quad -1 < x < 1\,; \quad U_x(\pm 1) = 0\,; \qquad Q(U) \equiv U \log U\,. \tag{2.1}$$

A simple analysis of (2.1) in the phase-plane (cf. (16)) shows that (2.1) admits spike-type solutions, where each spike is closely approximated by a homoclinic solution of $\varepsilon^2 U_{xx} + Q(U) = 0$ on the infinite line. We are interested in constructing a one-spike solution to (2.1) with the spike located at some $x_0 \in (-1, 1)$.

For the different nonlinearity $Q(U) = U^2 - 1$, Carrier (see pages 202-205 of (7)) showed the difficulty in determining spike locations from a straightforward application of the method of matched asymptotic expansions. For a one-spike solution it is clear by reflection symmetry that $x_0 = 0$. However, the analytical problem of determining x_0 is exponentially ill-conditioned, and requires the matching of exponentially small terms. Many formal analytical methods have been proposed for the determination of spike locations for (2.1) under a smooth nonlinearity $Q(u)$, including, the matching of exponentially small terms (cf. (10)), a variational method (cf. (8)), and a projection method based on a limiting solvability condition (cf. (22)). For a survey of the projection method see (23). More recently, the spike solutions constructed in (22) have been rigorously established in (17) and (18) using an analytical shooting method. A rigorous method based on Green's functions has been used in (9) to construct a one-spike solution. Multi-spike solutions for the case where $Q = Q(u, x)$ have been constructed formally in (13), and rigorously in (2).

A key difference between these previous analyses and the analysis of (2.1) is that $Q(U) = U \log U$ is not smooth at $U = 0$. This leads to a different far-field behavior of the homoclinic solution than that given in previous studies, and complicates the analysis considerably. In §2.1 we give a formal boundary-layer analysis for the asymptotic construction of a one-spike quasi-equilibrium solution of (2.1). A more rigorous approach for the equilibrium solution is given in §2.2.

On the infinite-line, let $U_c(y)$ be the unique homoclinic solution to

$$U_c'' + U_c \log U_c = 0\,, \quad -\infty < y < \infty\,; \quad U_c(0) > 0\,, \quad U_c'(0) = 0\,, \tag{2.2a}$$
$$U_c \to 0\,, \qquad |y| \to \infty\,. \tag{2.2b}$$

A simple calculation shows that the exact solution to (2.2) is

$$U_c(y) = e^{1/2} e^{-y^2/4}\,. \tag{2.2c}$$

We then look for a one-spike solution to (2.1) in the form

$$U(x) \sim U_{c0} \equiv U_c\left[\varepsilon^{-1}(x - x_0)\right]\,. \tag{2.3}$$

Here x_0 is the location of the spike. Substituting (2.3) into (1.8), we calculate that the corresponding flame-front interface has the concave parabolic form

$$S \sim -\frac{1}{2}(x - x_0)^2 + \varepsilon^2(1 - t) + \frac{1}{6}\int_0^t (1 + 3x_0^2)\, d\tau\,. \tag{2.4}$$

In this way, we say that a one-spike solution to (2.1) maps onto a concave parabolic flame-front interface for (1.1).

However, for any $x_0 \in (-1, 1)$, the quasi-equilibrium solution U_{c0} is not a true equilibrium solution in that it fails to satisfy the boundary conditions in (2.1) by asymptotically exponentially small terms as $\varepsilon \to 0$. Therefore, we must construct boundary layers near $x = \pm 1$ of exponentially small amplitude. To determine the boundary layer scaling, we linearize (2.1) around (2.3) by writing $U = U_{c0} + \tilde{U}$, where $\tilde{U} \ll 1$. Using $Q'(U) = 1 + \log U$, we calculate

$$\varepsilon^4 \tilde{U}_{xx} + \left[\frac{3\varepsilon^2}{2} - \frac{(x - x_0)^2}{4}\right] \tilde{U} = 0\,. \tag{2.5}$$

This form suggests that there are boundary layers near $x = \pm 1$ of width $O(\varepsilon^2)$.

We will only consider the left boundary layer near $x = -1$ since the analysis near $x = 1$ is very similar. We introduce the new variables $W = W(y)$ and y by

$$U = e^W\,, \qquad y = \varepsilon^{-2}(x + 1)\,. \tag{2.6}$$

Substituting (2.6) into (2.1), we get that W satisfies

$$W_{yy} + W_y^2 + \varepsilon^2 W = 0\,, \qquad 0 \le y < \infty\,; \qquad W_y(0) = 0\,. \tag{2.7a}$$

The far-field condition as $y \to \infty$, obtained by matching W to the quasi-equilibrium solution U_{c0}, is that

$$W = \frac{1}{2} - \frac{\left(-1+\varepsilon^2 y - x_0\right)^2}{4\varepsilon^2} \sim -\frac{(1+x_0)^2}{4\varepsilon^2} + \frac{(1+x_0)y}{2} + \frac{1}{2} - \frac{\varepsilon^2 y^2}{4} + \cdots . \tag{2.7b}$$

We then expand the solution to (2.7a) as

$$W = -\frac{(1+x_0)^2}{4\varepsilon^2} + w_0 + \varepsilon^2 w_1 + \cdots . \tag{2.8}$$

Substituting (2.8) into (2.7), and collecting powers of ε, we obtain the following problems for w_0 and w_1:

$$w_{0yy} + w_{0y}^2 = \frac{(1+x_0)^2}{4}, \qquad 0 \le y < \infty; \qquad w_{0y}(0) = 0, \tag{2.9a}$$

$$w_0 \sim \frac{1}{2} + \frac{(1+x_0)y}{2}, \qquad y \to \infty, \tag{2.9b}$$

$$w_{1yy} + (2w_{0y})\, w_{1y} = -w_0, \qquad 0 \le y < \infty; \qquad w_{1y}(0) = 0, \tag{2.9c}$$

$$w_1 \sim -\frac{y^2}{4} + o(1), \qquad y \to \infty. \tag{2.9d}$$

The solution for $w_0(y)$ is readily calculated as

$$w_0(y) = \log\left[\kappa \cosh(\alpha y)\right], \qquad \kappa \equiv 2e^{1/2}, \qquad \alpha \equiv (1+x_0)/2. \tag{2.10}$$

Although the linear equation for w_1 can be reduced to quadrature, an explicit form for w_1 is very complicated.

In §3 below, we require an asymptotic estimate of $U(-1) = e^{W(0)}$. Since $w_0(0) = \frac{1}{2} + \log 2$, we calculate from (2.8) that

$$U(-1) \sim 2e^{1/2} e^{-(1+x_0)^2/(4\varepsilon^2)} \left[1 + \varepsilon^2 w_1(0)\right]. \tag{2.11}$$

Therefore, we need only determine $w_1(0)$ from the solution to (2.9c) and (2.9d). The solution w_1 can be decomposed as

$$w_1 = -y^2/4 + w_p(y), \tag{2.12}$$

where $w_p(y)$ satisfies

$$Lw_p \equiv w_{pyy} + (2w_{0y})\, w_{py} = -w_0 + \frac{1}{2} + y w_{0y}, \qquad 0 \le y < \infty, \tag{2.13a}$$

$$w_p(0) = \gamma; \qquad w_{py}(0) = 0; \qquad w_p \to 0, \quad y \to \infty. \tag{2.13b}$$

We want to determine γ so that $w_p \to 0$ as $y \to \infty$.

