Asymptotics of Nonlinear Biharmonic Eigenvalue Problems of MEMS

Alan E. Lindsay and Michael J. Ward (UBC)

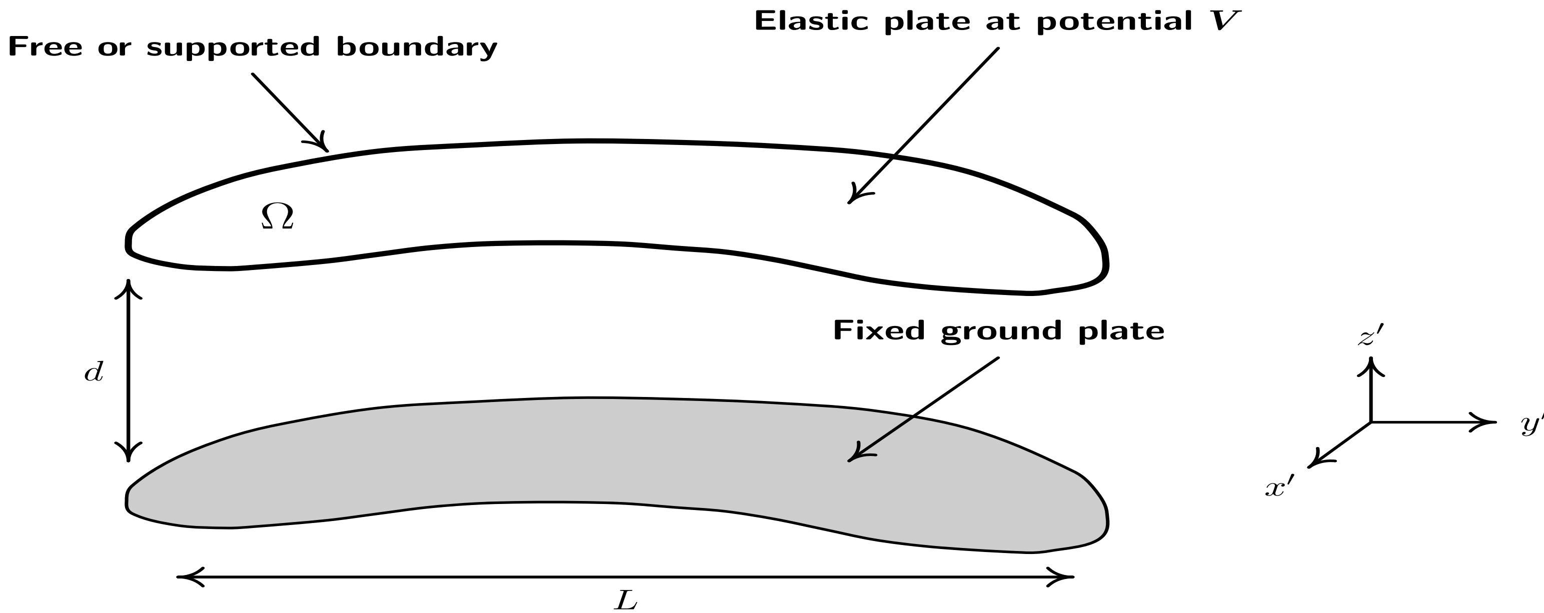
CAIMS 2010: July 18, 2010

Outline of the Talk

Nonlinear Biharmonic Eigenvalue Problems of MEMS

- Overview of Nonlinear Eigenvalue Problems of MEMS
- Calculation of the Pull-In Voltage Threshold. This has practical engineering applications.
- Concentration Behavior and Asymptotics of the Maximal Solution Branch. Of more mathematical interest in PDE.

A MEMS Capacitor



- Beam or plate deflecting in the presence of an electric field.
- Top plate will contact with lower plate (i.e. touchdown) when $V > V^*$.
- Device can act as a switch, valve or just capacitor.
- if $V > V^*$, then no stable steady-state solutions. The threshold V^* is called the pull-in voltage threshold.

The Mathematical Model of Pelesko

For small aspect ratio, the plate deflection satisfies Pelesko (2000):

$$u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}\left(1 + \beta |\nabla u|^2\right) \,, \quad x \in \Omega \in \mathbb{R}^2 \,,$$
$$u = u_n = 0 \,, \quad x \in \partial\Omega \,.$$

- singular nonlinearity represents a Coulomb attractive force between the deflectable surface and the fixed ground plate.
- nonlinear eigenvalue parameter λ is proportional to V^2.
- parameter β represents fringing-field effect due to the finite length of capacitor (Pelesko, Driscoll, J. Eng. Math, (2005).)
- Parameter δ represents bending rigidity.

Main Questions:

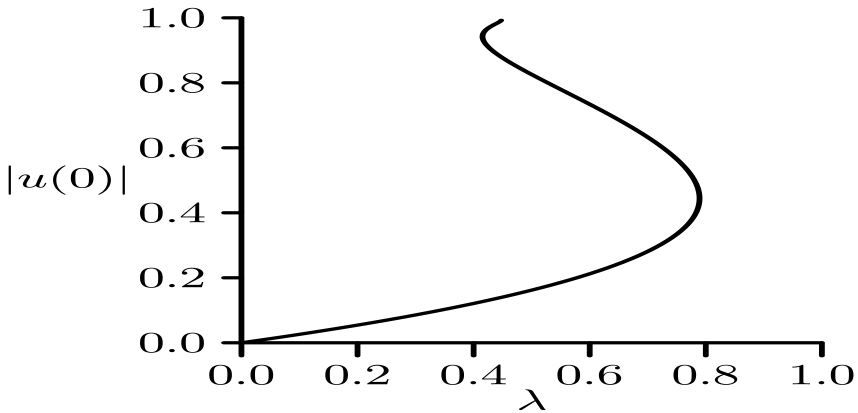
- **Pull-in Threshold:** Of importance for applications is the saddle-node point at the end of the minimal solution branch for $|u|_\infty$ vs. λ..
- **Solution Multiplicity:** how does the global bifurcation diagram depend on δ and on β?

