The Stability of Hot Spot Patterns for Reaction-Diffusion Models of Urban Crime

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PIMS Hot Topics: Computational Criminology and UBC (IAM), September 2012

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Modeling Urban Crime I

Multidisciplinary efforts to model patterns of urban crime lead by UCLA group; A. Bertozzi, P. Brantingham, L. Chayes, M. Short, etc.. (since 2008); field data from LA police; What is best policing strategy?

UC MASC Project: http://paleo.sscnet.ucla.edu/ucmasc.html

Fig. 1. Dynamic changes in residential burglary hotspots for two consecutive three-month periods beginning June 2001 in Long Beach, CA. These density maps were created using ArcGIS.

Modeling Urban Crime II

Key References:

- M. B. Short, P. J. Brantingham, A. L. Bertozzi and G. E. Tita (2010), *Dissipation and displacement of hotpsots in reaction-diffusion models of crime*, PNAS, **107**(9) pp. 3961-3965. Made the cover of PNAS.
- M. B. Short, M. R. D'Orsogna, V. B. Pasour, G. E. Tita, P. J. Brantingham, A. L. Bertozzi and L. B. Chayes (2008), *A statistical model of criminal behavior*, M3AS, **18**, Suppl. pp. 1249–1267.
- M. B. Short, A. L. Bertozzi and P. J. Brantingham (2010), *Nonlinear patterns in urban crime - hotpsots, bifurcations, and suppression*, SIADS, **9**(2), pp. 462–483.

Observations: Criminal activity concentrates non-uniformly (good versus bad neighborhoods). Often "hot-spots" of crime are observed. Need to incorporate near-repeat victimization and elevated risk of re-victimization in a short time period.

An Agent-Based Model I

Agent Based Models: City is represented by a square lattice. At each lattice site there is an "attractiveness" $A(x,t)$ and a "number" $N(x,t)$ of criminals. Criminals exhibit biased random walk and are more likely to move to a neighboring site with a higher attractiveness.

Criminal Behavior: Burglarize the house at site x between times t and $t+\delta t$ with probablity

$$p_v(x,t) = 1 - e^{-A(x,t)\delta t}\,.$$

- If site x is robbed, the burglar is removed from the lattice. After the attractiveness is updated, burglars are re-introduced at each lattice site at a rate Γ.
- If a burglary at x does not occur, the burglar moves to a neighbouring site x' with probability

$$p_m(x';t,x) = \frac{A(x',t)}{\sum_{x''\sim x} A(x'',t)}\,.$$

An Agent-Based Model II

Modeling attractiveness: It has a static and dynamic component:

$$A(x,t) = A^0 + B(x,t)\,.$$

Elevated risk of re-victimization in a short time-period with decay rate ω and with a parameter $\eta \ll 1$ that models how attractiveness is spread to its neighbours:

$$B(x,t+\delta t) = \left[(1-\eta)B(x,t) + \frac{\eta}{4}\sum_{x'\sim x} B(x',t)\right](1-\omega\,\delta t) + \theta E(x,t)\,.$$

Here $\theta > 0$ and $E(x,t)$ is the number of burglary events at site x in a time interval $(t, t+\delta t)$. Then, update attractiveness

$$A(x,t+\delta t) = A^0 + B(x,t+\delta t)$$

Remark: Expected value(E)$= N(x,t)p_v(x,t)$.

Numerics (Agent-Based): (stationary hot-spots, moving hot-spots, creation of new spots, etc..) Agent Based Simulation (M. Short et al:) (Movie)

The Basic Urban RD Crime Model

In the continuum limit, the resulting dimensionless PDE RD model with no flux b.c. is (Short et al., M3AS, (2008)):

$$A_t = \varepsilon^2 \Delta A - A + PA + \alpha\,, \quad x \in \Omega\,,$$

$$\tau P_t = D\nabla \cdot \left(\nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha\,, \quad x \in \Omega\,.$$

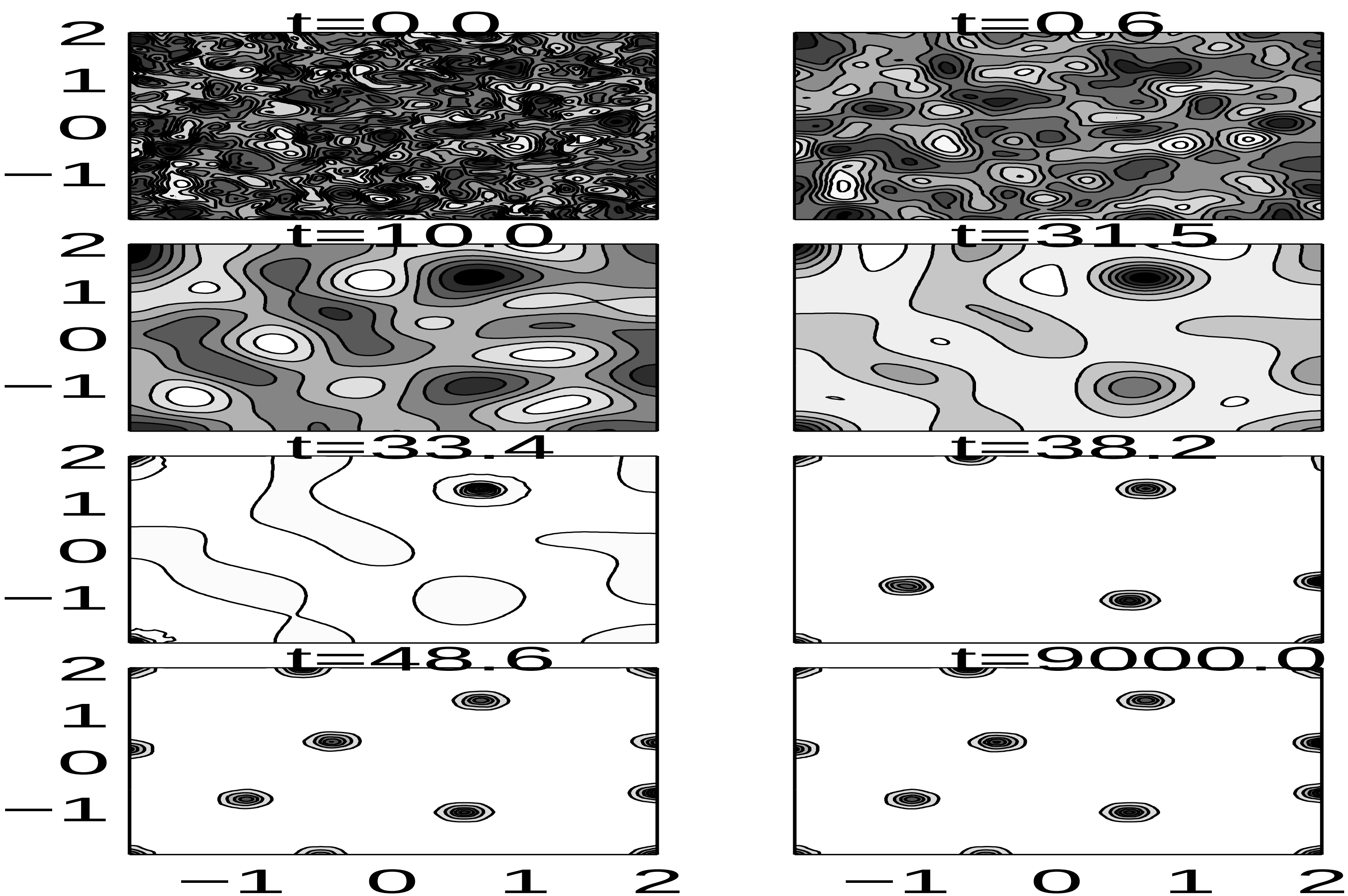
- $\varepsilon \ll 1$ (results from $\eta \ll 1$)
- $P(x,t)$ is criminal density; $A(x,t)$ is "attractiveness" to burglary.
- The chemotactic drift term $-2D\nabla \cdot \left(P\frac{\nabla A}{A}\right)$ represents the tendency of criminals to move towards sites with a higher attractiveness.
- Here α is the baseline attractiveness, while $\gamma - \alpha > 0$ is the constant rate of re-introduction of criminals after a burglary.
- The spatially homogeneous equilibrium state is

$$P_e = (\gamma - \alpha)/\gamma\,, \qquad A_e = \gamma\,.$$

Numerical: Formation of 2-D Hot-Spots

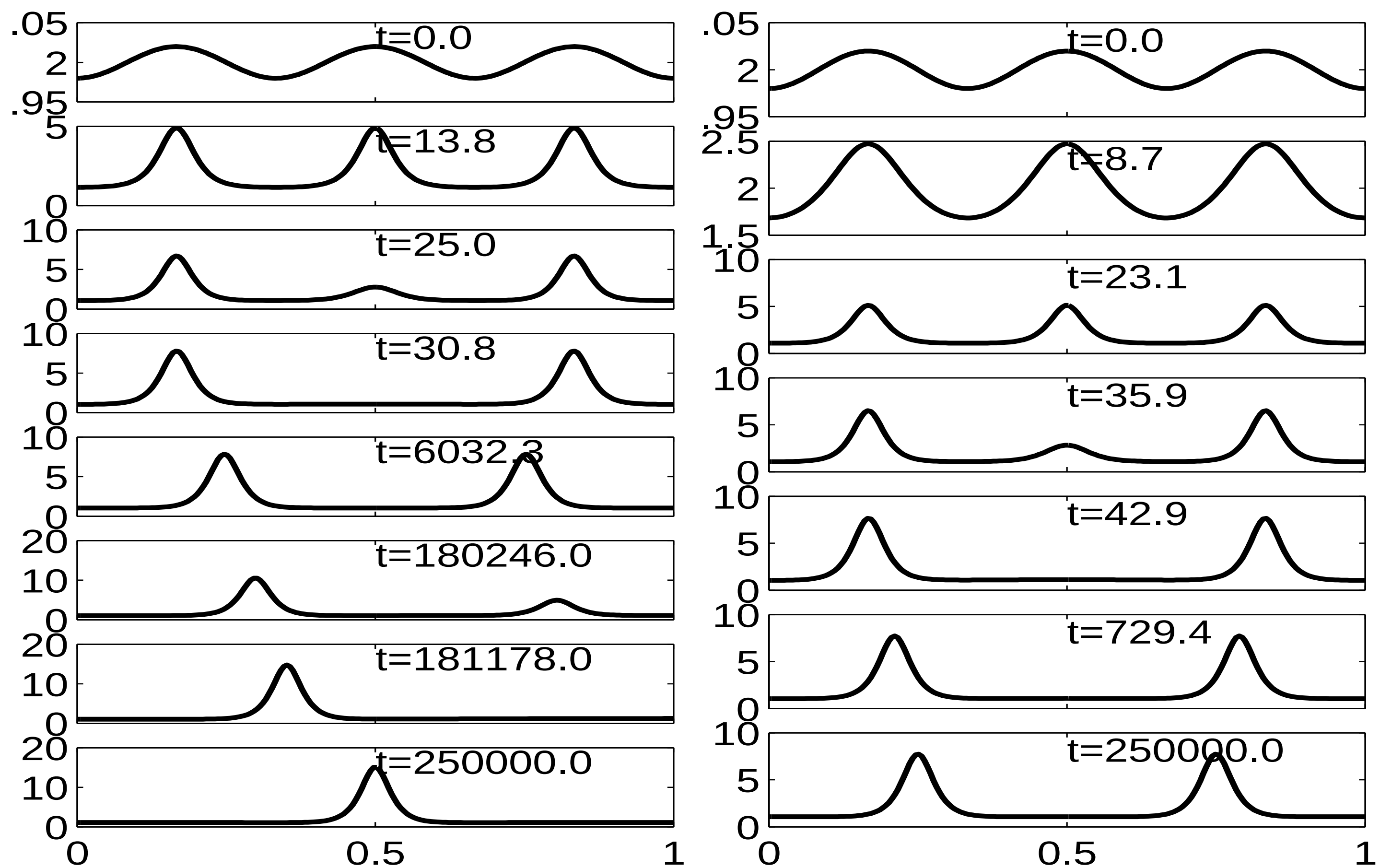
2-D Numerics: Take $P(x,0) = P_e$, $A(x,0) = \gamma(1 + \text{rand} * 0.001)$ in a square domain of width 4 with $\alpha = 1$, $\gamma = 2$, $\varepsilon = 0.08$, $\tau = 1$, and $D = 1$.

