# THE DYNAMICS OF LOCALIZED SOLUTIONS OF NON-LOCAL REACTION-DIFFUSION EQUATIONS

Michael J. Ward[1]
*Department of Mathematics, University of British Columbia*
*Vancouver, B.C. V6T 1Z2, Canada*

**Abstract**

Many classes of singularly perturbed reaction-diffusion equations possess localized solutions where the gradient of the solution is large only in the vicinity of certain points or interfaces in the domain. The problems of this type that are considered are an interface propagation model from materials science and an activator-inhibitor model of morphogenesis. These two models are formulated as non-local partial differential equations. Results concerning the existence of equilibria, their stability, and the dynamical behavior of localized structures in the interior and on the boundary of the domain are surveyed for these two models. By examining the spectrum associated with the linearization of these problems around certain canonical solutions, it is shown that the non-local term can lead to the existence of an exponentially small principal eigenvalue for the linearized problem. This eigenvalue is then responsible for an exponentially slow, or metastable, motion of the localized structure.

## 1 Introduction

Certain time-dependent singularly perturbed partial differential equations exhibit a phenomenon known as dynamic metastability, whereby the solution evolves on an asymptotically exponentially long time interval as the singular perturbation parameter $\epsilon$ tends to zero. Metastable dynamics has been observed and analyzed over the past decade for several classes of problems in a one-spatial dimensional setting. Many of these results are surveyed in [21]. Although metastable behavior occurs more commonly for problems in one spatial dimension, it has now been recognized (cf. [1], [3], [4], [13], [22]) that it can also occur in a multi-spatial dimensional context.

In this article we survey some results for the metastable behavior of localized solutions for two different reaction-diffusion equations in a multi-dimensional domain. The two specific problems that we consider are the constrained Allen-Cahn equation with applications to materials science and the shadow problem that arises from an activator-inhibitor model of morphogenesis in the limit of large inhibitor diffusivity. A critical common feature of these two models is that they are *non-local* reaction-diffusion models that can be formulated as

[1]This work was supported by NSERC grant 5-81541. This article is based on the Coxeter-James lecture for the CMS given in Victoria, B. C. in Dec. 1997.

special cases of

$$u_t = \epsilon^2 \Delta u + Q(u\,;\sigma_\epsilon)\,, \qquad \mathbf{x} \in D\,, \tag{1.1a}$$
$$\partial_n u = 0\,, \qquad \mathbf{x} \in \partial D\,, \tag{1.1b}$$
$$\sigma_\epsilon \equiv \int_D g(u\,;\epsilon)\,d\mathbf{x}\,. \tag{1.1c}$$

Here $\epsilon \ll 1$, $D$ is a bounded domain in $\mathbf{R^N}$ with smooth boundary $\partial D$, and $\partial_n u$ indicates the outward normal derivative. Before discussing the outline of the paper in detail, we give a qualitative overview of the analysis and the type of asymptotic results that are obtained.

Depending on the specific conditions imposed on $Q$ and $g$, (1.1) can possess localized solutions where the gradient of the solution is large only in the vicinity of certain points or interfaces of dimension $N-1$ in the domain. For the activator-inhibitor problem, the special type of solution we consider is a spike. A spike is a radially symmetric function localized at some specific point in the domain, with the solution decaying exponentially away from the center of the spike. For the constrained Allen-Cahn equation, we consider a localized solution that is asymptotically piecewise constant but that varies rapidly across a spherical interface of a given radius that separates the two constant states. The internal layer solution decays exponentially onto these two states. This type of solution is referred to as a bubble. The specific conditions on $Q$ and $g$ that yield these solutions are given in §2 and §4 below.

Thus, a common feature of these two problems is that in all of $\mathbf{R^N}$, (1.1a), (1.1c) admits a canonical radially symmetric solution of the form $u_q\left(\epsilon^{-1}r\,;\,\epsilon\right)$ as $\epsilon \to 0$, where $u_q$ is exponentially localized and $r \equiv |\mathbf{x} - \mathbf{x}_0|$. Here $\mathbf{x}_0 \in D$ is the center of the localized structure. In terms of $u_q$, the goal is to construct as $\epsilon \to 0$ an equilibrium solution to (1.1) of the form $u(\mathbf{x}) \sim u_q$, for some $\mathbf{x}_{0e} \in D$ to be determined. For the time-dependent problem we wish to determine whether (1.1) has a solution of the form $u(\mathbf{x}, t) \sim u_q$, where $\mathbf{x}_0 = \mathbf{x}_0(t)$ satisfies a differential equation that evolves exponentially slowly in time as $\epsilon \to 0$. The essential feature that is needed for the existence of such a metastable behavior is to show that the principal eigenvalue of the non-local eigenvalue problem associated with linearizing (1.1) around $u_q$ is exponentially small.

The non-local term in (1.1) is critical for ensuring this condition on the spectrum of the linearization. To see this, suppose that $Q$ in (1.1) is independent of $\sigma_\epsilon$ and that the resulting problem is capable of supporting a radially symmetric solution in all of $\mathbf{R^N}$ that has exponential decay. For such a problem, there will always be one eigenvalue bounded above from zero as well as $N$ zero eigenvalues due to translation invariance. As a result of the exponential localization of $u_q$, these eigenvalues get perturbed by only exponentially small terms by the presence of the finite domain. Thus, for the local problem in a finite domain, there is still a positive principal eigenvalue, which eliminates the possibility of metastability. However, when the non-local effect $\sigma_\epsilon$ is present in (1.1), the corresponding eigenvalue problem that determines the stability of the localized solution is non-local. Under certain simple conditions on $g(u)$, the non-local term in this eigenvalue problem has only an exponentially small effect on the eigenspace associated with the exponentially small eigenvalues. Thus, the non-local eigenvalue problem still retains the exponentially small eigenvalues associated with the local problem. The key step in the analysis then is reduced to ascertaining whether the non-local term in the eigenvalue problem is sufficiently strong to push the positive eigenvalue associated with the local problem into the left half-plane. If this occurs, then the exponentially small eigenvalues will be the principal eigenvalues associated with the linearization, and metastable dynamics will occur. This hypothesized behavior of the spectrum of the linearized problem is precisely what has been found for the activator-inhibitor problem and for the constrained Allen-Cahn equation.

To characterize the metastable dynamics, we then study the quasi-steady linearization of (1.1) around $u_q$, where we let $\mathbf{x}_0 = \mathbf{x}_0(t)$. Then, the solution to this problem must satisfy the limiting solvability conditions that the 'residual" is orthogonal to the eigenspace associated with the exponentially small eigenvalues as $\epsilon \to 0$. From this projection step and from certain key asymptotic exponential estimates for the translation eigenfunctions on the boundary of the domain, an explicit asymptotic differential equation for $\mathbf{x}_0(t)$, characterizing the metastable dynamics, is derived. For the two specific problems considered below, $\mathbf{x}_0$ moves exponentially slowly in the direction of the closest point on the boundary of the domain, whenever that point is uniquely defined. It then eventually attaches to the boundary.

When the localized structure becomes attached to the boundary, the dynamics is often of a very different nature. In particular, when the boundary is curved, a spike solution will creep along the boundary driven by the variation in the curvature in two dimensions and by the gradient of the mean curvature in three dimensions. The spike then reaches a stable equilibrium point at a local maxima of these curvatures. However, if the spike becomes attached to a flat portion of the boundary, it will again move only in response to exponentially weak or, metastable, forces. Similar qualitative features occur for the motion of a bubble for the constrained Allen-Cahn equation that is attached to the boundary.

The organization of this paper is as follows. In §2 we analyze the metastable dynamics of bubble solutions for the constrained Allen-Cahn equation. The motion of bubbles that are attached to the boundary of the domain is considered in §3. In §4, we analyze metastable spike dynamics for the activator-inhibitor model. Finally, in §5 we study spike dynamics for this model when the spike is attached to the boundary of the domain.