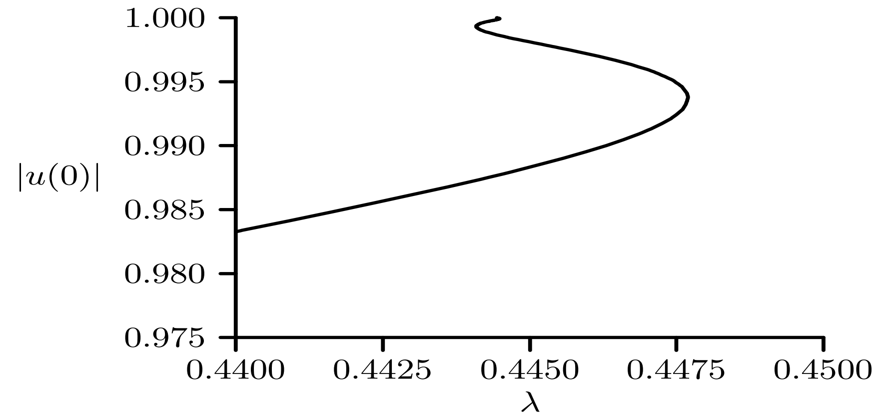
The Basic Membrane Problem

Pelesko, SIAP, (2000) considered the basic membrane problem

$$\Delta u = \frac{\lambda}{(1+u)^2}\,, \quad x \in \Omega\,; \quad u = 0\,, \quad x \in \partial\Omega\,.$$

For the unit disk the numerically computed bifurcation diagram is:





Left: Bifurcation diagram **Right:** Zoom of left figure.

Key Features:

- In unit disk there is an infinite number of fold points with limiting behavior $\lambda \to 4/9$ as $u(0) + 1 = \varepsilon \to 0^+$.
- In contrast, for the unit slab there is either zero, one, or two steady-state solutions.

Membrane Problem in General Domains

For a general 2-D domain, the following are rigorous results:

- **Theorem [Pelesko, SIAP, (2002)]**: *Let μ_0 be the first eigenvalue of the Laplacian, then there is no steady-state solution for $\lambda > \lambda_*$, where*

$$\lambda_* \leq \bar{\lambda}_1 \equiv \frac{4\mu_0}{27} \,.$$

- The lower (minimal) solution branch is linearly stable (N. Ghoussoub, Y. Guo, SIMA, (2007)).
- For $\lambda \ll 1$, there is a unique solution, and there are an infinite number of fold points for λ (Z. Guo, J. Wei, J. Lond. Math. Soc., (2008) (with no guarantee of clustering at some critical value of λ).
- extensions to N-dimensions with radial symmetry (Ghoussoub, Guo).

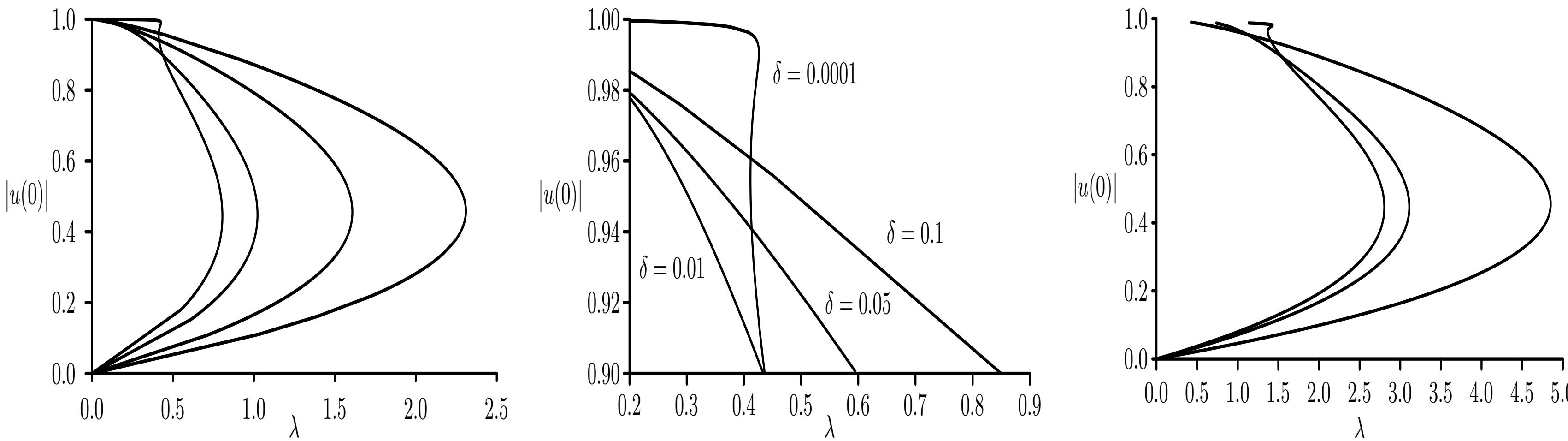
Open Question in 2-D: Does $\lambda \to \lambda_c$ as $|u|_\infty \to 1^-$, with an arbitrarily large number of fold points in a sufficiently small neighbourhood of some λ_c? If so, calculate λ_c, and x_0 for which $u(x_0) + 1 = \varepsilon \to 0^+$, and describe the asymptotics of the solution branch for λ near λ_c.

Remark: This concentration question has nothing to do with critical points of regular part of a Green's function.

Perturbation of the Membrane Problem by δ

$$u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}\,, \quad x \in \Omega\,; \quad u = u_n = 0\,, \quad x \in \partial\Omega\,.$$

Numerical computations with either shooting or psuedo-arclength yield:



Left and Middle (Unit Disk): $\delta = 0.0001,\ 0.01,\ 0.05,\ 0.1$. **Right (Unit Square):** $\delta = 0.0001,\ 0.001,\ 0.01$.