A hot-spot pattern emerges on an $O(1)$ time-scale, which then persists.



Numerical: Formation of 1-D Hot-Spots

1-D Numerics: Take $P(x,0) = P_e$, $A(x,0) = \gamma(1 - 0.01\cos(6\pi x))$ on an interval of length 1 with $\alpha = 1$, $\gamma = 2$, $\varepsilon = 0.02$, $\tau = 1$. Plot $A(x,t)$ at different t. Left: $D = 1.0$; Right: $D = 0.5$. Lowering D has increased the number of stable hot spots. (Explained in more detail later).



Outline and Perspective I

Mathematical Modeling Remarks:

- **Formulation:** formulation of stochastic agent-based model based on observational trends in crime, behavior of criminals, etc...... easy to incorporate many effects...
- **Age of Discovery:** numerical realizations of the agent-based model to observe qualitatively interesting phenomena (stationary hot-spots, moving hot-spots, creation of new spots, etc..)
- **Continuum Limit:** Derivation of a “simpler” PDE reaction–diffusion (RD) system. Usual First Step: Numerical simulations of PDE system, Turing and weakly nonlinear analysis of patterns.....
- **Rigorous PDE Analysis:** existence, regularity, and bifurcation-theoretic results for the PDE model (Rodriguez, Cantrell, Cosner and Manasevich).
- **Model Validation:** matching model predictions with field observations. Improving the model. Developing new models (some game-theoretic)... (Berestycki-Nadal, Pilcher, Short and D’Orsogna).

Outline and Perspective II

However: many seemingly “simple-looking” RD systems can exhibit extremely complex dynamics in the nonlinear regime in different parameter ranges; i.e witness the Gray-Scott model of chemical physics (1996–date).

Our Approach: For the RD model of urban crime, use a combination of formal asymptotics, rigorous analysis, and computation, to obtain analytical results for stability thresholds, delineating in parameter space where different solution behaviors occur, etc...

Specific Goal: Analyze the existence and stability of localized patterns of criminal activity for this model in the limit $\varepsilon \to 0$. We also consider extensions of the basic model to include the effect of “police”. These patterns are “far-from-equilibrium” (Y. Nishiura..), and not amenable to standard Turing stability analysis.

- **Particle-like solutions to PDE’s; vortices, skyrmions, hot-spots, ...**

Turing-Stability Analysis

The uniform state A_e, P_e on $\mathbb{R}^1$ is unstable for $\varepsilon \to 0$ when $\gamma > 3\alpha/2$. With an $e^{imx+\lambda t}$ perturbation, the Turing instability band is

$$D^{-1/2}\gamma(2\gamma - 3\alpha)^{-1/2} \sim m_{\text{lower}} < m < m_{\text{upper}} \sim \varepsilon^{-1}\gamma^{-1/2}(2\gamma - 3\alpha)^{1/2}\,.$$

For $\varepsilon \to 0$, the maximum growth rate is $\lambda_{\text{max}} \sim O(1)$, with the most unstable mode

$$m_{\text{max}} \sim \varepsilon^{-1/2} D^{-1/4} \gamma^{-1/2} \left[(\gamma - \alpha)(3\gamma^2 + 2\tau(2\gamma - 3\alpha)\right]^{1/4}\,.$$

Remark 1: For perturbations of the uniform state the preferred pattern has a characteristic half-length $l_{\text{turing}} \sim \pi/m_{\text{max}}$, where

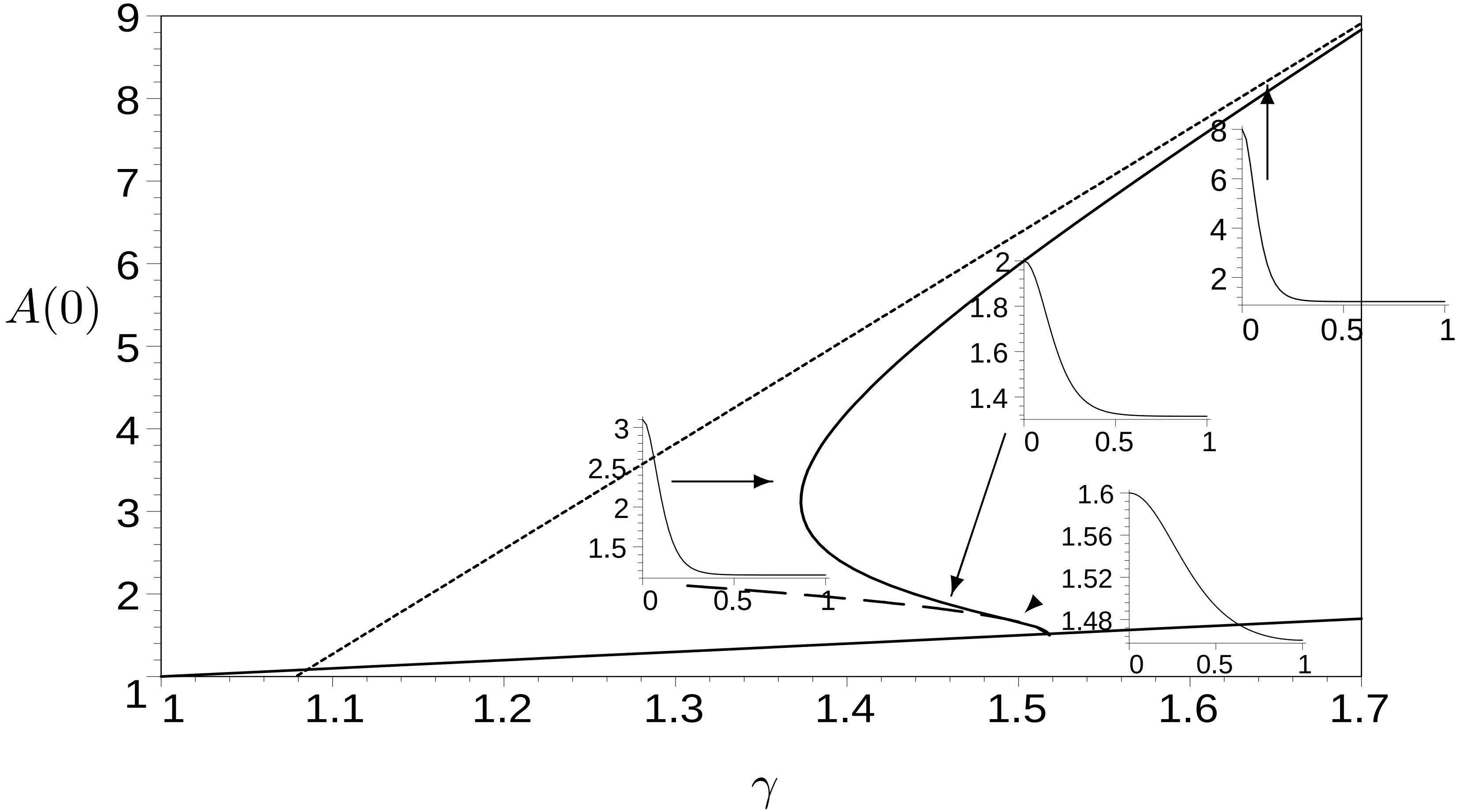
$$l_{\text{turing}} \sim \varepsilon^{1/2} D^{1/4} \gamma^{1/2} \left[(\gamma - \alpha)(3\gamma^2 + 2\tau(2\gamma - 3\alpha))\right]^{-1/4} \pi\,.$$

Notice that $l_{\text{turing}} = O(1)$ when $D = O(\varepsilon^{-2})$.

Remark 2: Turing and weakly nonlinear analysis in 1-D and 2-D given in Short et al. (SIADS 2010), together with full numerical computations.

Global Bifurcation Diagram in 1-D

Bifurcation Diagram: $A(0)$ versus γ on $0 < x < 1$ for $\alpha = 1$, $\varepsilon = 0.05$, $D = 2$. Notice that a localized hot-spot occurs when $A(0)$ is large.



Caption: $A_e = \gamma$ is the solid line with shallow slope; The hot-spot asymptotics (to be derived) is the dotted line $A(0) \sim 2(\gamma - \alpha)/(\varepsilon\pi)$. Subcritical Turing occurs at $\gamma = 3\alpha/2 + O(\varepsilon) \approx 1.5$, with weakly nonlinear asymptotics (dashed parabolic curve). Inserts plot $A(x)$ versus x.

Basic RD Crime Model: Qualitative I

There are two key parameter regimes: $D \gg 1$ **and** $D = O(1)$

Regime 1: $D \gg 1$

- Localized patterns in the form of pulses or spikes are readily constructed using singular perturbation techniques for the regime $O(1) \ll D \leq O(\varepsilon^{-2})$ in 1-D and $O(1) \ll D \leq O(\varepsilon^{-4})$ in 2-D.
- The stability threshold for these patterns occurs when $D = O(\varepsilon^{-2})$ in 1-D and $D = O(\varepsilon^{-4})$ in 2-D.
- The stability theory is based primarily on an exactly solvable nonlocal eigenvalue problem.
- A further stability threshold in D with respect to instabilities developing over a long time scale $t = O(\varepsilon^{-2})$ must also be calculated.

Implication: The stability threshold in terms of D determines the mininum spacing between localized elevated regions of criminal activity that allows for a stable pattern, i.e. If D_{crit} is large, stable hot-spots are closely spaced.

Localized Hot-Spot Patterns in 1-D: History

A rather extensive literature on the stability of pulses for two-component RD systems without gradient terms. Prototypical is

$$v_t = \varepsilon^2 v_{xx} - v + v^2/u\,, \quad \tau u_t = D u_{xx} - u + \varepsilon^{-1} v^2\,, \quad \text{(GM model)}.$$

- **History:** NLEP stability theory (1999-date) (Iron, Kolokolnikov, Ward, Wei, Winter; Doelman, Gardner, Kaper, Van der Ploeg,..).
- Let $w'' - w + w^2 = 0$ be the homoclinic. To study the stability on an $O(1)$ time-scale need to analyze the spectrum of the NLEP

$$\Phi_{yy} - \Phi + 2w\Phi - \chi(\tau\lambda) w^2 \frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w^2\, dy} = \lambda\Phi\,; \quad \Phi \to 0 \text{ as } |y| \to \infty.$$

  - If $D > D_c$, $K > 1$, and $\tau < \tau_c$, then there is a sign-fluctuating instability of the spike amplitudes due to a positive real eigenvalue; this is a competition instability (Iron-Ward-Wei (Physica D, 2001)).
  - If $D < D_c$, $K > 1$, but $\tau > \tau_c$, then there is a synchronous oscillatory instability of the spike amplitudes due to a Hopf bifurcation. (Ward-Wei, (J. Nonl. Sci, 2003)).