To do so, we consider the following adjoint problem for h_1:

$$L^{\dagger} h_1 \equiv h_{1yy} - \left(2w_{0y} h_1\right)_y = 0, \qquad 0 \le y < \infty; \qquad h_1(\infty) = 1, \qquad h_{1y}(\infty) = 0. \tag{2.14}$$

By using Green's identity on w_p and h_1, we derive

$$\int_0^\infty h_1 L w_p \, dy = \left(h_1 w_{py} - h_{1y} w_p \right) \big|_0^\infty + 2 h_1 w_{0y} w_p \big|_0^\infty \,. \tag{2.15}$$

Using the conditions $w_{0y}(0) = 0$, $w_p(0) = \gamma$, $w_{py}(0) = 0$, $w_p(\infty) = 0$, $h_1(\infty) = 1$, and $h_{1y}(\infty) = 0$, we get that (2.15) reduces to

$$\gamma h_{1y}(0) = \int_0^\infty h_1 L w_p \, dy \,. \tag{2.16}$$

Next, we use (2.10) in (2.14) to calculate $h_1(y)$ explicitly as

$$h_1(y) = 2\cosh^2(\alpha y) \left[1 - \tanh(\alpha y) \right] . \tag{2.17}$$

Noting that $h_{1y}(0) = -2\alpha$, we then substitute (2.17) and (2.13a) for $L w_p$ into (2.16) to get an equation for γ

$$\gamma = -\frac{1}{\alpha} \int_0^\infty \cosh^2(\alpha y) \left[1 - \tanh(\alpha y) \right] \left[\alpha y \tanh(\alpha y) - \log\left(2\cosh(\alpha y) \right) \right] \, dy \,. \tag{2.18}$$

Changing variables by $x = \alpha y$, we then evaluate γ as

$$\begin{aligned}
\gamma &= -\frac{1}{\alpha^2} \int_0^\infty \cosh^2 x \left[1 - \tanh x \right] \left[x \tanh x - \log\left(2\cosh x \right) \right] \, dx \,, \\
&= \frac{1}{\alpha^2} \int_0^\infty x \left(\cosh x - \sinh x \right)^2 \, dx - \frac{1}{\alpha^2} \int_0^\infty \left(\cosh x - \sinh x \right) \cosh x \left(x - \log[2\cosh x] \right) \, dx \,, \\
&= \frac{1}{\alpha^2} \int_0^\infty x e^{-2x} \, dx + \frac{1}{2\alpha^2} \int_0^\infty \left(1 + e^{-2x} \right) \log\left(1 + e^{-2x} \right) \, dx \,.
\end{aligned} \tag{2.19}$$

Setting $t = e^{-2x}$ in the second integral in (2.19), we can write γ as

$$\gamma = \frac{1}{\alpha^2} \int_0^\infty x e^{-2x} \, dx + \frac{1}{2\alpha^2} \left[\frac{1}{2} \int_0^1 \frac{\log(1+t)}{t} \, dt + \frac{1}{2} \int_0^1 \log(1+t) \, dt \right] . \tag{2.20}$$

Next, we substitute $\int_0^1 t^{-1} \log(1+t) \, dt = \pi^2/12$, $\int_0^1 \log(1+t) \, dt = 2\log 2 - 1$, and $\int_0^\infty x e^{-2x} \, dx = 1/4$, into (2.20). Recalling from (2.10) that $\alpha = (1+x_0)/2$, we get

$$\gamma = \frac{1}{(1+x_0)^2} \left(\frac{\pi^2}{12} + 2\log 2 \right) . \tag{2.21}$$

Finally, recalling (2.11) for $U(-1)$, and noting that $w_1(0) = \gamma$ from (2.12), we obtain the following two-term estimate for $U(-1)$ for $\varepsilon \ll 1$:

$$U(-1) \sim 2 e^{1/2} e^{-(1+x_0)^2/(4\varepsilon^2)} \left[1 + \left(\frac{\pi^2}{12} + 2\log 2 \right) \frac{\varepsilon^2}{(1+x_0)^2} \right] . \tag{2.22a}$$

A very similar boundary layer analysis can be done to estimate $U(1)$. Omitting the details of the calculation, we obtain for $\varepsilon \ll 1$ that

$$U(1) \sim 2e^{1/2} e^{-(1-x_0)^2/(4\varepsilon^2)} \left[1 + \left(\frac{\pi^2}{12} + 2\log 2 \right) \frac{\varepsilon^2}{(1-x_0)^2} \right] . \tag{2.22b}$$

By reflection symmetry, we must have $x_0 = 0$ for the true equilibrium solution.

2.2 Asymptotic Analysis of the Equilibrium Solution

We now construct a one-spike equilibrium solution to (2.1) using a rigorous shooting method similar to that of (18). In addition, we give an alternative method to obtain (2.22a). We begin by considering (2.1) with

$$U(-1) = a \,, \qquad U'(-1) = 0 \,, \tag{2.23}$$

where a is a real constant. We assume that $0 < a < e^{-1}$. When a is not in this interval it is easy to see that there is no solution $U(x,a)$ to (2.1) with $U_x(\pm 1, a) = 0$.

By integrating (2.1) from -1 to x, we get

$$\frac{\varepsilon^2}{2} U_x^2 + \int_a^U s \log s \, ds = 0 \,. \tag{2.24}$$

It follows from (2.24) that there exists a point, say (x_a, U_a), such that $U(x_a, a) = U_a$, and $U_x(x_a, a) = 0$. At this point, we have

$$\int_a^{U_a} s \log s \, ds = 0 \,, \tag{2.25}$$

and $\int_a^U s \log s \, ds < 0$ for any U in $a < U < U_a$. By integrating (2.25), we get

$$\frac{\left(U_a e^{-1/2}\right)^2}{2} \log \left(U_a e^{-1/2}\right) = e^{-1} \left(\frac{a^2}{2} \log a - \frac{a^2}{4} \right) . \tag{2.26}$$

For $a \ll 1$, we calculate from (2.26) that

$$U_a \sim e^{1/2} + e^{-1/2} \left(a^2 \log a - \frac{a^2}{2} \right) . \tag{2.27}$$

Next, we define $T(a, \varepsilon)$ to be the distance between -1 and x_a. Then, from (2.24), we get

$$T(a,\varepsilon) = \varepsilon B(a) \,, \qquad B(a) \equiv \int_a^{U_a} \frac{du}{\sqrt{(a^2 \log a - a^2/2) - (u^2 \log u - u^2/2)}} \,. \tag{2.28}$$

We now show that we can choose a such that $T(a,\varepsilon) = \varepsilon B(a) = 1$. This choice of a corresponds to a one-spike solution for (2.1) on $|x| < 1$ centered at $x = 0$.

We first show that such a value of a must necessarily be small. To show this, we write the expression in the integrand in (2.28) as

$$\left(a^2 \log a - a^2/2\right) - \left(u^2 \log u - u^2/2\right) = (u-a)(U_a - u)F(a,u)\,.$$

For any given small positive constant δ, independent of ε, it is easy to see that if $a \geq \delta$, then $F(a,u)$ is positive and bounded below by some positive constant. Therefore, $B(a)$ in (2.28) is also bounded. This implies that $T(a,\varepsilon) = O(\varepsilon)$ for $a \geq \delta$. Therefore, we must consider the case where a is sufficiently small. The result will provide an alternative verification of the formal result (2.22a).

Next, we introduce the new variables v, b, and V_b, by

$$v = ue^{-1/2}\,, \qquad b = ae^{-1/2}\,, \qquad V_b = U_a e^{-1/2}\,. \tag{2.30}$$

Then, (2.28) is transformed to

$$T(a;\varepsilon) = \varepsilon T_b\,, \qquad T_b \equiv \int_b^{V_b} \frac{dv}{\sqrt{b^2 \log b - v^2 \log v}}\,. \tag{2.31}$$

Using $V_b = U_a e^{-1/2}$ and (2.27), we obtain for $b \ll 1$ that

$$V_b \sim 1 + b^2 \log b\,. \tag{2.32}$$

In (2.31) we introduce a new integration variable θ defined by $v = b\cos^2\theta + V_b \sin^2\theta$. This yields,

$$T_b = 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{G(\theta,b)}}\,, \tag{2.33a}$$

where

$$G(\theta,b) = \frac{b^2 \log b - (b\cos^2\theta + V_b \sin^2\theta)^2 \log(b\cos^2\theta + V_b\sin^2\theta)}{(V_b - b)^2 \sin^2\theta \cos^2\theta}\,. \tag{2.33b}$$

By differentiating (2.33b) with respect to b, we obtain

$$\frac{\partial G(\theta,b)}{\partial b} = \frac{H(\theta,b)}{(V_b - b)^3}\,, \qquad H(\theta,0) = -1 - \frac{2\log\left(\sin^2\theta\right)}{\cos^2\theta}\,. \tag{2.34}$$

A simple calculation shows that $H(\theta,0) > 0$ on $0 < \theta < \pi/2$, so that $G(\theta,b)$ is increasing in b for $b \ll 1$. Therefore, from (2.33a), we have that T_b is decreasing in b when b is sufficiently small. Finally, we calculate from (2.31) that

$$T_b \geq \int_b^{1/2} \frac{dv}{\sqrt{-v^2 \log v}} = 2\sqrt{-\log b} - 2\sqrt{-\log(1/2)}\,. \tag{2.35}$$

This shows that $\lim_{b\to 0} T_b = +\infty$.

