The Existence and Stability of Spike Patterns in a Chemotaxis Model

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Abstract

In the limit of small chemo-attractant diffusivity ϵ, the existence, stability, and dynamics of spiky patterns in a chemotaxis model is studied in a bounded multi-dimensional domain. In this model, the transition probability density function $\Phi(w)$ is assumed to have a power law form $\Phi(w) = w^p$, and the production of chemo-attractant w is assumed to saturate according to a Michaelis-Menten kinetic function. In the limit $\epsilon \to 0$, it is proved that there is a steady-state single boundary spike solution located at the maximum of the mean curvature of the boundary. Moreover, a steady-state interior spike solution is proved to concentrate at a maximum of the distance function. The single interior spike solution is shown to be metastable for certain ranges of p and the dimension N. The stability of a single boundary spike solution is also analyzed in detail. Finally, a formal asymptotic analysis is used to characterize the metastable interior spike dynamics in both a one-dimensional and a multi-dimensional domain.

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1 Introduction

In this paper, we investigate qualitative properties of a class of solutions to a special case of the following generalized chemotaxis system. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega$. We seek solutions, $P \in \mathbb{R}$, and $W \in \mathbb{R}^{m+1}$, of the system

$$P_t = D\nabla \cdot (P\nabla(\log(P/\Phi(W)))) \,, \qquad W_t = d\Delta W + F(P, W) \,, \qquad (x,t) \in \Omega \times (0,T) \,, \tag{1.1}$$

subject to the 'no-flux' boundary condition

$$P\nabla(\log P/\Phi(W)) \cdot \nu(x) = 0 \,, \qquad \nabla W \cdot \nu(x) = 0 \,. \tag{1.2}$$

Here $\nu(x)$ denotes the inward pointing normal to $\partial\Omega$. To close the system we prescribe the initial conditions

$$P(x,0) = P_0(x) > 0 \,, \qquad W(x,0) = W_0(x) \geq 0 \,, \qquad \text{for } x \in \overline{\Omega} \,.$$

In this system, D is a constant diffusion coefficient, d is a positive semi-definite diagonal matrix, P is a population density, $\log \Phi(W)$ is the chemotactical sensitivity function, and W is a vector of nutrients or chemicals whose dynamics influences the movement of P. The function $\Phi(W)$ is a prescribed "transition probability function". This general system includes the so-called Keller-Segel model of biology [17]. To biologically motivate our study

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we begin by outlining two themes. One basic to developmental biology and the other from angiogenesis. Our contribution to the mathematical analysis of chemotactic systems is discussed at the end of the introduction.

A basic quality of all living systems is that they sense the environment in which they live and respond to it. The response often involves movement toward or away from an external stimulus. The mechanism for the response is called taxis. Any taxis involves two components: an external signal, and the response of the organism to the signal. The response also involves two steps: the detection of the signal, and the transduction of the external signal into an internal signal that triggers the response.

In many mathematical models analyzing taxis the signal is transported by diffusion, convection, or some other mechanism. There are however instances in which the organism seems to modify its environment in a strictly local manner and there is little or no transport of the modifying substance. A typical example of this is myxobacteria which produce slime over which their cohorts can easily move. Myxobacteria are soil bacteria which glide on suitable surfaces or at air-water interfaces. Under starvation conditions they tend to move close together forming complex patterns. Finally they aggregate to build fruiting bodies. Inside the fruiting bodies they survive as dormant myxospores. An account of this intriguing sequence of events can be found in [1].

A novel approach to the modeling of aggregation in myxobacteria is due to Othmer and Stevens [29]. Their inspiration and motivation comes from the work of Davis [6] concerning reinforced random walks. The models developed by Othmer and Stevens are of the form (1.1), (1.2) with $d = 0$ and have been studied in depth by Levine and Sleeman [19] who were able to provide some understanding of the numerical findings in [29] particularly with regard to aggregation, blow-up and collapse of solutions.

More recently the idea of mathematical modeling based on the idea of reinforced random walks has been developed to gain some understanding of tumor angiogenesis. Angiogenesis is a morphogenetic process whereby new blood vessels are induced to grow out of a pre-existing vasculature. It is fundamental to the formation of any new blood vessels during embryonic development and contributes to the maintenance of tissue functionality in the adult (e.g. placental growth). Angiogenesis is also an important feature of various pathological processes such as wound healing and cancer progression. We are particularly interested in tumor angiogenesis.

Capillaries which are the main microvessels involved in tumor angiogenesis are composed of three components: (a) the basement membrane, which is a complex extra cellular matrix (ECM) encircling and supporting the cellular components, (b) the endothelial cells (EC) which form a mono-layer of flattened and extended cells lining the lumen and resting on the basement membrane and (c) pericyte cells which form a periendothelial cellular network embedded within the basement membrane.

The first event of tumor-induced angiogenesis is triggered by the secretion of a number of chemicals, collectively called tumor angiogenesis factors (TAFs) (cf. [8], [9]) from a colony of cancerous cells of a solid tumor. These factors diffuse through the tissue creating a chemotactic gradient which eventually reaches neighboring capillaries and other blood vessels. In response to TAF the EC in nearby capillaries appear to thicken (i.e. aggregate) and produce a proteolytic enzyme which in turn degrades the basement membrane.

In response to the TAF the normally smooth edothelial cell surface begins to develop pseudopodia which penetrate the weakened basement membrane. Capillary sprouts are formed by the accumulation of EC from the parent vessel. The sprouts grow in length, proliferate, form loops leading to micro-circulation of blood (i.e. anastomoses) and also branch successively. The resulting capillary network continues to progress through the tissue extra cellular matrix (ECM) forming a micro-vasculature and eventually invades the tumor colony leading to rapid growth and metastasis. The means of progress through the tissue ECM is via chemotaxis and haptotaxis.

Chemotaxis is the response of EC to chemical gradients set up by the TAF. A major component of the tissue ECM is fibronectin. It has been verified experimentally that fibronectin stimulates directional migration of EC by establishing an adhesive gradient, i.e. via haptotaxis.

Mathematical modeling of the complex processes involved in tumor angiogenesis has been vigorously pursued in recent years. For recent overviews of some of the modeling ideas see [36] and [33].

Driven by the need to understand the underlying biochemistry and also to attempt to bridge the gap between micro and macro-cellular events, Sleeman together with Levine, Pamuk, Nilsen-Hamilton [20], [21], [22], Holmes [14], Plank [31] and Wallis [35] has modeled angiogenesis on the basis of reinforced random walks and Michaelis-Menten kinetics. In these modeling ideas systems of equations of the form (1.1) play a crucial role.

Systems of the form (1.1) enjoy very rich dynamics. Consider for example the following two-component system in one space dimension

$$
\begin{gathered}
P_t = D\left(P_{xx} - a(P\frac{W_x}{W})_x\right), \quad W_t = \lambda PW - \mu W, \qquad \text{for } 0 < x < \ell,\ t > 0, \\
\frac{P_x}{P} - a\frac{W_x}{W} = 0, \qquad \text{for } x = 0, \ell,\ t > 0. \\
P(x,0) = P_0(x) > 0, \qquad W(x,0) = W_0(x) > 0, \qquad \text{for } 0 \le x \le \ell.
\end{gathered} \tag{1.4}
$$

In [19] it is shown, among other things, that when $a = 1$ there are solution pairs (P, W) for which $P > 0$ but for which P blows up in finite time and that the power spectrum converges to that of the delta function in finite time. Indeed it is possible to construct an explicit family of such solutions. When $a = -1$ there exist solution pairs (P, W) for which $P > 0$ and P collapses to a constant in infinite time but exponentially fast.

In this context we mention the related work of Rascle and Ziti [34]. In our notation they considered the following system in $\mathbb{R}^N$:

$$
P_t = D_1 \Delta P - \nabla \cdot [PW^{-\alpha}\nabla W], \quad W_t = D_2 \Delta W - kW^m P. \tag{1.5}
$$

Here all the constants are positive unless otherwise specified. They constructed similarity solutions of the form $(P, W) = ((T-t)^a P(\xi), (T-t)^b w(\xi))$, where $\xi = (T-t)^{-1}|x|^2$ for $x \in \mathbb{R}^N$ in one, two, or three space dimensions, when $0 < m < \alpha = 1$ and $D_2 = 0$. (Here $a, b \ge 0$). When $D_1 = 0$ as well, they construct such solutions which blow up in infinite time in one and two dimensions. In the case $D_1 > 0$ they are able to construct only global self-similar solutions.

An important question which again is motivated by the need to understand how new capillaries sprout via angiogenesis from a pre-existing vasculature is: Can we expect solutions of the system (1.1) (with $d = 0$) to possess spatially non-constant, piecewise constant aggregating solutions? Here we define aggregation as follows: $P(x,t)$ as a solution to (1.1), aggregates if it converges to a non-constant steady state in finite or infinite time.

From numerical experiments, Othmer and Stevens [29] show that $P(x,t)$ can evolve to an aggregating solution through the formation of a "shock". In their experiments they consider the system (1.1) with

$$
\Phi(W) = \left(\frac{\beta + W}{\gamma + W}\right)^a, \quad F(P, W) = \frac{PW}{1 + \nu W} - \mu W + \gamma_r \frac{P}{1 + P}. \tag{1.6}
$$

In [19] it is argued that the seeds of such shock formation are already present in the case of the simpler system (1.4).

During the initiation of angiogenesis, as outlined above, the EC in capillaries near the tumor produce a proteolytic enzyme in response to the TAF. While a detailed analysis of the process involved in angiogenesis initiation has been discussed in [20], a simple model involving only the EC and the proteolytic enzyme can be formulated. Such a model is of the form (1.1) in which P represents EC density and W is enzyme concentration. In this model d is small since in the capillary diffusion takes place on a much longer time scale than the kinetic reactions. This of course is not the case for the developing angiogenesis in the ECM

It has been demonstrated recently by Holash et al. (cf. [13]) that once a tumor has become vascularised the resulting capillary network may undergo periods of dramatic collapse and remodeling. Paradoxically, the coopted vasculature does not undergo angiogenesis to support the growing tumor, but instead regresses via a process that involves disruption of EC/smooth muscle cell interactions and EC apoptosis (i.e. programmed cell death). This vessel regression in turn results in necrosis within the central part of the tumor. However vigorous angiogenesis is initiated at the tumor boundary rescuing the surviving tumor and supporting further growth. This behavior could be modeled by systems of the form (1.1) and suggests the existence of point-condensation solutions or spike-type patterns.

It is the purpose of this paper to investigate the existence, stability and dynamics of the spike patterns in the following variant of (1.1) and (1.6). That is we consider the system

$$\begin{cases} P_t = D_1 \nabla \cdot (P \nabla (\log \frac{P}{\Phi(W)})) \,, \quad W_t = D_2 \Delta W - \mu W + \frac{PW}{1+\gamma W} \quad \text{in} \ \ \Omega \times (0, +\infty) \,, \\ \frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega \times (0, +\infty) \,, \quad P(x,0) = P_0(x) \geq 0 \,, \quad W(x,0) = W_0(x) \geq 0 \,, \end{cases} \tag{1.7}$$

where $D_1 > 0$ and $D_2 > 0$ are two diffusion coefficients, $P(x,t)$ is the particle density of a particular species, $W(x,t)$ is the concentration of the "active agent", $\Omega \subset \mathbb{R}^N (N \leq 3)$ is a smooth and bounded domain, μ, γ are positive constants, and $\nu = \nu(x)$ is the unit normal derivative at $x \in \partial\Omega$. The term $\frac{W}{1+\gamma W}$ is a typical Michaelis-Menten saturating function. Throughout the paper, we take $\Phi(W) = W^p$ where $p > 1$, which corresponds to a logarithmic chemotactical sensitivity function.

We will show that the inclusion of a **small diffusion coefficient** D_2 can produce **stable** spiky patterns. By non-dimensionalizing (1.7), we may assume, without loss of generality, that

$$\mu = 1 \,, \qquad D_1 = 1, \qquad D_2 = \epsilon^2 << 1 \,. \tag{1.8}$$

In this new setting, (1.7) becomes

$$\begin{cases} P_t = \nabla \cdot (P \nabla (\log \frac{P}{W^p})), \quad W_t = \epsilon^2 \Delta W - W + \frac{PW}{1+\gamma W} \quad \text{in} \ \ \Omega \times (0, +\infty) \,, \\ \frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega \times (0, +\infty) \,, \quad P(x,0) = P_0(x) \geq 0 \,, W(x,0) = W_0(x) \geq 0 \,. \end{cases} \tag{1.9}$$

For $\gamma \gg 1$, (1.9) is the Keller-Segel model with a *logarithmic* chemotactical sensitivity function. A survey of the existence and regularity properties of solutions of the classic Keller-Segel model with a *linear* chemotactical sensitivity function, and for certain extensions of the basic model, is given in [15]. Results for global existence of solutions for the limiting model (1.9) where $\gamma \gg 1$ are surveyed in §6.1.1 of [15]. Our results for the stability of spike-type patterns of (1.9) for $\gamma > 0$ are, to our knowledge, new. In particular, for certain ranges of p, we prove that an interior spike solution for (1.9) is metastable. The existence of metastable phenomena is known to occur in certain reaction-diffusion systems, including the shadow Gierer-Meinhardt model (cf. [16] and [5]), but to our knowledge has never been shown previously in a chemotaxis system. After submission of this article, the occurrence of metastability has been shown asymptotically and numerically in [32] for the volume-filling chemotaxis model of [12] and [30].