Regime 1: Hot-Spot Equilibria in 1-D: I

Basic Cell Problem: Consider WLOG the interval $|x| < l$.

Since $P_x - \frac{2P}{A} A_x = (P/A^2)_x A^2$, we let $V = P/A^2$. Then, on $|x| < l$

$$A_t = \varepsilon^2 A_{xx} - A + V A^3 + \alpha \,,$$
$$\tau \left(A^2 V\right)_t = D \left(A^2 V_x\right)_x - V A^3 + \gamma - \alpha \,.$$

For $D = O(\varepsilon^{-2})$, the correct scaling is

$$V = \varepsilon^2 v \,, \qquad D = D_0/\varepsilon^2 \,.$$

Therefore, our re-scaled RD system on the basic cell $|x| < l$ is

$$A_t = \varepsilon^2 A_{xx} - A + \varepsilon^2 v A^3 + \alpha \,; \qquad A_x(\pm l, t) = 0 \,,$$
$$\varepsilon^2 \tau \left(A^2 v\right)_t = D_0 \left(A^2 v_x\right)_x - \varepsilon^2 v A^3 + \gamma - \alpha \,; \qquad v_x(\pm l, t) = 0 \,.$$

Remark: $A = O(\varepsilon^{-1})$ in the core of a hot-spot while $A = O(1)$ away from the core. We obtain that $v = O(1)$ globally.

Regime 1: Hot-Spot Equilibria in 1-D: II

From a matched asymptotic analysis:

Principal Result: *Let* $\varepsilon \to 0$ *and* $O(1) \ll D \leq O(\varepsilon^{-2})$ *with* $D_0 \equiv \varepsilon^2 D$*. Then, for a one-hot-spot solution centered at* $x = 0$ *on* $|x| \leq l$*, the leading-order uniform asymptotics for* A *and* P *are*

$$A \sim \frac{1}{\sqrt{2}} \left(\frac{2l(\gamma - \alpha)}{\pi \varepsilon} - \alpha \right) w(x/\varepsilon) + \alpha \,, \qquad P \sim \left[w(x/\varepsilon) \right]^2 .$$

Here $w(y) = \sqrt{2}\text{sech}\,(y)$ *is the homoclinic of* $w'' - w + w^3 = 0$*. The inner and outer approximations for* v *are*

$$v \sim v_0 + \varepsilon v_1 \,, \quad |x| = O(\varepsilon) \,; \quad v \sim \frac{\zeta}{2} \left((l - |x|)^2 - l^2 \right) + v_0 \,, \quad O(\varepsilon) < |x| < l \,,$$

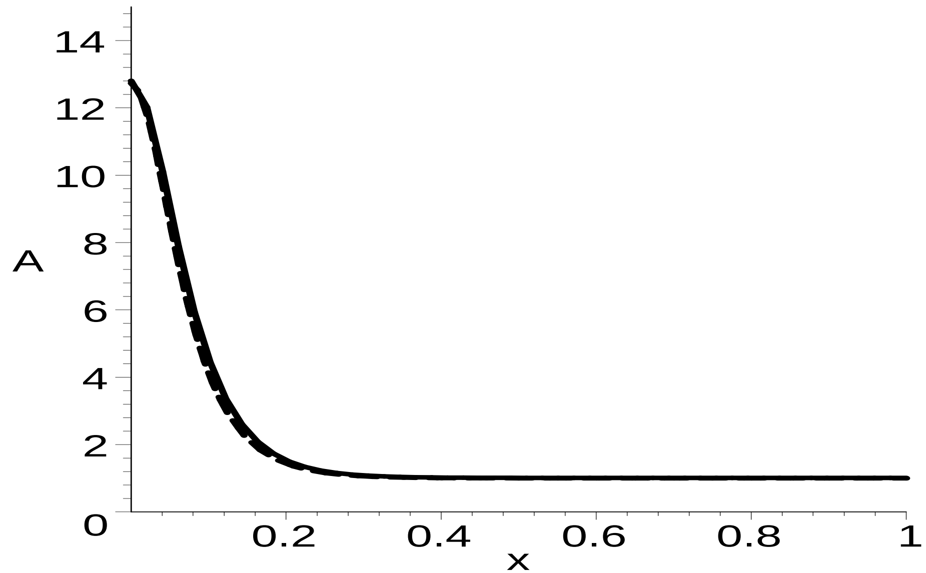
Here $\zeta \equiv (\alpha - \gamma)/(D_0 \alpha^2) < 0$ *and* $v_0 = \pi^2 \left[2l^2 (\gamma - \alpha)^2 \right]^{-1}$*.*

Remarks:

- $A(0) \sim 2(\gamma - \alpha)/(\varepsilon \pi)$, as plotted previously on bifurcation diagram.
- For a symmetric K-hot-spot pattern with spots of equal height on a domain of length S, simply set $l = S/2K$ and use gluing.

Regime 1: Hot-Spot Equilibria in 1-D: III

Comparison with Full Numerics: with $D_0 = 1$, $\gamma = 1$, $\varepsilon = 0.05$, and $\alpha = 2$:



ε	$A(0)$ (num)	$A(0)$ (asy1)	$v(0)$ (num)	$v(0)$ (asy1)	$v(0)$ (asy2)
0.1	6.281	6.366	3.5844	4.935	2.961
0.05	12.805	12.732	4.1474	4.935	3.948
0.025	25.628	25.465	4.4993	4.935	4.441
0.0125	51.145	50.930	4.7039	4.935	4.688

Remark: A one-term approximation for $A(0)$ is accurate even for $\varepsilon = 0.1$.

NLEP Stability Analysis: I

Let A_e, v_e be one-spike solution on the basic cell $|x| < l$. We introduce

$$A = A_e + \phi e^{\lambda t}\,, \quad v = v_e + \varepsilon \psi e^{\lambda t}\,.$$

The singularly perturbed eigenvalue problem is

$$\varepsilon^2 \phi_{xx} - \phi + 3\varepsilon^2 v_e A_e^2 \phi + \varepsilon^3 A_e^3 \psi = \lambda \phi\,,$$

$$D_0 \left(\varepsilon A_e^2 \psi_x + 2 A_e v_{ex} \phi\right)_x - 3\varepsilon^2 A_e^2 v_e \phi - \varepsilon^3 \psi A_e^3 = \lambda \tau \varepsilon^2 \left(\varepsilon A_e^2 \psi + 2 A_e v_e \phi\right)\,.$$

For z complex, we impose the Floquet boundary conditions

$$\phi(l) = z\phi(-l)\,, \quad \phi'(l) = z\phi'(-l)\,, \quad \psi(l) = z\psi(-l)\,, \quad \psi'(l) = z\psi'(-l)\,,$$

To obtain the spectrum of a K-spike pattern on a domain of length $2Kl$ subject to periodic boundary conditions we set $z^K = 1$, so that

$$z_j = e^{2\pi i j/K}\,, \qquad j = 0, \ldots, K-1\,.$$

Remark: An NLEP is derived for the periodic b.c. problem by using asymptotics to determine jump conditions for ψ across $x = 0$. Then, the Neumann spectra is extracted from Periodic spectra.

NLEP Stability Analysis: II

An asymptotic analysis shows that $\phi \sim \Phi(y)$ with $y = \varepsilon^{-1} x$, where $\Phi(y)$ on $-\infty < y < \infty$ satisfies

$$L_0 \Phi \equiv \Phi'' - \Phi + 3w^2 \Phi = -v_0^{-3/2} w^3 \psi(0) + \lambda \Phi \, ; \qquad \Phi \to 0 \text{ as } |y| \to \infty .$$

Then, for $\tau = O(1)$, $\psi(0)$ is determined from

$$\psi_{xx} = 0 \, , \quad 0 < |x| \leq l \, ; \qquad \psi(l) = z \psi(-l) \, , \quad \psi'(l) = z \psi'(-l) \, ,$$

subject to $\psi(0^+) = \psi(0^-) \equiv \psi(0)$ and the jump condition across $x = 0$:

$$a_0 \left[\psi_x \right]_0 + a_1 \psi(0) = a_2 \, ;$$

$$a_0 \equiv D_0 \alpha^2 \, , \qquad a_1 = -v_0^{-3/2} \int_{-\infty}^{\infty} w^3 \, dy \qquad a_2 = 3 \int_{-\infty}^{\infty} w^2 \Phi \, dy$$

Remark: $\psi(0)$ involves one nonlocal term.

NLEP Stability Analysis: III

Principal Result: *Consider a $K > 1$ equilibrium hot-spots on an interval of length S with no-flux conditions. For $\varepsilon \to 0$, $\tau = O(1)$, and with $D_0 = \varepsilon^2 D$, the stability of this solution with respect to the "large" eigenvalues $\lambda = O(1)$ of the linearization is determined by the spectrum of the NLEP*

$$L_0\Phi - \chi_j w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \lambda \Phi \, ; \qquad \Phi \to 0 \quad \text{as} \quad |y| \to \infty \, ,$$

$$\chi_j = 3 \left[1 + \frac{D_0 \alpha^2 \pi^2 K^4}{4(\gamma - \alpha)^3} \left(\frac{2}{S} \right)^4 \left(1 - \cos\left(\pi j / K\right)\right) \right]^{-1} \, , \quad j = 0, \ldots, K-1 .$$

For $K = 1$ we have $\chi_0 = 3$. Here L_0 (local operator) is

$$L_0 \Phi \equiv \Phi'' - \Phi + 3 w^2 \Phi$$

Remark: In contrast to the NLEP's for the GM and GS models, the discrete spectrum for this NLEP is explicitly available.

NLEP Stability Analysis: IV

Lemma: *Let c be real, and consider the NLEP on $-\infty < y < \infty$:*

$$L_0\Phi - cw^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy = \lambda \Phi \, ; \qquad \Phi \to 0 \quad \text{as} \quad |y| \to \infty \, ,$$

corresponding to $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$. On the range Re$(\lambda) > -1$, there is a unique discrete eigenvalue given by

$$\lambda = 3 - c \int_{-\infty}^{\infty} w^5 \, dy \, .$$

Thus, λ is real and $\lambda < 0$ when $c > 3/\int w^5 \, dy$.

Idea of Proof: We use the key identity $L_0 w^2 = 3w^2$ and Green's theorem $\int_{-\infty}^{\infty} \left(w^2 L_0 \Phi - \Phi L_0 w^2 \right) \, dy = 0$ with $L_0 \Phi = cw^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy + \lambda \Phi$. Thus,

$$\left(\lambda - 3 + c \int_{-\infty}^{\infty} w^5 \, dy \right) \int_{-\infty}^{\infty} w^2 \Phi \, dy = 0 \, ,$$

which yields the result.