This leads to the following result in terms of a: For any small positive constant a in some interval, say $(0,\delta]$ we have $B(\delta) = O(1)$, $\lim_{a\to 0} B(a) = \infty$, and $B(a)$ is strictly decreasing in $(0,\delta]$. Therefore, for sufficiently small ε, there exists exactly one point a in $(0,\delta]$ such that $T(a,\varepsilon) = \varepsilon B(a) = 1$. This proves that there exists exactly one solution to the boundary value problem (2.1) having a spike-layer centered at the origin.

Next, for a one-spike solution centered at $x=0$, we asymptotically calculate $U(-1)$ in terms of ε. Since $T(a,\varepsilon) = \varepsilon T_b = 1$, we have $T_b = 1/\varepsilon$. From (2.31), this gives an equation for b

$$T_b \equiv \int_b^{V_b} \frac{dv}{\sqrt{b^2 \log b - v^2 \log v}} = \varepsilon^{-1} . \tag{2.36}$$

Since $U(-1) = b e^{1/2}$, we must determine b in terms of ε by expanding the integral in (2.36) as $b \to 0$. This analysis will provide an alternative method for deriving the formal result (2.22a).

To do so, it is convenient to introduce the new variables t, c, c_b, and y, by

$$t = -\log v\,, \qquad c = -\log b\,, \qquad c_b = -\log V_b\,, \qquad y = (t-c)/(c_b - c)\,. \tag{2.37}$$

Then, (2.36) transforms to

$$T_b = \int_{c_b}^{c} \frac{dt}{\sqrt{t - c e^{-2(c-t)}}} = \frac{(c-c_b)}{\sqrt{c}} I_1\,, \qquad I_1 \equiv \int_0^1 \frac{dy}{\sqrt{1 - (1-\frac{c_b}{c})y - e^{-2(c-c_b)y}}}\,. \tag{2.38}$$

From (2.32), we calculate for $b \ll 1$ that

$$c_b = -\log V_b \sim -b^2 \log b \sim -e^{-2c} c\,. \tag{2.39}$$

Therefore, c_b is exponentially small in the limit $c \to \infty$. For $c \gg 1$, a leading-order estimate of I_1 is simply

$$I_1 \sim \int_0^1 \frac{dy}{\sqrt{1-y}} = 2\,. \tag{2.40}$$

To obtain higher order terms in the expansion of T_b for $b \to 0$, we need to rigorously estimate the difference between I_1 and $\int_0^1 \frac{dy}{\sqrt{1-y}}$ for $c \gg 1$.

To do so, we begin by writing

$$I_1 - \int_0^1 \frac{dy}{\sqrt{1-y}} = I_2 + \bar{I}_2\,, \qquad I_2 \equiv \int_0^\delta F(y;c,c_b)\,dy\,, \qquad \bar{I}_2 \equiv \int_\delta^1 F(y;c,c_b)\,dy\,, \tag{2.41a}$$

where $F(y;c,c_b)$ and δ are defined by

$$F(y;c,c_b) \equiv \frac{\sqrt{1-y} - \sqrt{1-(1-\frac{c_b}{c})y - e^{-2(c-c_b)y}}}{\sqrt{1-y}\sqrt{1-(1-\frac{c_b}{c})y - e^{-2(c-c_b)y}}}\,, \qquad \delta \equiv \frac{3\log(2c)}{2c}\,. \tag{2.41b}$$

We want to calculate I_2 and $\bar{I}_2$ for $c \gg 1$ with an error of $o\left(c^{-2}\right)$. A simple calculation shows that $\bar{I}_2 = O\left(c^{-3}\right)$. For $c \gg 1$, we estimate I_2 as

$$I_2 = \int_0^\delta F(y;c,0)\,dy + O(c^{-3}) = \int_0^\delta \frac{e^{-2cy}\,dy}{\sqrt{1-y}\sqrt{1-y-e^{-2cy}}\left(\sqrt{1-y}+\sqrt{1-y-e^{-2cy}}\right)} + O(c^{-3})\,. \tag{2.42}$$

Next, we let $t = e^{-2cy}$, so that $y = -(2c)^{-1}\log t$. Then (2.42) becomes

$$\begin{aligned} I_2 &= \frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{dt}{\sqrt{1+\frac{\log t}{2c}}\sqrt{1+\frac{\log t}{2c}-t}\left(\sqrt{1+\frac{\log t}{2c}}+\sqrt{1+\frac{\log t}{2c}-t}\right)} + O\left(c^{-3}\right)\,, \\ &= \frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{(1-\frac{1}{4c}\log t)\,dt}{\sqrt{1+\frac{\log t}{2c}-t}\,\left(\sqrt{1+\frac{\log t}{2c}}+\sqrt{1+\frac{\log t}{2c}-t}\right)} + O\left(c^{-3}\right)\,, \\ &= I_3 + I_4 + O\left(c^{-3}\right)\,. \end{aligned} \tag{2.43}$$

Here I_3 and I_4 are defined by

$$I_3 = \frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{dt}{\sqrt{1+\frac{\log t}{2c}-t}\,\left(\sqrt{1+\frac{\log t}{2c}}+\sqrt{1+\frac{\log t}{2c}-t}\right)}\,, \tag{2.44a}$$

$$I_4 = -\frac{1}{8c^2}\int_{e^{-2c\delta}}^1 \frac{\log t}{\sqrt{1+\frac{\log t}{2c}-t}\,\left(\sqrt{1+\frac{\log t}{2c}}+\sqrt{1+\frac{\log t}{2c}-t}\right)}\,dt\,. \tag{2.44b}$$

Let $c \to \infty$ in I_4. This leads to an integral that is readily evaluated

$$\begin{aligned} I_4 &= -\frac{1}{8c^2}\int_0^1 \frac{\log t}{\sqrt{1-t}(1+\sqrt{1-t})}\,dt + o\left(c^{-2}\right)\,, \\ &= -\frac{1}{8c^2}\int_0^1 \frac{2\log(1-u^2)}{1+u}\,du + o(c^{-2}) = -\frac{1}{8c^2}\left(-\frac{\pi^2}{6}+2\log^2 2\right) + o\left(c^{-2}\right)\,. \end{aligned} \tag{2.45}$$

Next, we decompose I_3 in (2.44a) as

$$I_3 = I_5 + I_6\,. \tag{2.46a}$$

Here I_5 and I_6 are defined by

$$I_5 = \frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{dt}{\sqrt{1+\frac{\log t}{2c}}\sqrt{1+\frac{\log t}{2c}-t}}\,, \tag{2.46b}$$

$$I_6 = -\frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{dt}{\sqrt{1+\frac{\log t}{2c}}\left(\sqrt{1+\frac{\log t}{2c}}+\sqrt{1+\frac{\log t}{2c}-t}\right)}\,. \tag{2.46c}$$

Using the Binomial Theorem on $\sqrt{1+(2c)^{-1}\log t}$ in the integrand in I_5, we obtain

$$I_5 = \frac{1}{2c}\int_{e^{-2c\delta}}^1 \frac{dt}{\sqrt{1+\frac{\log t}{2c}-t}} - \frac{1}{8c^2}\int_{e^{-2c\delta}}^1 \frac{\log t}{\sqrt{1+\frac{\log t}{2c}-t}}\,dt + o\left(c^{-2}\right)\,. \tag{2.47}$$