# 2 Metastable Bubble Motion

As shown in [18], a simple model for the phase separation of a binary mixture is the constrained Allen-Cahn equation

$$u_t = \epsilon^2 \Delta u + F(u) - \sigma_\epsilon\,, \qquad \mathbf{x} \in D \subset \mathbf{R}^{\mathbf{N}}\,, \tag{2.1a}$$

$$\partial_n u = 0\,, \qquad \mathbf{x} \in \partial D\,, \tag{2.1b}$$

$$\sigma_\epsilon = \frac{1}{|D|} \int_D F(u)\, d\mathbf{x}\,. \tag{2.1c}$$

Here $\epsilon \ll 1$, $|D|$ is the volume of $D$, and $F(u) = -V'(u)$, where $V(u)$ is a double-well potential with wells of equal depth located at $u = s_+ > 0$ and $u = s_- < 0$. More specifically, we assume that $F(u)$ is smooth and has exactly three zeroes on the interval $[s_-, s_+]$ located at $u = 0$ and $u = s_\pm$, with

$$F'(s_\pm) < 0\,, \qquad F'(0) > 0\,, \qquad V(s_\pm) = 0\,. \tag{2.2}$$

Prototypical is $F(u) = 2(u - u^3)$. For (2.1), the key feature is that the mass $m = \int_D u\, d\mathbf{x}$ is conserved. We now describe the different stages of the dynamics for (2.1) that have been studied in detail in [1], [2], [3], [4], [18], [19] and [22]. The various stages of the dynamics of a single closed interface for (2.1) are shown qualitatively in Fig. 1.

The first stage is a transient phase. Starting from arbitrary initial data, in an $O(1)$ time interval the solution to (2.1) develops internal layers of width $O(\epsilon)$ separating the two minima of the potential well $V(u)$. Thus, as $\epsilon \to 0$, we can approximate the details of the internal layer solution by a sharp interface $\Gamma$, whose motion is to be determined. For the sharp interface model, the asymptotic analysis of [18] showed that the normal velocity $v$ of a single closed interface $\Gamma$ satisfies the constrained mean curvature flow

$$v \sim \epsilon^2 \left( K - \frac{1}{|\Gamma|} \int_\Gamma K\, ds \right) . \tag{2.3}$$

Here $K$ is the mean curvature of $\Gamma$. This law holds for closed interfaces in the interior of $D$ and for interfaces that are connected to $\partial D$ orthogonally. As shown in [7], a single closed convex interface evolving according to (2.3) will tend to a sphere that encloses the same volume.

For a bubble solution to (2.1), $\Gamma$ is a sphere of radius $r_b$ and from (2.3) we get $v = 0$. Thus, the asymptotic analysis leading to (2.3) gives no indication of the nature of the motion of a bubble contained in $D$. The evolution of such a bubble is more subtle and arises from exponentially small metastable forces. The existence of such a metastable bubble motion has been proved in [1], [3], and [4]. An explicit differential equation for the evolution of a bubble has been derived in [22] using formal asymptotic analysis.

There are three key ingredients steps in the asymptotic analysis of [22]. The first step is to construct a bubble solution $u_q$ to (2.1a), (2.1c) in all of $\mathbf{R^N}$. Then, we show that the principal eigenvalue associated with the linearization of (2.1) around $u_q$ is exponentially small. The non-local conservation of mass condition in (2.1c) is essential for this conclusion. Finally, for the time-dependent problem we impose a limiting solvability condition on the solution to the quasi-steady linearization of (2.1) around $u_q$. This condition ensures that the linearized solution has no component in the eigenspace associated with the exponentially small eigenvalues. In this way, an explicit differential equation for the motion of the bubble is obtained.

The first result summarizes the asymptotic construction of the bubble in $\mathbf{R^N}$:

**Proposition 2.1** (Bubble Solution [22]): *A bubble solution to (2.1a) and (2.1c) of radius $r_b$ in $\mathbf{R^N}$, with $u \sim s_-$ inside the bubble and $u \sim s_+$ outside the bubble, is given asymptotically for $\epsilon \to 0$ by $u \sim u_q(\epsilon^{-1} r; \epsilon)$ where*

$$u_q \sim \begin{cases} S_+(\epsilon) - a_+ (r/r_b)^{(1-N)/2} e^{-\nu_+^\epsilon \epsilon^{-1}(r - r_b)}\,, & r > r_b\,, \\ u_0[\epsilon^{-1}(r - r_b)] + O(\epsilon)\,, & \epsilon^{-1}(r - r_b) = O(1)\,, \\ S_-(\epsilon) + a_- (r/r_b)^{(1-N)/2} e^{-\nu_-^\epsilon \epsilon^{-1}(r_b - r)}\,, & 0 \le r < r_b\,. \end{cases} \tag{2.4a}$$

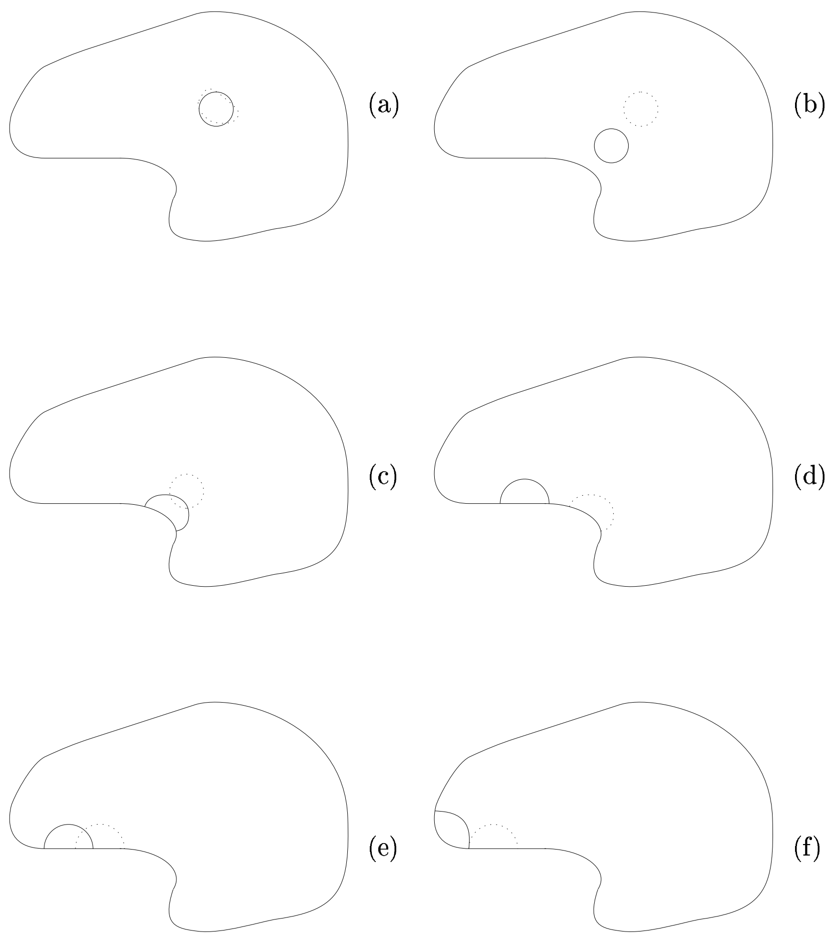


Figure 1: Evolution of a small convex interface inside a two-dimensional domain $D$. (a) The convex interface evolves by (2.3) into a circle. (b) The circular interface drifts, satisfying (2.11a), towards the closest point on $\partial D$. (c) The interface attaches to $\partial D$, intersecting orthogonally. (d) The interface moves along $\partial D$ satisfying (2.3). (e) If the interface encounters a flat portion of $\partial D$, it moves along this flat portion according to (3.5). (f) When a curved part of $\partial D$ is reached, the interface again evolves by (2.3) until a steady state is attained, which is centered near a local maximum of the curvature of $\partial D$. The motion of the center of a drop satisfies (3.1).