NLEP Stability Analysis: V

By applying this Lemma to our NLEP, we get:

Principal Result: *On a domain of length S with Neumann b.c., for $\tau = O(1)$ and $D_0 = \varepsilon^2 D$ a one-hot-spot solution is stable on an $O(1)$ time-scale $\forall D_0 > 0$. For $K > 1$ it is stable on an $O(1)$ time-scale iff $D_0 < D_{0K}^L$, where*

$$D_{0K}^L \equiv \frac{2(\gamma-\alpha)^3 \left(S/(2K)\right)^4}{\alpha^2 \pi^2 \left[1+\cos\left(\pi/K\right)\right]} \,.$$

In terms of the original diffusivity D, given by $D = \varepsilon^{-2} D_0$, the stability threshold is $D_K^L = \varepsilon^{-2} D_{0K}^L$ when $K > 1$.

Small Eigenvalues: There are "small" $o(1)$ eigenvalues in the linearization that are difficult to asymptotically calculate directly. The threshold in D for this critical spectrum is obtained indirectly by determining the value D_K^S of D for which a asymmetric K-hot-spot equilibrium branch bifurcates from a symmetric K-hot-spot branch. We readily calculate that

$$D_K^S = \frac{(\gamma-\alpha)^3}{\varepsilon^2 \pi^2 \alpha^2} \left(\frac{S}{2K}\right)^4 \,.$$

NLEP Stability Analysis: VI

Summary: The small and large (NLEP) eigenvalue stability thresholds are

$$D^S_{\mathsf{K}} \sim \left(\frac{S}{2K}\right)^4 \frac{(\gamma-\alpha)^3}{\varepsilon^2\pi^2\alpha^2}\,, \qquad D^L_{\mathsf{K}} = D^S_{\mathsf{K}}\left(\frac{2}{1+\cos\left(\pi/K\right)}\right) > D^S_{\mathsf{K}}\,.$$

Remark 1: Thus, we have stability wrt both classes of eigenvalues when $D < D^S_K$; a weak translational instability when $D^S_K < D < D^L_K$; a fast $O(1)$ time-scale when $D > D^L_K$.

Remark 2 (KEY): For stability, we need the inter-hot-spot spacing l to satisfy $l > l_c$, where

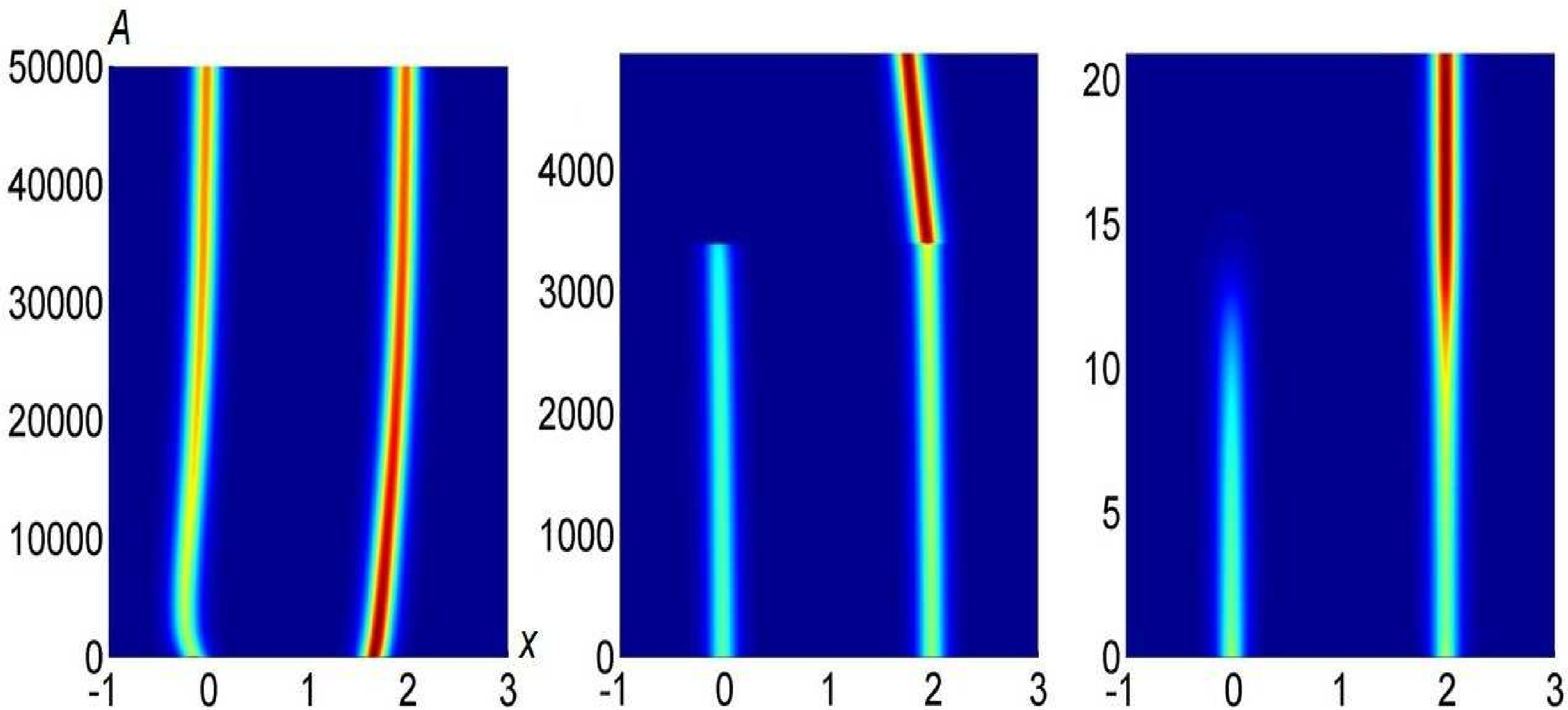
$$l_c \sim \sqrt{\pi} D^{1/4} \varepsilon^{1/2} \alpha^{1/2} (\gamma-\alpha)^{-3/4}$$

Remark 3 (KEY): Since l_c and l_{turing} are both $O(1)$ when $D = O(\varepsilon^{-2})$, the maximum number of stable hot-spots corresponds (roughly) to the most unstable Turing mode.

Numerical Validation I

Full Numerics: Let $\varepsilon = 0.07$, $\alpha = 1$, $\gamma = 2$, $S = 4$, $\tau = 1$, and $K = 2$, so that $D_{\mathsf{K}}^{S} \approx 20.67$ and $D_{\mathsf{K}}^{L} \approx 41.33$. The results below confirm the stability theory.

Left: $D = 15$ Middle: $D = 30$ Right: $D = 50$



Qualitative Implications: Stability Analysis

On an interval of length S, the stability properties of a K-hot-spot equilibrium pattern can be phrased in terms of the maximum number of hot-spots:

- Unstable wrt a competition instability developing on an $O(1)$ time scale if $K > K_{c+}$, where $K_{c+} > 0$ is the unique root of

$$K\left(1+\cos\left(\pi/K\right)\right)^{1/4} = \left(\frac{S}{2}\right)\left(\frac{2}{D}\right)^{1/4}\frac{(\gamma-\alpha)^{3/4}}{\sqrt{\pi\varepsilon\alpha}}\,.$$

- Stable with respect to slow translational instabilities developing on an $O(\varepsilon^{-2})$ time-scale if $K < K_{c-} < K_{c+}$, where

$$K_{c-} = \left(\frac{S}{2}\right)D^{-1/4}\frac{(\gamma-\alpha)^{3/4}}{\sqrt{\pi\varepsilon\alpha}}\,.$$

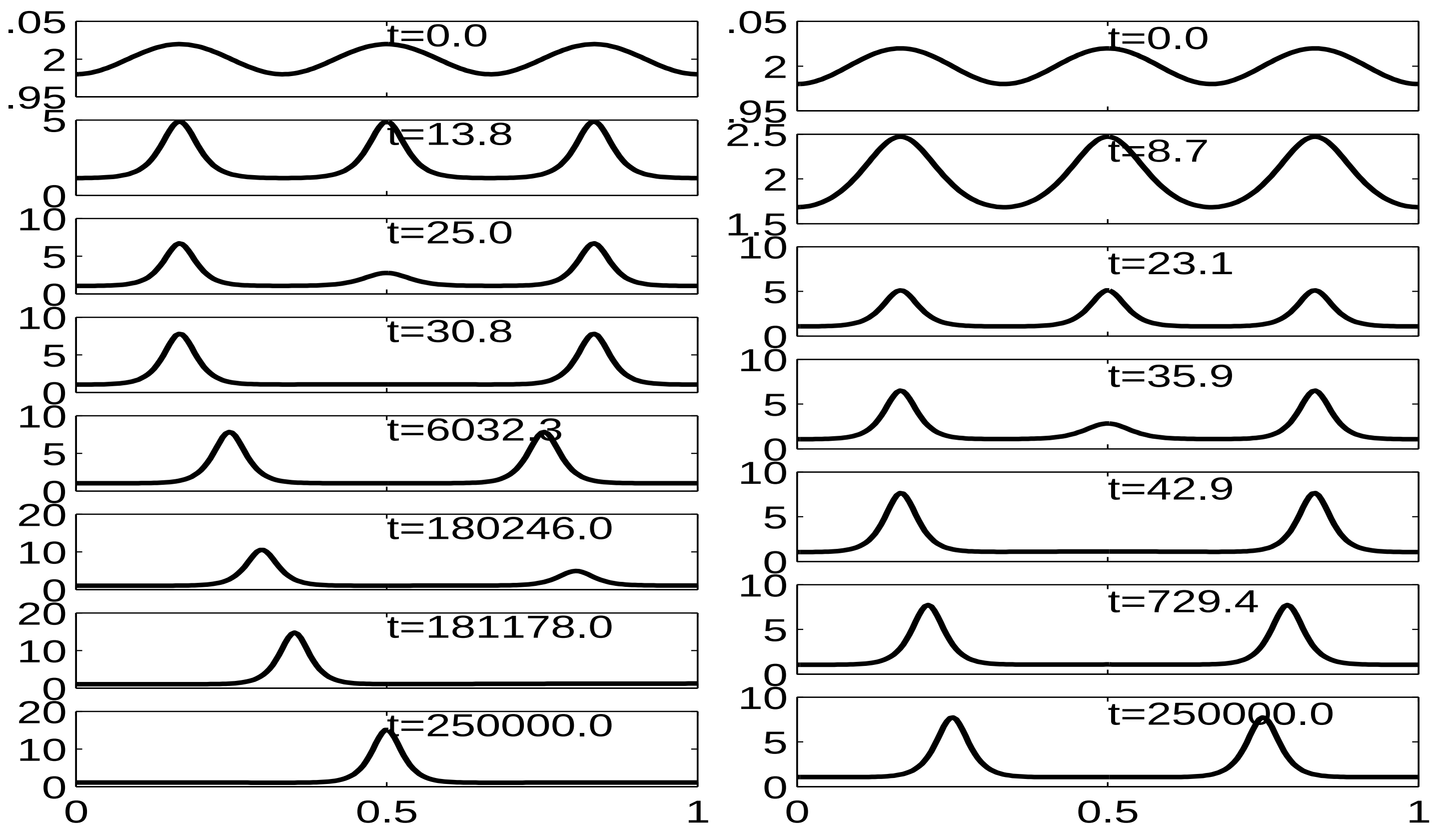
- **Summary:** stability when $K < K_{c-}$; stability wrt $\mathcal{O}(1)$ time-scale instabilities but unstable wrt slow translation instabilities when $K_{c-} < K < K_{c+}$; a fast $\mathcal{O}(1)$ time-scale instability when $K > K_{c+}$.