We then write (2.47) as

$$I_5 = \frac{1}{2c}\left[\int_{e^{-2c\delta}}^{1} \frac{dt}{\sqrt{1-t}} + \int_{e^{-2c\delta}}^{1}\left(\frac{1}{\sqrt{1+\frac{\log t}{2c}-t}} - \frac{1}{\sqrt{1-t}}\right) dt\right] - \frac{1}{8c^2}\int_{e^{-2c\delta}}^{1} \frac{\log t}{\sqrt{1+\frac{\log t}{2c}-t}}\, dt\,. \tag{2.48}$$

Now using the identity $A^{-1/2} - B^{-1/2} = (AB)^{-1/2}(B-A)/(\sqrt{B}+\sqrt{A})$ on the middle integral in (2.48), and letting $c \to \infty$, we get

$$I_5 = \frac{1}{2c}\left[\int_0^1 \frac{dt}{\sqrt{1-t}} - \frac{1}{4c}\int_0^1 \frac{\log t}{(1-t)^{3/2}}\, dt\right] - \frac{1}{8c^2}\int_0^1 \frac{\log t}{\sqrt{1-t}}\, dt + o\left(c^{-2}\right)\,. \tag{2.49}$$

The integrals in (2.49) are given explicitly by

$$\int_0^1 \frac{\log t}{\sqrt{1-t}}\, dt = -4 + 4\log 2\,, \qquad \int_0^1 \frac{\log t}{(1-t)^{3/2}}\, dt = -4\log 2\,. \tag{2.50}$$

Finally, substituting (2.50) into (2.49), we obtain

$$I_5 = \frac{1}{c} + \frac{1}{2c^2} + o\left(c^{-2}\right)\,. \tag{2.51}$$

Finally, we calculate I_6 in (2.46c). Using the Binomial Theorem, we obtain

$$\begin{aligned} I_6 &= -\frac{1}{2c}\int_{e^{-2c\delta}}^{1} \frac{1}{1+\sqrt{1-t/(1+\frac{\log t}{2c})}}\left(1-\frac{\log t}{2c}\right) dt + o\left(c^{-2}\right)\,, \\ &= -\frac{1}{2c}\int_{e^{-2c\delta}}^{1} \frac{1}{1+\sqrt{1-t/(1+\frac{\log t}{2c})}}\, dt + \frac{1}{4c^2}\int_0^1 \frac{\log t}{1+\sqrt{1-t}}\, dt + o\left(c^{-2}\right)\,. \end{aligned} \tag{2.52}$$

The second integral in (2.52) is evaluated as

$$\frac{1}{4c^2}\int_0^1 \frac{\log t}{1+\sqrt{1-t}}\, dt = \frac{1}{4c^2}\left(-4 + \frac{\pi^2}{6} - 2\log^2 2 + 4\log 2\right)\,. \tag{2.53}$$

The first integral in (2.52) is calculated as

$$\begin{aligned} &-\frac{1}{2c}\int_{e^{-2c\delta}}^{1} \frac{1}{1+\sqrt{1-t/(1+\frac{\log t}{2c})}}\, dt \\ &= -\frac{1}{2c}\int_{e^{-2c\delta}}^{1} \frac{1}{1+\sqrt{1-t}}\, dt - \frac{1}{2c}\int_{e^{-2c\delta}}^{1}\left(\frac{1}{1+\sqrt{1-t/(1+\frac{\log t}{2c})}} - \frac{1}{1+\sqrt{1-t}}\right) dt\,, \\ &= -\frac{1}{2c}(2-\log 2) + \frac{1}{4c^2}\int_0^1 \frac{t\log t}{2(1+\sqrt{1-t})^2\sqrt{1-t}}\, dt + o\left(c^{-2}\right)\,, \\ &= -\frac{1}{2c}(2-\log 2) + \frac{1}{4c^2}\left(2 - \frac{\pi^2}{6} - 2\log 2 + 2\log^2 2\right) + o\left(c^{-2}\right)\,. \end{aligned}$$

Combining this last expression with (2.53), we obtain from (2.52) that

$$I_6 = -\frac{(1-\log 2)}{c} + \frac{1}{2c^2}(-1+\log 2) + o\left(c^{-2}\right) . \tag{2.54}$$

Therefore, combining (2.41a), (2.43), and (2.46a), we get

$$I_1 = 2 + I_2 + \bar{I}_2 = 2 + I_3 + I_4 + \bar{I}_2 = 2 + I_5 + I_6 + I_4 + \bar{I}_2 . \tag{2.55}$$

Finally, substituting (2.45), (2.51), and (2.54), for I_4, I_5, and I_6, respectively, we obtain

$$I_1 = 2 + \frac{\log 2}{c} + \frac{\omega}{c^2} + o\left(c^{-2}\right) , \qquad \omega \equiv \frac{1}{8}\left(\frac{\pi^2}{6} + 4\log 2 - 2\log^2 2\right) . \tag{2.56}$$

For $c \gg 1$, we have that c_b is exponentially small. Therefore, from (2.38) and (2.56), we get

$$T_b = \frac{(c-c_b)}{\sqrt{c}} I_1 \sim \sqrt{c} I_1 \sim \sqrt{c}\left(2 + \frac{\log 2}{c} + \frac{\omega}{c^2} + o\left(c^{-2}\right)\right) . \tag{2.57}$$

Setting $T_b = \varepsilon^{-1}$, we get an equation for c in terms of ε

$$\varepsilon^{-2} = 4c\left(1 + \frac{\log 2}{2c} + \frac{\omega}{2c^2}\right)^2 . \tag{2.58}$$

For $\varepsilon \ll 1$, the solution to (2.58) is

$$c \sim \frac{1}{4\varepsilon^2} - \log 2 - \left(\frac{\pi^2}{12} + 2\log 2\right)\varepsilon^2 . \tag{2.59}$$

Recalling that $c = -\log b$ from (2.37), and $U(-1) = a = e^{1/2} b$, we conclude for $\varepsilon \ll 1$ that

$$U(-1) \sim 2e^{1/2} e^{-1/(4\varepsilon^2)}\left[1 + \left(\frac{\pi^2}{12} + 2\log 2\right)\varepsilon^2\right] . \tag{2.60}$$

By symmetry, we have $U(1) = U(-1)$. We observe that (2.60) clearly agrees with the formally obtained result (2.22a) with $x_0 = 0$.

3 The Spectral Problem

In this section we analyze the eigenvalue problem associated with linearizing (2.1) around an equilibrium one-spike solution. Let U denote the equilibrium one-spike solution of §2. The associated eigenvalue problem is

$$L_\varepsilon \phi \equiv \varepsilon^2 \phi_{xx} + (\log U + 1)\,\phi = \lambda\phi , \qquad -1 < x < 1 ; \qquad \phi_x(\pm 1) = 0 . \tag{3.1}$$

If we use only the outer approximation $U \sim U_{c0} = e^{1/2} e^{-x^2/(4\varepsilon^2)}$, we obtain that (3.1) becomes

$$\varepsilon^2 \phi_{xx} + \left(\frac{3}{2} - \frac{x^2}{4\varepsilon^2} \right) \phi = \lambda \phi \,, \qquad -1 < x < 1 \,; \qquad \phi_x(\pm 1) = 0 \,. \tag{3.2}$$

We now look for eigenfunctions of (3.2) that decay exponentially away from $x = 0$. To do so, we introduce $\Phi(y)$ and y by $\Phi(y) = \phi(\varepsilon y)$ and $y = \varepsilon^{-1} x$. Then, (3.2) becomes

$$\Phi_{yy} + \left(\frac{3}{2} - \frac{y^2}{4} \right) \Phi = \lambda^\infty \Phi \,, \qquad -\infty < y < \infty \,; \qquad \Phi \to 0 \,, \quad |y| \to \infty \,. \tag{3.3}$$

Setting $\Phi(y) = e^{-y^2/4} H(y)$, we obtain that $H(y)$ satisfies Hermite's equation

$$H_{yy} - y H_y = (\lambda^\infty - 1) \,, \qquad -\infty < y < \infty \,. \tag{3.4}$$