- Infinite fold point structured destroyed when $\delta > 0$.
- Maximal solution branch with $\lambda \to 0$ and $|u|_\infty \to 1^-$.
- Pull-in voltage increases with δ
- Similar phenomena under effect of fringing field with $\beta > 0$

Bounds for the Pull-In Voltage

Ref: [LW1]: A. Lindsay, MJW, *Asymptotics of Some Nonlinear Eigenvalue Problems for a MEMS Capacitor: Part I: Fold Point Asymptotics*, Methods and Applications of Analysis, (2008), (28 pages)

Theorem [LW1]: *Let Ω be the unit slab or the unit disk, and let $\mu_0 > 0$ be the first eigenvalue of*

$$-\delta \Delta^2 \phi + \Delta \phi = -\mu \phi \,, \quad x \in \Omega \,; \qquad \phi = \partial_n \phi = 0 \,, \quad x \in \partial\Omega \,.$$

Then, there is no steady-state solution for $\lambda > \lambda_*$, where $\lambda_* \leq \bar{\lambda} \equiv 4\mu_0/27$.

- **Key:** proof requires positivity of first eigenfunction, which is guaranteed for slab and disk, but not other domains.
- For the unit disk, we have

$$\phi = J_0(\xi_- r) - \frac{J_0(\xi_-)}{I_0(\xi_+)} I_0(\xi_+ r) \,, \qquad \xi_\pm \equiv \sqrt{\frac{\pm 1 + \sqrt{1 + 4\mu\delta}}{2\delta}} \,,$$

  where $\mu_0 > 0$ is the smallest root of $\xi_+ I_1(\xi_+) + \xi_- \frac{I_0(\xi_+)}{J_0(\xi_-)} J_1(\xi_-) = 0$.
- Similar formula can be derived for the unit slab.

Asymptotics for the Pull-In Voltage

	Slab		Unit Disk	
δ	$\bar{\lambda}$	λ_c	$\bar{\lambda}$	λ_c
0.25	20.3576	19.249	4.886	4.395
0.5	38.900	36.774	8.754	7.871
1.0	75.979	71.823	16.486	14.826
2.0	150.137	141.918	31.948	28.704

Asymptotics of the Pull-In Threshold for $\delta \ll 1$: Let $\alpha = ||u||_\infty$. Assume that we know the fold point $(\lambda_0(\alpha_0), \alpha_0)$ for the unperturbed problem

$$\Delta u_0 = \frac{\lambda_0}{(1+u_0)^2}\,, \quad x \in \Omega\,; \quad u_0 = 0 \quad x \in \partial\Omega\,.$$

- **Goal:** Derive formulae for the corrections to the fold point for an arbitrary 2-D domain. How does it depend on curvature of $\partial\Omega$?
- Then, calculate coefficients in this expansion for slab and disk.

Asymptotics for the Pull-In Voltage

For $\delta \ll 1$, there is a $\mathcal{O}(\delta^{1/2})$ boundary layer near $\partial\Omega$. Away from $\partial\Omega$ we expand

$$u = u_0 + \delta^{1/2} u_1 + \delta u_2 + \cdots \,, \qquad \lambda = \lambda_0 + \delta^{1/2}\lambda_1 + \delta\lambda_2 + \cdots \,,$$

to derive PDE's for u_1 and u_2 with effective boundary conditions from matching to the boundary layer solution. For $\delta \ll 1$, the fold point location, defined by $d\lambda/d\alpha = 0$, is

$$\lambda_c = \lambda_{0c} + \delta^{1/2}\lambda_1(\alpha_0) + \delta\left[\lambda_2(\alpha_0) - \frac{\lambda_{1\alpha}^2(\alpha_0)}{2\lambda_{0\alpha\alpha}(\alpha_0)}\right] + \mathcal{O}(\delta^{3/2})\,.$$

At the unperturbed fold point $\alpha_0, \lambda_{0c} \equiv \lambda_0(\alpha_0)$, the linearized operator

$$\mathcal{L}\phi \equiv \Delta\phi + \frac{2\lambda_0}{(1+u_0)^3}\phi = \frac{\lambda_{0\alpha}}{(1+u_0)^2}\,,$$

has the one-dimensional nullspace $\phi = u_{0\alpha}$. By invoking solvability conditions to evaluate the various terms:

$$\lambda_c = 1.4 + 5.6\delta^{1/2} + 25.45\delta + \mathcal{O}(\delta^{3/2})\,, \quad \text{(Unit Slab)}$$

$$\lambda_c = 0.789 + 1.578\delta^{1/2} + 6.26\delta + \mathcal{O}(\delta^{3/2})\,, \quad \text{(Unit Disk)}$$

Asymptotics for the Pull-In Voltage

For $\delta \to 0$ in an arbitrary 2-D domain, we obtain

Principal Result: [LW1]: *Let Ω have a smooth boundary. Then, for $\delta \ll 1$,*

$$\lambda_c = \lambda_{0c} + 3\lambda_0 \delta^{1/2} \left(\frac{\int_{\partial\Omega} (\partial_n u_0)(\partial_n u_{0\alpha}) \, dx}{\int_{\partial\Omega} \partial_n u_{0\alpha} \, dx} \right) + \delta \Lambda_2 + \mathcal{O}(\delta^{3/2}) \,,$$

where Λ_2 involves the curvature of the boundary $\partial\Omega$.