Integrating the equation for P over Ω, and using the Divergence theorem, we obtain

$$\int_\Omega P(x,t) = \int_\Omega P(x,0) = m\,. \tag{1.10}$$

This conservation of mass condition plays a central role in stabilizing nontrivial spatial patterns. To simplify the computations, we will assume that $m = 1$. The steady-state problem for (1.9) becomes

$$\begin{cases} \nabla\cdot(P\nabla(\log\frac{P}{W^p})) = 0 \qquad \text{in } \Omega\,, \\ \epsilon^2\Delta W - W + \frac{PW}{1+\gamma W} = 0 \qquad \text{in } \Omega\,, \\ \frac{\partial P}{\partial\nu} = \frac{\partial W}{\partial\nu} = 0 \qquad \text{on } \partial\Omega\,, \quad \int_\Omega P(x) = 1\,. \end{cases} \tag{1.11}$$

Note that both P and W must be nonnegative. From the equation for P in (1.11), and the condition that $m = 1$, we get

$$P(x) = \frac{1}{\int_\Omega W^p} W^p(x)\,. \tag{1.12}$$

Defining the new function $\hat{W}$ by

$$W(x) = \frac{1}{\gamma\int_\Omega \hat{W}^p}\hat{W}, \tag{1.13}$$

and substituting (1.12) and (1.13) into (1.11), we obtain an equation for $\hat{W}$

$$\epsilon^2\Delta\hat{W} - \hat{W} + \frac{\hat{W}^{p+1}}{\int_\Omega \hat{W}^p + \hat{W}} = 0 \qquad \text{in } \Omega\,, \quad \hat{W} > 0 \quad \text{in } \Omega\,, \quad \frac{\partial\hat{W}}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{1.14}$$

Equation (1.14) without the nonlocal term $\int_\Omega \hat{W}^p$ has a variational structure, and has been studied by numerous authors. For results on the existence of boundary spike solutions see [2], [27], [28], [40], [43], and the references therein. Results for the existence of interior spike solutions are given in [11], [38], [39], and the references therein. A survey of some of these previous results is given in [26].

Throughout the paper $C > 0$ is a generic constant, which is independent of ϵ, that may change from line to line. The notation $O(A), o(A)$ means that $|O(A)| \le C|A|$, $\lim_{\epsilon\to 0}\frac{o(A)}{|A|} = 0$.

The organization of the paper is as follows. In Section 2, we state our main results. In Section 3, we construct both single boundary and single interior spikes. In Sections 4–6, we analyze the spectrum of the linearized problem. In the spectrum there are eigenvalues that are $O(1)$ as $\epsilon \to 0$, called the large eigenvalues, and eigenvalues that tend to zero as $\epsilon \to 0$, called the small eigenvalues. In Section 4, we study the linearized eigenvalue problem and reduce the study of the large eigenvalues to a nonlocal eigenvalue problem. In Section 5, we analyze this nonlocal eigenvalue problem. In Section 6, we analyze the small eigenvalues. In Section 7, we derive the dynamical law for the motion of an interior spike and present some numerical results. Finally, a brief discussion in given in Section 8.

2 Statements of Main Results

We first state our main results on the existence of steady-state solutions.

Theorem 2.1 *Assume that*

$$1 < p < +\infty, \quad \textit{if } N = 1,2; \qquad 1 < p < 5, \quad \textit{if } N = 3\,. \tag{2.1}$$

Then, for $\epsilon << 1$, there exists a steady-state solution for (1.11) of the following form

$$(P_\epsilon, W_\epsilon) = \left(\frac{1}{\int_\Omega \hat{W}_\epsilon^p}\hat{W}_\epsilon^p\,, \frac{\hat{W}_\epsilon}{\gamma\int_\Omega \hat{W}_\epsilon^p}\right), \quad \textit{where} \quad \hat{W}_\epsilon = w\left(\frac{x-Q_\epsilon}{\epsilon}\right) + O(\epsilon)\,. \tag{2.2}$$

Here $w(y)$ is the unique solution of

$$\Delta w - w + w^p = 0\,, \qquad w > 0 \quad \textit{in } \mathbb{R}^N\,, \quad w(0) = \max_{y\in\mathbb{R}^N} w(y)\,, \quad w(y) \to 0 \quad \textit{as } |y| \to +\infty\,. \tag{2.3}$$

The point Q_ϵ is classified either by

*(a) (**single boundary spike**) $Q_\epsilon \in \partial\Omega$, $H(Q_\epsilon) \to \max_{Q\in\partial\Omega} H(Q)$, where $H(Q)$ is the mean curvature function at $Q \in \partial\Omega$,*

*or (b) (**single interior spike**) $Q_\epsilon \in \Omega$, $d(Q_\epsilon, \partial\Omega) \to \max_{Q\in\Omega} d(Q,\partial\Omega)$, where $d(Q,\partial\Omega)$ is the distance function at $Q \in \Omega$.*

Next, we study the linearized stability of the solutions constructed in Theorem 2.1. To this end, we linearize (1.9) around (P_ϵ, W_ϵ), as given in Theorem 2.1, to obtain the following linearized eigenvalue problem:

$$\nabla\cdot(\psi_\epsilon\nabla\log(\frac{P_\epsilon}{W_\epsilon^p})) + \nabla\cdot(P_\epsilon\nabla(\frac{\psi_\epsilon}{P_\epsilon} - p\frac{\phi_\epsilon}{W_\epsilon})) = \lambda_\epsilon\psi_\epsilon \quad \text{in } \Omega\,, \tag{2.4}$$

$$\epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + \frac{P_\epsilon}{(1+\gamma W_\epsilon)^2}\phi_\epsilon + \frac{W_\epsilon}{1+\gamma W_\epsilon}\psi_\epsilon = \lambda_\epsilon\phi_\epsilon \quad \text{in } \Omega\,, \tag{2.5}$$

$$\frac{\partial\phi_\epsilon}{\partial\nu} = \frac{\partial\psi_\epsilon}{\partial\nu} = 0\,, \quad \text{on } \partial\Omega\,, \tag{2.6}$$

where $\lambda_\epsilon \in \mathcal{C}$-the set of complex numbers.

Note that (2.4)-(2.6) is **not self-adjoint**, and so complex eigenvalues are expected. We say that (P_ϵ, W_ϵ) is **linearly stable**, if for all eigenvalues λ_ϵ of (2.4)-(2.6), we have $Re(\lambda_\epsilon) < 0$. We say that (P_ϵ, W_ϵ) is **linearly unstable**, if there exists an eigenvalue λ_ϵ of (2.4)-(2.6) such that $Re(\lambda_\epsilon) > 0$. We say that (P_ϵ, W_ϵ) is **metastable**, if for all eigenvalues λ_ϵ of (2.4)-(2.6), we have either $Re(\lambda_\epsilon) < 0$ or $|\lambda_\epsilon| = O(e^{-d/\epsilon})$ for some $d > 0$ independent of $\epsilon > 0$. With these definitions, we now give our main results classifying the stability of (P_ϵ, W_ϵ).

Theorem 2.2 *Assume that*

$$1 < p < +\infty \quad \textit{if } N = 1; \qquad 2 \le p \le 5 \quad \textit{if } N = 2; \qquad 2 \le p \le 3 \quad \textit{if } N = 3. \tag{2.7}$$

Let (P_ϵ, W_ϵ) be the solution given in Theorem 2.1.

(a) (metastability) The single interior spike is metastable.

(b) (stability) If $N = 1$, then the single boundary spike is linearly stable.

(c) (stability) If $\Omega = B_R(0) = \{x||x| < R\}$ and $(P(x,t), W(x,t)) = (P(|x|,t), W(|x|,t))$, then the single interior spike is linearly stable.

(d) (stability) If $N = 2, 3$ and Q_0 is a nondegenerate global maximum point of $H(P)$, where $Q_\epsilon \to Q_0$, then the single boundary spike is linearly stable.

Remarks: 2.1). The condition on the exponent p given in (2.7) is needed for the analysis in section 5 of a nonlocal eigenvalue problem. Certainly it is not optimal. We conjecture that the same conclusion holds if p satisfies (2.1) only.

2.2). As $\epsilon \to 0$, we have $\int_\Omega \hat{W}_\epsilon^p \sim \epsilon^N$. Therefore, we see from (2.2) that $P_\epsilon(Q_\epsilon) \sim \epsilon^{-N}$, $W_\epsilon(Q_\epsilon) \sim \epsilon^{-N}$ and $P_\epsilon(x), W_\epsilon(x) \to 0$ for all $x \in \Omega$ with $|x - Q_\epsilon| \geq \delta > 0$.

2.3). Theorems 2.1 and 2.2 remain true for the following Keller-Segel model with logarithmic growth ([17], [23], [24]):

$$\begin{cases} P_t = \nabla \cdot (P \nabla (\log \frac{P}{W^p})), \quad W_t = \epsilon^2 \Delta W - W + P, \qquad \text{in } \Omega \times (0, +\infty), \\ \frac{\partial P}{\partial \nu} = \frac{\partial W}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, +\infty), \ P(x,0) = P_0(x) \geq 0, \qquad W(x,0) = W_0(x) \geq 0. \end{cases} \tag{2.8}$$

2.4). In Theorems 2.1 and 2.2, we have assumed that $D_1 = 1$, where D_1 is the diffusion coefficient of P. Theorem 2.1 holds true for any $D_1 > 0$. It is not difficult to see that Theorem 2.2 also holds provided that

$$\frac{\epsilon^2}{D_1} << 1. \tag{2.9}$$

Biologically, this means that if the active agent diffuses more slowly than the species, the species will move toward the boundary and form nontrivial stable spiky patterns. It is unclear what happens if $\frac{\epsilon^2}{D_1} \sim 1$.

2.5). There may be solutions with multiple spikes. We will not discuss this case here as most likely multiple spike solutions are unstable.

The existence of spiky patterns for the steady-states of Keller-Segel model (2.8) has been established in [23], [27], [28]. Theorem 2.1 establishes the existence of spiky patterns for more general case (1.9). As far as the authors know, Theorem 2.2 is the first rigorous result on the stability of spiky patterns for a chemotaxis system.

3 Construction of the Steady-State: Proof of Theorem 2.1

In this section we construct steady-state solutions for (1.9) and prove Theorem 2.1. By the transformation leading to (1.14), we need to find a $\hat{W}$ satisfying (1.14). We do this in two steps.

In Step 1, we fix δ small and solve the following problem:

$$\epsilon^2 \Delta \hat{W} - \hat{W} + \frac{\hat{W}^{p+1}}{\delta + \hat{W}} = 0, \quad \hat{W} > 0 \ \text{ in } \Omega, \qquad \frac{\partial \hat{W}}{\partial \nu} = 0 \ \text{ on } \partial\Omega. \tag{3.1}$$

This yields a solution $\hat{W}_{\epsilon,\delta}$. In Step 2, we solve the algebraic equation

$$\delta = \int_\Omega \hat{W}_{\epsilon,\delta}^p. \tag{3.2}$$

The first step is more or less standard, but we have to treat the dependence of $\hat{W}$ on δ. In the second step we have to make sure that the function on the right-hand side of (3.2) is continuous in δ. We begin with the following simple but important observation.

Lemma 3.1 *There exists a unique solution to*

$$\Delta w - w + \frac{w^{p+1}}{\delta + w} = 0\,, \quad w > 0 \quad in \quad \mathbb{R}^N\,, \quad w(0) = \max_{y \in \mathbb{R}^N} w(y)\,, \qquad w(y) \to 0 \quad as \; |y| \to +\infty\,. \tag{3.3}$$

We call such a solution $w_\delta(y)$*. As* $\delta \to 0$*, we have*

$$|w_\delta(y) - w(y)| \le C\delta e^{-\min(1,p-1)|y|}\,, \tag{3.4}$$

where C *is independent of* $\delta > 0$*, and* w *is the unique solution of (2.3).*

Proof: By the well-known theorem of Gidas-Ni-Nirenberg, all solutions to (3.3) are radially symmetric. Let $f_\delta(u) = \frac{u^{p+1}}{\delta+u}$. Then, we have $(\frac{f_\delta(u)}{u})' \ge 0$. By [18], there exists a unique solution, called $w_\delta(|y|)$, to (3.3). Since p is subcritical, and $f_\delta(u) \le u^p$, we see that w_δ is uniformly bounded in δ. By compactness and the uniqueness of w_δ, it follows that as $\delta \to 0$, $w_\delta \to w(y)$, where $w(y)$ is the unique solution of (2.3). This implies that

$$w_\delta(y) < Ce^{-|y|}\,, \tag{3.5}$$

where C is independent of $\delta > 0$. Next we consider $w_\delta = w + \delta \hat{w}_\delta$. It is easy to see that $\hat{w}_\delta$ satisfies

$$\Delta_y \hat{w}_\delta - \hat{w}_\delta + pw^{p-1}\hat{w}_\delta + \delta^{-1}\Big(\frac{(w+\delta\hat{w}_\delta)^{p+1}}{\delta + w_\delta} - w^p - p\delta w^{p-1}\hat{w}_\delta\Big) = 0\,, \tag{3.6}$$

where by (3.5)

$$\Big|\delta^{-1}\Big(\frac{(w+\delta\hat{w}_\delta)^{p+1}}{\delta + w_\delta} - w^p - p\delta w^{p-1}\hat{w}_\delta\Big)\Big| \le Cw_\delta^{p-1} \le Ce^{-(p-1)|y|}\,. \tag{3.7}$$

Since the operator $L_0 := \Delta - 1 + pw^{p-1}$ is invertible from $H^2_r(\mathbb{R}^N) = H^2(\mathbb{R}^N) \cap \{u(y) = u(|y|)\}$ to $L^2_r(\mathbb{R}^N) = L^2(\mathbb{R}^N) \cap \{u(y) = u(|y|)\}$ (see Lemma 4 of [45]), we see from (3.6) and (3.7) that

$$\|\hat{w}_\delta\|_{H^2(\mathbb{R}^N)} \le C\,, \quad \text{and} \quad |\hat{w}_\delta| \le Ce^{-\min(1,p-1)|y|}.$$

■

To analyze single boundary spikes we proceed as follows. For each $\delta > 0$ small, we define

$$J_{\epsilon,\delta}[u] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u^2 - \int_\Omega F_\delta(u)\,, \qquad \text{for } u \in H^1(\Omega)\,, \tag{3.8}$$

where $F_\delta(u) = \int_0^u \frac{s^{p+1}}{s+\delta}ds$. By taking a function $e(x) \equiv k$ for some constant k in Ω, and by choosing k large enough, we have $J_{\epsilon,\delta}(e) < 0$ for all $\epsilon \in (0,1)$ and $\delta \in (0,1)$. Then, for each $\epsilon \in (0,1)$ and $\delta \in (0,1)$, we can define the so-called mountain-pass value

$$c_{\epsilon,\delta} = \inf_{h\in\Gamma}\max_{0\le t\le 1} J_{\epsilon,\delta}[h(t)]\,, \tag{3.9}$$

where $\Gamma = \{h : [0,1] \to H^1(\Omega) | h(t) \text{ is continuous, } h(0) = 0,\ h(1) = u_0\}$.

