Delayed Reaction Kinetics and The Stability of Spikes

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Outline

For $\epsilon \to 0$ spatially localized solutions can occur for

$$v_t = \epsilon^2 v_{xx} + g(u,v)\,; \quad \tau u_t = u_{xx} + f(u,v)\,, \qquad x \in \mathbb{R}\,.$$

Gene Expression Time Delays: In applications to morphogenesis, one might need to account for gene expression time delays whereby $g(u,v) \to g(u_T, v_T)$ and $f(u,v) \to f(u_T, v_T)$ with $u_T = u(x, t-T)$ and $v_T = v(x, t-T)$, where $T > 0$ is fixed delay.

E. A. Gaffney, N. A. M. Monk, *Gene expression time delays and Turing pattern formation systems*, Bull. Math. Bio., **68**, (2006).
S. Lee, E. A. Gaffney, N. A. M. Monk, *The influence of gene expression time delays on Gierer-Meinhardt...*, Bull. Math. Bio., **72**(8), (2010).

Our Goal: Analyze GM Model and Spike Stability with Delay

- [Iron, Khalil] (pulse stability and dynamics with delay).
- **[FWW1]:** N. Fadai, M. Ward, J. Wei, *Delayed Reaction-Kinetics and the Stability of Spikes for the Gierer-Meinhardt Model*, SIAP (2017).
- **[FWW2]:** N. Fadai, M. Ward, J. Wei, *A Time Delay in the Activator Kinetics........*, under review, DCDS-B, (2017).

1-D GM Infinite-Line Model With Delay I

Consider a 1-spike steady-state solution to the GM RD model for $x \in \mathbb{R}$ with delay:

$$v_t = \epsilon^2 v_{xx} - v + v_T^p / u_T^q \,, \qquad \tau u_t = u_{xx} - u + \epsilon^{-1} v_T^m / u_T^s$$

where $u_T \equiv u(x, t-T)$, $v_T \equiv v(x, t-T)$.

Remarks: Assume $0 < \epsilon^2 \ll 1$, $D = \mathcal{O}(1)$, and $\tau > 0$.

- Assume the usual conditions on (p, q, m, s) that

$$p > 1 \,, \quad q > 0 \,, \quad m > 1 \,, \quad s \geq 0 \,, \quad \xi \equiv \frac{qm}{p-1} - s - 1 > 0 \,.$$

- Prototypical exponent set $(p, q, m, s) = (2, 1, 2, 0)$.
- Explicitly solvable NLEP set $p = 2m - 3$, $m > 2$, where the spectral problem is highly tractable when delay in only inhibitor kinetics.

Goal: Understand the Effect of Delay in Different Terms

- **Case I:** Delay in inhibitor only v^p / u_T^q and v^m / u_T^s.
- **Case II:** Delay in inhibitor and activator v_T^p / u_T^q and v_T^m / u_T^s.
- **Case III:** Delay in activator only v_T^p / u^q and v_T^m / u^s.

Infinite-Line Model: Equilibrium

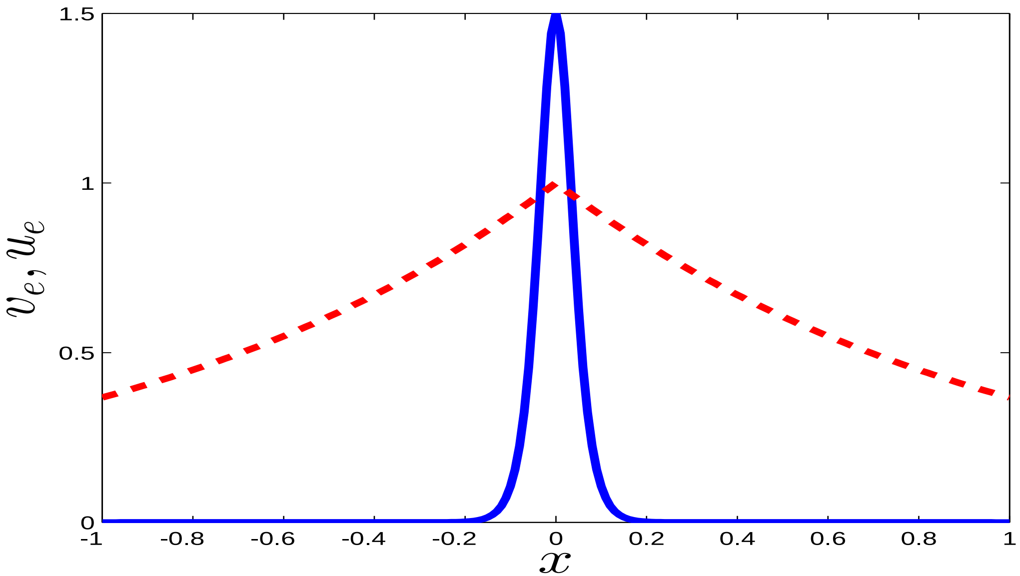
Steady-State [IWW]: *For* $\epsilon \to 0$, *a one-spike steady-state solution is*