Bounded solutions to (3.4) exist when $\lambda_N^\infty = 1 - N$, where $N = 0, 1, 2, \ldots$, and they are given by the Hermite polynomials $\mathrm{He}_N(y)$. We normalize the solution by setting $\int_{-1}^{1} \phi_N^2 \left[\varepsilon^{-1} x\right] dx = 1$. By recalling the well-known formula $\int_{-\infty}^{\infty} \left[\mathrm{He}_N(y)\right]^2 e^{-y^2/2} \, dy = \sqrt{2\pi} N!$, we obtain that the normalized eigenfunctions and eigenvalues are

$$\Phi_N(y) = \frac{(2\pi)^{-1/4}}{\sqrt{\varepsilon N!}} \mathrm{He}_N(y) e^{-y^2/4} \,, \qquad \lambda_N^\infty = 1 - N \,, \qquad N = 0, 1, 2, \ldots \,. \tag{3.5}$$

The first few Hermite polynomials are $\mathrm{He}_0(y) = 1$, $\mathrm{He}_1(y) = y$, and $\mathrm{He}_2(y) = y^2 - 1$. Hermite polynomials are also known to play a key role in the well-known boundary-layer resonance phenomena (cf. (4), (11)).

In particular, the result (3.5) for the infinite-line eigenvalue problem (3.3) shows that the only non-negative eigenvalues are $\lambda_0^\infty = 1$ and $\lambda_1^\infty = 0$. Since $\Phi_N(y)$ decays exponentially as $|y| \to \infty$, the effect of the finite domain boundary conditions at $y = \pm\varepsilon^{-1}$, will be to perturb the eigenvalues in (3.5) by exponentially small terms as $\varepsilon \to 0$. Since $\lambda_1^\infty = 0$, this suggests that the finite domain problem will have an asymptotically exponentially small eigenvalue.

We now derive a leading-order estimate for λ_N by solving (3.2) in terms of special functions. We emphasize that the eigenvalue problem (3.2) uses only the outer solution and neglects the boundary layers in U near $x = \pm 1$. We begin by recalling that Weber's equation is $\phi_{yy} - (a + y^2/4)\, \phi = 0$. For $a \neq -N - 1/2$, where $N = 0, 1, 2, \ldots$, two linearly independent solutions are the parabolic cylinder functions $U(a, y)$ and $U(a, -y)$ (cf. (1)). Their asymptotic behavior is (cf. (1)),

$$U(a, y) \sim y^{-a-\frac{1}{2}} e^{-y^2/4} \,, \qquad U'(a, y) \sim -\frac{1}{2} y^{-a+1/2} e^{-y^2/4} \,, \qquad y \to \infty \,, \tag{3.6a}$$

$$U(a, y) \sim \frac{\sqrt{2\pi}}{\Gamma(a + \frac{1}{2})} |y|^{a-\frac{1}{2}} e^{y^2/4} \,, \qquad U'(a, y) \sim -\frac{\sqrt{2\pi}}{2\Gamma(a + \frac{1}{2})} |y|^{a+\frac{1}{2}} e^{y^2/4} \,, \qquad y \to -\infty \,. \tag{3.6b}$$

Here $\Gamma(z)$ is the Gamma function. The solution to (3.2) is $\phi(y) = c_1 U(a, y) + c_2 U(a, -y)$, where $a = -\frac{3}{2} + \lambda$ and $y = x/\varepsilon$. In order to satisfy the boundary conditions $\phi_y = 0$ at $y = \pm\varepsilon^{-1}$, we require that the determinant vanishes

$$\begin{vmatrix} U'(a, -\frac{1}{\varepsilon}) & U'(a, \frac{1}{\varepsilon}) \\ U'(a, \frac{1}{\varepsilon}) & U'(a, -\frac{1}{\varepsilon}) \end{vmatrix} = \left(U'(a, -\frac{1}{\varepsilon})\right)^2 - \left(U'(a, \frac{1}{\varepsilon})\right)^2 = 0\,. \tag{3.7}$$

For $\varepsilon \ll 1$, we substitute (3.6) into (3.7) to derive the asymptotic eigenvalue relation

$$\frac{1}{\Gamma(\lambda - 1)} \sim \frac{1}{\sqrt{2\pi}} \varepsilon^{2\lambda - 3} e^{-1/2\varepsilon^2}\,. \tag{3.8}$$

Since $\Gamma(z)$ is unbounded at the non-negative integers, we conclude from (3.8) that λ must be close to $1, 0, -1, -2\ldots$.

Therefore, let $h \ll 1$ and set $\lambda = 1 - N - h$, where $N = 0, 1, 2, \ldots$, in (3.8). To estimate $\Gamma(-N-h)$ we use the identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ to derive

$$\Gamma(-N - h) \sim \frac{(-1)^{N+1}}{h\Gamma(N+1)} = \frac{(-1)^{N+1}}{hN!}\,, \qquad h \to 0\,. \tag{3.9}$$

Using (3.9) in (3.8), we calculate h. In this way, we obtain the following leading-order estimate for the eigenvalues of the finite-domain problem (3.2):

$$\lambda_N \sim \lambda_N^{\infty} + \frac{(-1)^N}{\sqrt{2\pi} N!} \varepsilon^{-2N-1} e^{-1/2\varepsilon^2}\,, \qquad N = 0, 1, 2, \ldots\,. \tag{3.10}$$

Here $\lambda_N^{\infty} = 1 - N$. Notice that λ_1 is asymptotically exponentially small as $\varepsilon \to 0$, with an estimate $\lambda_1 = O\left(\varepsilon^{-3} e^{-1/(2\varepsilon^2)}\right)$.

3.1 An Asymptotically Exponentially Small Eigenvalue

We now calculate the exponentially small eigenvalue λ_1 more precisely. A key point is that, since (3.10) is based on using only the outer approximation U_{c0} to the equilibrium solution, (3.10) provides at most a leading-order estimate for λ_1. By incorporating the effects of the boundary layers near $x = \pm 1$ in U, we now show that $\lambda_1 = O\left(\varepsilon^{-3} e^{-1/(2\varepsilon^2)}\right)$ as $\varepsilon \to 0$, but with a different pre-exponential factor than in (3.10).

In this subsection, we simplify the notation by letting ϕ denote the eigenfunction of (3.1) with exponentially small eigenvalue λ_1. We now calculate λ_1 for the quasi-equilibrium solution of §2.

As shown below, to estimate λ_1 we must first calculate $\phi(\pm 1)$ for $\varepsilon \ll 1$ using a boundary layer analysis. In the left boundary layer we introduce

$$\phi_l(y) = \phi(-1+\varepsilon^2 y) \qquad y = \varepsilon^{-2}(1+x)\,. \tag{3.11}$$

Using $\log U = W$, where W is given in (2.8), we get that (3.1) is transformed to

$$\phi_l'' + \left(-\alpha^2 + \varepsilon^2\left[1+w_0(y)\right] + O(\varepsilon^4)\right)\phi_l = \varepsilon^2 \lambda_1 \phi_l\,, \quad 0 \le y < \infty\,; \qquad \phi_l'(0) = 0\,. \tag{3.12}$$

Here $w_0(y)$ is the boundary-layer function of (2.10), and $\alpha \equiv (1+x_0)/2$. In the outer region, away from the boundary layers, the outer eigenfunction Φ_1 is obtained by setting $N=1$ and $y = (x-x_0)/\varepsilon$ in (3.5). Near the left boundary, where $x = -1+\varepsilon^2 y$, this expression becomes

$$\Phi_1 \sim N_0 e^{\alpha y}\left[1 - \varepsilon^2\left(\frac{y^2}{4} + \frac{y}{2\alpha}\right)\right]\,, \qquad N_0 \equiv -2\,(2\pi)^{-1/4}\,\alpha\,\varepsilon^{-3/2}\,e^{-\alpha^2/\varepsilon^2}\,. \tag{3.13}$$