Asymptotics of the Pull-In Threshold for $\delta \gg 1$: We first write

$$-\Delta^2 u + \frac{1}{\delta}\Delta u = \frac{\tilde{\lambda}}{(1+u)^2} \,, \quad x \in \Omega \,; \quad u = \partial_n u = 0 \,, \quad x \in \partial\Omega \,,$$

where $\tilde{\lambda} \equiv \lambda/\delta$. With u_0 satisfying pure Biharmonic, we expand

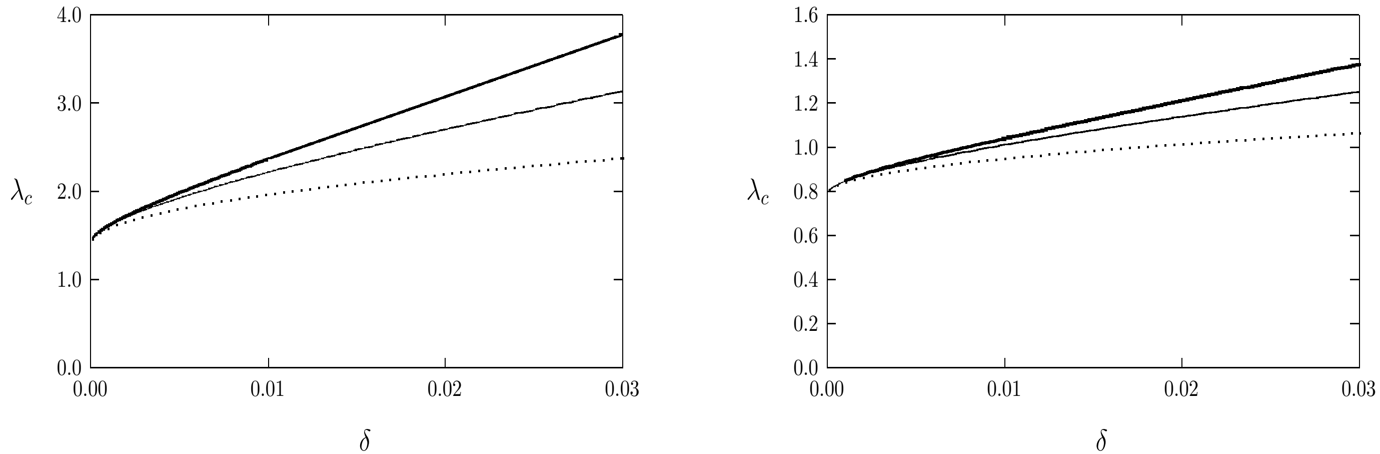
$$u = u_0 + \delta^{-1} u_1 + \cdots \,; \quad \tilde{\lambda} = \tilde{\lambda}_0 + \delta^{-1}\tilde{\lambda}_1 + \cdots \,, \quad \delta \to \infty \,.$$

By invoking appropriate solvability conditions, we get for $\delta \gg 1$ that

$\lambda_c \sim 70.1\delta + 1.7 \,,$ (Unit Slab) ; $\lambda_c \sim 15.4\delta + 1.0 \,,$ (Unit Disk) .

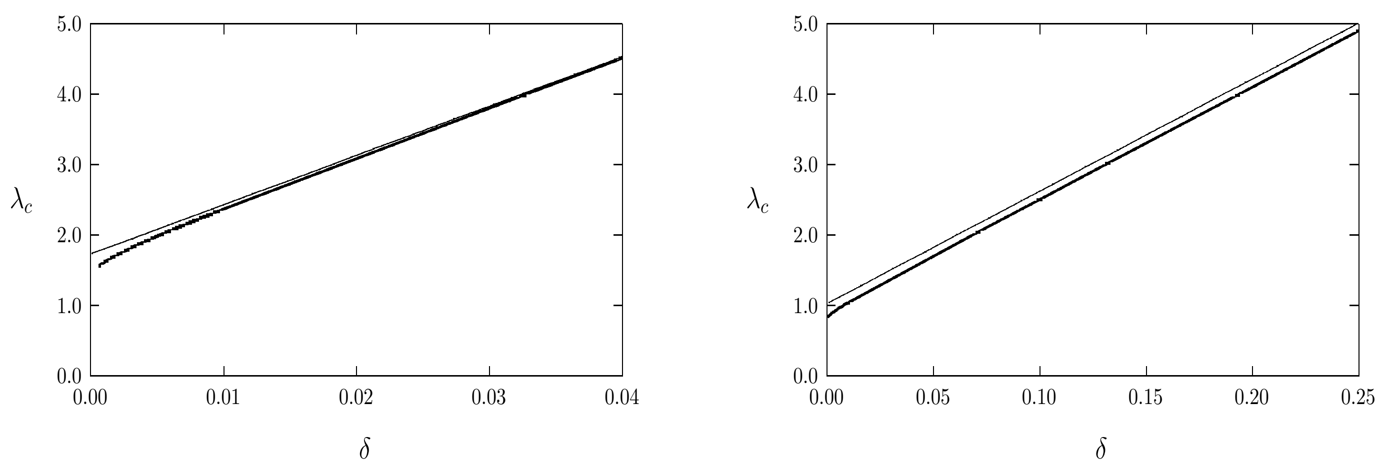
Comparison of Asymptotics and Numerics

For $\delta \ll 1$ comparison of asymptotics and numerics for λ_c



Left: Unit Slab; **Right:** Unit Disk. Numerics (heavy solid), two-term (dashed) and three-term (solid) asymptotics.

For $\delta \gg 1$ comparison of asymptotics and numerics for λ_c



Concentration Phenomena in Unit Disk

Construct the limiting asymptotics of the maximal solution branch to $\Delta^2 u = -\lambda/(1+u)^2$, with $u = u_r = 0$ on $r = 1$, for which $\lambda \to 0$ as $u(0) + 1 = \varepsilon \to 0^+$.