Since p satisfies (2.1), p is subcritical. Furthermore, $(\frac{f_\delta(u)}{u})' \ge 0$. So f_δ satisfies all the assumptions in [27]. Similar to the analysis in Section 2 of [27], $c_{\epsilon,\delta}$ is attained by some function $\hat{W}_{\epsilon,\delta}$, which satisfies (3.1), and $c_{\epsilon,\delta}$ is the least among all nonzero critical values of $J_{\epsilon,\delta}$. Furthermore, the analysis in Section 3 of [27] shows that for ϵ sufficiently small, and uniformly in δ, $\hat{W}_{\epsilon,\delta}$ has a unique maximum point $Q_{\epsilon,\delta}$, with $Q_{\epsilon,\delta} \in \partial\Omega$ and $H(Q_{\epsilon,\delta}) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$. This completes the first step in the proof.

Next we define the following set:

$$\mathcal{S}_{\epsilon,\delta} = \{W \mid W \text{ satisfies (3.1) and } \; J_{\epsilon,\delta}[W] = c_{\epsilon,\delta}\}\,. \tag{3.10}$$

In other words, $\mathcal{S}_{\epsilon,\delta}$ contains the set of all least energy solutions of (3.1). By Step 1, $\mathcal{S}_{\epsilon,\delta}$ is not empty. Moreover, $\mathcal{S}$ is a compact set (uniformly in δ and ϵ small) since

$$\sup_{W\in\mathcal{S}_{\epsilon,\delta}} \|W\|_\epsilon \le C\,, \tag{3.11}$$

where C is independent of ϵ and δ, and

$$\|W\|_\epsilon^2 := \epsilon^{-N}(\epsilon^2 \int_\Omega |\nabla W|^2 + \int_\Omega W^2). \tag{3.12}$$

(In fact, by Lemma 3.1 and by using a test function, we have $c_{\epsilon,\delta} \le C\epsilon^N$, where C is independent of ϵ and δ. Integrating the equation gives (3.11).)

We consider the algebraic equation

$$\beta(\delta) = \delta - \rho(\delta) = 0\,, \qquad \text{where} \qquad \rho(\delta) := \inf_{W\in\mathcal{S}_{\epsilon,\delta}} \int_\Omega W^p\,. \tag{3.13}$$

Note that for any $W \in \mathcal{S}_{\epsilon,\delta}$, the same asymptotic analysis for mountain-pass solutions hold (since they have the energy level). So we have

$$\int_\Omega W^p = \epsilon^N \left(\frac{1}{2}\int_{\mathbb{R}^N} w_\delta^p + o(1)\right)\,, \tag{3.14}$$

where $o(1) \to 0$ as $\epsilon \to 0$ uniformly in δ. We also remark that by the compactness of the set $\mathcal{S}_{\epsilon,\delta}$, $\rho(\delta)$ is attained and is a continuous function in δ.

Note that $\beta(0) < 0$ and $\beta(\epsilon^N \int_{\mathbb{R}^N} w^p) > 0$. Therefore, by the mean-value theorem, there exists a $\delta_\epsilon \in (0, \epsilon^N \int_{\mathbb{R}^N} w^p)$ such that $\beta(\delta_\epsilon) = 0$. That is there exists a $\hat{W}_{\epsilon,\delta_\epsilon} \in \mathcal{S}_{\epsilon,\delta_\epsilon}$ such that $\delta_\epsilon = \int_\Omega \hat{W}^p_{\epsilon,\delta_\epsilon}$.

Let $\hat{W}_\epsilon = \hat{W}_{\epsilon,\delta_\epsilon}$ and $Q_\epsilon = Q_{\epsilon,\delta_\epsilon}$. Then, $\hat{W}_\epsilon$ is a single boundary spike and it satisfies the properties stated in Theorem 2.1. This completes Step 2 for single boundary spikes.

Finally, we use a similar method to discuss the case of single interior spikes. We follow the analysis in [11]. By [11], there exists solution $\hat{W}_{\epsilon,\delta}$ to (3.1), with a single interior spike for ϵ small (uniformly for δ small). Moreover, $\hat{W}_{\epsilon,\delta}$ has a unique local maximum point $Q_{\epsilon,\delta}$ such that $d(Q_{\epsilon,\delta}, \partial\Omega) \to \max_{Q\in\Omega} d(Q, \partial\Omega)$. Now we fix one such solution $\bar{W}_{\epsilon,\delta}$, and consider the following set:

$$\mathcal{S}^{'}_{\epsilon,\delta} = \{u | u \text{ satisfies (3.1) and } \|u - \bar{W}_{\epsilon,\delta}\|_\epsilon \le \epsilon\}\,. \tag{3.15}$$

Similarly $\mathcal{S}^{'}_{\epsilon,\delta}$ is a non-empty compact set and the following problem has a solution δ_ϵ:

$$\beta^{'}(\delta) = \delta - \rho^{'}(\delta) = 0, \qquad \text{where} \qquad \rho^{'}(\delta) := \inf_{\hat{W}\in\mathcal{S}^{'}_{\epsilon,\delta}} \int_\Omega \hat{W}^p\,. \tag{3.16}$$

Let $\hat{W}_\epsilon = \hat{W}_{\epsilon,\delta_\epsilon}$ and $Q_\epsilon = Q_{\epsilon,\delta_\epsilon}$. Then, $\hat{W}_\epsilon$ is a single interior spike solution and satisfies the properties stated in Theorem 2.1. ■

Remarks: 3.1). If $N=1$, both the single boundary spike solution and the single interior spike solution are unique.

3.2). If $\Omega=B_R(0)$, (P_ϵ, W_ϵ) can be chosen to be radially symmetric if we restrict to the class of radially symmetric functions.

We list several properties of $\hat{W}_\epsilon$ for later use. Their proofs can be found in [27], [28] and [11].

Lemma 3.2 *Let $\hat{W}_\epsilon$ be given in Theorem 2.1. Then, we have*

(1) $\delta_\epsilon=\int_\Omega \hat{W}_\epsilon^p=\begin{cases}\epsilon^N(\int_{\mathbb{R}^N}w^p+o(1)), & \text{for } Q_\epsilon\in\Omega,\\ \epsilon^N(\frac{1}{2}\int_{\mathbb{R}^N}w^p+o(1)), & \text{for } Q_\epsilon\in\partial\Omega.\end{cases}$

(2) $\hat{W}_\epsilon(x)\le Ce^{-c|x-Q_\epsilon|/\epsilon}$ *for some constants $C, c>0$.*

(3) $\epsilon\frac{|\nabla_x\hat{W}_\epsilon(x)|}{\hat{W}_\epsilon(x)}\ge\sqrt{1-\eta}$ *for $|x-Q_\epsilon|>\epsilon R$, where $0<\eta<1$ is a fixed constant, R is large, and ϵ is sufficiently small.*

(4) $\hat{W}_\epsilon=w_{\delta_\epsilon}(\frac{x-Q_\epsilon}{\epsilon})+\begin{cases}O(e^{-d/\epsilon}), & \text{for } Q_\epsilon\in\Omega,\\ O(\epsilon), & \text{for } Q_\epsilon\in\partial\Omega,\end{cases}$ *. Here w_{δ_ϵ} is the unique solution of (3.3), and $d>0$ is some constant (independent of ϵ).*

4 Study of the Linearized Eigenvalue Problem

In this section, we begin to study the stability of (P_ϵ, W_ϵ). We consider the interior spike case first. The boundary spike case will be treated later.

We introduce a perturbation around (P_ϵ, W_ϵ) of the following:

$$P(x,t)=P_\epsilon(x)+\eta e^{\lambda_\epsilon t}\psi_\epsilon\,,\qquad W(x,t)=W_\epsilon(x)+\eta e^{\lambda_\epsilon}\phi_\epsilon\,. \tag{4.1}$$

Substituting (4.1) into (1.9) and discarding higher order terms, we obtain the following linearized eigenvalue problem

$$\nabla\cdot(P_\epsilon\nabla(\frac{\psi_\epsilon}{P_\epsilon}-p\frac{\phi_\epsilon}{W_\epsilon}))=\lambda_\epsilon\psi_\epsilon\qquad\text{in }\ \Omega\,, \tag{4.2}$$

$$\epsilon^2\Delta\phi_\epsilon-\phi_\epsilon+\frac{P_\epsilon}{(1+\gamma W_\epsilon)^2}\phi_\epsilon+\frac{W_\epsilon}{1+\gamma W_\epsilon}\psi_\epsilon=\lambda_\epsilon\phi_\epsilon\qquad\text{in }\ \Omega\,, \tag{4.3}$$

$$\frac{\partial\phi_\epsilon}{\partial\nu}=\frac{\partial\psi_\epsilon}{\partial\nu}=0\,,\qquad\text{on }\ \partial\Omega\,, \tag{4.4}$$

where $\lambda_\epsilon\in\mathcal{C}$. Note that the conservation of mass (1.10) requires that $\int_\Omega\psi_\epsilon=0$.

Recall from (2.2) that

$$P_\epsilon=\frac{1}{\int_\Omega W_\epsilon^p}W_\epsilon^p=\frac{1}{\int_\Omega\hat{W}_\epsilon^p}\hat{W}_\epsilon^p\,,\qquad W_\epsilon=\frac{1}{\gamma\int_\Omega\hat{W}_\epsilon^p}\hat{W}_\epsilon\,, \tag{4.5}$$

where $\hat{W}_\epsilon$ satisfies (1.14) and has all the properties listed in Lemma 3.2.

We begin by simplifying (4.2) and (4.3). We introduce $\tilde{\psi}_\epsilon$ by

$$\psi_\epsilon=p\frac{P_\epsilon}{W_\epsilon}\phi_\epsilon-\eta_\epsilon P_\epsilon+\tilde{\psi}_\epsilon\,, \tag{4.6}$$

where η_ϵ is a constant, and $\tilde{\psi}_\epsilon$ satisfies

$$\int_\Omega \tilde{\psi}_\epsilon = 0\,. \tag{4.7}$$

Since $\int_\Omega \psi_\epsilon = 0$, from (4.6), we obtain that $\eta_\epsilon = (\int_\Omega P_\epsilon)^{-1} p \int_\Omega \frac{P_\epsilon}{W_\epsilon}\phi_\epsilon$. Therefore, using (4.5), we get

$$\psi_\epsilon = p\gamma \hat{W}_\epsilon^{p-1}\phi_\epsilon - p\gamma \frac{\int_\Omega \hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_\Omega \hat{W}_\epsilon^p}\hat{W}_\epsilon^p + \tilde{\psi}_\epsilon\,. \tag{4.8}$$

Substituting (4.5) and (4.8) into (4.2), we obtain that

$$\nabla\cdot\left(P_\epsilon\nabla\left(\frac{\tilde{\psi}_\epsilon}{P_\epsilon}\right)\right) = \lambda_\epsilon\left(p\gamma\hat{W}_\epsilon^{p-1}\phi_\epsilon - p\gamma\frac{\int_\Omega \hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_\Omega \hat{W}_\epsilon^p}\hat{W}_\epsilon^p + \tilde{\psi}_\epsilon\right)\,. \tag{4.9}$$

We introduce local coordinates y and $\hat{\psi}_\epsilon$ by

$$x = Q_\epsilon + \epsilon y\,, \qquad y \in \Omega_\epsilon := \{y|\epsilon y + Q_\epsilon \in \Omega_\epsilon\}\,, \qquad \tilde{\psi}_\epsilon(x) = \epsilon^2\lambda_\epsilon\gamma\hat{W}_\epsilon^{p/2}\hat{\psi}_\epsilon(y)\,, \tag{4.10}$$

where Q_ϵ is the unique maximum point of $\hat{W}_\epsilon$. We still use $\hat{W}_\epsilon, \phi_\epsilon, etc.$ to denote the functions $\hat{W}_\epsilon, \phi_\epsilon, etc.$ under the new coordinate y.