*Here $u_0(z)$ is the unique solution to*

$$u_0^{''} + F(u_0) = 0, \qquad -\infty < z < \infty; \qquad u_0(0) = 0, \tag{2.4b}$$

$$u_0(z) \sim s_\pm - a_\pm e^{\mp \nu_\pm z}, \quad \text{as} \quad z \to \pm\infty, \tag{2.4c}$$

*where $\nu_\pm \equiv \left[-F^{'}(s_\pm)\right]^{1/2}$ and $a_\pm$ are defined by (2.4c). The constant $\sigma_\epsilon$ is given asymptotically by*

$$\sigma_\epsilon = \epsilon\sigma_1 + O(\epsilon^2), \qquad \sigma_1 \equiv \frac{\beta(N-1)}{(s_+ - s_-)r_b}, \qquad \beta \equiv \sqrt{2}\int_{s_-}^{s_+} [V(u)]^{1/2}\, du, \tag{2.4d}$$

*and the constants $S_\pm(\epsilon)$, $\nu_\pm^\epsilon$ are given by*

$$S_\pm(\epsilon) = s_\pm - \epsilon\sigma_1\nu_\pm^{-2} + O(\epsilon^2); \qquad \nu_\pm^\epsilon = \nu_\pm\left[1 + \frac{\epsilon\sigma_1 F^{''}(s_\pm)}{2\nu_\pm^4} + O(\epsilon^2)\right]. \tag{2.4e}$$

Next, the quasi-steady linearization of (2.1) is obtained by setting

$$u(\mathbf{x},t) = u_q\left(\epsilon^{-1}|\mathbf{x} - \mathbf{x_0}(t)|;\, \epsilon\right) + v(\mathbf{x},t), \tag{2.5}$$

where $v \ll u_q$, $v_t \ll \partial_t u_q$. This leads to the linearized problem

$$L_\epsilon v \equiv \epsilon^2\Delta v + F^{'}(u_q)v - \frac{1}{|D|}\int_D F^{'}(u_q)v\, d\mathbf{x} = \partial_t u_q, \qquad \mathbf{x} \in D, \tag{2.6a}$$

$$\partial_n v = -\partial_n u_q, \qquad \mathbf{x} \in \partial D. \tag{2.6b}$$

Let $\mathbf{x}_0$ be fixed and let $\lambda_j$, $\phi_j$ for $j \geq 1$ be the eigenpairs of the associated non-local eigenvalue problem

$$L_\epsilon\phi = \lambda\phi, \quad \mathbf{x} \in D; \quad \partial_n\phi = 0, \quad \mathbf{x} \in \partial D; \quad (\phi,\phi) \equiv \int_D \phi^2\, d\mathbf{x} = 1. \tag{2.7}$$

If the non-local term in (2.6) were deleted it is easy to show that the resulting principal eigenvalue of (2.7) would satisfy $\lambda_1 = \epsilon^2/r_b^2$ as $\epsilon \to 0$, and the corresponding eigenfunction $\phi_1$ would have the form $\phi_1 \sim u_q^{'}$ away from the boundary of $D$. This is consistent with the observation that if mass was not conserved, the bubble would shrink to a point under a mean curvature flow on a time scale $|\lambda_1^{-1}| = O(\epsilon^{-2})$ (see [8]). However, this positive eigenvalue is only present for the local problem, as the non-local term in (2.6) eliminates this unstable mode. The following result for the principal eigenvalue of the non-local problem (2.7) was obtained in [22]:

**Proposition 2.2** (Exponentially Small Eigenvalues [22]): *Let $\mathbf{x}_0 \in D$ be fixed and assume that $|\mathbf{x} - \mathbf{x}_0| - r_b > 0$ for $\mathbf{x} \in \partial D$. Then, for $\epsilon \to 0$, the principal eigenvalue of (2.7) is exponentially small and there are $N$ such exponentially small eigenvalues given asymptotically for $j = 1, \ldots, N$ by*

$$\lambda_j \sim \frac{a_+^2\nu_+^3 N}{\beta\Omega_N}\int_{\partial D} r^{1-N} e^{-2\nu_+^\epsilon\epsilon^{-1}(r-r_b)}\left(\frac{x_j - x_{0j}}{r}\right)^2 \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\,[1 + \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}]\, dS. \tag{2.8a}$$

*Here $\Omega_N$ is the surface area of the unit $N$-ball, $\hat{\mathbf{r}} = (\mathbf{x}-\mathbf{x}_0)/r$, $\hat{\mathbf{n}}$ is the unit outward normal to $\partial D$, and $x_j$ and $x_{0j}$ are the $j^{th}$ coordinates of $\mathbf{x}$ and $\mathbf{x}_0$, respectively. The corresponding unnormalized eigenfunctions have the form*

$$\phi_j \sim \partial_{x_j} u_q + \phi_{Lj}\,, \qquad j = 1, \ldots, N\,, \tag{2.8b}$$

*where $\phi_{Lj}$ is a boundary layer function localized near $\partial D$ that allows the boundary condition in (2.1b) to be satisfied.*

This result then implies that the bubble will translate without change of shape. The final step is multiply both sides of (2.6a) by the eigenfunctions in (2.8b), integrate by parts, and use the estimates for $\lambda_j$ in (2.8a), to derive tht

$$(\partial_t u_q, \phi_j) \sim -\epsilon^2 \int_{\partial D} \phi_j \partial_n u_q \, dS\,, \tag{2.9}$$

where $(f,g) \equiv \int_D fg\, d\mathbf{x}$. This equation can be viewed as resulting from a projection of $v$ against the eigenspace associated with the exponentially small eigenvalues. By obtaining explicit formulae for the translation eigenfunctions (2.8b) on the boundary of the domain, the following result was obtained in [22]:

**Proposition 2.3** (Metastability [22]): *Assume that $|\mathbf{x}-\mathbf{x}_0(t)| - r_b > 0$ for $\mathbf{x} \in \partial D$ .Then, for $\epsilon \to 0$, the metastable bubble dynamics for (2.1) is characterized by $u(\mathbf{x},t) \sim u_q(\epsilon^{-1}|\mathbf{x} - \mathbf{x}_0(t)|;\, \epsilon)$, where $\mathbf{x}_0(t)$ satisfies the differential equation*

$$\dot{\mathbf{x}}_0 \sim \frac{\epsilon N a_+^2 \nu_+^2}{\beta \Omega_N} I(\mathbf{x}_0)\,, \qquad I(\mathbf{x}_0) \equiv \int_{\partial D} r^{1-N} e^{-2\nu_+^{\epsilon}\epsilon^{-1}(r-r_b)} \hat{\mathbf{r}}\,[1 + \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}]\,\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\, dS\,. \tag{2.10}$$

*Here $\dot{\mathbf{x}}_0 \equiv d\mathbf{x}_0/dt$, $u_q$ is given in Prop. 2.1, and we have the same notation as in Prop. 2.2.*

Then, we can characterize an unstable equilibrium solution by

**Corollary 2.1** (Equilibrium [22]): *Under the conditions of Prop. 2.3, (2.1) has an unstable equilibrium solution of the form $u \sim u_q(\epsilon^{-1}|\mathbf{x} - \mathbf{x}_{0e}|;\, \epsilon)$, where the center $\mathbf{x}_{0e}$ of the bubble satisfies $I(\mathbf{x}_{0e}) = 0$. For a strictly convex domain $D$, $\mathbf{x}_{0e}$ is located at an $O(\epsilon)$ distance from the center of the unique largest inscribed sphere that can be inserted in $D$.*

For $\epsilon \ll 1$, the surface integral in (2.10) is dominated asymptotically by the contributions from the points on the boundary of the domain that are closest to the center of the bubble. When, there is only one such closest point we can readily evaluate the integral using a multi-dimensional Laplace's method to obtain the following explicit dynamics:

**Corollary 2.2** (Explicit Dynamics [22]): *Assume that at $t = 0$, $\mathbf{x}_m$ is the unique point on $\partial D$ which is closest to $\mathbf{x}_0(0)$. Then, the motion of the center of the bubble is in the direction of the closest point and the distance $r_m(t) \equiv |\mathbf{x}_m - \mathbf{x}_0(t)|$ satisfies the asymptotic ODE*

$$\dot{r}_m \sim -\zeta r_m \left(\frac{\epsilon}{r_m}\right)^{(N+1)/2} H(r_m)\, e^{-2\nu_+^{\epsilon}\epsilon^{-1}(r_m - r_b)}\,, \quad \zeta \equiv \frac{2N a_+^2 \nu_+^2}{\Omega_N \beta} \left(\frac{\pi}{\nu_+^{\epsilon}}\right)^{(N-1)/2}\,, \tag{2.11a}$$

*where $H(r_m)$ is defined by*

$$H(r_m) \equiv (1 - r_m/R_1)^{-1/2} (1 - r_m/R_2)^{-1/2} \ldots (1 - r_m/R_{N-1})^{-1/2}\,. \tag{2.11b}$$

*Here $R_j > r_m$ for $j = 1, .., N-1$, are the principal radii of curvature of $\partial D$ at $\mathbf{x}_m$. The constants in (2.11a) are defined in (2.4). This result is valid provided that $r_m - r_b > 0$, and it breaks down when $r_m - r_b = O(\epsilon)$.*

The final result gives the time that it takes for the bubble to touch the boundary.