$$v_e(x) \sim U_0^{q/(p-1)} w\left(\epsilon^{-1} x\right) ; \quad u_e(x) \sim U_0 \frac{G_0(x)}{G_0(0)} ; \quad U_0 = \left(\frac{1}{2} \int_{-\infty}^{\infty} w^m \, dy \right)^{-1/\xi} ,$$

where $w(y)$ *is the homoclinic satisfying*

$$w'' - w + w^p = 0 \, , \quad w(0) > 0 \, , \quad w'(0) = 0 \, , \quad w \to 0 \quad \text{as} \quad |y| \to \infty \, .$$

Here $G_0(x) = e^{-|x|}/2$ *is the Green's function satisfying* $G_{0xx} - G_0 = -\delta(x)$ *with* $G_0 \to 0$ *as* $|x| \to \infty$.



Infinite-Line Model: Linearization

Linearize: Assume delay in both activator and inhibitor:

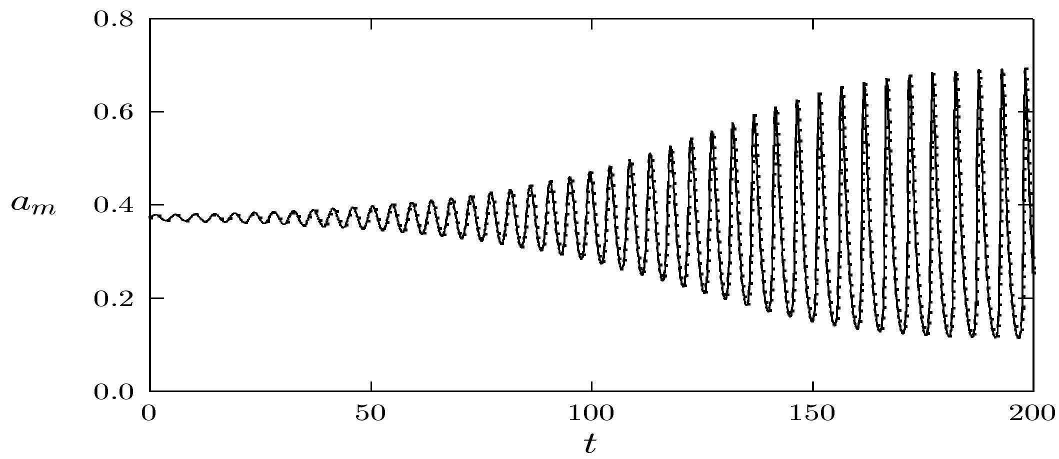
$$v = v_e + e^{\lambda t}\Phi(x/\epsilon)\,, \qquad u = u_e + e^{\lambda t}\eta\,.$$

Case II: We get the NLEP spectral problem for $\Phi(y)$ for $\Phi \in H^1(\mathbb{R})$:

$$L_\mu \Phi - \frac{mq\mu^2}{\sqrt{1+\tau\lambda} + s\mu} w^p \frac{\int_{-\infty}^{\infty} w^{m-1}\Phi\, dy}{\int_{-\infty}^{\infty} w^m\, dy} = \lambda\Phi\,, \quad -\infty < y < \infty\,,$$

$$L_\mu \Phi \equiv \Phi'' - \Phi + p w^{p-1}\mu\Phi\,, \quad \mu \equiv e^{-\lambda T}\,.$$

We refer to L_μ as the delayed local operator and L_1 as the local operator which corresponds to setting $T = 0$.
The NLEP with L_1 and $\mu = 1$ has been extensively studied (**[IWW2001], [WW2003]**) and a Hopf bifurcation can occur when τ exceeds a threshold.