A simple computation shows that (4.7) and (4.9) become

$$\int_{\Omega_\epsilon} \hat{W}_\epsilon^{\frac{p}{2}}\hat{\psi}_\epsilon(y) = 0, \tag{4.11}$$

$$\Delta_y\hat{\psi}_\epsilon - h(\hat{W}_\epsilon)\hat{\psi}_\epsilon - \epsilon^2\lambda_\epsilon\hat{\psi}_\epsilon = p\hat{W}_\epsilon^{\frac{p}{2}-1}\phi_\epsilon - p\frac{\int_\Omega \hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_\Omega \hat{W}_\epsilon^p}\hat{W}_\epsilon^{\frac{p}{2}}\,, \qquad y\in\Omega_\epsilon\,, \tag{4.12}$$

where $h(\hat{W}_\epsilon) = \frac{\Delta_y \hat{W}_\epsilon^{p/2}}{\hat{W}_\epsilon^{p/2}}$. Substituting (4.5) and (4.8) into (4.3), we obtain that ϕ_ϵ satisfies

$$\Delta_y\phi_\epsilon - \phi_\epsilon + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\phi_\epsilon - p\frac{\int_{\Omega_\epsilon}\hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_{\Omega_\epsilon}\hat{W}_\epsilon^p}\frac{\hat{W}_\epsilon^{p+1}}{\delta_\epsilon + \hat{W}_\epsilon} = \lambda_\epsilon\left(\phi_\epsilon - \frac{\epsilon^2\hat{W}_\epsilon^{1+p/2}\hat{\psi}_\epsilon}{\delta_\epsilon + \hat{W}_\epsilon}\right)\,, \qquad y\in\Omega_\epsilon\,, \tag{4.13}$$

where $f'_{\delta_\epsilon}(\hat{W}_\epsilon)$ is defined by

$$f'_{\delta_\epsilon}(\hat{W}_\epsilon) = \frac{p\hat{W}_\epsilon^p}{\delta_\epsilon + \hat{W}_\epsilon} + \frac{\delta_\epsilon\hat{W}_\epsilon^p}{(\delta_\epsilon + \hat{W}_\epsilon)^2}\,. \tag{4.14}$$

The boundary condition (4.4) becomes

$$\frac{\partial\phi_\epsilon}{\partial\nu_\epsilon} = \frac{\partial\hat{\psi}_\epsilon}{\partial\nu_\epsilon} = 0\,, \qquad \text{on } \partial\Omega_\epsilon\,, \tag{4.15}$$

where ν_ϵ is the unit normal derivative of $\partial\Omega_\epsilon$ at y.

We now need to solve the reformulated eigenvalue problem (4.12) and (4.13), subject to (4.11) and (4.15). We begin with the following simple observation:

Lemma 4.1 *There exists a constant $C > 0$ such that for ϵ sufficiently small we have,*

$$\int_{\Omega_\epsilon}(|\nabla\psi|^2 + h(\hat{W}_\epsilon)\psi^2) \geq C\int_{\Omega_\epsilon}\psi^2\,, \quad \forall\psi\in H^1(\Omega_\epsilon) \quad \textit{such that} \quad \int_{\Omega_\epsilon}\hat{W}_\epsilon^{p/2}\psi = 0. \tag{4.16}$$

Proof: We just need to show that the principal eigenvalue ν_1^ϵ of

$$\Delta_y \psi^\epsilon - h(\hat{W}_\epsilon)\psi^\epsilon = \nu_1^\epsilon \psi^\epsilon \quad \text{in} \quad \Omega_\epsilon \,, \ \int_{\Omega_\epsilon} \hat{W}_\epsilon^{p/2}\psi^\epsilon = 0 \,, \quad \frac{\partial \psi^\epsilon}{\partial \nu_\epsilon} = 0 \quad \text{on} \ \partial\Omega_\epsilon \,, \tag{4.17}$$

satisfies $\nu_1^\epsilon < -C < 0$. Suppose not. Multiplying (4.17) by ψ_ϵ and integrating over Ω_ϵ, we obtain

$$\nu_1^\epsilon \int_{\Omega_\epsilon} (\psi^\epsilon)^2 = \int_{\Omega_\epsilon} (\nabla \cdot (P_\epsilon \nabla \left(\frac{\psi^\epsilon}{\sqrt{P_\epsilon}} \right))) \frac{\psi^\epsilon}{\sqrt{P_\epsilon}} = - \int_{\Omega_\epsilon} P_\epsilon |\nabla \cdot (\frac{\psi^\epsilon}{\sqrt{P_\epsilon}})|^2 \,. \tag{4.18}$$

Hence $\nu_1^\epsilon \leq 0$ and $\nu_1^\epsilon = 0$ if and only if $\frac{\psi^\epsilon}{\sqrt{P_\epsilon}}$ is identically a constant. Since $\int_{\Omega_\epsilon} \psi^\epsilon \sqrt{P_\epsilon} = 0$, we see that $\nu_1^\epsilon < 0$. Suppose now that as $\epsilon \to 0$ we have $\nu_1^\epsilon < 0$ with $\nu_1^\epsilon \to 0$. We proceed to derive a contradiction.

We calculate

$$h(\hat{W}_\epsilon) = \frac{p}{2}(\frac{p}{2} - 1)\frac{|\nabla \hat{W}_\epsilon|^2}{\hat{W}_\epsilon^2} + \frac{p}{2} - \frac{p}{2}\frac{\hat{W}_\epsilon^p}{\delta_\epsilon + \hat{W}_\epsilon} \,. \tag{4.19}$$

By Lemma 3.2 we see that, for ϵ sufficiently small, the inequalities

$$\frac{|\nabla \hat{W}_\epsilon|}{\hat{W}_\epsilon} \geq \sqrt{1-\eta} \,, \qquad \frac{\hat{W}_\epsilon^p}{\delta_\epsilon + \hat{W}_\epsilon} < \eta \,, \tag{4.20}$$

hold for any η small and $|y|$ large. Hence, for $|y|$ large, we have

$$h(\hat{W}_\epsilon) \geq \frac{p}{2}\left(\frac{p}{2} - 1\right)(1-\eta) + \frac{p}{2} - \frac{p}{2}\eta = \frac{p^2}{4}(1-\eta) \,.$$

Therefore, by the Maximum Principle, we get

$$|\psi^\epsilon(y)| \leq C \|\psi^\epsilon\|_{H^1(\Omega_\epsilon)} e^{-\delta |y|} \,, \tag{4.21}$$

where $\delta = \frac{p^2}{8}(1-\eta)$. By (4.21) and a compactness argument we can now take a subsequence $\epsilon \to 0$ such that $\psi^\epsilon \to \psi^0$ in $H^1(\Omega_\epsilon)$, and ψ^0 satisfies

$$\Delta_y \psi^0 - h(w^{p/2})\psi^0 = 0 \,, \qquad \psi^0 \in H^1(\mathbb{R}^N) \,, \qquad \int_{\mathbb{R}^N} w^{p/2}\psi^0 = 0 \,.$$

This is impossible by the same reasoning leading to (4.18). (Note that $\psi^0 \in H^1(\mathbb{R}^N)$ and w decays exponentially fast.) ■

From Lemma 4.1, we have

Lemma 4.2 *Let λ_ϵ be such that $Re(\lambda_\epsilon) \geq 0$. Then, there exists a constant $C > 0$ such that $|\lambda_\epsilon| \leq C$, uniformly for ϵ small.*

Proof: Multiplying (4.12) by the conjugate function of $\hat{\psi}_\epsilon$, labeled by $\overline{\hat{\psi}_\epsilon}$, and integrating over Ω_ϵ, we obtain

$$\left| \int_{\Omega_\epsilon} (|\nabla \hat{\psi}_\epsilon|^2 + h(\hat{W}_\epsilon)|\hat{\psi}_\epsilon|^2 + \epsilon^2 \lambda_\epsilon |\hat{\psi}_\epsilon|^2) \right|$$
$$= \left| p \int_{\Omega_\epsilon} \hat{W}_\epsilon^{p/2-1}\phi_\epsilon \overline{\hat{\psi}_\epsilon} - p \frac{\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_{\Omega_\epsilon} \hat{W}_\epsilon^p} \int_{\Omega_\epsilon} \hat{W}_\epsilon^{p/2}\overline{\hat{\psi}_\epsilon} \right| \leq C \|\hat{\psi}_\epsilon\|_{L^2(\Omega_\epsilon)} \|\phi_\epsilon\|_{L^2(\Omega_\epsilon)} \,. \tag{4.22}$$

Now applying Lemma 4.1, and the fact that $Re(\lambda_\epsilon) \geq 0$, we arrive at

$$\|\hat{\psi}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C\|\phi_\epsilon\|_{L^2(\Omega_\epsilon)}\,. \tag{4.23}$$

Multiplying (4.13) by $\bar{\phi}_\epsilon$, the conjugate function of ϕ_ϵ, and integrating over Ω_ϵ, we get

$$|\int_{\Omega_\epsilon}(|\nabla\phi_\epsilon|^2 + |\phi_\epsilon|^2 + \lambda_\epsilon|\phi_\epsilon|^2)| \leq C(\int_{\Omega_\epsilon}|\phi_\epsilon|^2 + \epsilon^2|\lambda_\epsilon|\int_{\Omega_\epsilon}|\phi_\epsilon||\hat{\psi}_\epsilon|)\,. \tag{4.24}$$

This yields that $|\lambda_\epsilon| \leq C$, using (4.23). ■

A corollary of Lemma 4.2 is the following:

Corollary 4.3 *Let λ_ϵ be such that $Re(\lambda_\epsilon) \geq 0$. Assume that for a subsequence $\epsilon \to 0$ and $\lambda_\epsilon \to \lambda_0$. Then, λ_0 is an eigenvalue of the nonlocal eigenvalue problem*

$$\Delta_y\phi_0 - \phi_0 + pw^{p-1}\phi_0 - p\frac{\int_{\mathbb{R}^N} w^{p-1}\phi_0}{\int_{\mathbb{R}^N} w^p}w^p = \lambda_0\phi_0\,, \quad \phi_0 \in H^1(\mathbb{R}^N)\,, \qquad \lambda_0 \in \mathcal{C}\,, \qquad Re(\lambda_0) \geq 0\,. \tag{4.25}$$

Proof: By Lemma 4.2, we have $|\lambda_\epsilon| \leq C$. We may assume that for a subsequence $\epsilon \to 0$ and $\lambda_\epsilon \to \lambda_0$. Assume that $\|\phi_\epsilon\|_{H^1(\Omega_\epsilon)} = 1$. Then, from Lemma 4.1, we have

$$\|\hat{\psi}_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C, \qquad \|\hat{\psi}_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C.$$

By taking a limit in (4.13), we see that $\phi_\epsilon \to \phi_0$ in $H^1(\Omega_\epsilon)$ and $\lambda_\epsilon \to \lambda_0$, where (λ_0, ϕ_0) satisfies (4.25). ■

Finally we discuss the boundary spike case. Let $Q_\epsilon \in \partial\Omega$ be the global maximum point $\hat{W}_\epsilon$. Without loss of generality, we may assume from now on that $Q_\epsilon = 0$ and that the normal derivative at Q_ϵ is $\nu(Q_\epsilon) = (0, ..., -1)$. Similar as to the interior spike case, we have

Corollary 4.4 *Let (P_ϵ, W_ϵ) be a single boundary spike at Q_ϵ. Let λ_ϵ be such that $Re(\lambda_\epsilon) \geq 0$. Assume that for a subsequence $\epsilon \to 0$ and $\lambda_\epsilon \to \lambda_0$. Then, λ_0 is an eigenvalue of the nonlocal eigenvalue problem*

$$\Delta_y\phi_0 - \phi_0 + pw^{p-1}\phi_0 - p\frac{\int_{\mathbb{R}^N_+} w^{p-1}\phi_0}{\int_{\mathbb{R}^N_+} w^p}w^p = \lambda_0\phi_0 \quad in \ \mathbb{R}^N_+, \quad \frac{\partial\phi_0}{\partial y_N} = 0 \quad on \ \partial\mathbb{R}^N_+\,, \quad \phi_0 \in H^1(\mathbb{R}^N_+)\,, \tag{4.26}$$

with $Re(\lambda_0) \geq 0$, where $\mathbb{R}^N_+ = \{(y', y_N) \in \mathbb{R}^N | y_N > 0\}$.

Let ϕ_0 be a solution of (4.26) on $\mathbb{R}^N_+$. By an even extension of ϕ_0 to $\mathbb{R}^N$, it is easy to see that the new function, denoted by $\tilde{\phi}_0$, satisfies (4.25).