Equation (3.13) provides the far-field behavior of the boundary layer function ϕ_l. We then expand $\phi_l(y)$ as

$$\phi_l(y) = N_0\left[\phi_{l0}(y) + \varepsilon^2\phi_{l1}(y) + \cdots\right]\,. \tag{3.14}$$

Substituting (3.14) into (3.12) and (3.13), assuming that λ_1 is exponentially small, and collecting powers of ε, we obtain the following problems for ϕ_{l0} and ϕ_{l1} on $0 \le y < \infty$:

$$L\phi_{l0} \equiv \phi_{l0}'' - \alpha^2\phi_{l0} = 0\,; \qquad \phi_{l0}'(0) = 0\,; \qquad \phi_{l0} \sim e^{\alpha y}\,, \quad y \to \infty\,, \tag{3.15a}$$

$$L\phi_{l1} = -(1+w_0)\phi_{l0}\,; \qquad \phi_{l1}'(0) = 0\,; \qquad \phi_{l1} \sim -e^{\alpha y}\left(\frac{y^2}{4} + \frac{y}{2\alpha}\right)\,, \quad y \to \infty\,. \tag{3.15b}$$

The solution to (3.15a) is

$$\phi_{l0} = 2\cosh(\alpha y)\,. \tag{3.16}$$

Next, we calculate $\phi_{l1}(0)$ from the solution to (3.15b). To do so, we introduce $f(y)$ defined by

$$\phi_{l1}(y) = -e^{\alpha y}\left(\frac{y^2}{4} + \frac{y}{2\alpha}\right) + f(y)\,. \tag{3.17}$$

Substituting (3.17) into (3.15b), and using (2.10) and (3.16) for w_0 and ϕ_{l0}, respectively, we obtain, in terms of the operator L of (3.15a), that $f(y)$ satisfies

$$Lf = \chi(y) \equiv -\left(3 + 2\log\left[2\cosh(\alpha y)\right]\right)\cosh(\alpha y) + e^{\alpha y}\left(\alpha y + \frac{3}{2}\right)\,, \qquad 0 \le y < \infty\,, \tag{3.18a}$$

$$f(0) = \gamma\,, \qquad f'(0) = \frac{1}{2\alpha}\,, \tag{3.18b}$$

with f bounded as $y \to \infty$. A simple calculation shows that $\chi(y)$ is bounded as $y \to \infty$. To determine $\phi_{l1}(0)$ we use an integral identity. Let $h(y)$ be the solution to $Lh = 0$, with $h(0) = 1$, and $h'(0) = -\alpha$. Therefore, $h(y) = e^{-\alpha y}$. Using Green's identity on f and h, we then derive

$$\int_0^\infty hLf\,dy = f(0)h'(0) - h(0)f'(0) = -\alpha f(0) - f'(0)\,. \tag{3.19}$$

In (3.19) we set $f(0)=\gamma$, and $f'(0)=1/(2\alpha)$, and we use (3.18a) for Lf. This determines γ as

$$\gamma = -\frac{1}{2\alpha^2} - \frac{1}{\alpha}\int_0^\infty e^{-\alpha y}\chi(y)\,dy\,. \tag{3.20}$$

Then, using (3.18a) for $\chi(y)$, and letting $x=\alpha y$, we find that (3.20) reduces to

$$\gamma = -\frac{1}{2\alpha^2} + \frac{1}{\alpha^2}\left[\int_0^\infty \left(\frac{3}{2}+x\right)e^{-2x}\,dx + \int_0^\infty \left(1+e^{-2x}\right)\log\left(1+e^{-2x}\right)\,dx\right]. \tag{3.21}$$

The second integral in (3.21) was evaluated in (2.19) and (2.20). In this way, we get

$$\gamma = \frac{1}{2\alpha^2}\left(\frac{\pi^2}{12}+2\log 2\right). \tag{3.22}$$

Finally, we can estimate $\phi(-1)$ from (3.14). Using $\alpha=(1+x_0)/2$, $\phi_{l0}(0)=2$, and $\phi_{l1}(0)=\gamma$, where γ is given in (3.22), we obtain for $\varepsilon \ll 1$ that

$$\phi(-1) \sim -2\,(2\pi)^{-1/4}\,\varepsilon^{-3/2}e^{-(1+x_0)^2/(4\varepsilon^2)}(1+x_0)\left[1+\frac{\varepsilon^2}{(1+x_0)^2}\left(\frac{\pi^2}{12}+2\log 2\right)+\cdots\right]. \tag{3.23a}$$

A similar boundary layer analysis can be done near the right boundary at $x=1$ to estimate $\phi(1)$. For $\varepsilon \ll 1$, we find that

$$\phi(1) \sim 2\,(2\pi)^{-1/4}\,\varepsilon^{-3/2}e^{-(1-x_0)^2/(4\varepsilon^2)}(1-x_0)\left[1+\frac{\varepsilon^2}{(1-x_0)^2}\left(\frac{\pi^2}{12}+2\log 2\right)+\cdots\right]. \tag{3.23b}$$

Next, we obtain our estimate for λ_1. Notice that U_x is a solution of (3.1) with $\lambda=0$. Therefore, using Green's identity on U_x and ϕ, and letting L_ε be the operator in (3.1), we obtain that

$$\int_{-1}^{1}\left(\phi L_\varepsilon U_x - U_x L_\varepsilon \phi\right)\,dx = -\lambda_1\int_{-1}^{1}U_x\phi\,dx = \varepsilon^2\left(\phi U_{xx}-U_x\phi_x\right)\big|_{-1}^{1}\,. \tag{3.24}$$

Using $\phi_x(\pm 1)=0$ and $\varepsilon^2 U_{xx}=-U\log U$, we obtain from (3.24) that

$$\lambda_1 J = \phi(1)U(1)\log U(1) - \phi(-1)U(-1)\log U(-1)\,, \qquad J \equiv \int_{-1}^{1}U_x\phi\,dx\,. \tag{3.25}$$

The right-hand side of (3.25) is evaluated from our estimates of $U(\pm 1)$ and $\phi(\pm 1)$ given in (2.22) and (3.23), respectively. A little calculation using (2.22) shows, for $\varepsilon \ll 1$, that

$$U(\pm 1)\log U(\pm 1) \sim -\frac{(1\mp x_0)^2}{2\varepsilon^2}e^{1/2}e^{-(1\mp x_0)^2/(4\varepsilon^2)}\left[1+\frac{\varepsilon^2}{(1\mp x_0)^2}\left(\frac{\pi^2}{12}-2\log 2-2\right)\right]. \tag{3.26}$$

To calculate J in (3.25), we notice that the support of the integrand in J is concentrated in a narrow zone of width $O(\varepsilon)$ near $x=x_0$. In this region we use $U \sim U_{c0} = e^{1/2}e^{-(x-x_0)^2/(4\varepsilon^2)}$ and

ε	$U(-1)$ (asy)	$U(-1)$ (num)	λ_1 (asy)	λ_1 (num)
0.250	0.68732×10^{-1}	0.81006×10^{-1}	0.33500×10^{-1}	0.42713×10^{-1}
0.230	0.32639×10^{-1}	0.35991×10^{-1}	1.0109×10^{-2}	0.11750×10^{-1}
0.210	0.12490×10^{-1}	0.13223×10^{-1}	0.20206×10^{-2}	0.22008×10^{-2}
0.190	0.34988×10^{-2}	0.36165×10^{-2}	0.22180×10^{-3}	0.23234×10^{-3}
0.180	0.15746×10^{-2}	0.16147×10^{-2}	0.53710×10^{-4}	0.55572×10^{-4}
0.170	0.61401×10^{-3}	0.62585×10^{-3}	0.98491×10^{-5}	0.10098×10^{-4}
0.160	0.19994×10^{-3}	0.20284×10^{-3}	0.12715×10^{-5}	0.12950×10^{-5}
0.150	0.51730×10^{-4}	0.52288×10^{-4}	0.10477×10^{-6}	0.10617×10^{-6}
0.145	0.23652×10^{-4}	0.23871×10^{-4}	0.24410×10^{-7}	0.24687×10^{-7}
0.140	0.99337×10^{-5}	0.10012×10^{-4}	0.48151×10^{-8}	0.48614×10^{-8}

Table 1
Comparison of the asymptotic results for $U(-1)$ and λ_1 given in (2.22a) and (3.29), respectively, with corresponding full numerical results.