Case I: Inhibitor Delay Only

Case I (Delay in Inhibitor Only): Suppose $v_T = v$ in reaction kinetics. The NLEP problem for $y \in \mathbb{R}$ with $\Phi \in H^1(\mathbb{R})$ is (with $\mu = e^{-\lambda T}$):

$$L_1 \Phi - \chi w^p \frac{\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} = \lambda \Phi \, , \qquad \chi \equiv \frac{mq\mu}{\sqrt{1+\tau\lambda} + s e^{-\lambda T}} \, .$$

Question: Under what conditions is R$e(\lambda) > 0$ in the τ versus T plane due to a Hopf Bifurcation crossing? Easiest: "Explicitly Solvable One".

Lemma [NW], [FWW] *Set $p = 2m - 3$ and $m > 2$. Then, for $\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy \neq 0$, any eigenvalues in Re$(\lambda) > 0$ must satisfy*

$$\lambda = (m^2 - 2m) - \frac{m}{2} \chi \, .$$

The other eigenpairs are simply $(\Phi, \lambda) = (w', 0)$ and any negative real eigenvalues of L_1.

Proof: *Uses the key identity $L_1 \left(w^{m-1} \right) = (m^2 - 2m)(w^{m-1})$ together with Green's second identity.*

Case I: Inhibitor Delay Only (Explicit)

When there is *no delay* $T = 0$ we can have a HB.

Main Result: [NW] *When $T = 0$, $\exists$ a unique Hopf bifurcation value τ_H^0, with corresponding eigenvalue $\lambda = i\lambda_{IH}^0$, given by*

$$\tau_H^0 = \frac{(m^2 q)^2}{2\zeta^2}\left(\beta - \frac{2s}{m^2 q}\right), \qquad \lambda_{IH}^0 = \sqrt{\zeta - \beta^2},$$

where $\zeta > \beta^2$ with $\beta \equiv m^2 - 2m$ is the smallest root of the quadratic

$$4(s^2-1)\zeta^2 - \left[(m^2 q)^2 + 4\beta s(m^2 q)\right]\zeta + 2\beta^2 (m^2 q)^2 = 0\,.$$

This result yields the following special cases:

- For $(p, q, m, s) = (3, 2, 3, 1)$, then $\tau_H^0 = 2.5$ and $\lambda_{IH}^0 = 3/\sqrt{5} \approx 1.34$.
- For $(p, q, m, s) = (3, 2, 3, 0)$, then $\tau_H^0 = [13 + 3\sqrt{17}]/12 \approx 2.114$ and $\lambda_{IH}^0 \approx 3\sqrt{3\sqrt{17} - 11}/\sqrt{2} \approx 2.482$.

[NW], Y. Nec, MJW, *An Explicitly Solvable NLEP...* [2013].

Case I: Inhibitor Delay Only (Explicit)

Question: Is there a HB value of T even when $\tau = 0$? If so, problem is more unstable with delay.

Main Result: [FWW] *When $\tau = 0$, then only if $0 < s < 1$, $\exists$ a minimum HB threshold $T = T^{f} > 0$ and corresponding eigenvalue λ_{IH}^{f} given by*

$$T^{\mathsf{f}} \equiv \frac{1}{\lambda_{IH}^{\mathsf{f}}} \sin^{-1}\left((1-s^2)^{1/2} \frac{\left[(\xi+1)^2 - 1\right]^{1/2}}{(\xi+1)-s} \right),$$

$$\lambda_{IH}^{\mathsf{f}} \equiv \frac{\beta}{\sqrt{1-s^2}} \left[(\xi+1)^2 - 1\right]^{1/2}.$$

where

$$\beta = m^2 - 2m\,, \quad \xi \equiv \frac{mq}{2m-4} - (1+s)\,.$$

In particular, for $(p, q, m, s) = (3, 2, 3, 0)$, where $\beta = 3$ and $\xi = 2$, we get

$$\lambda_{IH}^{\mathsf{f}} = 6\sqrt{2}\,, \qquad T^{\mathsf{f}} = \frac{1}{6\sqrt{2}} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \approx 0.145\,.$$

Case I: Inhibitor Delay Only (Explicit)

Parameterization: We parameterize the HB threshold in the τ versus T plane to determine the HB boundary in parameter space. Let $\lambda = i\lambda_{IH}$, where $\lambda_{IH} \equiv \beta\omega_0$. We calculate that:

Main Result: [FWW] *In terms of $\lambda_{IH} = \beta\omega_0$ with $\beta = m^2 - 2m$, we have that a HB occurs on the boundary*

$$\tau = \frac{\sqrt{\alpha^2 - 1}}{\beta\omega_0}, \qquad \alpha = \frac{(\xi+1)^2 + s^2\omega_0^2}{1+\omega_0^2}.$$

$$T = \frac{1}{\beta\omega_0}\sin^{-1}\left[\frac{B_+\omega_0(\xi+s+1) + B_-(s\omega_0^2 - \xi - 1)}{\alpha(1+\omega_0^2)}\right],$$

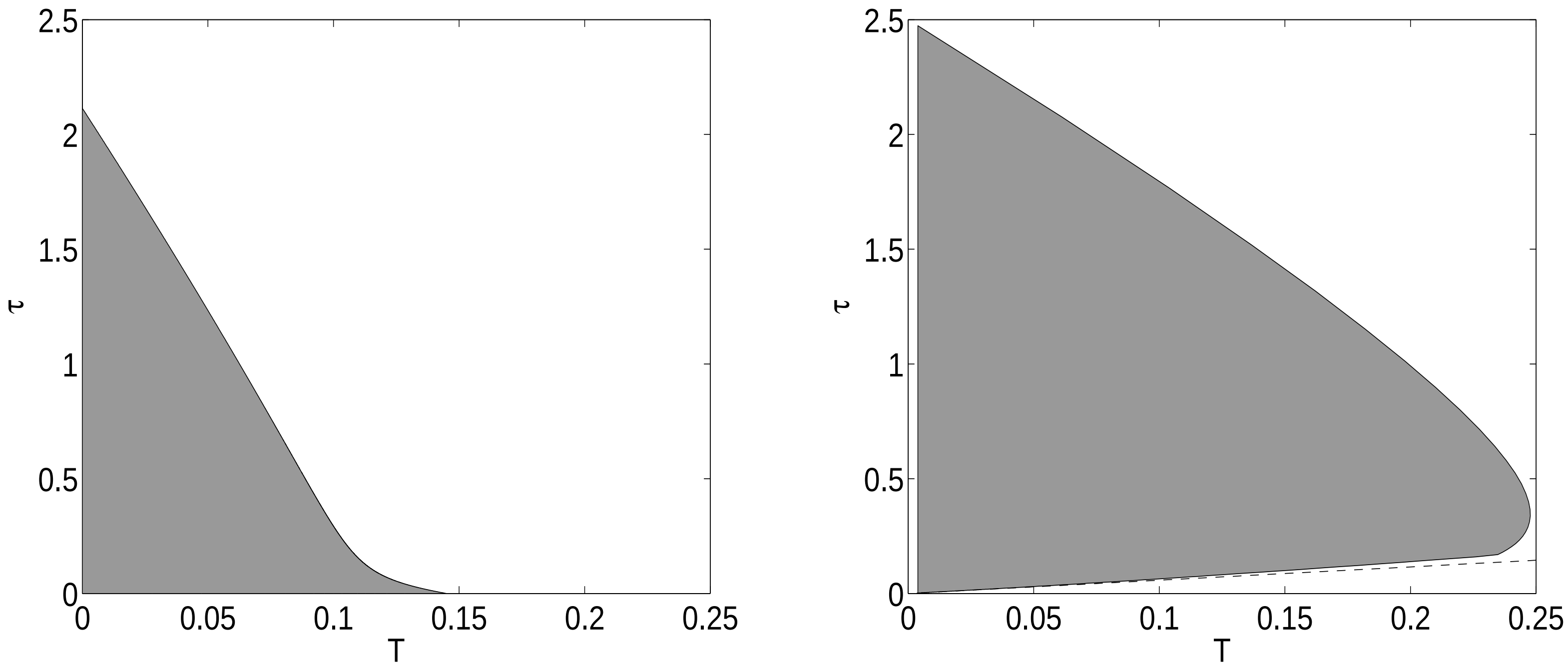
where

$$B_\pm \equiv \sqrt{\frac{\alpha \pm 1}{2}}, \qquad \alpha \equiv \sqrt{1 + \tau^2\beta^2\omega_0^2\lambda_{IH}^2}.$$