5 Study of a Nonlocal Eigenvalue Problem

In this section, we study the following nonlocal eigenvalue problem derived in Corollary (4.3)

$$L\phi := \Delta_y\phi - \phi + pw^{p-1}\phi - p\frac{\int_{\mathbb{R}^N} w^{p-1}\phi}{\int_{\mathbb{R}^N} w^p}w^p = \lambda_0\phi\,, \qquad \phi \in H^1(\mathbb{R}^N)\,, Re(\lambda_0) \geq 0 \tag{5.1}$$

where w is the unique solution of (2.3), and $\lambda_0 \in \mathcal{C}$ is the set of complex numbers. Nonlocal eigenvalues of this type have been studied in several papers. For the case of $N = 1$, we refer to [7] and [41]. For the case of $p = 2$, we refer to [5] and [41]. In the general (p, N) case, we follow an approach in [45].

We first characterize the kernel of L:

Lemma 5.1 *We have*

$$\{\phi \in H^1(\mathbb{R}^N) | L\phi = 0\} = K_0 := span\ \left\{\frac{\partial w}{\partial y_j}, j = 1, ..., N\right\},$$

$$\{\phi \in H^1(\mathbb{R}^N_+) | L\phi = 0,\ \frac{\partial \phi}{\partial y_N} = 0\ on\ \partial\mathbb{R}^N_+\} =\ span\ \left\{\frac{\partial w}{\partial y_j}, j = 1, ..., N-1\right\}.$$

Proof: The proof is similar to that of Lemma 5.1 of [41]. We omit the details. ∎

From Lemma 5.1, we may assume that $\lambda_0 \neq 0$. The following result was proved in [44]:

Lemma 5.2 *Assume that $N = 1$ and $1 < p < +\infty$. Then, for any nonzero eigenvalue λ_0 of (5.1), we have $Re(\lambda_0) \leq -C < 0$ for some constant $C > 0$.*

We are left with the case of $N = 2, 3$. We now use a continuation argument, similar to [45], where p is a continuation parameter. For $p = 2$, (5.1) was studied in [41]. Applying Theorem 1 of [45], we have

Theorem 5.3 *Suppose that p satisfies (2.7) in Theorem 2.2. Then, for any nonzero eigenvalue λ_0 of (5.1), we have $Re(\lambda_0) \leq -C < 0$ for some constant $C > 0$.*

Proof: Let $r = p$ and $\gamma = \frac{p}{p-1}$. Using (2.7), it is easy to see that $F(p) = 1 - \frac{p-1}{2p}N \geq 0$. Applying Theorem 1 of [45], we just need to check that

$$F(p) \geq \frac{\gamma - 2}{\gamma} F(p+1) + \frac{|\gamma - 2|}{\gamma} \sqrt{F(p+1)(F(p+1) - F(2))}, \tag{5.2}$$

where $F(r) = 1 - \frac{p-1}{2r}N$. Note that for $2 \leq p$, we have $\frac{\gamma-2}{\gamma} = \frac{2-p}{p} \leq 0$.

If $N = 2$, we have $F(p) = \frac{1}{p}, F(p+1) = \frac{2}{p+1}, F(2) = \frac{3-p}{2}$. By simple computations, (5.2) is equivalent to $p^2 - 6p + 5 \leq 0$ which holds if $2 \leq p \leq 5$.

If $N = 3$, we have $F(p) = \frac{3-p}{2p}, F(p+1) = \frac{5-p}{2(p+1)}, F(2) = \frac{7-3p}{4}$. By simple computations, $\frac{\gamma-2}{\gamma}F(p+1) + \frac{|\gamma-2|}{\gamma}\sqrt{F(p+1)(F(p+1) - F(2))} \leq 0 \leq F(p)$ if $2 \leq p \leq 3$. ∎

6 Study of the Small Eigenvalues: Proof of Theorem 2.2

In this section, we study the asymptotic behavior of the small eigenvalues that tend to zero as $\epsilon \to 0$. We also prove Theorem 2.2.

Suppose that p satisfies (2.7) and that λ_ϵ is an eigenvalue with $Re(\lambda_\epsilon) \geq 0$. Then, by Corollaries (4.3), (4.4), Lemma 5.1, and Theorem 5.3, we must have that $\lim_{\epsilon \to 0} \lambda_\epsilon = \lambda_0 = 0$. Namely, if $Re(\lambda_\epsilon) \geq 0$, then necessarily, $\lambda_\epsilon \to 0$. By Lemma 5.1, we have $\phi_0 \in$ span $\{\frac{\partial w}{\partial y_j}, j = 1, ..., N\}$ for $Q_\epsilon \in \Omega$, and $\phi_0 \in$ span $\{\frac{\partial w}{\partial y_j}, j = 1, ..., N-1\}$ for $Q_\epsilon \in \partial\Omega$. This result is summarized as follows:

Lemma 6.1 *Suppose that p satisfies (2.7). Let λ_ϵ be an eigenvalue of (4.2) and (4.3) with $Re(\lambda_\epsilon) \geq 0$. Then, for $\epsilon \to 0$ and $y \in \Omega_\epsilon$, we must have*

$$\phi_\epsilon = \begin{cases} \sum_{j=1}^{N} a_j^\epsilon \frac{\partial w}{\partial y_j}(y) + o(1), & if \quad Q_\epsilon \in \Omega, \\ \sum_{j=1}^{N-1} a_j^\epsilon \frac{\partial w}{\partial y_j}(y) + o(1), & if \quad Q_\epsilon \in \partial\Omega. \end{cases} \tag{6.1}$$

Lemma 6.1 immediately implies that for $N = 1$ the single boundary spike is linearly stable. Next, consider the radially symmetric case where $(P(x,t), W(x,t)) = (P(|x|,t), W(|x|,t))$. Then, $\phi_0(y) = \phi(|y|)$, and by Lemma 6.1 we conclude that $\phi_0 = 0$. Hence, there are no small eigenvalues. Moreover, if p satisfies (2.7), then $Re(\lambda_0) < 0$ for $\lim_{\epsilon\to 0}\lambda_\epsilon = \lambda_0 \neq 0$. Therefore, we conclude that in the radially symmetric case (P_ϵ, W_ϵ) is linearly stable. This proves (b) and (c) of Theorem 2.2.

Now we prove (a) of Theorem 2.2. Let (P_ϵ, W_ϵ) be a single interior spike solution. We now show that (P_ϵ, W_ϵ) is metastable. Namely we need to show that for $Re(\lambda_\epsilon) \geq 0$ we must have $|\lambda_\epsilon| = O(e^{-d/\epsilon})$.

Suppose now that $Re(\lambda_\epsilon) \geq 0$. From (4.13), the equation for ϕ_ϵ becomes

$$\Delta_y \phi_\epsilon - \phi_\epsilon + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\phi_\epsilon - p\frac{\int_{\Omega_\epsilon}\hat{W}_\epsilon^{p-1}\phi_\epsilon}{\int_{\Omega_\epsilon}\hat{W}_\epsilon^p}\frac{\hat{W}_\epsilon^{p+1}}{\delta_\epsilon + \hat{W}_\epsilon} = \lambda_\epsilon\left(\phi_\epsilon - \epsilon^2\frac{\hat{W}_\epsilon^{1+p/2}}{\delta_\epsilon + \hat{W}_\epsilon}\hat{\psi}_\epsilon\right), \qquad y \in \Omega_\epsilon. \tag{6.2}$$

We introduce the new function $\hat{\phi}_\epsilon$ by

$$\phi_\epsilon = \hat{\phi}_\epsilon + c_\epsilon \hat{W}_\epsilon, \qquad \text{where} \qquad c_\epsilon = -p\frac{\int_{\Omega_\epsilon}\hat{W}_\epsilon^{p-1}\hat{\phi}_\epsilon}{\int_{\Omega_\epsilon}\hat{W}_\epsilon^p}. \tag{6.3}$$

Then, it is easy to see that $\hat{\phi}_\epsilon$ satisfies

$$\Delta_y \hat{\phi}_\epsilon - \hat{\phi}_\epsilon + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\hat{\phi}_\epsilon + c_\epsilon\frac{\delta_\epsilon\hat{W}_\epsilon^{p+1}}{(\delta_\epsilon + \hat{W}_\epsilon)^2} = \lambda_\epsilon\left(\hat{\phi}_\epsilon + c_\epsilon\hat{W}_\epsilon - \epsilon^2\frac{\hat{W}_\epsilon^{1+p/2}}{\delta_\epsilon + \hat{W}_\epsilon}\hat{\psi}_\epsilon\right). \tag{6.4}$$

Let $\eta(x)$ be a smooth cut-off function such that $\eta(x) = 1$ for $|x| \leq 1$, and $\eta(x) = 0$ for $|x| > 2$. Set $r = \frac{1}{4}d(Q_\epsilon, \partial\Omega)$. Consider the following functions:

$$\phi_{\epsilon,j}(y) = \frac{\partial w_{\delta_\epsilon}}{\partial y_j}(y)\eta(\frac{\epsilon y}{r}), \qquad j = 1, ..., N, \qquad y \in \Omega_\epsilon. \tag{6.5}$$

Then, by Lemma 3.2, we have

$$\Delta_y \phi_{\epsilon,j} - \phi_{\epsilon,j} + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\phi_{\epsilon,j} = O(e^{-d/\epsilon}), \quad \int_{\Omega_\epsilon}\phi_{\epsilon,j}\phi_{\epsilon,k} = O(e^{-d/\epsilon}), \qquad \text{for } j \neq k. \tag{6.6}$$

Next, we decompose $\hat{\phi}_\epsilon$ as follows:

$$\hat{\phi}_\epsilon = \sum_{j=1}^N c_j^\epsilon \phi_{\epsilon,j} + \hat{\phi}_\epsilon^\perp,$$

where $\hat{\phi}_\epsilon^\perp \perp \phi_{\epsilon,j}$ for $j = 1, ..., N$, and $\sum_{j=1}^N |c_j^\epsilon|^2 = 1$. Lemma 6.1 implies $\|\hat{\phi}_\epsilon^\perp\|_\epsilon \to 0$ as $\epsilon \to 0$.

Note that from Lemma 3.2 we have

$$|\int_{\Omega_\epsilon}\hat{W}_\epsilon^{p-1}\phi_{\epsilon,j}| + |\int_{\Omega_\epsilon}\hat{W}_\epsilon^p\phi_{\epsilon,j}| = O(e^{-d/\epsilon}), \quad j = 1, ..., N. \tag{6.7}$$

The proof of part (a) of Theorem 2.2 proceeds in two steps. First, we obtain the estimates for $\hat{\phi}_\epsilon^\perp$. Then, we deduce the equation for λ_ϵ. Similar arguments have been used in Section 4 of [41].

It is easy to see that $\hat{\phi}_\epsilon^\perp$ satisfies

$$\Delta_y \hat{\phi}_\epsilon^\perp - \hat{\phi}_\epsilon^\perp + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\hat{\phi}_\epsilon^\perp = -c_\epsilon\frac{\delta_\epsilon\hat{W}_\epsilon^{p+1}}{(\delta_\epsilon + \hat{W}_\epsilon)^2} \tag{6.8}$$

$$+\sum_{j=1}^{N} c_j^\epsilon O(e^{-d/\epsilon}) + \lambda_\epsilon \left(\hat{\phi}_\epsilon + \sum_{j=1}^{N} c_j^\epsilon \phi_{\epsilon,j} + c_\epsilon \hat{W}_\epsilon - \epsilon^2 \frac{\hat{W}_\epsilon^{1+p/2}}{\delta_\epsilon + \hat{W}_\epsilon} \hat{\psi}_\epsilon \right) .$$

Note that from the definition of c_ϵ, we have that

$$c_\epsilon = -p \frac{\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1} (\sum_{j=1}^{N} c_j^\epsilon \phi_{\epsilon,j} + \hat{\phi}_\epsilon^\perp)}{\int_{\Omega_\epsilon} \hat{W}_\epsilon^p} = O\left(e^{-d/\epsilon} + |\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1} \hat{\phi}_\epsilon^\perp| \right) . \tag{6.9}$$

Let us define

$$\mathcal{K}_\epsilon = \text{span}\ \{\phi_{\epsilon,j}, j = 1, ..., N\} \subset H^2_{\nu_\epsilon}(\Omega_\epsilon), \quad \mathcal{C}_\epsilon = \text{span}\ \{\phi_{\epsilon,j}, j = 1, ..., N\} \subset L^2(\Omega_\epsilon), \tag{6.10}$$

where $H^2_{\nu_\epsilon}(\Omega_\epsilon) = H^2(\Omega_\epsilon) \cap \{\frac{\partial u}{\partial \nu_\epsilon} = 0 \text{ on } \partial\Omega_\epsilon\}$. Let $\mathcal{K}_\epsilon^\perp$ and $\mathcal{C}_\epsilon^\perp$ be the orthogonal space of $\mathcal{K}_\epsilon$ and $\mathcal{C}_\epsilon$ under the $L^2(\Omega_\epsilon)$ inner product, respectively. Put

$$L_\epsilon := \Delta_y \phi - \phi + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\phi : \qquad H^2_{\nu_\epsilon}(\Omega_\epsilon) \to L^2(\Omega_\epsilon) . \tag{6.11}$$

Then, as in Lemma 2.3 of [41], we have that the map $\mathcal{L}_\epsilon = \pi_\epsilon \circ L_\epsilon : \mathcal{K}_\epsilon^\perp \to \mathcal{C}_\epsilon^\perp$ is invertible and the inverse is bounded uniformly in ϵ. Here π_ϵ is the projection from $L^2(\Omega_\epsilon)$ into $\mathcal{C}_\epsilon^\perp$.