**Corollary 2.3** (Collapse Time [22]): *For the ODE (2.11), suppose that $r_m(0) = r_0 > r_b$. Then, under the conditions of Corollary 2.2, the bubble will first touch the boundary $\partial D$ when $r_m(t_c) = r_b$, where*

$$t_c \sim \left(\frac{\epsilon}{r_0}\right)^{(1-N)/2} \frac{[H(r_0)]^{-1}}{2\zeta\nu_+^\epsilon} \left[1 - \frac{\epsilon}{4\nu_+^\epsilon}\left(\frac{N-1}{r_0} - \sum_{i=1}^{N-1} \frac{1}{(R_i - r_0)}\right)\right] e^{2\nu_+^\epsilon \epsilon^{-1}(r_0 - r_b)} . \tag{2.12}$$

# 3 Boundary Bubble Motion

In this section we restrict ourselves to two-dimensional domains $D$. Once the bubble hits $\partial D$, it quickly becomes attached to $\partial D$ orthogonally and its boundary becomes a circular arc in order to minimize its perimeter. Its subsequent evolution is then given by (2.3). However, if the length scale of the interface is sufficiently small compared to the radius of curvature of $\partial D$, the interface will become approximately semi-circular in shape. The motion of such a semi-circular *drop* of radius $\delta \ll 1$ has been studied in [2]. It was proved in [2] that the center of a such a small drop of radius $\delta$, with $\delta \ll 1$ but $0 < \epsilon < \delta^3$, satisfies the asymptotic ODE

$$s_0'(\tau) \sim \frac{4\delta}{3\pi}\kappa'[s_0] , \qquad \tau = \epsilon^2 t . \tag{3.1}$$

Here $s_0(\tau)$ is an arclength parameter for $\partial D$ corresponding to the center of the drop, $\delta$ is the radius of the drop, and $\kappa$ is the curvature of the smooth boundary $\partial D$ (positive for a convex domain $D$). The drop will reach a stable equilibrium, where the interface can have minimum perimeter. This occurs near local maxima of $\kappa$ (cf. [1], [2], [6]).

In [19], the small drop result (3.1) was verified numerically and the motion of bubbles attached to the boundary evolving under the constrained mean curvature flow (2.3) was computed by extending the front-tracking code developed in [5]. The numerical trajectories of the center of drops of different radii $\delta$ obtained using this code, were compared to the asymptotic result (3.1) for several different boundary curves. To compare with the full numerical result, the differential equation (3.1) was solved numerically using fourth order Runge Kutta for the given boundary.

For example, consider the evolution of an interface that intersects the boundary of an ellipse with major axis 2 and minor axis 1. This boundary curve can be parameterized by $x(\theta) = 2\cos\theta, y(\theta) = \sin\theta$ for $0 \le \theta \le 2\pi$. In terms of $\theta$, we can determine the curvature $\kappa(\theta)$. Then, (3.1) becomes

$$\theta'(\tau) = -\frac{24\delta}{\pi}\frac{\sin\theta\cos\theta}{(4\sin^2\theta + \cos^2\theta)^{7/2}} . \tag{3.2}$$

In the numerical front-tracking procedure, we started with the initial data of a small semi-circle centered around the point on $\partial D$ where $\theta(0) = \pi/4$. In [19] it was observed that these drops move along the boundary in the direction of increasing curvature until a steady state was reached at the local maximum of $\kappa$ when $\theta = 0$. The trajectories of $\theta(\tau)$ obtained from the numerical method and from the asymptotic differential equation (3.2) are compared for several different drop radii in Fig. 2. Notice that, as expected, the numerical trajectory gets closer to the asymptotic trajectory as $\delta$ is decreased. For very small radii, both trajectories are very similar.

For a semi-circular interface that becomes attached to a flat portion of a boundary, the relation (2.3) and the drop result (3.1) predicts that there is no motion. This case, in which metastable behavior occurs, has also been studied in [19].

We now outline the metastability analysis for a semi-circular interface located on the straight-line boundary segment joining the points $(x_L, 0)$ and $(x_R, 0)$ as shown in Fig. 3. The flat portion of $\partial D$ is taken to be the straight-line segment between $(x_L, 0)$ and $(x_R, 0)$. The interface is centered around $\mathbf{x_0} = (x_0, 0)$ where $x_L < x_0 < x_R$. We decompose $\partial D = \partial D_c \cup \partial D_s$ where $\partial D_s$ refers to the straight-line segment of the boundary and $\partial D_c$ denotes the remaining curved part of $\partial D$. The distance between the interface and $\partial D_c$ is assumed to be a minima at either of the two corners $(x_L, 0)$ or $(x_R, 0)$.

Near the corner points, $\partial D$ is assumed to have the local behavior

$$\text{near } (x_L, 0)\,;\ y = \psi_L(x), \qquad \psi_L'(x) \sim -K_L(x_L - x)^{\alpha_L}, \ \text{ as } x \to x_L^-, \tag{3.3a}$$

$$\text{near } (x_R, 0)\,;\ y = \psi_R(x), \qquad \psi_R'(x) \sim K_R(x - x_R)^{\alpha_R}, \ \text{ as } x \to x_R^+, \tag{3.3b}$$

where $\alpha_L > 0$ and $\alpha_R > 0$. When $\alpha_L = \alpha_R = 1$, $K_L$ and $K_R$ are proportional to the curvatures of $\partial D_c$ at the corners.

The analysis in [19] proceeds as follows. First, we construct a radially symmetric bubble solution as described in (2.4). The bubble solution is then centered at a point $(x_0, 0)$ on the straight line interface. We then linearize (2.1) around $u_q$ as in (2.5), where now $\mathbf{x_0}(t) = (x_0(t), 0)$. This yields the linearized problem (2.6). In two dimensions, due to the exponential decay and the near translation invariance in the horizontal direction $x$, there is exactly one exponentially small eigenvalue corresponding to the approximate eigenfunction $\phi_1 \sim \partial_x u_q$. By imposing a limiting solvability condition on the solution to (2.6), which requires that $v$ is orthogonal to the eigenfunction associated with the exponentially small eigenvalue, the following result was obtained in [19]:

$$x_0'(t) \sim \frac{2\epsilon a_+^2 \nu_+^2}{\pi\beta} \int_{\partial D} \frac{(x - x_0)}{r^2} e^{-2\nu_+^\epsilon \epsilon^{-1}(r - r_b)} \hat{\mathbf{r}} \cdot \hat{\mathbf{n}}\, ds\,. \tag{3.4}$$

Here $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{x_0})/r$, $\mathbf{x_0} = (x_0, 0)$, $r = |\mathbf{x} - \mathbf{x_0}|$ and $\hat{\mathbf{n}}$ is the unit outward normal to $\partial D$.

Next, since $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$ are orthogonal on the straight-line segment $\partial D_s$ of $\partial D$, the integral in (3.4) reduces to an integral over the curved segment $\partial D_c$. For $\epsilon \to 0$, the dominant contribution to this integral arises from the corner regions $(x_L, 0)$ and $(x_R, 0)$. Laplace's method and the local boundary information (3.3) can then be used to obtain an explicit result for the metastable dynamics.