For $s > 1$, we can show that $T \to 0$ as $\tau \to 0$, with asymptotics:

$$T \sim \frac{\tau}{\sqrt{s^4-1}}\sin^{-1}\left(\frac{\sqrt{s^2-1}}{\sqrt{2}s}\right), \qquad \lambda_{IH} \sim \frac{\sqrt{s^4-1}}{\tau}, \qquad s > 1,$$

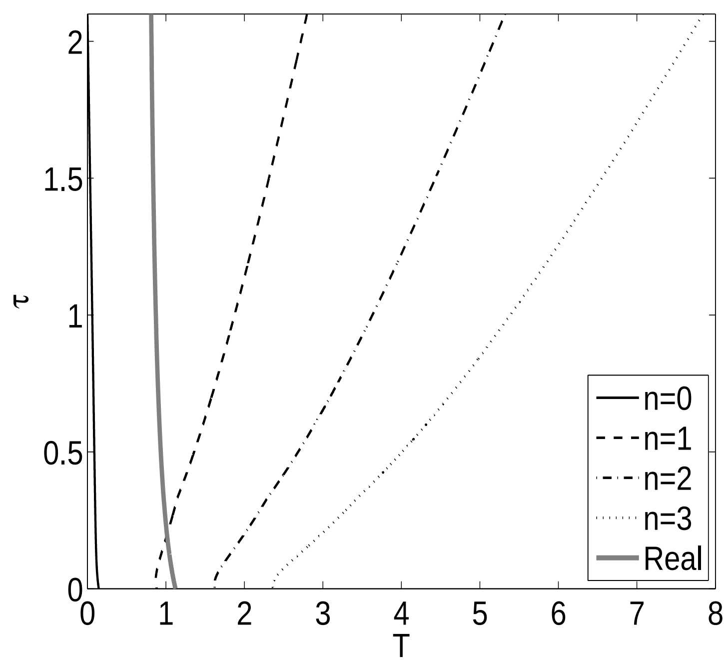
Case I: Inhibitor Delay Only (Explicit)

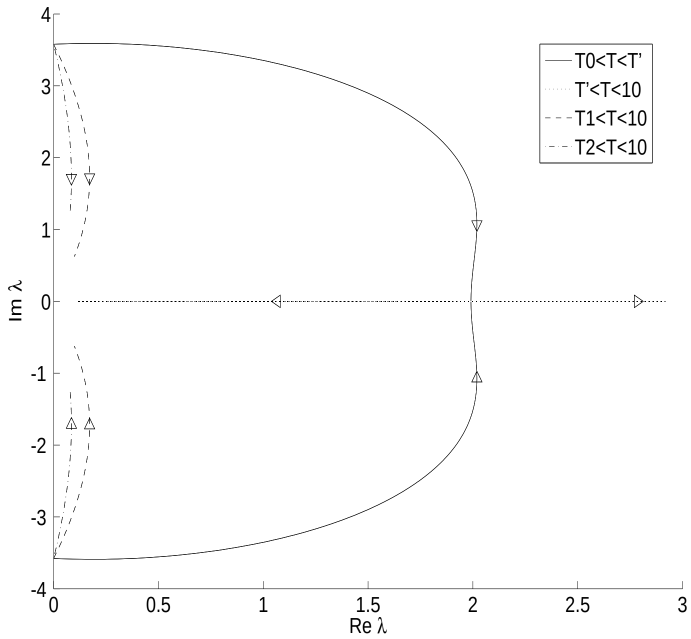


Caption: The shaded region in the τ versus T plane is linearly stable. A HB occurs on the boundary. Left: $(p, q, m, s) = (3, 2, 3, 0)$. Right: $(p, q, m, s) = (3, 2, 3, 1)$. The dashed line is the limiting approximation.

Conclusion (Explicit Case): With only inhibitor delay, the spike is more unstable than with no delay.

Case I: Inhibitor Delay Only (Explicit)