From (6.8) and (4.23) we obtain

$$\|\hat{\phi}_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C \left(|c_\epsilon| \delta_\epsilon + |c_\epsilon||\lambda_\epsilon| + e^{-d/\epsilon} + \epsilon^2 |\lambda_\epsilon| \right) .$$

From (6.9), this implies that

$$\|\hat{\phi}_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C(e^{-d/\epsilon} + \epsilon^2|\lambda_\epsilon|) . \tag{6.12}$$

Multiplying (6.8) by $\phi_{\epsilon,k}$, and integrating over Ω_ϵ, we obtain

$$\lambda_\epsilon \left(\sum_{j=1}^{N} c_j^\epsilon \int_{\Omega_\epsilon} \phi_{\epsilon,j}\phi_{\epsilon,k} + O(\epsilon^2) \right) = O(e^{-d/\epsilon}) + \int_{\Omega_\epsilon} \phi_{\epsilon,k} \left(\Delta \hat{\phi}_\epsilon^\perp - \hat{\phi}_\epsilon^\perp + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\hat{\phi}_\epsilon^\perp \right) = O(e^{-d/\epsilon}) . \tag{6.13}$$

This shows that $|\lambda_\epsilon| = O(e^{-d/\epsilon})$, which proves part (a) of Theorem 2.2.

Finally we prove (d) of Theorem 2.2. Let us assume that $Q_\epsilon \to Q_0$, where Q_0 is a nondegenerate global maximum point of $H(P)$. For single boundary spikes, we have from Theorem 1.3 of [42] or Theorem 2.2 of [3] that the following eigenvalue problem

$$\Delta_y \phi - \phi + f'_{\delta_\epsilon}(\hat{W}_\epsilon)\phi = \tau_\epsilon \phi \quad \text{in} \ \ \Omega_\epsilon , \qquad \frac{\partial \phi}{\partial \nu_\epsilon} = 0 , \quad \text{on} \ \ \partial\Omega_\epsilon , \tag{6.14}$$

has $N-1$ normalized eigenfunctions $\{\phi_{\epsilon,j}, j = 1, ..., N-1\}$ with N eigenvalues (multiplicity is allowed)

$$\tau_1^\epsilon \le ... \le \tau_{N-1}^\epsilon , \qquad \tau_j^\epsilon = c_0 \epsilon^2 \lambda_j + o(\epsilon^2) . \tag{6.15}$$

Here λ_j is the eigenvalue of the matrix $(\nabla^2 H(Q_0))$, and $c_0 > 0$ is a generic constant. Moreover, we have

$$\int_{\Omega_\epsilon} \phi_{\epsilon,j}^2 = 1 , \qquad \int_{\Omega_\epsilon} \phi_{\epsilon,j}\phi_{\epsilon,k} = 0 , \quad \text{for } j \ne k , \quad \phi_{\epsilon,j}(y) = \sum_{k=1}^{N-1} a_{jk} \frac{\partial w}{\partial y_k} + O(\epsilon) , \qquad j = 1, ..., N-1 , \tag{6.16}$$

for some constants a_{jk}.

Similar to the analysis above, we now have from Lemma 3.2 that

$$\hat{W}_\epsilon(y) = w_{\delta_\epsilon}(y) + O(\epsilon)\,, \quad |\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1}\phi_{\epsilon,j}| + |\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p}\phi_{\epsilon,j}| = O(\epsilon)\,, \quad j = 1, ..., N-1. \tag{6.17}$$

We decompose

$$\hat{\phi}_\epsilon = \sum_{j=1}^{N-1} c_j^\epsilon \phi_{\epsilon,j} + \hat{\phi}_\epsilon^\perp\,, \qquad \hat{\phi}_\epsilon^\perp \perp \phi_{\epsilon,j}\,, \quad j = 1, ..., N-1\,.$$

Hence, we have

$$c_\epsilon = -p\frac{\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1}(\sum_{j=1}^N c_j^\epsilon \phi_{\epsilon,j} + \hat{\phi}_\epsilon^\perp)}{\int_{\Omega_\epsilon} \hat{W}_\epsilon^p} = O\left(\epsilon + |\int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1}\hat{\phi}_\epsilon^\perp|\right)\,. \tag{6.18}$$

From (6.8) we obtain that

$$\Delta_y \hat{\phi}_\epsilon^\perp - \hat{\phi}_\epsilon^\perp + f_\delta'(\hat{W}_\epsilon)\hat{\phi}_\epsilon^\perp = O(\epsilon\delta_\epsilon + \delta_\epsilon \int_{\Omega_\epsilon} \hat{W}_\epsilon^{p-1}|\hat{\phi}_\epsilon^\perp|) \tag{6.19}$$

$$-\sum_{j=1}^{N-1} c_j^\epsilon \tau_j^\epsilon \phi_{\epsilon,j} + \lambda_\epsilon(\sum_{j=1}^{N-1} c_j^\epsilon \phi_{\epsilon,j} + \hat{\phi}_\epsilon^\perp + c_\epsilon \hat{W}_\epsilon - \epsilon^2 \frac{\hat{W}_\epsilon^{1+p/2}}{\delta_\epsilon + \hat{W}_\epsilon}\hat{\psi}_\epsilon)\,.$$

Similar arguments leading to (6.12) imply that

$$\|\hat{\phi}_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\left(\epsilon\delta_\epsilon + \epsilon^2|\lambda_\epsilon| + |c_\epsilon||\lambda_\epsilon|\right)\,. \tag{6.20}$$

Multiplying (6.19) by $\phi_{\epsilon,k}$ and integrating over Ω_ϵ, we obtain

$$\lambda_\epsilon(\sum_{j=1}^{N-1} c_j^\epsilon \int_{\Omega_\epsilon} \phi_{\epsilon,j}\phi_{\epsilon,k} + O(\epsilon^2)) = O(\epsilon\delta_\epsilon) + \sum_{j=1}^{N-1} c_j^\epsilon \tau_j^\epsilon \int_{\Omega_\epsilon} \phi_{\epsilon,j}\phi_{\epsilon,k}. \tag{6.21}$$

Therefore,

$$\lambda_\epsilon(c_k^\epsilon + O(\epsilon^2)) = O(\epsilon^{N+1}) + c_k^\epsilon \tau_k^\epsilon\,. \tag{6.22}$$

Since $\sum_{k=1}^N |c_k^\epsilon|^2 = 1$, we see that there exists some k such that $\lambda_\epsilon = \tau_k^\epsilon + o(\epsilon^2)$. Since $\tau_k^\epsilon < 0$, we see that $Re(\lambda_\epsilon) \le -c_1\epsilon^2 < 0$, where $c_1 > 0$ is a generic constant. This finishes the proof of part (d) of Theorem 2.2. ∎

7 Metastable Dynamics and Numerical Results

In this section, we assume that p satisfies the conditions (2.7) under Theorem 5.3. We then use a formal asymptotic analysis to characterize the metastable dynamics of an interior spike solution for (1.9). The metastability analysis is similar to that given in [16] for an interior spike solution to the Gierer-Meinhardt model of [10]. We look for a solution to (1.9) in the form

$$W(x,t) = W_\epsilon\left(\epsilon^{-1}|x - x_0|\right) + R(x,t)\,, \qquad P(x,t) = P_\epsilon\left(\epsilon^{-1}|x - x_0|\right) + H(x,t)\,. \tag{7.1}$$

Here (W_ϵ, P_ϵ) satisfy the PDE's of (1.11), and the error terms R and W are such that $R \ll W_\epsilon$ and $H \ll P_\epsilon$. In (7.1), $x_0 = x_0(t) \in \Omega$ is the unknown location of the center of the spike. Our goal is to derive an equation of motion for $x_0(t)$. We will only consider the long-time evolution of the spike, and do not discuss the transient process by which a spike forms from initial data. Therefore, we assume that at $t = 0$, we have $x_0(0) = x_0^0 \in \Omega$ and $R(x,0) = H(x,0) = 0$. Since the linearized problem has an exponentially small principal eigenvalue, we expect that the speed $x_0^{'}$ is exponentially small as $\epsilon \to 0$.

Substituting (7.1) into (1.9), and linearizing the resulting system, we obtain

$$\nabla \cdot \left(P_\epsilon \nabla \left(\frac{H}{P_\epsilon} - \frac{pR}{W_\epsilon} \right) \right) = \partial_t P_\epsilon + \partial_t H \,, \quad \text{in } \Omega \,; \qquad \frac{\partial R}{\partial \nu} = -\frac{\partial W_\epsilon}{\partial \nu} \,, \quad \text{on } \Omega \,, \tag{7.2a}$$

$$\epsilon^2 \Delta R - R + \frac{P_\epsilon}{(1+\gamma W_\epsilon)^2} R + \frac{W_\epsilon}{1+\gamma W_\epsilon} H = \partial_t W_\epsilon + \partial_t R \,; \qquad \frac{\partial H}{\partial \nu} = -\frac{\partial P_\epsilon}{\partial \nu} \,, \quad \text{on } \partial\Omega \,. \tag{7.2b}$$

Since both $\partial H / \partial \nu$ on $\partial\Omega$ and the right-hand side of the PDE in (7.2a) are small, we can asymptotically solve (7.2a) for H to get

$$H \sim \left(\frac{pP_\epsilon}{W_\epsilon} \right) R - pP_\epsilon \int_\Omega \frac{P_\epsilon R}{W_\epsilon} \,. \tag{7.3}$$

Substituting (7.3) into (7.2b), we obtain

$$\epsilon^2 \Delta R - R + f^{'}(W_\epsilon) R - \frac{pW_\epsilon P_\epsilon}{1+\gamma W_\epsilon} \int_\Omega \frac{P_\epsilon R}{W_\epsilon} = \partial_t W_\epsilon + \partial_t R \,, \quad \text{in } \Omega \,; \qquad \frac{\partial R}{\partial \nu} = -\frac{\partial W_\epsilon}{\partial \nu} \,, \quad \text{on } \partial\Omega \,. \tag{7.4}$$

Here $f^{'}(W_\epsilon)$ is defined by

$$f^{'}(W_\epsilon) = \frac{P_\epsilon}{(1+\gamma W_\epsilon)^2} + \frac{pP_\epsilon}{1+\gamma W_\epsilon} \,. \tag{7.5}$$

Next, we use (2.2) to write (7.4) as

$$\mathcal{L}_\epsilon R \equiv \epsilon^2 \Delta R - R + f^{'}_{\delta_\epsilon}(\hat{W}_\epsilon) R - \frac{p\hat{W}_\epsilon^{p+1}}{\delta_\epsilon + \hat{W}_\epsilon} \frac{\int_\Omega \hat{W}_\epsilon^{p-1} R}{\int_\Omega \hat{W}_\epsilon^p} = \partial_t W_\epsilon + \partial_t R \,, \quad \text{in } \Omega \,, \tag{7.6a}$$

$$\frac{\partial R}{\partial \nu} = -\frac{\partial W_\epsilon}{\partial \nu} \,, \quad \text{on } \partial\Omega \,. \tag{7.6b}$$

Here δ_ϵ and $f^{'}_{\delta_\epsilon}(\hat{W}_\epsilon)$ are defined in Lemma 3.2 and 4.14, respectively, and $\hat{W}_\epsilon$ satisfies (1.14).

We define the local operator in (7.6) as

$$L_\epsilon R \equiv \epsilon^2 \Delta R - R + f^{'}_{\delta_\epsilon}(\hat{W}_\epsilon) R \,. \tag{7.7}$$

By translation invariance, we find upon differentiating (1.14) that

$$L_\epsilon \left(\partial_{x_j} \hat{W}_\epsilon \right) = 0 \,, \qquad j = 1, .., N \,. \tag{7.8}$$

In addition, since $x_0 \in \Omega$ and $\hat{W}_\epsilon$ is locally radially symmetric near x_0, then $\mathcal{L}_\epsilon \left(\partial_{x_j} \hat{W}_\epsilon \right)$ is exponentially small as $\epsilon \to 0$. Moreover, $\partial_{x_j} \hat{W}_\epsilon$ is exponentially small on $\partial\Omega$ for $j = 1, .., N$.

From part 4 of Lemma 3.2, we recall that

$$\hat{W}_\epsilon = w_{\delta_\epsilon} \left[\epsilon^{-1} |x - x_0| \right] + O\left(e^{-d/\epsilon} \right) \,, \tag{7.9}$$

where $d > 0$ is some constant independent of ϵ. Here w_{δ_ϵ} satisfies (3.3), with $\delta_\epsilon = \int_\Omega \hat{W}_\epsilon^p = O(\epsilon^N)$. The far-field behavior of the solution, valid for $|x - x_0| \gg O(\epsilon)$, is

$$\hat{W}_\epsilon \sim a \left(\frac{|x - x_0|}{\epsilon} \right)^{(1-N)/2} e^{-|x - x_0|/\epsilon} \,. \tag{7.10}$$

Here a is a positive constant that depends on N, p, and ϵ. However, as $\epsilon \to 0$ we have $a \to a_0 > 0$, where a_0 is determined from the far-field behavior of (2.3), which corresponds to setting $\delta = 0$ in (3.3). In §7.1 we derive an ODE for $x_0(t)$ for the multi-dimensional case where $N \geq 2$. The one-dimensional case is studied in §7.2.