- For $\varepsilon \to 0^+$ it is a singular perturbation problem since $\lambda/(1+u)^2 \to 0$ except in a narrow zone near $r = 0$ where $u = -1 + \mathcal{O}(\varepsilon)$.
- Leading-order term u_0 in outer region satisfies $\Delta^2 u_0 = 0$ in $0 < r < 1$ with $u_0 = u_{0r} = 0$ on $r = 1$. We must impose the point constraint $u_0(0) = -1$ in order to match to inner solution. Thus,

$$u_0 = -1 + r^2 - 2r^2 \log r \, .$$

- If we expand $u = u_0 + \nu u_1$ and $\lambda = \nu \lambda_0$, then $\Delta^2 u_1 = -\lambda_0/(1+u_0)^2$, for which $u_{1p} \sim \frac{\lambda_0}{16} \log(-\log r)$ as $r \to 0$. This divergence of the particular solution as $r \to 0$ requires the inclusion of switchback terms.
- To find boundary layer width set $\rho = r/\gamma$ to obtain $u \sim (-\gamma^2 \log \gamma)(2\rho^2) + \gamma^2(\rho^2 - 2\rho^2 \log \rho)$. Set $u = -1 + \varepsilon v(\rho)$ in inner, to obtain that γ is given implicitly by $\varepsilon = -\gamma^2 \log \gamma$.

Biharmonic BVP and Point Constraints

Model Problem: Consider the Biharmonic BVP in an Annulus

$$\Delta^2 u = 0\,, \quad \varepsilon < r < 1\,,$$
$$u = 1\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,; \quad u = u_r = 0\,, \quad r = \varepsilon\,.$$

Since u is a linear combination of $\{r^2, r^2 \log r, \log r, 1\}$, then

$$u = A\left(r^2 - 1\right) + B r^2 \log r - (2A + B) \log r + 1\,,$$

for some $A(\varepsilon)$ and $B(\varepsilon)$. By expanding exact solution for $\varepsilon \to 0$ then

$$u \sim u_0(r) + \varepsilon^2 \left(\log \varepsilon\right)^2 u_1(r) + \mathcal{O}(\varepsilon^2 \log \varepsilon)\,,$$
$$u_0(r) = 1 - 16\pi G(r;0) \equiv r^2 - 2r^2 \log r\,, \quad u_1 = 4\left(r^2 - 1\right) - 8r^2 \log r\,,$$

where $G(r;0)$ is the Biharmonic Green's function; $\Delta^2 G = \delta(x)$ and $G = G_r = 0$ on $r = 1$, given by $G(r;0) = \left(r^2 \log r - r^2/2 + 1/2\right)/(8\pi)$.

Remark: In fact, Biharmonic spline interpolation is based on solving linear systems of the form in $\mathbb{R}^2$:

$$u_0(x_j) = \sum_{i=1}^{N} f_i G(x_j; x_i)\,.$$

Concentration Behavior in Unit Disk

Ref: [LW2]: A. Lindsay, MJW, *Asymptotics of Some Nonlinear Eigenvalue Problems for a MEMS Capacitor: Part II: Multiple Solutions and Singular Asymptotics*, under review, EJAM, (2010), (34 pages)

Consider $\Delta^2 u = -\lambda/(1+u)^2$, with $u = u_r = 0$ on $r = 1$.

Principal Result: [LW2]: *For $\varepsilon \equiv u(0) + 1 \to 0^+$, the limiting asymptotics of the maximal solution branch in the outer region, away from $r = 0$, is*

$$u = u_0 + \frac{\varepsilon}{\sigma} \log \sigma \, u_{1/2} + \frac{\varepsilon}{\sigma} u_1 + \varepsilon \log \sigma \, u_{3/2} + \varepsilon u_2 + \mathcal{O}(\varepsilon \sigma \log \sigma) \,,$$

$$\lambda = \frac{\varepsilon}{\sigma} \left[\lambda_0 + \sigma \lambda_1 + \mathcal{O}(\sigma^2) \right] \,, \quad \lambda_0 = 32 \,, \quad \lambda_1 = 16 \left(\log 2 - \frac{\pi^2}{6} \right) .$$

where $\sigma = -1/\log \gamma$ and the boundary layer width γ is determined implicitly by $-\gamma^2 \log \gamma = \varepsilon$. The point constraint $u_0(0) = -1$ holds, and

$$u_0 = -1 + r^2 - 2r^2 \log r \,, \quad u_{1/2} = -\frac{\lambda_0}{16} u_0 \,, \quad u_{3/2} = -\frac{\lambda_1}{16} u_0 \,.$$

Note: $u_{1/2}$ and $u_{3/2}$ are switchback terms.

Concentration Behavior in Unit Disk

In addition, u_1 *and* u_2 *are the unique solutions of*

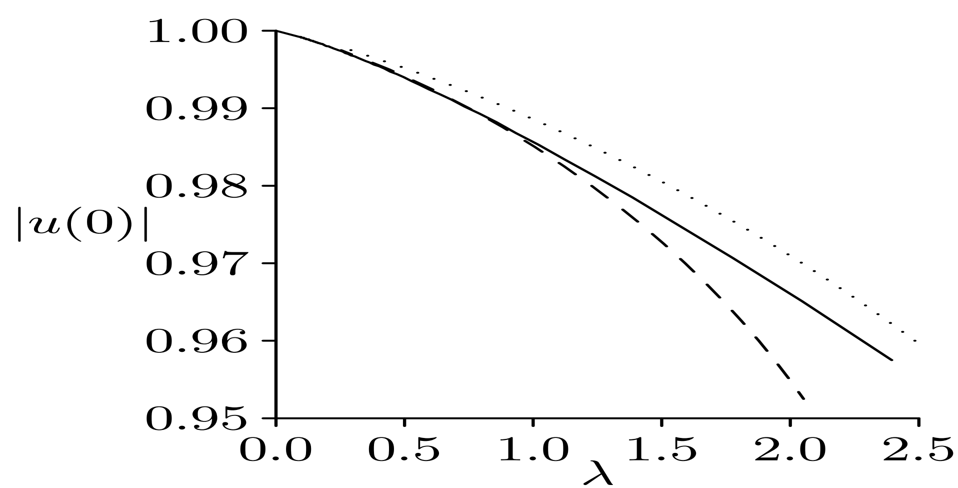
$$\Delta^2 u_1 = -\frac{\lambda_0}{(1+u_0)^2}\,, \quad 0 < r < 1\,; \qquad u_1(1) = u_{1r}(1) = 0\,,$$