Numerical Validation II

1-D Numerics (Revisited): Take $\alpha = 1$, $\gamma = 2$, and $\varepsilon = 0.02$.

Left ($D = 1$): $K_{c+} \approx 2.27$ and $K_{c-} \approx 1.995$. Thus, 3-spots are NLEP unstable ($O(1)$ time-scale instability), and 2-spots (unstable on very long time-interval).

Right ($D = 0.5$): $K_{c+} \approx 2.61$ and $K_{c-} \approx 2.37$. Thus, 3-spots are NLEP unstable, but 2-spots are stable wrt NLEP and translations.



Quasi-Equilibria and NLEP Stability in 2-D

In a 2-D domain Ω with area $|\Omega|$.

Principal Result: *For $\varepsilon \to 0$ and $\tau = O(1)$, a symmetric K-hot-spot quasi-steady-state solution for $D = \varepsilon^{-4}\mathcal{D}_0/\sigma$ with $\sigma = -1/\log\varepsilon$, is characterized near the j-th hot-spot by*

$$A \sim \frac{w(\rho)}{\varepsilon^2\sqrt{v_0}}\,, \qquad P \sim [w(\rho)]^2\,,$$

with $\rho = \varepsilon^{-1}|x - x_j|$. Here $w(\rho)$ is the radially symmetric ground-state of $\Delta_\rho w - w + w^3 = 0$. In addition,

$$v_0 \equiv \frac{4\pi^2 b^2 K^2}{|\Omega|^2(\gamma-\alpha)^2} \qquad b = \int_0^\infty \rho w^2\, d\rho\,.$$

this quasi-steady-state for $K > 1$ is stable wrt $O(1)$ instabilities of the associated NLEP iff

$$D < \left(\frac{\varepsilon^{-4}}{\sigma}\right)\mathcal{D}_{0K}^L\,, \quad \text{where} \quad \mathcal{D}_{0K}^L \equiv \frac{|\Omega|^3\left(\gamma-\alpha\right)^3}{4\pi\alpha^2K^3\left(\int_{\mathbb{R}^2} w^3\,dy\right)^2}\,.$$

For $K = 1$ a single hot-spot is stable for all $\mathcal{D}_0$ independent of ε.

Basic RD Crime Model: Qualitative II

Regime 2: $D = O(1)$

$$A_t = \varepsilon^2 \Delta A - A + PA + \alpha \,, \quad x \in \Omega \,,$$

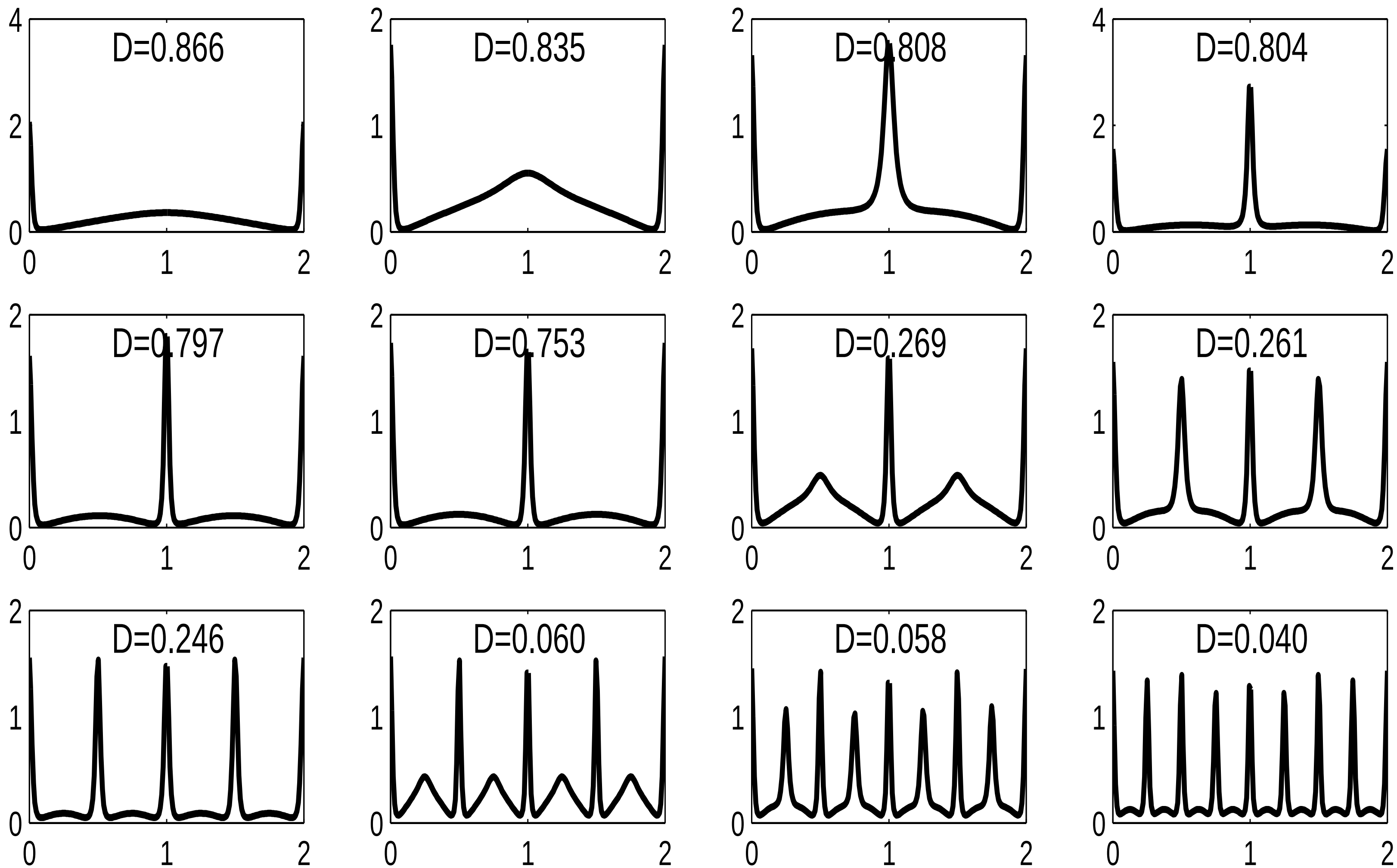
$$\tau P_t = D \nabla \cdot \left(\nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha \,, \quad x \in \Omega \,.$$

Localized hot-spots still exist, but

- In 1-D, localized regions of criminal activity can be nucleated in the region between neighboring hot-spots when the inter hot-spot spacing exceeds some threshold.
- Leads to the “spontaneous” creation of new hot-spots, i.e. new regions of elevated criminal activity.
- This is called peak-insertion in R-D theory, and it arises from a saddle-nose bifurcation point in terms of D.
- In 1-D the hot-spot dynamics is repulsive, and a reduction to finite-dimensional dynamics can be done.

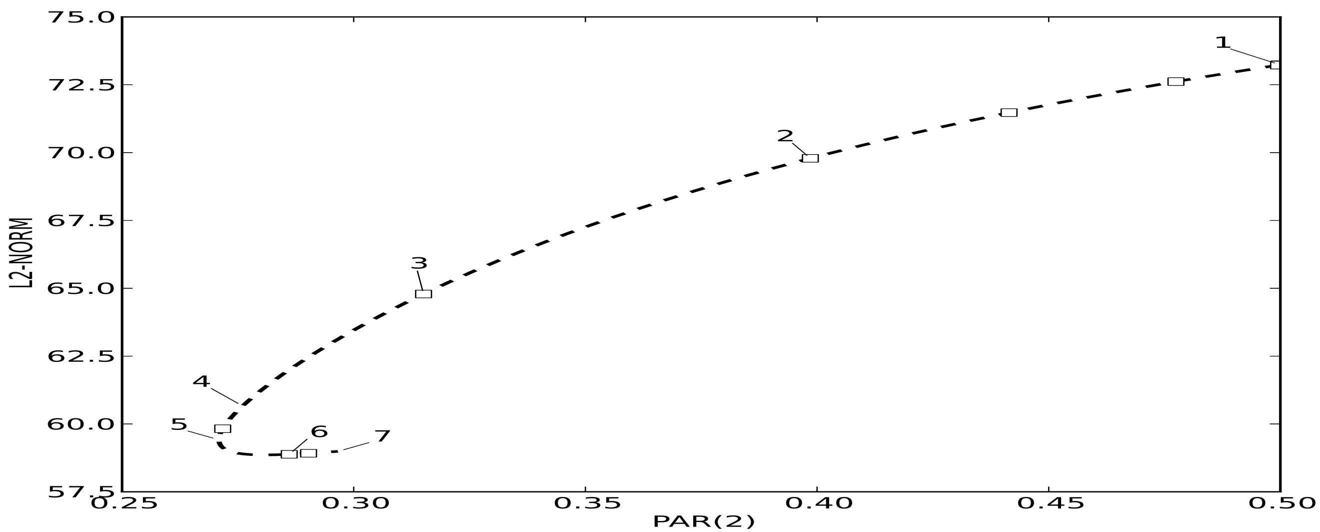
Peak-Insertion or Nucleation: $D = O(1)$

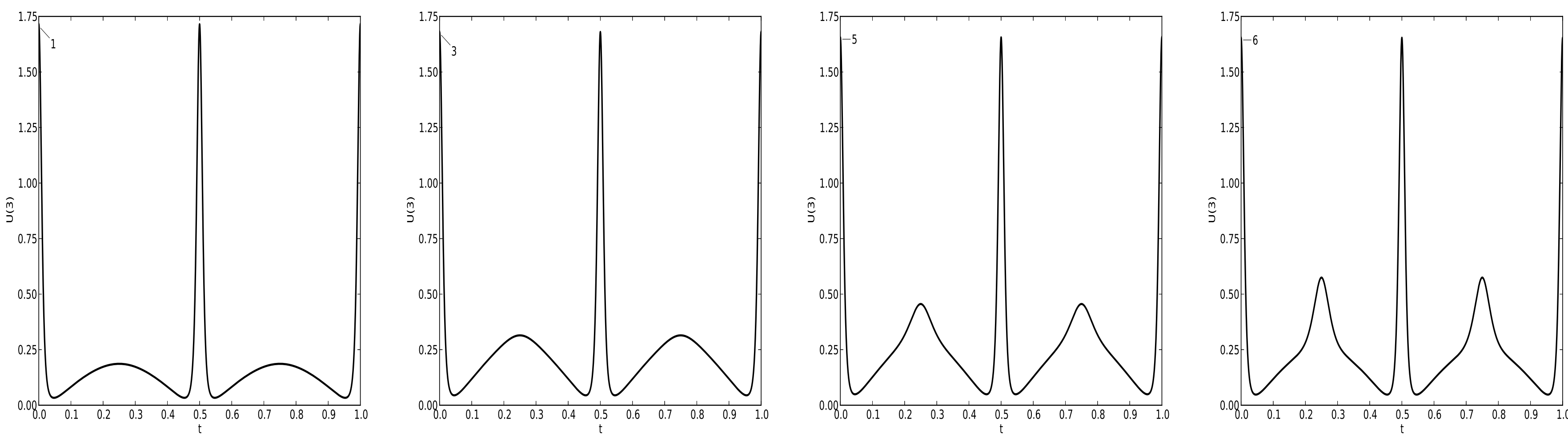
Numerics: $\varepsilon = 0.02, \alpha = 1, \gamma = 2$, $\tau = 1$, with $x \in (0, 2)$. when D is slowly decreasing in time as $D = 1/(1 + 0.01t)$. Plot of $P(x)$. Peak inserton occurs whenever D is quartered.