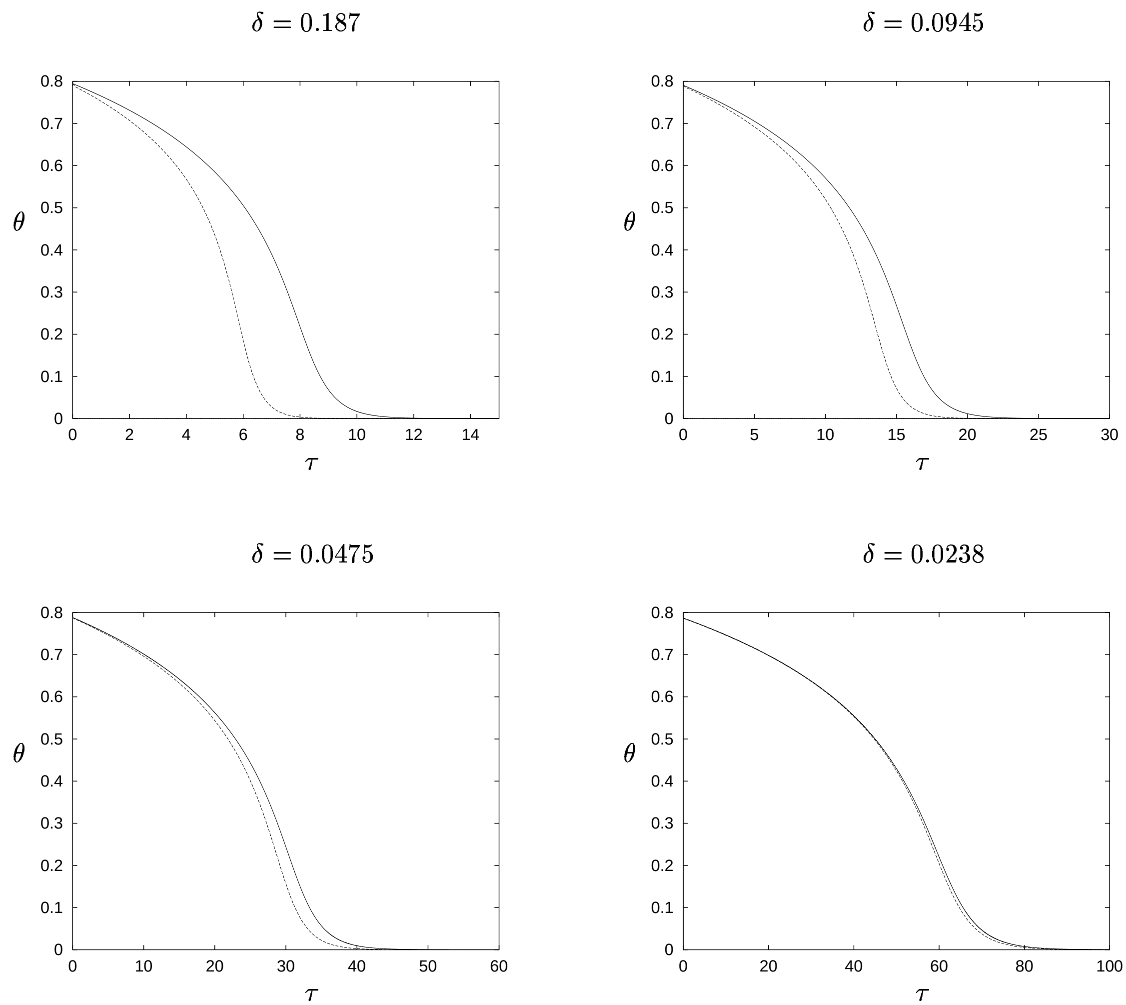


Figure 2: Plots of $\theta$ vs time for different $\delta$. The boundary curve is the ellipse defined above (3.2). The solid lines are the asymptotic result given by (3.2) and the dashed lines are the result from numerical motion by constrained mean curvature.

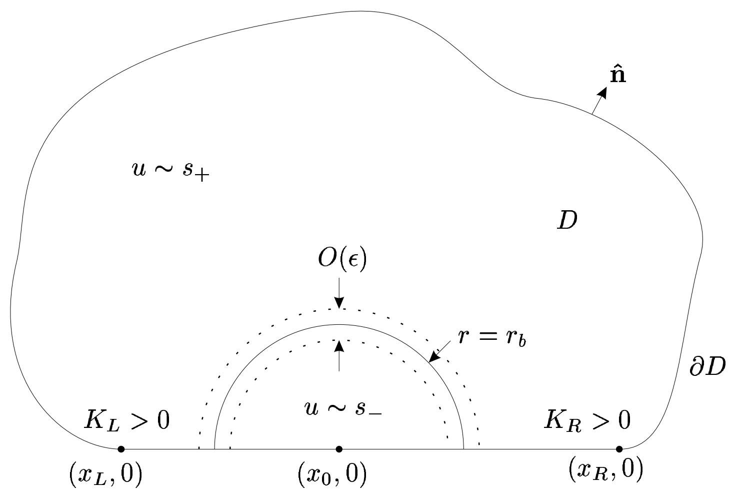


Figure 3: Plot of a two-dimensional domain $D$ with a flat boundary segment and a semi-circular interface of radius $r = r_b$ centered at $x_0$.

**Proposition 3.1** (Explicit Dynamics [19]): *Assume that the distance between $x_0$ and $\partial D_c$ is a minimum at either of the two corners $(x_L, 0)$ or $(x_R, 0)$. Then, for $\epsilon \to 0$, the center $x_0(t)$ of a semi-circular interface of radius $r_b$ that lies on the straight-line segment $y = 0, x_L \le x \le x_R$ satisfies the metastable dynamics*

$$\begin{aligned} x_0'(t) \sim \; & \frac{2\epsilon a_+^2 (\nu_+^\epsilon)^2}{\pi \beta} \left\{ \frac{K_R}{x_R - x_0} e^{-2\nu_+^\epsilon \epsilon^{-1}(x_R - x_0 - r_b)} \left( \frac{\epsilon}{2\nu_+^\epsilon} \right)^{\alpha_R + 1} \Gamma(\alpha_R + 1) \right. \\ & \left. - \frac{K_L}{x_0 - x_L} e^{-2\nu_+^\epsilon \epsilon^{-1}(x_0 - x_L - r_b)} \left( \frac{\epsilon}{2\nu_+^\epsilon} \right)^{\alpha_L + 1} \Gamma(\alpha_L + 1) \right\} . \end{aligned} \tag{3.5}$$

*Here $K_L$, $\alpha_L$ and $K_R$, $\alpha_R$ are defined in (3.3), the other constants are defined in (2.4), and $\Gamma(z)$ denotes the usual Gamma function.*

The following equilibrium result follows immediately:

**Corollary 3.1** (Equilibrium [19]): *When $K_L K_R > 0$, there is a unique steady-state solution $x_{0e}$, satisfying $x_L < x_{0e} < x_R$, that is given asymptotically by*

$$x_{0e} \sim \frac{x_L + x_R}{2} + \frac{\epsilon}{4\nu_+^\epsilon} \log \left[ \frac{K_L \Gamma(\alpha_L + 1)}{K_R \Gamma(\alpha_R + 1)} \left( \frac{\epsilon}{2\nu_+^\epsilon} \right)^{\alpha_L - \alpha_R} \right] + O(\epsilon^2) . \tag{3.6}$$

*This steady state is stable when $K_L < 0$, $K_R < 0$, and is unstable when $K_L > 0$, $K_R > 0$. Specifically, if $D$ is convex near $(x_L, 0)$ and $(x_R, 0)$, then there is no stable equilibrium location on $\partial D_s$.*

For the initial value, $x(0) = x_0^0$, the qualitative properties of the dynamics associated with (3.5) are as follows. When $K_L > 0$ and $K_R > 0$, $x_0(t)$ moves exponentially slowly towards $x_L$ if $x_0^0 < x_{0e}$, or towards $x_R$ if $x_0^0 > x_{0e}$. When $K_L < 0$ and $K_R < 0$, $x_0(t)$ will approach the stable steady state at $x_{0e}$ for any initial condition. If $K_L < 0$ and $K_R > 0$, then $x_0(t)$ will move towards $x_R$. Finally, $x_0(t)$ will move towards $x_L$ when $K_L > 0$ and $K_R < 0$. In each case, the interface moves in the direction that will allow its perimeter to decrease. When the interface reaches the corner points $(x_L, 0)$ or $(x_R, 0)$, the subsequent evolution of the interface is determined by (2.3).