7.1 The Multi-Dimensional Case

To derive an equation of motion for $x_0(t)$, we first must determine the eigenfunctions of $\mathcal{L}_\epsilon$ in (7.6) corresponding to the exponentially small eigenvalues. Let $(\lambda_{0j}, \phi_{0j})$, for $j = 1, .., N$, be the eigenpairs of $\mathcal{L}_\epsilon \phi_0 = \lambda_0 \phi_0$, where λ_{0j} is exponentially small as $\epsilon \to 0$. From (7.8), we note that these eigenfunctions are given asymptotically, in the interior of the domain, by $\phi_{0j} \sim \partial_{x_j} \hat{W}_\epsilon$ for $j = 1, .., N$. However, as in [16], we must insert a boundary layer correction term near Ω to ensure that ϕ_{0j} satisfies the homogeneous boundary condition $\partial \phi_{0j}/\partial \nu = 0$ on $\partial\Omega$. In order to resolve the boundary layer, we define a local coordinate system. Let $\hat{\eta}$ represent the distance from a point in Ω to $\partial\Omega$, where $\hat{\eta} < 0$ corresponds to the interior of Ω. Let s correspond to the other $N-1$ orthogonal coordinates. To localize the region near $\partial\Omega$, we let $\eta = \epsilon^{-1}\hat{\eta}$. The eigenfunction is then approximated by

$$\phi_{0j} \sim \partial_{x_j} \hat{W}_\epsilon + \hat{\phi}_j \,, \qquad j = 1, .., N \,. \tag{7.11}$$

Substituting (7.11) into (7.6a), we obtain that $\hat{\phi}_j$ satisfies

$$\partial_{\eta\eta} \hat{\phi}_j - \phi_j = 0 \,, \quad \eta < 0 \,; \quad \partial_\eta \hat{\phi}_j = -\epsilon \partial_{\hat{\eta}} \left(\partial_{x_j} \hat{W}_\epsilon \right) |_{\eta=0} \,, \quad \text{on} \quad \eta = 0 \,, \tag{7.12}$$

with $\hat{\phi}_j \to 0$ as $\eta \to -\infty$. Below, we require a formula for ϕ_{0j} on $\partial\Omega$. Solving (7.12), and using the far-field form (7.10), we substitute the resulting expression into (7.11) to get for $j = 1, .., N$, that

$$\phi_{0j} \sim -a\epsilon^{(N-3)/2} r^{-(1+N)/2} \left(x_j - x_{0j} \right) e^{-r/\epsilon} \left(1 + \hat{r} \cdot \hat{n} \right) \,, \qquad \text{on} \quad \Omega \,. \tag{7.13}$$

Here x_j denotes the j^{th} coordinate of x, $r \equiv |x - x_0|$, $\hat{r} = (x - x_0)/r$, and $\hat{n}$ is the unit outward normal to $\partial\Omega$. A similar calculation was done in [16] with regards to metastable behavior in the Gierer-Meinhardt model [10]. For further details of the calculation leading to (7.13) see [16].

Next, we multiply (7.6a) by ϕ_{0j} and integrate over Ω. Integrating the resulting equation by parts over Ω, and assuming that $\partial_t R$ is asymptotically small, we obtain

$$\left(\partial_t W_\epsilon, \phi_{0j} \right) = -\epsilon^2 \int_{\partial\Omega} \phi_{0j} \, \partial_\nu \hat{W}_\epsilon \, dS + \left(R, \mathcal{L}_\epsilon{}^* \phi_{0j} \right) \,. \tag{7.14}$$

We now evaluate the terms in (7.14) using $W_\epsilon = C\hat{W}_\epsilon$, for some constant C. The dominant contribution to the integral on the left-hand side of (7.14) arises from the region near $x = x_0$. Using $\phi_{0j} \sim \partial_{x_j} \hat{W}_\epsilon$, $W_\epsilon = C\hat{W}_\epsilon$, and (7.9) for $\hat{W}_\epsilon$, we calculate for $j = 1, .., N$ that

$$\left(\partial_t W_\epsilon, \phi_{0j} \right) \sim -C\epsilon^{N-2} x_{0j}^{'} \left(\frac{\omega_N \beta_N}{N} \right) \,, \qquad \beta_N \equiv \int_0^\infty \rho^{N-1} \left[w_{\delta_\epsilon}^{'}(\rho) \right]^2 \, d\rho \,. \tag{7.15}$$

Here $(u, v) \equiv \int_\Omega uv$, $x_{0j}^{'} \equiv dx_{0j}/dt$, $w_{\delta_\epsilon}(\rho)$ is the radially symmetric solution to (3.3), and ω_N is the surface area of the unit N-sphere. Next, we use (7.10) and (7.13) to estimate the boundary integral term in (7.14) as

$$-\epsilon^2 \int_{\partial\Omega} \phi_{0j} \, \partial_\nu \hat{W}_\epsilon \, dS \sim -Ca^2 \epsilon^{N-1} \int_{\partial\Omega} r^{1-N} e^{-2r/\epsilon} \frac{(x_j - x_{0j})}{r} \hat{r} \cdot \hat{n} \left(1 + \hat{r} \cdot \hat{n} \right) \, dS \,. \tag{7.16}$$

This expression shows that the boundary integral term in (7.14) is $O\left(\epsilon^q e^{-2r_0/\epsilon}\right)$, where q is some constant, and $r_0 = \text{dist}\,(x_0, \partial\Omega)$.

We now show that the inner product term on the right-hand side of (7.14) is asymptotically negligible in comparison with (7.16). We calculate, using $\delta_\epsilon = O(\epsilon^N)$ and symmetry, that

$$\mathcal{L}_\epsilon{}^* \phi_{0j} \sim -\frac{p\hat{W}_\epsilon^{p-1}}{\int_\Omega \hat{W}_\epsilon^p} \int_\Omega \frac{\hat{W}_\epsilon^{p+1}}{\delta_\epsilon + \hat{W}_\epsilon} \partial_{x_j} \hat{W}_\epsilon \sim -\frac{p\hat{W}_\epsilon^{p-1}}{N\,(p+1) \int_\Omega \hat{W}_\epsilon^p} \int_\Omega \nabla \cdot \hat{W}_\epsilon^{p+1} \,. \tag{7.17}$$

Then, using the Divergence theorem, and the far-field behavior (7.10), we obtain

$$\mathcal{L}_\epsilon{}^* \phi_{0j} \sim -\frac{p\hat{W}_\epsilon^{p-1}}{N \int_\Omega \hat{W}_\epsilon^p} \int_{\partial\Omega} \frac{\hat{W}_\epsilon^{p+1}}{p+1} = \hat{W}_\epsilon^{p-1}\, O\left(\epsilon^q e^{-(p+1)r_0/\epsilon}\right) . \tag{7.18}$$

Here q is a constant, and $r_0 = \text{dist}\,(x_0, \partial\Omega)$. Since $R \ll 1$ and $p > 1$, it follows that the exponentially small term $(R, \mathcal{L}_\epsilon{}^* \phi_{0j})$ is asymptotically negligible in comparison with the boundary integral term in (7.16).

Therefore, substituting (7.15) and (7.16) into (7.14), and neglecting $(R, \mathcal{L}_\epsilon{}^* \phi_{0j})$ in (7.14), we obtain

$$\frac{dx_0}{dt} \sim \frac{a^2 N \epsilon}{\beta_N \omega_N} \int_{\partial\Omega} r^{1-N} e^{-2r/\epsilon}\, \hat{r}\, (1 + \hat{r}\cdot\hat{n})\, \hat{r}\cdot\hat{n}\, dS \,. \tag{7.19}$$

Assuming that there is a unique point $x_m \in \partial\Omega$ closest to the initial center $x_0(0)$ of the spike, we can evaluate the surface integral in (7.19) using Laplace's method. This leads to the following explicit result:
Proposition 7.1: *For $\epsilon \ll 1$, a metastable spike solution for (1.9) is given by*

$$W_\epsilon \sim \frac{w_{\delta_\epsilon}\left[\epsilon^{-1}(x - x_0)\right]}{\gamma \epsilon^N \int_{\mathbb{R}^N} \left[w_{\delta_\epsilon}(y)\right]^p\, dy} \,, \qquad P_\epsilon \sim \frac{\left(w_{\delta_\epsilon}\left[\epsilon^{-1}(x - x_0)\right]\right)^p}{\epsilon^N \int_{\mathbb{R}^N} \left[w_{\delta_\epsilon}(y)\right]^p\, dy} \,. \tag{7.20a}$$

Here w_{δ_ϵ} satisfies (3.3). Let x_m be the point on $\partial\Omega$ closest to $x_0(0)$. Then, for $t > 0$, the spike moves in the direction of x_m, and the distance $r_m(t) = |x_m - x_0(t)|$ satisfies the first order nonlinear differential equation

$$\frac{dr_m}{dt} \sim -\xi r_m \left(\frac{\epsilon}{r_m}\right)^{(N+1)/2} K(r_m) e^{-2r_m/\epsilon} \,, \tag{7.20b}$$

where $\xi > 0$ and the function $K(r_m)$ are defined by

$$\xi \equiv \frac{2Na^2}{\omega_N \beta_N} \pi^{(N-1)/2} \,, \qquad K(r_m) \equiv (1 - r_m/R_1)^{-1/2} (1 - r_m/R_2)^{-1/2} \cdots (1 - r_m/R_{N-1})^{-1/2} \,. \tag{7.20c}$$

Here $R_j > 0$, for $j = 1, .., N-1$, are the principal radii of curvature of $\partial\Omega$ at x_m, ω_N is the surface area of the unit N-sphere, and a and β_N were defined in (7.10) and (7.15), respectively.

This result is valid up until the time when the spike approaches to within an $O(\epsilon)$ distance of x_m. If the initial condition for (7.20b) is $r_m(0) = r_0$, then the time T needed for $r_m(T) = 0$, is readily found to be

$$T \sim \frac{\epsilon^{(1-N)/2} r_0^{(N-1)/2}}{2K(r_0)\xi} e^{2r_0/\epsilon} \,. \tag{7.21}$$

7.2 The One-Dimensional Case

Let (λ_0, ϕ_0) be the principal eigenpair of $\mathcal{L}_\epsilon \phi_0 = \lambda_0 \phi_0$ with $\phi_0'(\pm 1) = 0$. Here $\mathcal{L}_\epsilon$ is defined in (7.6), and λ_0 is exponentially small. Similar to the analysis for the multi-dimensional case, ϕ_0 has the boundary layer form

$$\phi_0 \sim \partial_x \hat{W}_\epsilon + \phi_l \left[\epsilon^{-1}(1 + x)\right] + \phi_r \left[\epsilon^{-1}(1 - x)\right] . \tag{7.22}$$

Substituting (7.22) into (7.6), we obtain that the boundary layer correction terms $\phi_l(\eta)$ and $\phi_r(\eta)$ satisfy $v'' - v = 0$. Imposing that $\phi_0'(\pm 1) = 0$, and using the far-field behavior (7.10) with $N = 1$, we get

$$\phi_l(\eta) = a\epsilon^{-1} e^{-(1+x_0)/\epsilon} e^{-\eta}, \qquad \phi_r(\eta) = -a\epsilon^{-1} e^{-(1-x_0)/\epsilon} e^{-\eta}. \tag{7.23}$$

Combining (7.10), (7.22), and (7.23), and where a is defined in (7.10), we calculate

$$\phi_0(-1) \sim 2\epsilon^{-1} a e^{-(1+x_0)/\epsilon}, \qquad \phi_0(+1) \sim -2\epsilon^{-1} a e^{-(1-x_0)/\epsilon}. \tag{7.24}$$

To derive a differential equation for x_0, we proceed as in the multi-dimensional case. We multiply both sides of (7.6) by ϕ_0 and integrate over the domain. Assuming that $\partial_t R$ on the right-hand side of (7.6) is asymptotically small, we integrate the resulting expression by parts, and then use the boundary condition (7.6b), to obtain

$$(\phi_0, \partial_t W_\epsilon) = -\epsilon^2 \phi_0 \partial_x W_\epsilon |_{-1}^{1} + (R, \mathcal{L}_\epsilon{}^* \phi_0) \,. \tag{7.25}$$

Here $\mathcal{L}_\epsilon{}^*$ denotes the adjoint operator of $\mathcal{L}_\epsilon$. As in the multi-dimensional case, the second term on the right-hand side of (7.25) is asymptotically negligible in comparison to the other two terms in (7.25). Then, since $W_\epsilon = C\hat{W}_\epsilon$ for some constant C, (7.25) reduces to

$$\left(\phi_0, \partial_t \hat{W}_\epsilon\right) \sim -\epsilon^2 \phi_0 \hat{W}_{\epsilon x} |_{-1}^{1}. \tag{7.26}$$

Using (7.9), (7.10), (7.22) and (7.24), we calculate

$$\left(\phi_0, \partial_t \hat{W}_\epsilon\right) \sim -\epsilon^{-1} x_0' \int_{-\infty}^{\infty} \left[\hat{W}_\epsilon'(y)\right]^2 dy, \qquad \epsilon^2 \phi_0 \hat{W}_{\epsilon x}|_{-1}^{1} \sim 2a^2 \left(e^{-2(1-x_0)/\epsilon} - e^{-2(1+x_0)/\epsilon}\right). \tag{7.27}$$

Substituting (7.27) into (7.26), we obtain the ODE for the spike location. The corresponding asymptotic solution is obtained by combining (2.2) and part 4 of Lemma 3.2. We summarize the result as follows:

Proposition 7.2: *For $\epsilon \ll 1$, a metastable spike solution for (1.9) is given by*

$$W_\epsilon \sim \frac{w_{\delta_\epsilon}\left[\epsilon^{-1}(x - x_0)\right]}{\gamma\epsilon \int_{-\infty}^{\infty} \left[w_{\delta_\epsilon}(y)\right]^p \, dy}, \qquad P_\epsilon \sim \frac{\left(w_{\delta_\epsilon}\left[\epsilon^{-1}(x - x_0)\right]\right)^p}{\epsilon \int_{-\infty}^{\infty} \left[w_{\delta_\epsilon}(y)\right]^p \, dy}. \tag{7.28a}$$

Here w_{δ_ϵ} satisfies (3.3), and the spike location $x_0(t)$ satisfies the differential equation

$$\frac{dx_0}{dt} \sim \frac{2a^2\epsilon}{\beta} \left(e^{-2(1-x_0)/\epsilon} - e^{-2(1+x_0)/\epsilon}\right), \qquad x_0(0) = x_0^0; \qquad \beta \equiv \int_{-\infty}^{\infty} \left[w_{\delta_\epsilon}'(y)\right]^2 dy. \tag{7.28b}$$

The constant $a > 0$ is defined in (7.10).