Peak-Insertion Behavior: Global AUTO

Global Bifurcation Diagram (AUTO): A Two-Boundary and One-Interior Spike Solution for $\gamma = 2$, $\alpha = 1$, $\varepsilon = 0.02$, on a domain of length 2. Top: $|A|_2$ versus D: Bottom: $P(x)$.





Labels: 1 3 5 6

Analysis of Peak-Insertion Behavior: I

Basic Cell Problem: Consider WLOG a one-spike pattern centered at the midpoint of the interval $|x| < l$. Since $P_x - \frac{2P}{A} A_x = (P/A^2)_x A^2$, we define

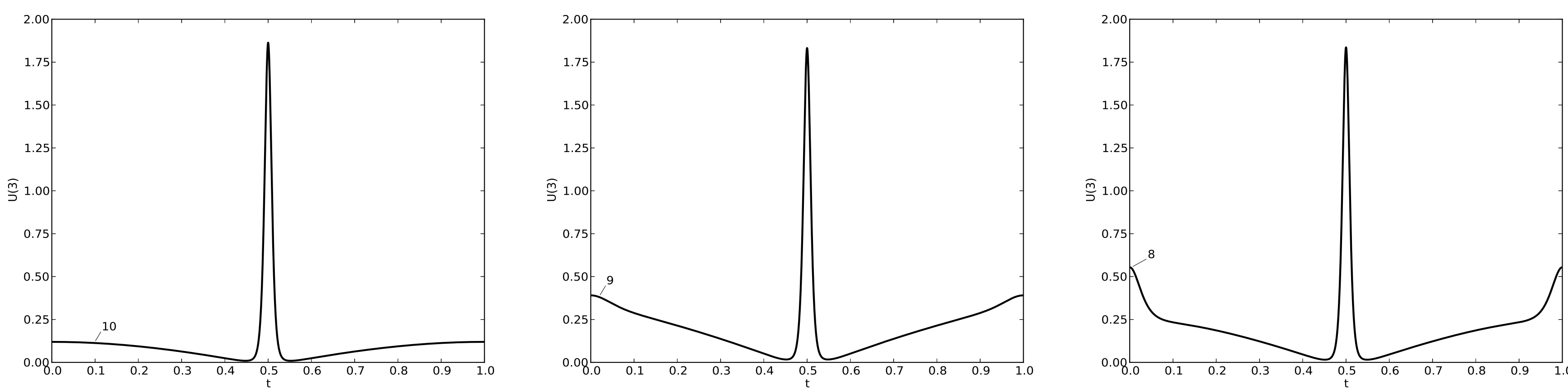
$$V = P/A^2 \,.$$

Then, on $|x| < l$, the equilibrium problem is

$$\varepsilon^2 A_{xx} - A + VA^3 + \alpha = 0 \,, \qquad A_x(\pm l) = 0,$$
$$D\left(A^2 V_x\right)_x - VA^3 + \gamma - \alpha = 0 \,, \qquad V_x(\pm l) = 0.$$

Goal: Use $\varepsilon \ll 1$ asymptotics to determine $A(l)$ versus $l/\sqrt{D}$.

As D decreases, peak insertion for $P(x)$ at the boundary occurs:



Analysis of Peak-Insertion Behavior: II

Inner Region: We set $y = x/\varepsilon$, and expand

$$A \sim \varepsilon^{-1} A_0 + A_1 + \cdots \,, \qquad V \sim \varepsilon^2 \tilde{v}_0 + \cdots$$

We obtain that

$$A_0 = w/\sqrt{\tilde{v}_0} \,, \qquad w = \sqrt{2}\text{sech}(y) \,,$$

where $w'' - w + w^3 = 0$. Here $\tilde{v}_0$ is to be found, and $A_1 \to \alpha$ as $y \to \pm\infty$.

Outer Region: WLOG consider $0^+ < x \leq l$. Key: The leading order outer problem is nonlinear. With, $A \sim a_0$ and $V \sim v_0$, then

$$v_0 a_0^3 = a_0 - \alpha \,, \qquad D \left(a_0^2 v_{0x}\right)_x = v_0 a_0^3 - (\gamma - \alpha) \,,$$

which leads to

$$D \left(f(a_0) a_{0x}\right)_x = a_0 - \gamma \,, \quad 0 < x \leq l \,; \quad a_0(0^+) = \alpha \,, \quad a_{0x}(l) = 0 \,,$$

where

$$f(a_0) \equiv a_0^{-2}(3\alpha - 2a_0) \,.$$

Analysis of Peak-Insertion Behavior: III

Remark 1: We have $f(a_0) > 0$ and $a_{0x} > 0$ when $a_0 < 3\alpha/2 < \gamma$. **Note:** for $\gamma > 3\alpha/2$ the spatially homogeneous steady-state is Turing unstable.

Remark 2: For the existence of a solution a_0 we require $a_0(l) \equiv \mu \leq 3\alpha/2$.

Upon integrating the BVP for a_0, we obtain an implicit relation for $\mu \equiv a_0(l)$ in terms of $l/\sqrt{D}$. Namely,

$$\sqrt{\frac{2}{D}}l = \chi(\mu) \equiv \frac{2}{\gamma - \alpha}\sqrt{G(\alpha;\mu)} + 2\int_{\alpha}^{\mu} \frac{1}{(\eta-\gamma)^2}\sqrt{G(\eta;\mu)}\, d\eta \,,$$

where

$$G(\eta;\mu) \equiv \int_{\eta}^{\mu} f(s)(\gamma - s)\, ds = 2(\mu - \eta) - (2\gamma + 3\alpha)\log\left(\frac{\mu}{\eta}\right) + 3\alpha\gamma\left(\frac{1}{\eta} - \frac{1}{\mu}\right) \,.$$

In addition, the unknown constant $\tilde{v}_0$ for the inner solution is

$$\tilde{v}_0 = \frac{\pi^2}{4D}\left[G(\alpha;\mu)\right]^{-1} \,.$$

Analysis of Peak-Insertion Behavior: IV

Key Monotonicity Properties: $G_\mu(\eta;\mu) > 0$ and $\chi'(\mu) > 0$.

Upshot: As $\mu = a_0(l)$ increases on $\alpha < \mu < 3\alpha/2$, then $l/\sqrt{D}$ increases.

Main Result: For l fixed, we require $D > D_{\text{min}}$, where

$$D_{\text{min}} = 2l^2 / \left[\chi(3\alpha/2)\right]^2 .$$

As $D \to D_{\text{min}}^+$, then $a_{0x}(l) \to +\infty$, and we predict peak insertion.

Corollary: For D fixed, we require $l < l_{\text{max}}$, where

$$l_{\text{max}} = \chi\left(3\alpha/2\right)\sqrt{D/2} .$$

As $l \to l_{\text{max}}^-$, then $a_{0x}(l) \to +\infty$, and we predict peak insertion.

Comparison: The analytical theory gives $D_{\text{min}} \approx 0.445$ for $l = 1/2$. Full numerics with AUTO gives $D_{\text{min}} \approx 0.27$ for $\varepsilon = 0.02$ and $D_{\text{min}} \approx 0.44$ for $\varepsilon = 0.0027$.

Remark The error in the outer approximation is $O(-\varepsilon \log \varepsilon)$.

Analysis of Peak-Insertion Behavior: V

Goal: perform a local analysis near $x = l$ when $D \approx D_{\text{min}}$ in order to describe the nucleation of the new peak.

- **Q1:** Can we analytically uncover solution multiplicity arising from a saddle-node bifurcation near D_{min}?
- **Q2:** If so, is there some normal form-type equation that can be rigorously analyzed describing the local behavior of solutions near the saddle-nose transition?

Local Analysis: Define the constants A_c, V_c, β, σ, and ζ by

$$A_c \equiv \frac{3\alpha}{2}\,, \qquad V_c \equiv \mathcal{F}(A_c)\,, \qquad \beta \equiv \frac{(\gamma - 3\alpha/2)}{2D_{\text{min}}A_c^2} > 0\,,$$

$$\sigma = \left(\frac{-2}{A_c^2 \mathcal{F}''(A_c)\beta} \right)^{1/6}, \quad \zeta = A_c^2 \beta \left(\frac{-2}{A_c^2 \mathcal{F}''(A_c)\beta} \right)^{2/3},$$

where $\mathcal{F}(A) \equiv (A - \alpha)/A^3$ with $\mathcal{F}''(A_c) < 0$.

Analysis of Peak-Insertion Behavior: VI

Main Result: *Then, near the endpoint $x = l$, we obtain the local approximation*

$$A \sim A_c - \varepsilon^{2/3}\zeta\, U(y)\,, \quad V \sim V_c - \varepsilon^{4/3}\beta\sigma^2\left(A^* + y^2\right)\,,$$

where $y = (l - x)/(\varepsilon^{2/3}\sigma)$. The function $U(y)$ on $y \geq 0$ satisfies the normal form non-autonomous ODE

$$U_{yy} = U^2 - A^* - y^2\,, \quad y \geq 0\,; \quad U_y(0) = 0\,; \quad U^{'} \to +1\,,\ y \to +\infty\,.$$

Remark: If $U(0) < 0$, then $A > A_c = 3\alpha/2$.