(3.5) for ϕ (with $N = 1$ and $y = \varepsilon^{-1}(x - x_0)$). Then, by using Laplace's method, we derive for $\varepsilon \ll 1$ that

$$J \sim -\frac{(2\pi)^{-1/4} e^{1/2}}{2\varepsilon^{3/2}} \int_{-\infty}^{\infty} \left(\frac{x - x_0}{\varepsilon} \right)^2 e^{-(x-x_0)^2/(2\varepsilon^2)} \, dx \sim -\frac{(2\pi)^{1/4} e^{1/2}}{2\sqrt{\varepsilon}} \,. \tag{3.27}$$

Finally, we substitute (3.23), (3.26), and (3.27), into (3.25). In this way, we obtain the following asymptotic estimate for λ_1 when $\varepsilon \ll 1$:

$$\lambda_1 \sim \sqrt{\frac{2}{\pi}} \varepsilon^{-3} \left((1 - x_0)^3 e^{-(1-x_0)^2/(2\varepsilon^2)} \left[1 + \frac{\varepsilon^2}{(1 - x_0)^2} \left(\frac{\pi^2}{6} - 2 \right) \right] \right. \\ \left. + (1 + x_0)^3 e^{-(1+x_0)^2/(2\varepsilon^2)} \left[1 + \frac{\varepsilon^2}{(1 + x_0)^2} \left(\frac{\pi^2}{6} - 2 \right) \right] \right) \,. \tag{3.28}$$

For the true equilibrium solution where $x_0 = 0$, (3.28) reduces to

$$\lambda_1 \sim 2\sqrt{\frac{2}{\pi}} \varepsilon^{-3} e^{-1/(2\varepsilon^2)} \left[1 + \varepsilon^2 \left(\frac{\pi^2}{6} - 2 \right) \right] \,. \tag{3.29}$$

The estimate (3.29) predicts again that $\lambda_1 = O\left(\varepsilon^{-3} e^{-1/(2\varepsilon^2)}\right)$, but the pre-exponential factor is different both in magnitude and sign from that given in (3.10). Since $\lambda_1 > 0$, but is exponentially small, we conclude that an equilibrium parabolic flame-front for (1.1), with the tip at the centerline of the channel, is unstable but with an exponentially slow growth rate.

We validate (2.22a) for $U(-1)$ and (3.29) for λ_1 by using the boundary value solver COLSYS (3) to numerically compute an equilibrium one-spike solution to (2.1) and the associated eigenvalue

λ_1 of (3.1). For small values of ε, in Table 1 we show a very favorable comparison between the asymptotic results for $U(-1)$ and λ_1 and corresponding full numerical results.

Finally, we can easily calculate the fundamental eigenvalue λ_0 corresponding to the linearization around the true equilibrium solution with a spike at the origin. By combining (3.1) with $\varepsilon^2 U_{xx} + U \log U = 0$, we readily conclude, up to a normalization constant, that

$$\phi_0 \equiv U\,, \quad \lambda_0 \equiv 1\,. \tag{3.30}$$

Recall that the estimate in (3.10), which was based on using only the outer approximation to the eigenfunction, gave $\lambda_0 \sim 1 + O\left(\varepsilon^{-1} e^{-1/(2\varepsilon^2)}\right)$.

3.2 *Dynamic Metastability*

We now derive an ODE for the location $x_0(t)$ of the tip of the parabolic flame-front. Since a parabolic flame-front is mapped onto a one-spike quasi-equilibrium solution for (1.7), we look for a time-dependent solution to (1.7) in the form

$$u(x,t) = U_\varepsilon\left[x; x_0(t)\right] + E(x,t)\,. \tag{3.31}$$

Here U_ε denotes the quasi-equilibrium solution with a spike-layer at some time-dependent location $x = x_0(t)$. We will assume that $x_0' \ll 1$ as $\varepsilon \to 0$. As shown in §2, $U_\varepsilon \sim U_{c0} \equiv e^{1/2} e^{-(x-x_0)^2/(4\varepsilon^2)}$ in the outer region, and U_ε has boundary layers of width $O(\varepsilon^2)$ near $x = \pm 1$. In (3.31), we assume that the error E satisfies $E \ll 1$. Substituting (3.31) into (1.7), we obtain that E satisfies

$$E_t = L_\varepsilon E - \partial_t U_\varepsilon + R\,, \tag{3.32a}$$

$$E_x(\pm 1, t) = -U_{\varepsilon x}\big|_{-1}^{1}\,, \qquad E(x,0) = 0\,. \tag{3.32b}$$

Here L_ε and the residual R are defined by

$$L_\varepsilon E \equiv \varepsilon^2 E_{xx} + \left(\log U_\varepsilon + 1\right) E\,, \qquad R \equiv \varepsilon^2 U_{\varepsilon xx} + U_\varepsilon \log U_\varepsilon\,. \tag{3.32c}$$

Recall from §3.1 that $L_\varepsilon \phi = \lambda \phi$, with $\phi_x(\pm 1) = 0$, has one $O(1)$ positive eigenvalue λ_0 and one asymptotically exponentially small eigenvalue λ_1.

Since $\lambda_0 > 0$ and $\lambda_0 = O(1)$ we might naively expect that the error E, satisfying (3.32), grows exponentially on an $O(1)$ time-scale for any $x_0(t)$. However, we now show that this is not the case. We decompose E in an eigenfunction expansion as $E(x,t) = \sum_{j=0}^{\infty} c_j(t)\phi_j$ for some coefficients $c_j(t)$ with $c_j(0) = 0$. We want to show that $c_0(t)$ does not grow exponentially fast in t. Introducing the inner product $(f,g) \equiv \int_{-1}^{1} fg\,dx$, we multiply (3.32a) by ϕ_0 and integrate over the domain.

Upon integrating by parts, we get

$$c_0' - \lambda_0 c_0 = -\left(\phi_0, \partial_t U_\varepsilon\right) - \varepsilon^2 \phi_0 U_{\varepsilon x}|_{-1}^{1} + \int_{-1}^{1} \left(\varepsilon^2 U_{\varepsilon xx} + U_\varepsilon \log U_\varepsilon\right) \phi_0 \, dx \,. \tag{3.33}$$

A further integration by parts yields

$$\begin{aligned} c_0' - \lambda_0 c_0 &= -\left(\phi_0, \partial_t U_\varepsilon\right) + \int_{-1}^{1} \left[\varepsilon^2 \phi_{0xx} + \left(\log U_\varepsilon\right) \phi_0\right] U_\varepsilon \, dx \,, \\ &= -\left(\phi_0, \partial_t U_\varepsilon\right) + \left(\lambda_0 - 1\right) \int_{-1}^{1} \phi_0 U_\varepsilon \, dx \,. \end{aligned} \tag{3.34}$$

In obtaining (3.34), we used the equation satisfied by ϕ_0. Since $\phi_0 \sim U_{c0}$ and $U_{\varepsilon t} \sim -\varepsilon^{-1} x_0' U_{c0}'$, it follows that $(\phi_0, \partial_t U_\varepsilon) \sim -\varepsilon^{-1} x_0' \left(U_{c0}, U_{c0}'\right)$. Since U_{c0} is localized near $x = x_0$ and is even, we conclude that $(\phi_0, \partial_t U_\varepsilon)$ is the product of x_0' and an exponentially small term. Furthermore, $\lambda_0 - 1$ is exponentially small. Hence, the right hand-side of (3.34) is exponentially small. Therefore, despite the fact that $\lambda_0 \sim 1$, there is no exponential growth in c_0 on an $O(1)$ time-scale.