# 4 Metastable Spike Motion

Turing [20] proposed a reaction-diffusion system of activator-inhibitor type to mathematically model morphogenesis. From a linear stability analysis he suggested that such a system could have stable spatially inhomogeneous solutions with isolated peaks in the activator concentration. Subsequent studies on the Gierer-Meinhardt activator-inhibitor model (e.g. [9], [12]), which have involved large-scale numerical computations, have shown that robust spike-type patterns in the activator concentration are possible when the activator diffuses much more slowly than the inhibitor. A survey of some mathematical results for the Gierer-Meinhardt model is given in [17].

In [13], the shadow problem for the Gierer-Meinhardt activator-inhibitor system, that results from assuming an infinite inhibitor diffusivity, was analyzed. This problem is formulated as the non-local problem

$$u_t = \epsilon^2 \triangle u - u + \frac{u^p}{\sigma_\epsilon^q}, \qquad \mathbf{x} \in D \subset \mathbf{R^N}, \tag{4.1a}$$

$$\partial_n u = 0, \qquad \mathbf{x} \in \partial D, \tag{4.1b}$$

$$\sigma_\epsilon = \frac{\epsilon^{-N}}{\mu |D|} \int_D u^m \, d\mathbf{x}. \tag{4.1c}$$

Here $\epsilon \ll 1$, $u(\mathbf{x}, t)$ is the activator concentration, $\mu > 0$, and $|D|$ is the volume of $D$. The exponents are assumed to satisfy

$$p > 1, \qquad q > 0, \qquad m > 0, \qquad \frac{p-1}{q} < m, \qquad p < p_c(N), \tag{4.1d}$$

where $p_c(N) = 0$ if $N \leq 2$ and $p_c(N)$ is the critical Sobolev exponent if $N \geq 3$.

The first simple result summarizes the asymptotic construction of a 1-spike equilibrium solution to (4.1a), (4.1c) in all of $\mathbf{R^N}$.

**Proposition 4.1** (Canonical Spike [13]): *A 1-spike solution to (4.1a) and (4.1c) in $\mathbf{R^N}$ is given asymptotically for $\epsilon \to 0$ by $u \sim u_q(\epsilon^{-1} r)$ where*

$$u_q(\rho) \sim \sigma_\epsilon^{q/(p-1)} w(\rho). \tag{4.2a}$$

Here $w(\rho)$ is the unique positive solution to

$$w'' + \frac{(N-1)}{\rho} w' - w + w^p = 0, \qquad 0 \leq \rho < \infty, \tag{4.2b}$$

$$w'(0) = 0; \qquad w(\rho) \sim a \rho^{(1-N)/2} e^{-\rho}, \quad \text{as} \quad \rho \to \infty, \tag{4.2c}$$

*for some $a > 0$. The constant $\sigma_\epsilon$ is given by*

$$\sigma_\epsilon = \left( \frac{\Omega_N}{\mu |D|} \int_0^\infty [w(\rho)]^m \rho^{N-1} d\rho \right)^{\frac{p-1}{(p-1)-qm}}, \tag{4.2d}$$

*Here $\Omega_N$ is the surface area of the unit $N$-ball.*

The asymptotic construction of an $M$-spike solution is given by

$$u(\mathbf{x}) \sim u^* [\mathbf{x}; \mathbf{x}_0 \ldots, \mathbf{x}_{M-1}] \equiv \sigma_\epsilon^{q/(p-1)} \sum_{j=0}^{M-1} w \left( \epsilon^{-1} |\mathbf{x} - \mathbf{x}_j| \right), \tag{4.3}$$

where $\sigma_\epsilon$ is given in (4.2d) with $\mu$ replaced by $\mu/M$. The goal is to construct an equilibrium solution to the finite domain problem (4.1) of the form (4.3), where the spike locations $\mathbf{x}_j \in D$ with $|\mathbf{x}_j - \mathbf{x}_k| = O(1)$ as $\epsilon \to 0$ for $j \neq k$ are to be determined. Under these conditions, it is clear that $u^*$ satisfies the equilibrium equation (4.1a) and the boundary condition (4.1c) up to exponentially small terms as $\epsilon \to 0$. Thus, determining the correct locations for the spikes requires exponential precision. This delicate problem has now been solved for the Neumann problem and for other boundary conditions in a series of papers (see [11], [15], [23], [24], [25] and the references therein). For a 1-spike solution these results have shown that the center of the spike is at an $O(\epsilon)$ distance from the maxima of $\text{dist}(\mathbf{x}, \partial D)$. For the $M$-spike case, the important papers of [11] and [15] have proved that the problem for the determination of the locations of the spikes is directly related to the geometric problem of the lattice packing of $M$ balls of equal radii inside $D$.

The results in [13] pertain to the dynamics of a 1-spike solution to (4.1). For this problem, it is shown that the dynamics is metastable and very similar results are obtained as compared with the motion of the bubble solution for the constrained Allen-Cahn equation considered in §2. The non-local term (4.1c) is again crucial for this conclusion. We now outline the analysis and summarize the results in [13].

We first introduce the quasi-steady linearization of (4.1) by setting

$$u(\mathbf{x}, t) = u_q \left( \epsilon^{-1} |\mathbf{x} - \mathbf{x_0}(t)| \right) + v(\mathbf{x}, t), \tag{4.4}$$

where $v \ll u_q$, $v_t \ll \partial_t u_q$, and $u_q$ is defined in (4.2). This leads to the linearized problem

$$L_\epsilon v \equiv \epsilon^2 \Delta v + (-1 + p w^{p-1}) v - \frac{m q \epsilon^{-N} w^p}{\beta_N \Omega_N} \int_D w^{m-1} v \, d\mathbf{x} = \partial_t u_q, \quad \mathbf{x} \in D, \tag{4.5a}$$

$$\partial_n v = -\partial_n u_q, \qquad \mathbf{x} \in \partial D. \tag{4.5b}$$

Here, $\beta_N$ is defined by

$$\beta_N \equiv \int_0^\infty w^{m-1} \rho^{N-1} \, d\rho. \tag{4.5c}$$

Let $\mathbf{x}_0$ be fixed and let $\lambda_j$, $\phi_j$ for $j \geq 1$ be the eigenpairs of the associated non-local eigenvalue problem

$$L_\epsilon \phi = \lambda \phi, \quad \mathbf{x} \in D; \quad \partial_n \phi = 0, \quad \mathbf{x} \in \partial D; \quad (\phi, \phi) \equiv \int_D \phi^2 \, d\mathbf{x} = 1. \tag{4.6}$$

Although this eigenvalue problem is not self-adjoint and is more difficult to analyze than its counterpart (2.7) for the bubble problem, it is easy to see that the spectrum of (4.6) contains $N$ exponentially small eigenvalues with corresponding eigenfunctions satisfying $\phi_j \sim \partial_{x_j} u_q$ away from $\partial D$ for $j = 1, \ldots, N$. This follows from the exponential decay of $u_q$, the near translation invariance, and the fact that by symmetry and exponential localization the non-local term is exponentially small for these functions.

The analysis then is reduced to determining the principal eigenpair of (4.6). When the non-local term in (4.6) is absent, it is well-known that the local problem has an $O(1)$ positive eigenvalue with a corresponding localized radially symmetric eigenfunction (cf. [23]). For specific parameter sets $(p, q, m)$ and for dimensions $N = 1, 2, 3$, a path-following method was used in [13] to numerically compute where this positive eigenvalue goes in the complex plane as the non-local effect is gradually introduced through the use of a homotopy parameter $\delta$. This parameter, satisfying $0 \leq \delta \leq 1$, was chosen to multiply the non-local integral term in (4.5a). It was shown in [13] that this eigenvalue branch, which emanates from the positive eigenvalue for the local problem when $\delta = 0$, has crossed into the left-half plane $\text{Re}(\lambda) < 0$ well before $\delta = 1$. Hence, for the parameter sets considered in [13], the effect of the non-local term is to eliminate the positive eigenvalue associated with the local problem.