Main Result: For $A^* \gg 1$, *there are two solutions $U^{\pm}(y)$ with $U' > 0$ for $y > 0$ given asymptotically by*

$$U^{+} \sim \sqrt{A^* + y^2}\,, \qquad U^{+}(0) \sim \sqrt{A^*}\,,$$

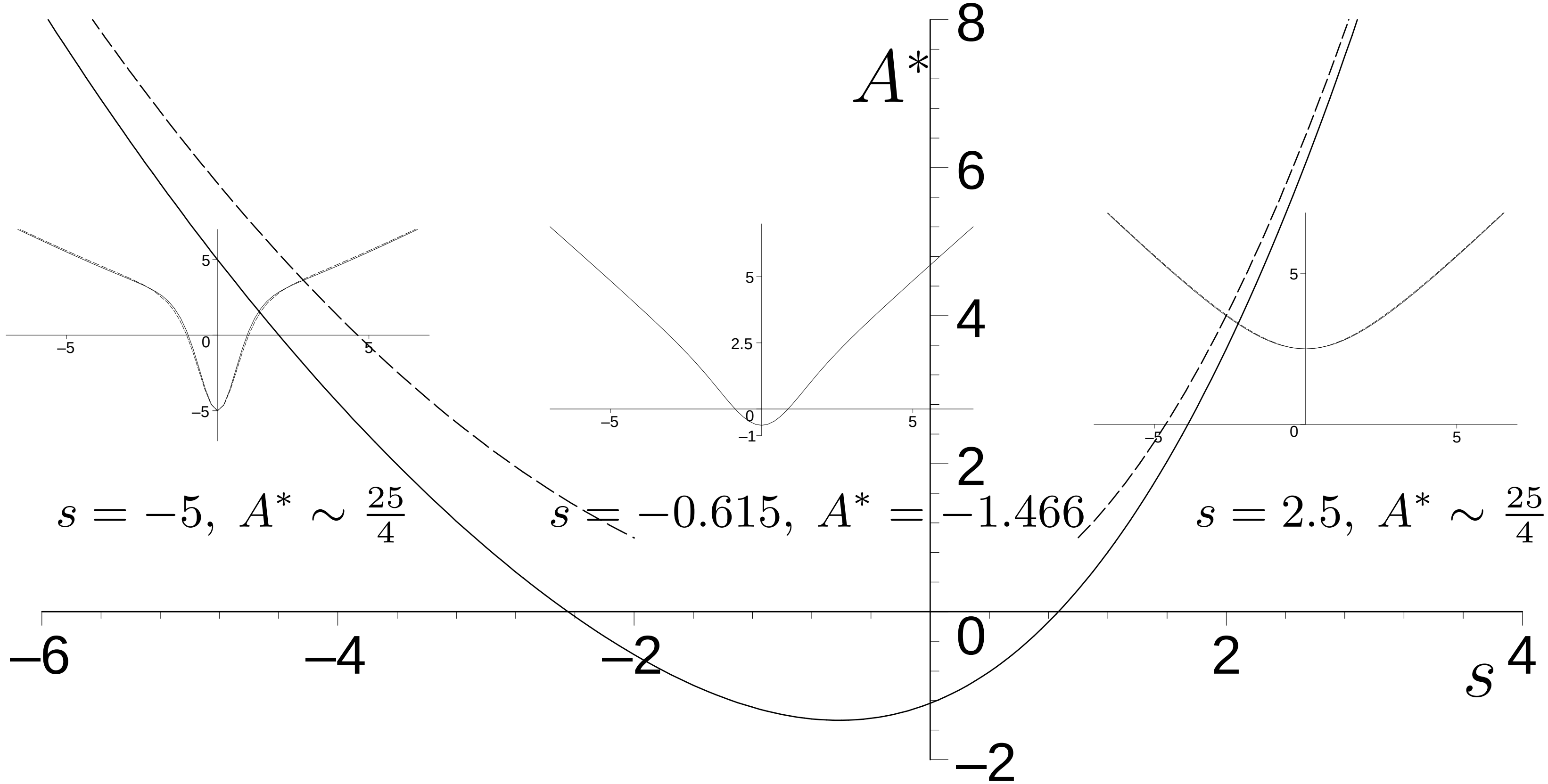
$$U^{-} \sim \sqrt{A^* + y^2}\left(1 - 3\ \mathrm{sech}^2\left(\frac{\sqrt{A^*}y}{\sqrt{2}}\right)\right)\,, \qquad U^{-}(0) \sim -2\sqrt{A^*}\,.$$

Rigorous: these solutions are connected via a saddle-node bifurcation.

Analysis of Peak-Insertion Behavior: VII

Remark 2: The same normal form ODE was derived for the analysis of self-replicating mesa patterns in reaction-diffusion systems: see KWW, Physica D **236**(2), (2007), pp. 104-122.

Plot of A^* versus $s \equiv U(0)$:



Finite-Dimensional Dynamics: $D = O(1)$: I

On the basic cell, $|x| \leq l$, one can construct a quasi-equilibrium one-spike solution centered at $x = x_0$, where $x_0 = x_0(\varepsilon^2 t)$ moves slowly in time.

Main Result: *Provided that no peak-insertion effects occur, then for $\varepsilon \to 0$ the dynamics on the slow time-scale $\sigma = \varepsilon^2 t$ is characterized by*

$$A \sim \varepsilon^{-1} w\,(y) \,/\, \sqrt{\tilde{v}_0}\,, \quad y = \varepsilon^{-1}(x - x_0(\sigma))\,,$$

$$\frac{dx_0}{d\sigma} \sim \frac{3}{8\alpha}\mathcal{F}(x_0)\,, \qquad \mathcal{F}(x_0) \equiv \left[a_{0x}(x_0^+) + a_{0x}(x_0^-)\right]\,.$$

Here $a_0(x)$ is the solution to the multi-point BVP

$$D\left[f(a_0)a_{0x}\right]_x = a_0 - \gamma\,, \quad 0 < |x| < l\,; \quad a_{0x}(\pm l) = 0\,, \quad a_0(0) = \alpha\,,$$

where $f(a_0) \equiv a_0^{-2}(3\alpha - 2a_0)$.

Remark 1: The derivation of this is delicate in that one must resolve a corner layer or knee region for V that allows for matching between the inner and outer approximations for V.

Finite-Dimensional Dynamics $D = O(1)$: II

From an integration of the BVP for a_0, one gets explicit dynamics:

$$\frac{dx_0}{d\sigma} = \frac{3}{8}\left(\frac{2}{D}\right)\left[\sqrt{G(\alpha;\mu_r)} - \sqrt{G(\alpha;\mu_l)}\right] \,,$$

where $\mu_r \equiv a_0(l)$ and $\mu_l \equiv a_0(-l)$ are determined implicitly by

$$\sqrt{\frac{2}{D}}\,(l - x_0) = \chi(\mu_r)\,, \qquad \sqrt{\frac{2}{D}}\,(l + x_0) = \chi(\mu_l)\,.$$

Qualitative I: The dynamics is repulsive: If $x_0(0) > 0$, then $x_0 \to 0$ as $t \to \infty$. By using reflection through the Neumann B.C., two adjacent hot-spots will repel.

Qualitative II: Since the hot-spot dynamics is repulsive, then dynamic peak-insertion events due to large inter-hot-spot separations are unlikely.

Remark: Peak insertion events in the presence of mutually attracting localized pulses, leads to spatial temporal chaos in a Keller-Segel model with logistic growth in 1-D (Painter and Hillen, Physica D 2011).

The Effect of Police: 3-Component Systems

In 1-D, an RD system incorporating police is $U = U(x,t)$:

$$A_t = \varepsilon^2 \Delta A - A + PA + \alpha\,, \quad x \in \Omega\,,$$

$$P_t = D\nabla \cdot \left(\nabla P - \frac{2P}{A} \nabla A \right) - PA + \gamma - \alpha - f\,, \quad x \in \Omega\,,$$

$$\tau_u U_t = D\nabla \cdot \left(\nabla U - \frac{qU}{A} \nabla A \right)\,, \quad x \in \Omega\,,$$

with $\partial_n (A, P, U) = 0$ on $\partial\Omega$. **Police are conserved;** $U_0 = \int_\Omega U\, dx > 0$ **for all** t**.**

- **Police Model I:** $f = U$ (**simple interaction**) L. Ricketson (UCLA).
- **Police Model II:** $f = UP$ (**standard "predator-prey" type interaction**)

Remark: The police drift velocity is $\mathcal{V} = \frac{d}{dx} \ln(A^q)$.

- If $q = 2$, police drift exhibits mimicry (L. Ricketson, UCLA) Referred to as "Cops on the Dots" (Jones, Brantingham, L. Chayes, M3As, 2011). Can be derived from an agent-based model.
- If $q > 2$, police focus more on attractive sites than do criminals.
- If $0 < q < 2$, the police are less focused, and more "diffusive".

The Effect of Police: Qualitative

<u>Question I:</u> Optimal Police Strategy: For a given U_0 find the optimal q (parameter in police drift velocity) that minimizes the NLEP stability threshold of D with $D = D_0/\varepsilon^2$. In this way, we maximize over q the distance between stable localized hot-spots.

- Investigate this question for **both** Police models I and II.
- Is the optimal strategy the same for both models?

<u>Question II:</u> For τ_u sufficiently large on the regime $D = O(\varepsilon^{-2})$, corresponding to police diffusivity D/τ_u, can a two-hot spot solution undergo a Hopf bifurcation leading to asynchronous temporal oscillations in the hot-spot amplitudes?

- If $\tau_u > 1$, then police 'diffuse" more slowly than do criminals.
- Typically, only synchronous oscillatory instabilities of the spike amplitudes occur in RD systems (GM, Gray-Scott, etc..)

<u>Question III:</u> Open: For $D = O(1)$ can police prevent or limit the nucleation of new hot-spots of criminal activity arising from peak-insertion?

Police Model I: Simple-Interaction Model

Principal Result (Equilibrium): *On the basic cell $|x| > l$, and with $q > 1$, the leading order asymptotics for A, P, and U, in the hot-spot region near $x = 0$ is*

$$A \sim \frac{1}{\varepsilon\sqrt{v_0}} w\left(x/\varepsilon\right) \,, \quad P \sim \left[w\left(x/\varepsilon\right)\right]^2 \,, \quad U \sim \frac{U_0}{\varepsilon b}\left[w\left(x/\varepsilon\right)\right]^q \,.$$

Here $b \equiv \int w^q \, dy$, and $w = w(y) = \sqrt{2}\text{sech}\,(y)$ is the homoclinic of $w'' - w + w^3 = 0$. The amplitude of the hot-spot is determined by v_0, where

$$\frac{1}{\sqrt{v_0}} \int w^3 \, dy = 2l(\gamma - \alpha) - U_0 \,.$$

Remark I: A hot-spot solution ceases to exist if the total number of police satisfies

$$U_0 \geq U_{0c} \equiv 2l(\gamma - \alpha)$$

Remark II: For $U_0 < U_{0c}$, for a K-spot equilibrium on a domain of length S, let $l = S/(2K)$ and replace $U_0 \to U_0/K$. Then, use a glueing technique to construct the multi-pulse pattern.

Police Model I: Stability I

For $q > 1$, a Floquet-based analysis leads to an NLEP with two nonlocal terms:

$$L_0\Phi - 3\chi_{0j} w^3 \frac{\int w^2 \Phi \, dy}{\int w^3 \, dy} - \chi_{1j} w^3 \int w^{q-1} \Phi \, dy = \lambda \Phi \,,$$

where for $j = 1, \ldots, K-1$,

$$\chi_{0j} \equiv \left[1 + v_0^{3/2} D_{j2} / \int w^3 \, dy \right]^{-1} , \qquad \chi_{1j} \equiv \frac{C_q(\lambda)}{\int w^3 \, dy} \chi_{0j} \,.$$

Here v_0, D_{jq}, and $C_q(\lambda)$ are defined by

$$\frac{\int w^3 \, dy}{\sqrt{v_0}} = \frac{S}{K}(\gamma - \alpha) - \frac{U_0}{K} \,, \quad D_{jq} \equiv D_0 \alpha^q \left(\frac{2K}{S} \right) \left(1 - \cos\left(\frac{\pi j}{k} \right) \right)$$

$$C_q(\lambda) \equiv \frac{q \kappa_p D_{jq}}{D_{jq} + \hat{\tau}\lambda} \,, \quad \hat{\tau} \equiv \varepsilon^{q-3} \tau_u \left(\frac{\int w^q \, dy}{v_0^{q/2}} \right) \,, \quad \kappa_p \equiv \frac{U_0 \sqrt{v_0}}{K \int w^q \, dy} \,.$$

Remark 1: For $q = 3$ it is an exactly solvable NLEP.
Remark 2: By using $L_0(w^2) = 3w^2$, the NLEP can be transformed into one with a single nonlocal term.