Next, since λ_1 is exponentially small, we have that E is quasi-steady. Hence, we set $E_t \ll 1$ in (3.32a). We then multiply (3.32a) by ϕ_1 and integrate over the domain to get

$$\left(\phi_1, L_\varepsilon E\right) = \varepsilon^2 \left(\phi_1 E_x - \phi_{1x} E\right) |_{-1}^{1} + \lambda_1 \left(E, \phi_1\right) \,. \tag{3.35}$$

Using (3.32a) for $L_\varepsilon E$, and $\phi_{1x}(\pm 1) = 0$, we then obtain

$$\left(E, \phi_1\right) = \lambda_1^{-1} \left[-\varepsilon^2 \phi_1 E_x|_{-1}^{1} + \left(\phi_1, \partial_t U_\varepsilon\right) - \left(R, \phi_1\right)\right] \,. \tag{3.36}$$

Since $\lambda_1 \to 0$ exponentially as $\varepsilon \to 0$, the limiting solvability condition is that the term in the square brackets in (3.36) vanishes as $\varepsilon \to 0$. From this condition, and using $E_x = -U_{\varepsilon x}$ at $x = \pm 1$, together with (3.32c) for R, we obtain

$$\left(\phi_1, \partial_t U_\varepsilon\right) \sim -\varepsilon^2 \phi_1 U_{\varepsilon x}|_{-1}^{1} + \int_{-1}^{1} \left(\varepsilon^2 U_{\varepsilon xx} + U_\varepsilon \log U_\varepsilon\right) \phi_1 \, dx \,. \tag{3.37}$$

Integrating by parts in the integral in (3.37), and using $\phi_{1x}(\pm 1) = 0$, we get

$$\left(\phi_1, \partial_t U_\varepsilon\right) \sim \int_{-1}^{1} \left(\varepsilon^2 \phi_{1xx} + \left(\log U_\varepsilon\right) \phi_1\right) U_\varepsilon \, dx \,. \tag{3.38}$$

Next, we use the equation for ϕ_1 to write (3.38) as

$$\left(\phi_1, \partial_t U_\varepsilon\right) \sim G(x_0) \equiv \left(\lambda_1 - 1\right) \int_{-1}^{1} \phi_1 U_\varepsilon \, dx \,. \tag{3.39}$$

This is an ODE for $x_0(t)$. In particular, the left-hand side is proportional to x_0', while each term on the right-hand side depends on x_0.

Finally, we derive a simple expression for $G'(x_0)$ in (3.39). By differentiating the equation for ϕ_1 with respect to x_0, we get

$$M\phi_{1x_0} \equiv \phi''_{1x_0} + (\log U_\mathcal{E})\,\phi_{1x_0} = -\phi_{1x_0} - (\log U_\mathcal{E})_{x_0}\,\phi_1 + \lambda_{1x_0}\phi_1 + \lambda_1\phi_{1x_0}\,, \tag{3.40}$$

with $\phi'_{1x_0}(\pm 1) = 0$. Here the primes indicate derivatives with respect to x. Since $MU_\mathcal{E} = 0$, we obtain from a solvability condition on (3.40) that the right hand-side must be orthogonal to $U_\mathcal{E}$ with respect to the inner product (u,v). This condition can be written as

$$-\lambda_1 \int_{-1}^{1} \phi_{1x_0} U_\mathcal{E}\,dx = \lambda_{1x_0}\int_{-1}^{1}\phi_1 U_\mathcal{E}\,dx - \int_{-1}^{1}(\phi_1 U_\mathcal{E})_{x_0}\,dx\,. \tag{3.41}$$

Now differentiating $G(x_0)$ in (3.39), we get

$$G'(x_0) = \lambda_{1x_0}\int_{-1}^{1}\phi_1 U_\mathcal{E}\,dx + (\lambda_1 - 1)\int_{-1}^{1}(\phi_1 U_\mathcal{E})_{x_0}\,dx\,. \tag{3.42}$$

By comparing (3.41) and (3.42), and recalling that λ_1 is exponentially small, we conclude from (3.39) that

$$(\phi_1, \partial_t U_\mathcal{E}) \sim G(x_0)\,, \qquad G'(x_0) \sim -\lambda_1 \int_{-1}^{1}\phi_{1x_0} U_\mathcal{E}\,dx\,. \tag{3.43}$$

Finally, we note that the dominant contribution from each of the two inner product terms in (3.43) arises from the region near $x = x_0$. In this region, we use $U_\mathcal{E} \sim e^{1/2} e^{-(x-x_0)^2/(4\mathcal{E}^2)}$, together with (3.5) for the outer approximation for ϕ_1, to calculate

$$(\phi_1, \partial_t U_\mathcal{E}) \sim \frac{(2\pi)^{1/4}}{2\varepsilon^{1/2}} e^{1/2} x'_0\,, \qquad (\phi_{1x_0}, U_\mathcal{E}) \sim -\frac{(2\pi)^{1/4}}{2\varepsilon^{1/2}} e^{1/2}\,. \tag{3.44}$$

Substituting (3.44) into (3.43), and using the condition $x'_0 = 0$ when $x_0 = 0$, we get the ODE

$$x'_0 \sim \int_0^{x_0} \lambda_1(s)\,ds\,. \tag{3.45}$$

Here $\lambda_1(x_0)$ is the eigenvalue for the quasi-equilibrium solution given in (3.28).

Finally, using (3.28) in (3.45), and evaluating the resulting integral for $\varepsilon \ll 1$, we obtain the following explicit asymptotic ODE for the motion of the tip of the parabolic flame-front:

$$x'_0 \sim \sqrt{\frac{2}{\pi\varepsilon^2}}\left[\left((1-x_0)^2 + \frac{\pi^2\varepsilon^2}{6}\right)e^{-(1-x_0)^2/2\mathcal{E}^2} - \left((1+x_0)^2 + \frac{\pi^2\varepsilon^2}{6}\right)e^{-(1+x_0)^2/2\mathcal{E}^2}\right]. \tag{3.46}$$

This is the ODE given in (1.2) of §1.

4 Conclusion

We have shown that the nonlinear transformation (1.4), followed by the time-dependent transformation (1.6), reduces the nonlocal flame-front evolution equation (1.1) to the quasilinear PDE (1.7). The steady-state problem for (1.7) is closely related to Carrier's original singular perturbation problem with spike-layer solutions (7). The metastable flame-front behavior for (1.1) was studied by first constructing a one-spike equilibrium and quasi-equilibrium solution to the transformed problem (1.7). The spectrum of the linearized problem was then analyzed for $\varepsilon \ll 1$.

A key feature of this approach is that, unlike the transformation (1.3), our transformation readily extends to the two-dimensional case. For a channel with a constant cross-section Ω in the $x \equiv (x_1, x_2)$ plane, the two-dimensional extension of the flame-front interface equation (1.1) is

$$S_t - \frac{1}{2}|\nabla S|^2 = \varepsilon^2 \triangle S + S - \frac{1}{|\Omega|}\int_\Omega S\, dx\,, \quad x \in \Omega\,; \quad \partial_n S = 0\,, \quad x \in \partial\Omega\,. \tag{4.1}$$

Here $|\Omega|$ is the area of the cross-section, and ∂_n is the outward normal derivative. For this model, the flame-front interface assumes a roughly paraboloidal shape and the tip of the paraboloid moves very slowly towards the wall of the channel. Introducing the change of variables

$$S(x,t) = 2\varepsilon^2 \log v(x,t)\,, \quad v(x,t) = f(t)u(x,t)\,, \quad f(t) \equiv \exp\left(-\int_0^t \langle \log u(x,\tau)\rangle\, d\tau\right)\,, \tag{4.2}$$

where $\langle w \rangle \equiv |\Omega|^{-1}\int_\Omega w\, dx$, it is readily shown that (4.1) transforms to the quasilinear PDE

$$u_t = \varepsilon^2 \triangle u + u \log u \quad x \in \Omega\,; \quad \partial_n u = 0\,, \quad x \in \partial\Omega\,. \tag{4.3}$$

The steady-state problem for (4.3) admits a spike solution, and is closely related to 'point-condensation' problems (cf. (15)). Work is in progress to analyze one-spike equilibria of (4.3) and the spectrum of the associated linearized operator.

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