The recent result of [25] provides the first key rigorous analytical result on the eigenvalue problem (4.6). There it is proved that the principal eigenvalue of (4.6) is exponentially small when either $m = 2$ and $1 < p \leq 1 + 4/N$ or when $m = p + 1$ and $1 < p < p_c(N)$, where $p_c$ is the critical Sobolev exponent for dimension $N \geq 3$. The exponentially small eigenvalues were not estimated in [25]. Although, there are still some gaps in our understanding of the spectrum of (4.6), it does appear for typical ranges of the exponents that the non-local term has pushed the positive eigenvalue associated with the local problem into the left half-plane $\text{Re}(\lambda) < 0$. The results given in Propositions 4.2, 4.3 and Corollaries 4.2, 4.3 follow when this condition on the spectrum holds.

**Proposition 4.2** (Exponentially Small Eigenvalues [13]): *Let $\mathbf{x}_0 \in D$ be fixed and assume that $dist(\mathbf{x}_0, \partial D) = O(1)$ as $\epsilon \to 0$. Then, for $\epsilon \to 0$, the principal eigenvalue of (4.6) is exponentially small and there are $N$ such exponentially small eigenvalues given asymptotically for $j = 1, \ldots, N$ by*

$$\lambda_j \sim \frac{a^2 N}{\hat{\beta}_N \Omega_N} \int_{\partial D} r^{1-N} e^{-2\epsilon^{-1} r} \left( \frac{x_j - x_{0j}}{r} \right)^2 \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \left[ 1 + \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \right] dS \,. \tag{4.7a}$$

*Here $\Omega_N$ is the surface area of the unit $N$-ball, $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{x}_0)/r$, $\hat{\mathbf{n}}$ is the unit outward normal to $\partial D$, and $x_j$ and $x_{0j}$ are the $j^{th}$ coordinates of $\mathbf{x}$ and $\mathbf{x}_0$, respectively. The constant $\hat{\beta}_N$ is defined by*

$$\hat{\beta}_N \equiv \int_0^\infty \left[ w'(\rho) \right]^2 \rho^{N-1} \, d\rho \,. \tag{4.7b}$$

*The corresponding unnormalized eigenfunctions have the form*

$$\phi_j \sim \partial_{x_j} u_q + \phi_{Lj} \,, \qquad j = 1, \ldots, N \,, \tag{4.7c}$$

*where $\phi_{Lj}$ is a boundary layer function localized near $\partial D$ that allows the boundary condition in (4.1b) to be satisfied.*

The final step to characterize the metastable dynamics is to impose a limiting solvability condition on the solution $v$ to the linearized problem (4.5). Since this problem is not self-adjoint we would normally be required to obtain key results for the small eigenvalues and their associated eigenfunctions of the adjoint problem. However, for this problem it is easy to show that the estimates given in (4.7) also pertain to the adjoint problem since the non-local term is asymptotically negligible in the asymptotic calculation of this subspace. With this observation, limiting solvability conditions can be imposed using the estimates given in (4.7) to obtain the following result:

**Proposition 4.3** (Metastability [13]): *Assume that dist*$(\mathbf{x}_0, \partial D) = O(1)$ *as* $\epsilon \to 0$. *Then, for* $\epsilon \to 0$, *the metastable spike dynamics for (4.1) is characterized by* $u(\mathbf{x}, t) \sim u_q(\epsilon^{-1}|\mathbf{x} - \mathbf{x}_0(t)|)$, *where* $\mathbf{x}_0(t)$ *satisfies the differential equation*

$$\dot{\mathbf{x}}_0 \sim \frac{\epsilon N a^2}{\hat{\beta}_N \Omega_N} I(\mathbf{x}_0)\,, \qquad I(\mathbf{x}_0) \equiv \int_{\partial D} r^{1-N} e^{-2\epsilon^{-1} r} \hat{\mathbf{r}} \left[1 + \hat{\mathbf{r}} \cdot \hat{\mathbf{n}}\right] \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \, dS\,. \tag{4.8}$$

*Here* $\dot{\mathbf{x}}_0 \equiv d\mathbf{x}_0/dt$, $u_q$ *is given in Prop. 4.1, and we have the same notation as in Prop. 4.2.*

As in §2, we get the following equilibrium result:

**Corollary 4.1** (Equilibrium [23]): *Under the conditions of Prop. 4.3, (4.1) has an unstable equilibrium solution of the form* $u \sim u_q(\epsilon^{-1}|\mathbf{x} - \mathbf{x}_{0e}|)$, *where the center* $\mathbf{x}_{0e}$ *of the spike satisfies* $I(\mathbf{x}_{0e}) = 0$. *For a strictly convex domain* $D$, $\mathbf{x}_{0e}$ *is located at an* $O(\epsilon)$ *distance from the center of the unique largest inscribed sphere that can be inserted in* $D$.

This result was proved rigorously in [24]. Next, by evaluating the surface integral in (4.8) asymptotically using Laplace's method we get the following explicit dynamics:

**Corollary 4.2** (Explicit Dynamics [13]): *Assume that at* $t = 0$, $\mathbf{x}_m$ *is the unique point on* $\partial D$ *which is closest to* $\mathbf{x}_0(0)$. *Then, for* $t > 0$, *the motion of the center of the spike is in the direction of the closest point and the distance* $r_m(t) = |\mathbf{x}_m - \mathbf{x}_0(t)|$ *satisfies the asymptotic ODE*

$$\dot{r}_m \sim -\zeta r_m \left(\frac{\epsilon}{r_m}\right)^{(N+1)/2} H(r_m) e^{-2\epsilon^{-1} r_m}\,, \qquad \zeta \equiv \frac{2Na^2}{\Omega_N \hat{\beta}_N} \pi^{(N-1)/2}\,. \tag{4.9}$$

*Here* $H(r_m)$ *is defined in (2.11b). The constants in (4.9) are defined in (4.2) and (4.7).*

This result is valid up until the time when the spike approaches to within an $O(\epsilon)$ distance of $\mathbf{x}_m$. The collapse time is given in the final result:

**Corollary 4.3** (Collapse Time [13]): *For the ODE (4.9), label* $r_m(0) = r_0$. *Then, under the conditions of Corollary 4.2, the spike will first touch the boundary* $\partial D$ *when* $r_m(t_c) = 0$, *where*

$$t_c \sim \left(\frac{\epsilon}{r_0}\right)^{(1-N)/2} \frac{[H(r_0)]^{-1}}{2\zeta} \left[1 - \frac{\epsilon}{4}\left(\frac{N-1}{r_0} - \sum_{i=1}^{N-1} \frac{1}{(R_i - r_0)}\right)\right] e^{2\epsilon^{-1} r_0}\,. \tag{4.10}$$

# 5 Boundary Spike Motion

In [14] it has been shown, using formal asymptotic analysis, that a spike for (4.1) on the boundary of a smooth domain moves in the direction of the gradient of the mean curvature until it reaches an equilibrium point where the mean curvature of the boundary has a local maximum. The existence of such equilibrium solutions, where the spike is located at these special points on the boundary, has been proved in [10] and [16].