$$u_1 = \frac{\lambda_0}{16}\log(-\log r) + \frac{\lambda_0}{16} + \mathcal{O}(\log^{-1} r)\,, \quad r \to 0\,,$$

$$\Delta^2 u_2 = -\frac{\lambda_1}{(1+u_0)^2}\,, \quad 0 < r < 1\,; \qquad u_2(1) = u_{2r}(1) = 0\,,$$

$$u_2 = \frac{\lambda_1}{16}\log(-\log r) + \frac{1}{16}\left(\lambda_0 + \lambda_1\right) - \log 2 + \frac{\lambda_0}{16}\log r + \mathcal{O}(\log^{-1} r)\,, \quad r \to 0\,.$$

Comparison of Asymptotics 1-term (dotted), 2-term (dashed) and Numerics (solid)



Concentration in Unit Disk: Mixed Biharmonic

Consider $-\delta\Delta^2 u + \Delta u = \lambda/(1+u)^2$, with $u = u_r = 0$ on $r = 1$:

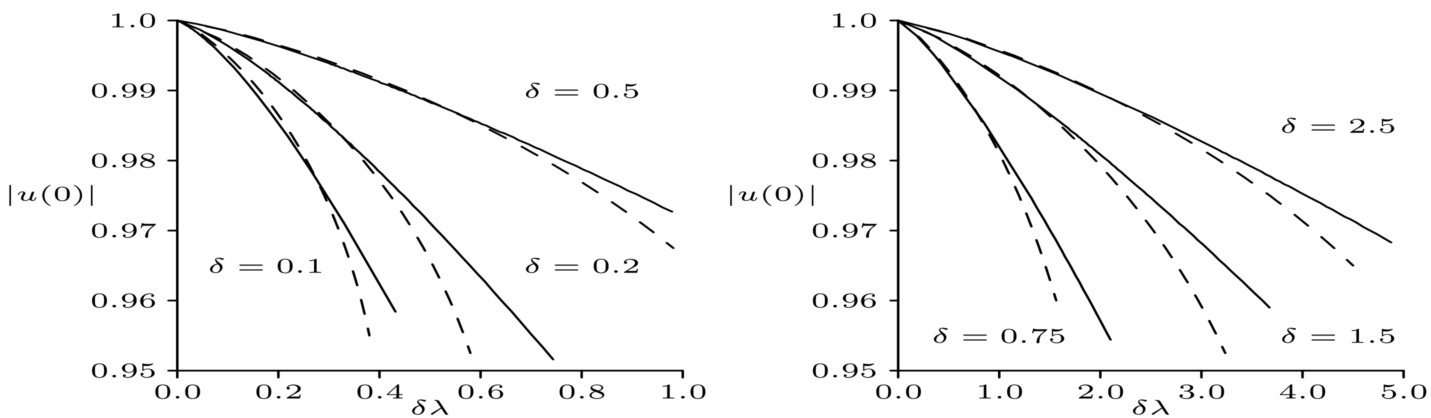
Then, u_0 with point constraint $u_0(0) = -1$ satisfies

$$-\delta\Delta^2 u_0 + \Delta u_0 = 0\,, \quad 0 < r < 1\,; \qquad u_0(1) = u_{0r}(1) = 0\,,$$
$$u_0 = -1 + \alpha r^2 \log r + \varphi r^2 + o(r^2)\,, \quad \text{as} \quad r \to 0\,.$$

We can calculate $\alpha < 0$ and φ in terms of modified Bessel functions. Then,

$$\lambda_0 = 8\alpha^2\,, \qquad \lambda_1 = -\frac{\lambda_0}{2}\left[\frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\varphi}{\alpha}\right)\right]\,.$$

Comparison of Asymptotics and Numerics



Left: Small δ **Right: Larger δ**

Concentration in Arbitrary 2-D Domain

Ref: [KLW]: M.C. Kropinski, A. Lindsay, MJW, *Asymptotic Analysis of Localized Solutions to Some Linear and Nonlinear Biharmonic Eigenvalue Problems*, to be submitted, Studies Appl. Math., (2010).

Consider $\Delta^2 u = -\lambda/(1+u)^2$, with $u = \partial_n u = 0$ on $\partial\Omega$.

Principal Result: [KLW: *For* $\varepsilon \equiv u(x_0) + 1 \to 0^+$*, the limiting asymptotics of the maximal solution branch in the outer region, away from* x_0*, and* λ *is*

$$u = u_0 + \mathcal{O}\left(\varepsilon\sigma^{-1}\log\sigma\right)\,, \quad \lambda = \frac{\varepsilon}{\sigma}\lambda_0 + \varepsilon\lambda_1 + \mathcal{O}(\varepsilon\sigma)\,,$$

where $\sigma = -1/\log\gamma$ *and the boundary layer width* γ *is given implicitly by* $-\gamma^2\log\gamma = \varepsilon$*. Here,*

$$u_0(x;x_0) = -\frac{G(x;x_0)}{R(x_0;x_0)}\,,$$

with point constraint $u_0(x_0) = -1$*, where* $G(x;\xi)$ *satisfies*

$$\Delta^2 G = \delta(x-\xi)\,, \quad x \in \Omega\,; \qquad G = \partial_n G = 0\,, \quad x \in \partial\Omega\,,$$

$$G(x,\xi) = \frac{1}{8\pi}|x-\xi|^2\log|x-\xi| + R(x;\xi)\,.$$

Concentration in Arbitrary 2-D Domain

To leading-order, the concentration point $x_0 \in \Omega$ *satisfies*

$$\nabla_x R(x;x_0)|_{x=x_0} = 0\,, \quad \text{provided that} \quad R(x_0;x_0) > 0\,.$$