Following ideas in [37], the homoclinic orbit constants a, β, and δ_ϵ, can be computed numerically in terms of the nonlinearity of (3.3). In this way, we obtain the explicit formulae for a and β

$$\log a \equiv \log w_m + \int_0^{w_m} \left(\frac{1}{\left[\eta^2 - 2Q(\eta)\right]^{1/2}} - \frac{1}{\eta} \right) d\eta, \qquad \beta = 2 \int_0^{w_m} \left[\eta^2 - 2Q(\eta)\right]^{1/2} d\eta. \tag{7.29a}$$

Here $w_m = w_{\delta_\epsilon}(0)$, which denotes the maximum of $w_{\delta_\epsilon}(y)$ on $y \geq 0$, satisfies the transcendental equation

$$w_m^2 = 2Q(w_m), \qquad \text{where} \qquad Q(w) = \int_0^w \frac{y^{p+1}}{\delta_\epsilon + y} \, dy. \tag{7.29b}$$

The nonlocal term δ_ϵ, defined in Lemma 3.2, can be rewritten by determining a formula for w'_{δ_ϵ} in terms of w_{δ_ϵ} for $y \geq 0$. In this way, we get

$$\delta_\epsilon = 2\epsilon \int_0^{w_m} \frac{w^p}{\sqrt{w^2 - 2Q(w)}}\, dw\,. \tag{7.29c}$$

We use Newton's method to solve the coupled system (7.29b) and (7.29c) for w_m and δ_ϵ for various values of ϵ. In terms of these values, we then calculate a and β from (7.29a). An important feature of these formulae are that they do not require an explicit pointwise expression for the solution $w_{\delta_\epsilon}(y)$ to (3.3).

ϵ	w_m	δ_ϵ	a	β
0.01	1.587	0.0625	5.34	1.38
0.02	1.679	0.137	5.27	1.57
0.03	1.776	0.225	5.28	1.79
0.04	1.880	0.331	5.34	2.03
0.05	1.993	0.458	5.45	2.31
0.06	2.115	0.610	5.59	2.64
0.07	2.248	0.793	5.77	3.01
0.08	2.396	1.017	5.98	3.45
0.09	2.559	1.286	6.22	3.97
0.10	2.741	1.627	6.51	4.60

Table 1: Numerical values for the homoclinic orbit constants at different values of ϵ when $p = 2$.

p	w_m	w_{m0}	δ_ϵ	δ_0	a	β
2	2.115	1.500	0.610	0.360	5.59	2.64
3	1.604	1.414	0.351	0.267	2.85	1.77
4	1.456	1.357	0.276	0.228	2.17	1.59
5	1.379	1.316	0.240	0.207	1.89	1.51

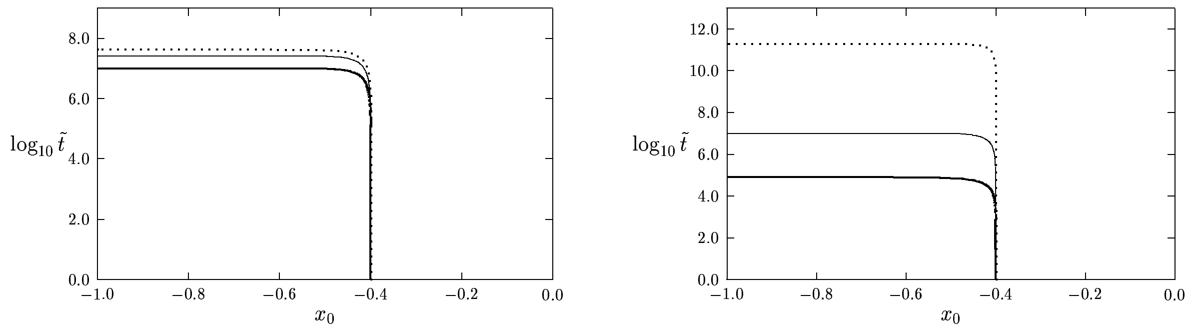
Table 2: Numerical values for the homoclinic orbit constants for different values of p when $\epsilon = 0.06$.

When $\epsilon \to 0$, then $\delta_\epsilon \to 0$. Consequently, in this limit, the solution to (3.3) is given to leading order by the solution to (2.3), which is given explicitly by

$$w(y) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \left(\cosh\left[\frac{(p-1)y}{2}\right]\right)^{-2/(p-1)}\,. \tag{7.30}$$

Therefore, as $\epsilon \to 0$, we have for $p = 2$, that $w_m \to w_{m0} = 1.5$, $\delta_\epsilon \to \delta_0 = \epsilon \int_{-\infty}^{\infty} w^2\, dy = 6\epsilon$, $a \to a_0 = 6$, and $\beta \to \beta_0 = 6/5$. In Table 1 we give numerical values for the homoclinic orbit constants for different values of ϵ when $p = 2$. In Table 2 numerical values are given for these constants for different values of p when $\epsilon = 0.06$. In this table we compare the numerical values for w_m and δ_ϵ with the numerical values for the corresponding leading order approximations w_{m0} and δ_0 at different values of p. These values show that the nonlocal term δ_ϵ is significant at this value of ϵ.

The ODE (7.28b) shows that the spike moves exponentially slowly towards the right or the left boundary when $x_0(0) > 0$ or $x_0(0) < 0$, respectively. In Fig. 1(a) we plot the numerical solution to (7.28b) in the form $\log_{10}(1+t)$ versus x_0 when $\epsilon = 0.06$ for $p = 2$, $p = 3$, and $p = 4$. The homoclinic orbit constants needed in



(a) $\log_{10}\tilde{t}$ versus x_0: $\tilde{t} = 1 + t$

(b) $\log_{10}\tilde{t}$ versus x_0: $\tilde{t} = 1 + t$

Figure 1: Left figure: Plot of asymptotic spike location computed from (7.28b) when $\epsilon = 0.06$, for $p = 2$ (heavy solid curve), $p = 3$ (solid curve), and $p = 4$ (dashed curve). Right figure: Similar plot when $p = 2$, for $\epsilon = 0.08$ (heavy solid curve), $\epsilon = 0.06$ (solid curve), and $\epsilon = 0.04$ (dashed curve).

(7.28b) are given in Table 2. The initial value for (7.28b) is $x_0(0) = -0.4$. The spike motion is found to be slower for larger values of p. For $p = 2$ and the initial value $x_0(0) = -0.4$, in Fig. 1(b) we plot $\log_{10}(1 + t)$ versus x_0 for $\epsilon = 0.04$, $\epsilon = 0.06$, and $\epsilon = 0.08$. Notice the dramatic change in the time-scale of the metastability for a small change in ϵ. For $p = 2$ and $\epsilon = 0.06$, in Fig. 2(a) and Fig. 2(b) we plot the leading order solutions for W_ϵ and P_ϵ, respectively, at different times. These leading order solutions are obtained by replacing w_{δ_ϵ} in (7.28a) by w as given in (7.30).

Finally, we compare full numerical results for the evolution of an interior spike for (1.9) with the asymptotic dynamical behavior given in (7.28). The routine D03PCF of the NAG library [25] is used to compute the numerical solution to (1.9). The initial condition for the numerical solution to (1.9) is

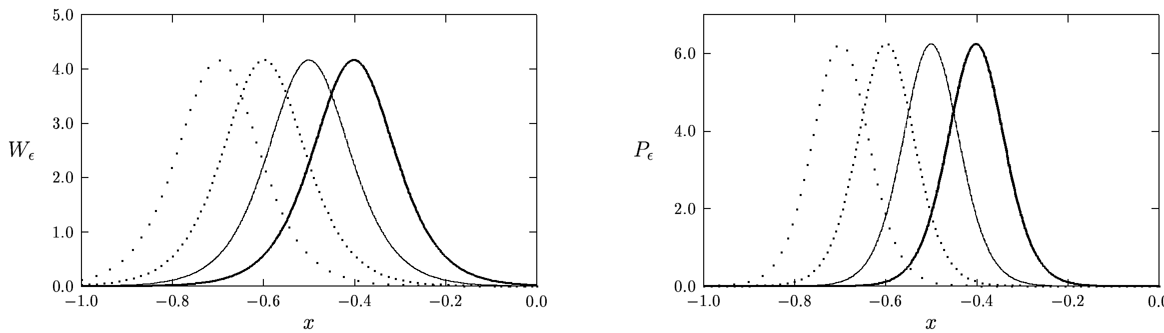
$$W(x, 0) = \frac{w\left[\epsilon^{-1}(x - x_0^0)\right]}{\gamma\epsilon \int_{-\infty}^{\infty} [w(y)]^p \, dy}, \qquad P(x, 0) = \frac{\left(w\left[\epsilon^{-1}(x - x_0^0)\right]\right)^p}{\epsilon \int_{-\infty}^{\infty} [w(y)]^p \, dy}, \tag{7.31}$$

where w is given in (7.30). In the computations, we choose $p = 2$, $\gamma = 1$, $\epsilon = 0.06$, and $x_0^0 = -0.4$. The spike location is determined numerically from the maximum of the numerical solution for W. In Fig. 3, we show a favorable comparison between the asymptotic and numerical results for x_0. For this value of ϵ, the relative error between the asymptotic and numerical predictions for the time at which the spike hits the boundary at $x = -1$ is about 3%.

8 Concluding Remarks

Motivated by the general system (1.1), (1.2) in which we have a single equation governing the population density P coupled with a system governing several substrates or nutrients we have focussed on the simpler system (1.6) in which the transition probability function is represented by a power law.

Under weak diffusion of the substrate, we have established the existence of spike solutions and investigated



(a) W_ϵ versus x

(b) P_ϵ versus x

Figure 2: Plot of the leading order solutions for W_ϵ and P_ϵ at four different times when $p = 2$, $\gamma = 1$, and $\epsilon = 0.06$. The heavy solid curve is for $t = 0$, the solid curve is for $t = 4.01 \times 10^5$, the dashed curve is for $t = 1.0214 \times 10^7$, and the widely spaced dots is for $t = 1.0227 \times 10^7$.

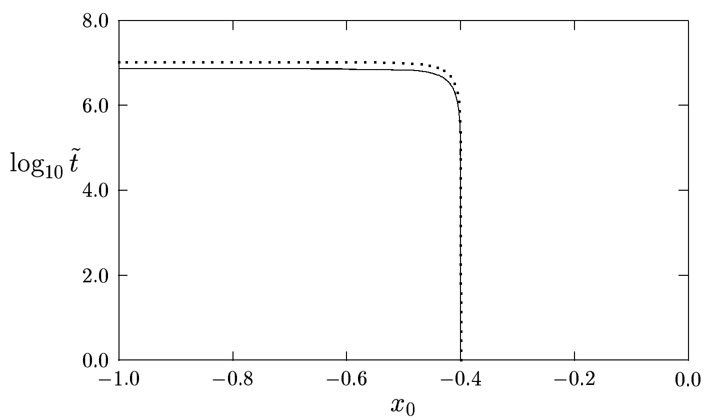


Figure 3: Comparison of numerical and asymptotic results for $\log_{10} \tilde{t}$, where $\tilde{t} = 1 + t$, versus x_0 when $p = 2$, $\gamma = 1$, $x_0^0 = -0.4$, and $\epsilon = 0.06$. The solid curve is the full numerical result computed from (1.9). The dashed curve is the asymptotic result computed from the ODE (7.28b).

their stability. Of particular interest is the existence of metastable spikes. With regards to the modeling of the movement of myxobacteria our results suggest that bacteria which aggregates when diffusion of the substrate is neglected are, in fact, metastable spikes under weak diffusion. A similar observation can be made regarding the initiation of capillary sprouts in tumor angiogenesis under the simple model indicated in §1. Furthermore, it is an important problem to investigate whether spike behavior can account for the observed vigorous angiogenesis in a vascularised tumor.

The investigations of this paper suggest a number of open problems. For example it is of interest to investigate whether spikes arise for more general transition probability function Φ. In addition we may ask whether spike patterns exist for systems of the form (1.1) in which there are several substrates or nutrients.

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