The asymptotic results obtained in [14] for the spike motion on the boundary of the domain are very similar to the small drop result given in (3.1) for the constrained Allen-Cahn equation. For the two-dimensional case, the following result was obtained in [14]:

**Proposition 5.1** (Two Dimensions [14]): *For $\epsilon \to 0$, a time-dependent spike solution for (4.1) where the spike is located on the smooth boundary of a two-dimensional domain is characterized by $u \sim u_q\left[\epsilon^{-1}(\eta^2 + (\xi - s_0(t))^2)^{1/2}\right]$,* where

$$\dot{s}_0(t) \sim \frac{4\epsilon^3 f}{3\pi}\,\kappa'(s_0)\,, \qquad f \equiv \frac{\int_0^\infty [w'(\rho)]^2 \rho^2\, d\rho}{\int_0^\infty [w'(\rho)]^2 \rho\, d\rho}\,. \tag{5.1}$$

*Here $u_q$ is defined in (4.2), $\eta$ is the distance from $\mathbf{x} \in D$ to $\partial D$, $\xi$ is arclength along $\partial D$, and $\kappa(\xi)$ is the curvature of $\partial D$.*

This result shows that the spike centered at $\eta = 0$, $\xi = s_0(t)$ will have a stable equilibrium at local maxima of $\kappa$. The corresponding result for the three-dimensional case was obtained in [14]:

**Proposition 5.2** (Three Dimensions [14]): *For $\epsilon \to 0$, a time-dependent spike solution for (4.1) where the spike is located on the smooth boundary of a three-dimensional domain is characterized by $u \sim u_q\left[\epsilon^{-1}(\eta^2 + (\xi_1 - s_1(t))^2 + (\xi_2 - s_2(t)))^{1/2}\right]$,* where

$$\dot{\mathbf{s}}(t) \sim \frac{3\epsilon^3 f}{4}\,\nabla H(\mathbf{s})\,, \qquad f \equiv \frac{\int_0^\infty [w'(\rho)]^2 \rho^3\, d\rho}{\int_0^\infty [w'(\rho)]^2 \rho^2\, d\rho}\,. \tag{5.2}$$

*Here $\mathbf{s}$ is the vector $\mathbf{s} = (s_1, s_2)$, $\eta$ is the distance from $\mathbf{x} \in D$ to $\partial D$, and $\xi_1$ and $\xi_2$ correspond locally to arclength for the two principal directions through $\xi_1 = s_1$ and $\xi_2 = s_2$. Here the function $H(\xi) \equiv H(\xi_1, \xi_2)$ denotes the mean curvature of $\partial D$.*

Finally, we provide a result for the metastable spike motion that occurs whenever the spike becomes attached to a flat portion of a two-dimensional boundary. In this case, a very similar result to (3.5) was obtained in [14]. The last result is obtained in [14] under the same qualitative geometrical setting as shown in Fig. 3 with the same assumptions as in (3.3) for the domain near the corner points $(x_L, 0)$ or $(x_R, 0)$:

**Proposition 5.3** (Explicit Dynamics [14]): *Under the same conditions as in Prop. 3.1, the center $x_0(t)$ of a spike solution constrained to the straight-line segment $y = 0, x_L \le x \le x_R$ satisfies the metastable dynamics*

$$\begin{aligned} \dot{x}_0(t) \quad \sim \quad & \frac{\epsilon a^2}{\pi \hat{\beta}_2}\left\{ \frac{K_R}{x_R - x_0} e^{-2\epsilon^{-1}(x_R - x_0)} \left(\frac{\epsilon}{2}\right)^{\alpha_R + 1} \Gamma(\alpha_R + 1) \right. \\ & \left. - \frac{K_L}{x_0 - x_L} e^{-2\epsilon^{-1}(x_0 - x_L)} \left(\frac{\epsilon}{2}\right)^{\alpha_L + 1} \Gamma(\alpha_L + 1) \right\}. \end{aligned} \tag{5.3}$$

*Here $K_L$, $\alpha_L$ and $K_R$, $\alpha_R$ are defined in (3.3), while $a$ and $\hat{\beta}_2$ are defined in (4.2) and (4.7b), respectively.* This differential equation has the same qualitative properties as discussed following (3.6).

## Acknowledgements

I am deeply honored at being chosen by the CMS to give the Coxeter-James lecture. It is my pleasure to acknowledge the contributions of my graduate students David Iron and Doug Stafford to some of the work described in this article.

## References

[1] N. Alikakos, L. Bronsard, G. Fusco, *Slow motion in the Gradient Theory of Phase Transitions via Energy and Spectrum*, Calc. Var. Part. Diff. Eq. Vol. 6, No. 1, (1998), pp. 39-66.

[2] N. Alikakos, X. Chen, G. Fusco, *Motion of a Drop by Surface Tension Along the Boundary*, preprint.

[3] N. Alikakos, G. Fusco, *Some Aspects of the Dynamics of the Cahn-Hilliard Equation*, Resenhas Vol. 1, No. 4, (1994), pp. 517-530.

[4] N. Alikakos, G. Fusco, *Slow Dynamics for the Cahn-Hilliard Equation in Higher Spatial Dimensions, Part 2: the Motion of Bubbles*, Arch. Rat. Mech. Anal. 141, (1998), pp. 1-61.

[5] L. Bronsard, B. Wetton, *A Numerical Method for Tracking Curved Networks Moving with Curvature Motion*, J. Comp. Phys. 120, No. 1, (1995), pp. 66-87.

[6] X. Chen, M. M. Kowalczyk, *Existence of Equilibria for the Cahn-Hilliard Equation via Local Minimizers of the Perimeter*, Comm. Part. Diff. Eq. 21, No. 7-8, (1996), pp. 1207-1233.

[7] M. Gage, *On an Area-Preserving Evolution For Plane Curves*, Contemp. Math. 51, (1986), pp. 51-62.

[8] M. Gage, R.S. Hamilton, *The Heat Equation Shrinking Convex Plane Curves*, J. Diff. Geom. 23, (1986), pp. 69-96.

[9] A. Gierer, H. Meinhardt, *A Theory of Biological Pattern Formation*, Kybernetik, Vol. 12, (1972), pp. 30-39.

[10] C. Gui, *Multi-Peak Solutions for a Semilinear Neumann Problem*, Duke Math. J. 84, No. 3, (1996), pp. 739-769.

[11] C. Gui, J. Wei, *Multiple Interior Peak Solutions for some Singularly Perturbed Neumann Problems*, accepted, J. Diff. Eq. (1998).

[12] D. Holloway, *Reaction-diffusion Theory of Localized Structures with Application to Vertebrate Organogenesis*, Ph. D Thesis (Chemistry), University of British Columbia, Vancouver, (1995).

[13] D. Iron, M. J. Ward, *A Metastable Spike Solution for a Non-Local Reaction-Diffusion Model*, accepted, SIAM J. Appl. Math. (1998).

[14] D. Iron, M. J. Ward, *The Motion of Boundary Spikes for a Non-Local Reaction-Diffusion Model*, submitted, SIAM J. Appl. Math. (1998).

[15] M. Kowalczyk, *Multiple Spike Layers in the Shadow Gierer-Meinhardt System: Existence of Equilibria and Approximate Invariant Manifold*, preprint (1998).

[16] W. M. Ni, *On the Shape of Least-Energy Solutions to a Semilinear Neumann Problem*, Comm. Pure Appl. Math, Vol. XLIV, (1991), pp. 819-851.

[17] W. M. Ni, *Diffusion, Cross-Diffusion, and their Spike-Layer Steady-States*, Notices of the AMS, Vol. 45, No. 1, (1998), pp. 9-18.

[18] J. Rubinstein, P. Sternberg, *Nonlocal Reaction-Diffusion Equations and Nucleation*, IMA J. Appl. Math. Vol. 48, (1992), pp. 249-264.

[19] D. Stafford, M. J. Ward, B. Wetton, *Motion of Attached Bubbles for the Constrained Allen-Cahn Equation*, in preparation.

[20] A. Turing, *The Chemical Basis of Morphogenesis*, Phil. Trans. Roy. Soc. B, 327, (1952), pp. 37-72.

[21] M. J. Ward, *Exponential Asymptotics and Convection-Diffusion-Reaction Models*, to appear, AMS short course series *Singular Perturbation Concepts of Differential Equations*, Baltimore, (1999).

[22] M. J. Ward, *Metastable Bubble Solutions for the Allen-Cahn Equation with Mass Conservation*, SIAM J. Appl. Math. Vol. 56, No. 5, (1996), pp. 1247–1279.

[23] M. J. Ward, *An Asymptotic Analysis of Localized Solutions for some Reaction-Diffusion Models in Multi-Dimensional Domains*, Stud. Appl. Math Vol. 97 No. 2, (1996), pp. 103-126.

[24] J. Wei, *On the Interior Peak Solution to a Singularly Perturbed Neumann Problem*, Tohoku Math. J. 50, (1998), pp. 159-178.

[25] J. Wei, *On Single Interior Spike Solutions of the Gierer Meinhardt System: Uniqueness and Spectrum Estimates*, submitted, Europ. J. Appl. Math. (1998).