As $x \to x_0$, *with* $r = |x - x_0|$, *we identify* α *and* β *by*

$$u_0 \sim -1 + \alpha r^2 \log r + r^2(\beta + a_c \cos 2\theta + a_s \sin 2\theta) + \cdots\,,$$

where $\alpha < 0$ *by assumption, and* β *(sign* $\pm$*) is*

$$\alpha \equiv \frac{-1}{8\pi R(x_0;x_0)}\,, \qquad \beta \equiv \frac{-1}{4R(x_0;x_0)}\left[\frac{\partial^2 R}{\partial x_1^2} + \frac{\partial^2 R}{\partial x_2^2}\right]_{x=x_0}\,.$$

Finally, λ_0 *and* λ_1 *are given by*

$$\lambda_0 = 8\alpha^2, \qquad \lambda_1 = -\frac{\lambda_0}{2}\left[\frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\beta}{\alpha}\right)\right]\,.$$

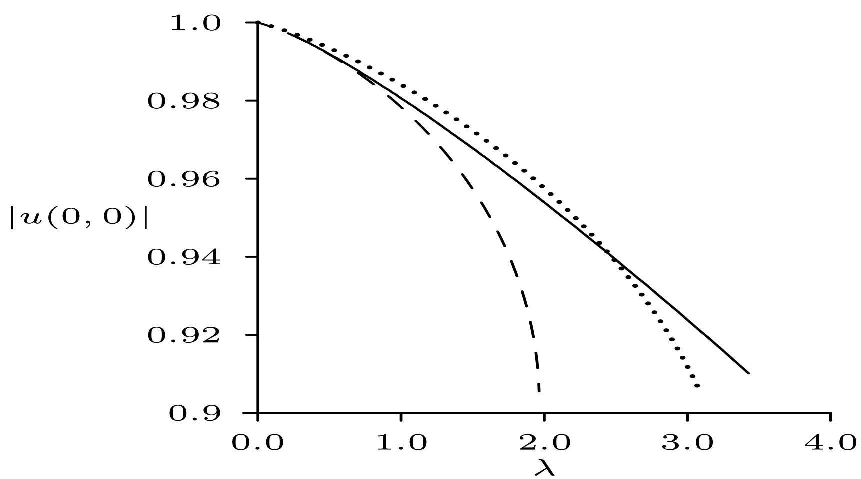
- Asymptotics of λ determined in terms of properties of the regular part of the biharmonic Green's function.
- **Note:** $R_{00} \equiv R(x_0;x_0)$ and the trace Trace $(\mathcal{R}_{00})$ computed by fast multipole methods for Low Reynolds number flow (Kropinski).

Numerics: Concentration in 2-D Domains

Comparison of Asymptotics and Numerics in Square Domain: For the square $[-1,1]^2$, then $x_0 = 0$, and we compute

$$R_{00} \approx 0.0226\ldots\,, \qquad \text{Trace}\,(\mathcal{R}_{00}) \approx -0.0892\ldots\,.$$

Numerics (solid); 1-term asymptotics (dots); 2-term asymptotics (dashed)



Class of Dumbell-Shaped Domains:

Let $z \in \mathcal{B}$, where $\mathcal{B}$ is the unit disk, and define the complex mapping

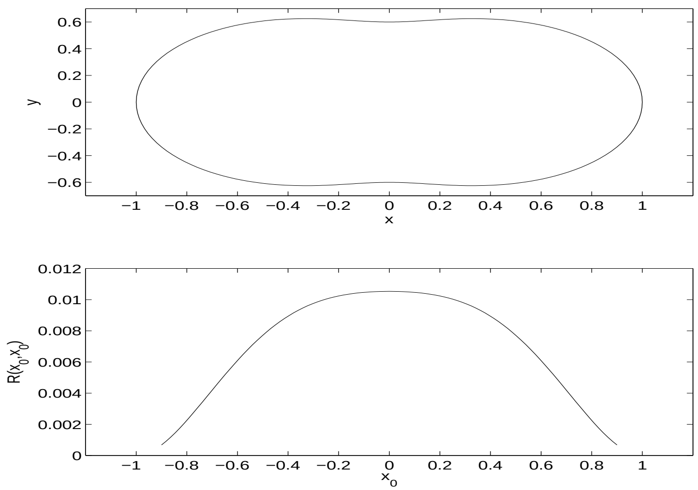
$$w = f(z;b) = \frac{(1-b^2)z}{z^2 - b^2}\,,$$

Numerics: Concentration in a Dumbbell

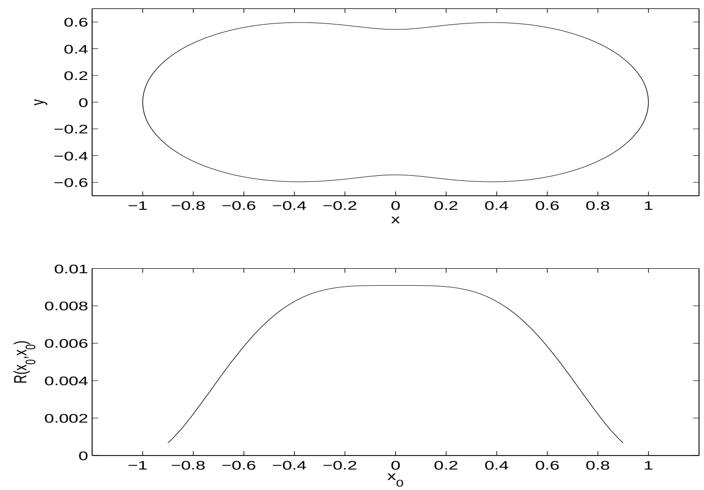
For various values of b, numerical values for $R(x_0; x_0)$ and Trace $(\mathcal{R}_{00})$ at the points $x_0 = (x_0, 0)$ where $dR/dx_0 = 0$.