Police Model I: Stability IV

Main Result: *For $q > 1$, the NLEP governing the stability of a K-hot-spot pattern is for $j = 1, \ldots, K-1$:*

$$L_0 \Phi - \frac{1}{\mathcal{C}_j(\lambda)} w^3 \frac{\int w^{q-1} \Phi \, dy}{\int w^q \, dy} = \lambda \Phi \,,$$

$$\mathcal{C}_j(\lambda) = \frac{1}{\chi_{1j} \int w^q \, dy} \left[1 - \frac{9 \chi_{0j}}{2(3-\lambda)} \right] \,.$$

The discrete eigenvalues are the roots of $g_j(\lambda) = 0$, where

$$g_j(\lambda) \equiv \mathcal{C}_j(\lambda) - \mathcal{F}(\lambda) \,, \qquad \mathcal{F}(\lambda) \equiv \frac{\int w^{q-1} \left(L_0 - \lambda \right)^{-1} w^3 \, dy}{\int w^q \, dy} \,.$$

Rigorous: If $\mathcal{C}_j(0) > 1/2$, then there exists an unstable real eigenvalue on $0 < \lambda < 3$. Setting $\mathcal{C}_j(0) = 0$, then gives

$$D_{j2} = \frac{1}{2} \left(1 + \frac{q U_0 \sqrt{v_0}}{K \int w^3 \, dy} \right) \frac{\int w^3 \, dy}{v_0^{3/2}} \,.$$

Police Model I: Stability V

Main Result: *For $q > 1$, a K-hot-spot equilibrium on an interval of length S is unstable on an $O(1)$ time-scale for any $\tau_u > 0$ when $D > D_K^L$, where*

$$D_K^L \equiv \frac{S^4}{8\varepsilon^2\pi^2\alpha^2K^4\left(1+\cos\left(\frac{\pi}{K}\right)\right)}\omega^3\left(1+\frac{qU_0}{\omega}\right).$$

Here ω is defined by

$$\omega = S\left(\gamma-\alpha\right) - U_0\,.$$

By a separate analysis involving the construction of asymmetric patterns, the stability threshold with respect to the $o(1)$ "small eigenvalues" is

$$D_K^S \equiv \frac{S^4}{16\varepsilon^2\pi^2\alpha^2K^4}\omega^3\left(1+\frac{qU_0}{\omega}\right) < D_K^L\,.$$

Notice that D_K^S is monotonically increasing in q. But, the optimal police strategy is one that minimizes the stability threshold.

Qualitative Result: For a fixed U_0, it is not optimal for the police to be overly focussed on drifting towards hot spots (observed numerically by L. Ricketson).

Police Model I: A Hopf Bifurcation I

Consider two hot-spots, and let $q = 3$ so that the NLEP is solvable. Does there exist a Hopf bifurcation when $D < D_K^L$ whereby the amplitude of the two hot-spots oscillate asynchronously on an $O(1)$ time-scale?

For $q = 3$ and two-hot-spots, the function $\mathcal{F}(\lambda)$ in the NLEP problem

$$g_1(\lambda) \equiv \mathcal{C}_j(\lambda) - \mathcal{F}(\lambda) \,, \qquad \mathcal{F}(\lambda) \equiv \frac{\int w^{q-1} \left(L_0 - \lambda\right)^{-1} w^3 \, dy}{\int w^q \, dy} \,,$$

is $\mathcal{F}(\lambda) = 3/\left[2(3-\lambda)\right]$, and so λ is a root of

$$\lambda = a + \frac{b}{1 + \hat{\tau}\lambda} \,, \quad a \equiv 3\left(1 - \frac{3\chi_{01}}{2}\right) \,, \quad b \equiv -\frac{9\chi_{01}\kappa_p}{2} \,.$$

Recall:

$$\hat{\tau} \equiv \varepsilon^{q-3}\tau_u \left(\frac{\int w^q \, dy}{v_0^{q/2}}\right) \,, \qquad \kappa_p \equiv \frac{U_0\sqrt{v_0}}{K\int w^q \, dy} \,.$$

(simply set $q = 3$ and $K = 2$).

Police Model I: A Hopf Bifurcation II

Main Result: *For $q = 3$ and $K = 2$, there exists a Hopf Bifurcation at $\tau_u = \tau_{uH}$ with $\lambda = \pm i\lambda_I$, when $0 < D_K^H < D < D_K^L$. We have,*

$$\tau_{uH} = \frac{v_0^{3/2} D_{13}}{a \int w^3 \, dy}, \qquad \lambda_I = \sqrt{-a(a+b)}\,.$$

There is an explicit formula for D_K^H. No Hopf bif. if $D < D_K^H$.

Qualitative: If $\tau_u > \tau_{uH}$ on $0 < D_K^H < D < D_K^L$, we predict asychronous oscillatory instability of the two spots. Numerically, the bifurcation looks supercritical. If $D < D_K^H$, then we have stability for all τ_u.

Interpretation: There is an intermediate range of police diffusivity (slower than criminals) for which the two hot-spots exhibit asynchronous oscillations. Maximum criminal activity is displaced periodically in time from one hot-spot to its neighbour.

Remark: Similar behavior for is expected for $q \neq 3$, but is more difficult to analyze. (no explicit analytical formulas available).

Police Model II: Predator-Prey Interaction

Principal Result (Equilibrium): *On the basic cell $|x| > l$, and with $q > 1$, the leading order asymptotics for A, P, and U, in the hot-spot region near $x = 0$ is*

$$A \sim \frac{1}{\varepsilon\sqrt{v_0}} w\left(x/\varepsilon\right) , \quad P \sim \left[w\left(x/\varepsilon\right)\right]^2 , \quad U \sim \frac{U_0}{\varepsilon b}\left[w\left(x/\varepsilon\right)\right]^q .$$

Here $b \equiv \int w^q \, dy$, and $w = w(y) = \sqrt{2}\text{sech}\,(y)$ is the homoclinic of $w'' - w + w^3 = 0$. The amplitude of the hot-spot is determined by v_0, where

$$\frac{1}{\sqrt{v_0}} \int w^3 \, dy = 2l(\gamma - \alpha) - U_0 \frac{\int w^{2+q} \, dy}{\int w^q \, dy} , \qquad \frac{\int w^{2+q} \, dy}{\int w^q \, dy} = \frac{2q}{q+1} .$$

Remark I: A hot-spot solution ceases to exist if

$$U_0 \geq U_{0c} \equiv (q+1)l(\gamma - \alpha)/q.$$

Remark II: For $U_0 < U_{0c}$, for a K-spot equilibrium on a domain of length S, let $l = S/(2K)$ and replace $U_0 \to U_0/K$. Then, use a glueing technique to construct the multi-pulse pattern.

Police Model II: Stability I

Let $q > 1$, and $U_0 < U_{0c}$ so that a K-hot-spot equilibrium exists. Then:

Small Eigenvalue Threshold: Asymmetric hot-spot equilibria bifurcate from the symmetric K-hot-spot branch at the threshold

$$D_{\mathsf{K}}^{S} \equiv \frac{\omega^3 S}{16\varepsilon^2\pi^2\alpha^2 K^4}\left(1 + \frac{2q^2 U_0}{(q+1)\omega}\right), \qquad \omega \equiv S(\gamma - \alpha) - \frac{2U_0 q}{q+1} > 0 \,.$$

(Note that the amplitude of the hot-spot is proportional to ω.)

NLEP Analysis: The NLEP now involves 3 separate nonlocal terms. When $\tau_u \ll O(\varepsilon^{q-3})$, the stability threshold for this NLEP is

$$D_{\mathsf{K}}^{L} = D_{\mathsf{K}}^{S}\left(\frac{2}{1+\cos\left(\pi/K\right)}\right) > D_{\mathsf{K}}^{S} \,.$$

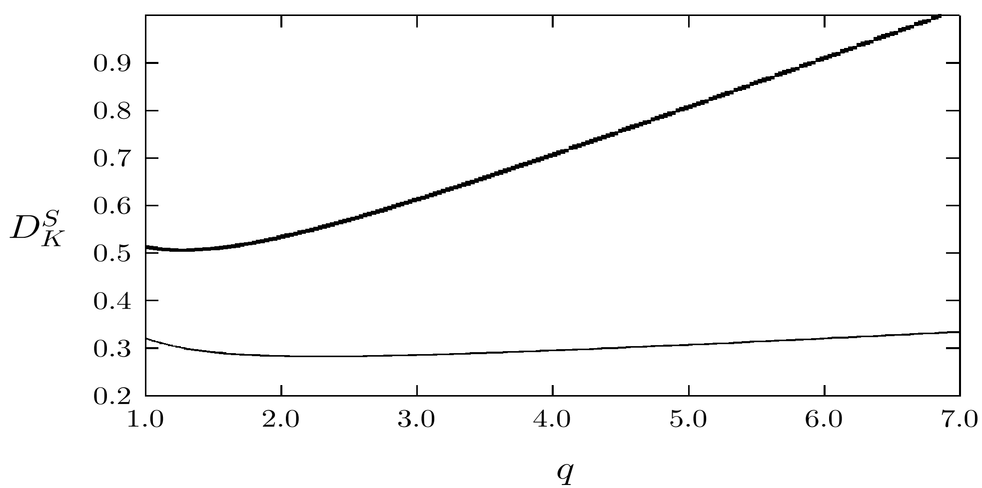
By simple calculus we study D_{K}^{S} as a function of q for U_0 fixed with $U_0 < U_{0c}$. Optimal strategy is to minimize the threshold.

Police Model II: Stability II

Key Qualitative Features (Optimal Strategy):

- For $q \to \infty$, then $D_{\mathsf{K}}^{S} \to \infty$, which gives a poor police strategy. Overly focused aggressive police can lead (paradoxically) to rather closely spaced stable hot-spots.
- D_{K}^{S} has a minimum wrt q at some interior point in $1 < q < \infty$. The minimum value can be rather small only if U_0 is relatively large. Thus, only if there is enough police should they really focus their effort towards hot-spots.

Example: $S = 1$**,** $\gamma = 2$**,** $\alpha = 1$**,** $K = 2$**,** $\varepsilon = 0.02$ **Top Curve:** $U_0 = 0.28$**, Bottom Curve** $U_0 = 0.44$**: Note:** $(U_{0c} = 0.5)$**.**



Further Directions

- For the basic urban crime model in 2-D with $D = O(1)$:
  - Investigate effect of spatial heterogeneity of γ, α.
  - Derive the scalar nonlinear elliptic PDE, associated with a saddle-node point, governing peak insertion.
  - **Dynamics in 1-D and 2-D:** Derive ODE's for the locations of centers of a collection of hot-spots (repulsive interactions?).
  - Are dynamic peak-insertion events for a collection of hot-spots possible? If hot-spots become too closely spaced, we anticipate annihilation. Can annihilation and creation events lead to spatial-temporal chaos?
- Analyze other models of the effect of police, incorporating dynamic deterrence, i.e. A. Pilcher, EJAM (2010).

References:

T. Kolokolnikov, M.J. Ward, J. Wei, *The Stability of Steady-State Hot-Spot Patterns for a Reaction-Diffusion Model of Urban Crime*, to appear DCDS-B, (2012), (34 pages).

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