b	x_0	$R(x_0, x_0)$	Trace $(\mathcal{R}_{00})$
2.00000	0.00000	1.05312×10^{-2}	-2.44476×10^{-2}
1.83995	0.00000	9.08917×10^{-3}	-2.44656×10^{-2}
1.50000	-0.39000	6.48716×10^{-3}	1.12095×10^{-2}
	0.00000	5.15298×10^{-3}	4.00099×10^{-2}
	0.39000	6.48716×10^{-3}	1.12095×10^{-2}
1.05000	-0.49450	4.94718×10^{-3}	3.11557×10^{-2}
	0.000000	9.59768×10^{-5}	0.379489×10^{-2}
	0.494500	4.94718×10^{-3}	3.11557×10^{-2}

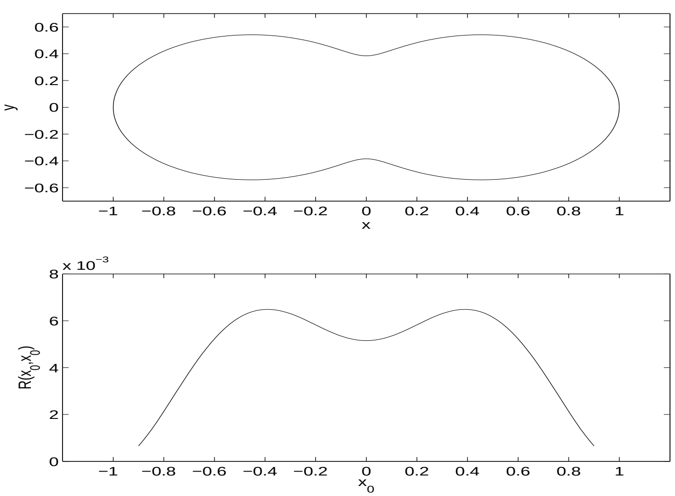
Numerics: R_{00} and the Dumbbell-Shape



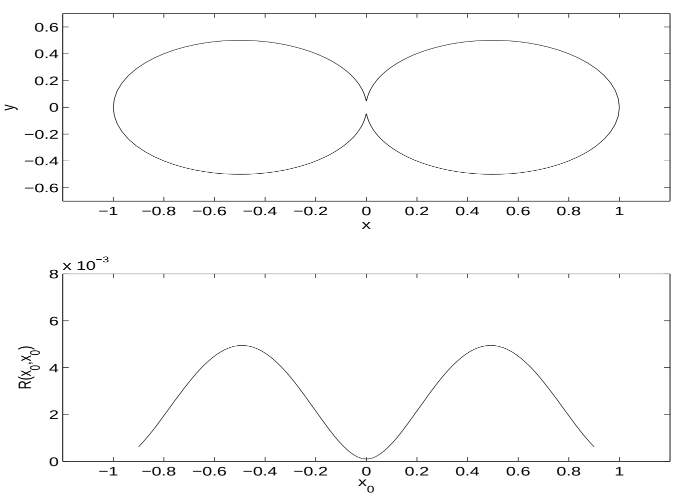
(a) $b = 2.0$



(b) $b = b_c = 1.83995$



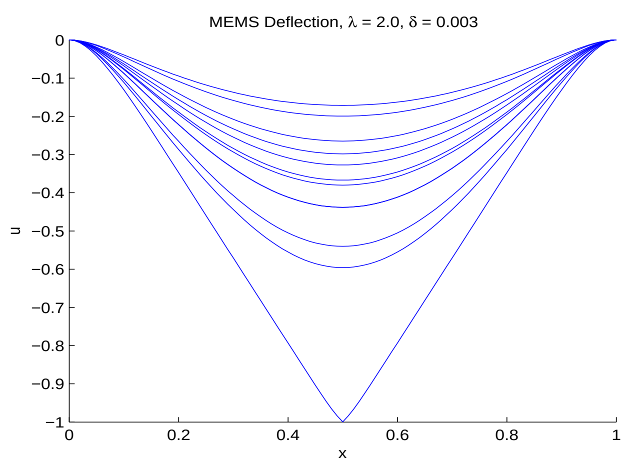
(c) $b = 1.5$



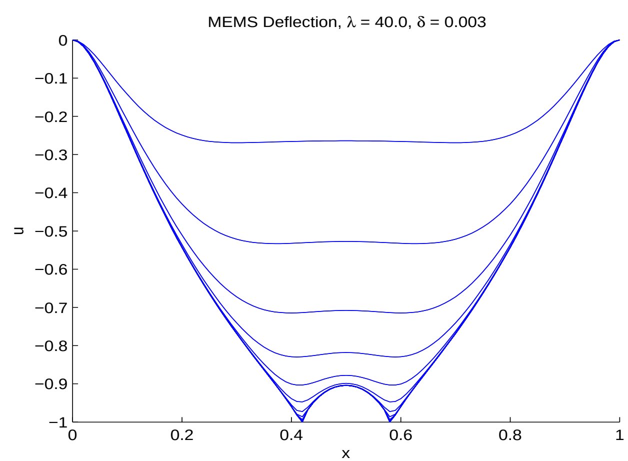
(d) $b = 1.05$

Further Directions

- For $\delta \to 0$, calculate the number of solutions for the mixed Biharmonic problem, and describe the breakup of infinite fold point structure analytically.
- For fringing-field problem:
  1. Describe breakup of infinite fold point structure for $\beta \to 0$.
  2. Calculate limiting asymptotics of maximal solution branch for $\beta = O(1)$.
- For pure biharmonic, find an Ω for which $\nabla_x R(x_0, x_0) = 0$ but $R(x_0; x_0) < 0$. Related to non-positivity properties of G.
- Time-dependent quenching behavior beyond the pull-in instability.



(e) $\lambda = 2.0$, $\delta = 0.003$



(f) $\lambda = 40.0$, $\delta = 0.003$