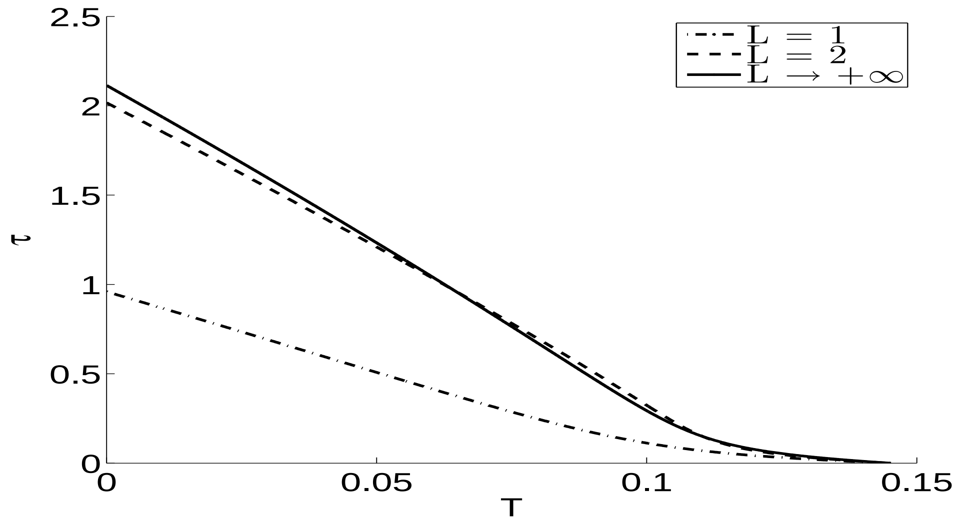


Caption: $(p, q, m, s) = (3, 2, 3, 0)$. **Left:** HB curves (dashed curves) where additional complex conjugate pairs, indexed by n, enter the region Re$(\lambda) > 0$. Thick line: when eigenvalues first appear as a double root on the positive real axis. **Right:** Eigenvalue paths for $\tau = 1$. Heavy solid curve gives the minimum threshold.

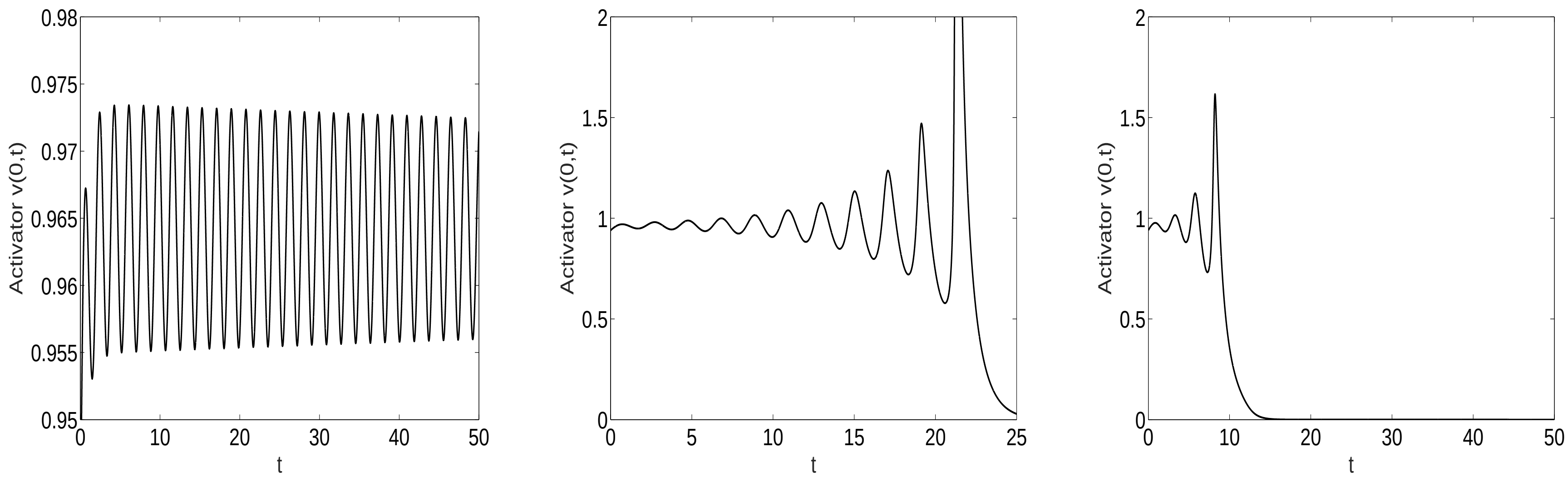
Eigenvalues near $\lambda = 0$: For $(p, q, m, s) = (3, 2, 3, 0)$

$$\lambda \sim \frac{[\ln 3 + 2n\pi i]}{T} \left(1 + \frac{1}{T}\left[\frac{1}{3} - \frac{\tau}{2}\right] + + \cdots\right).$$

Case I: Inhibitor Delay (Validation)



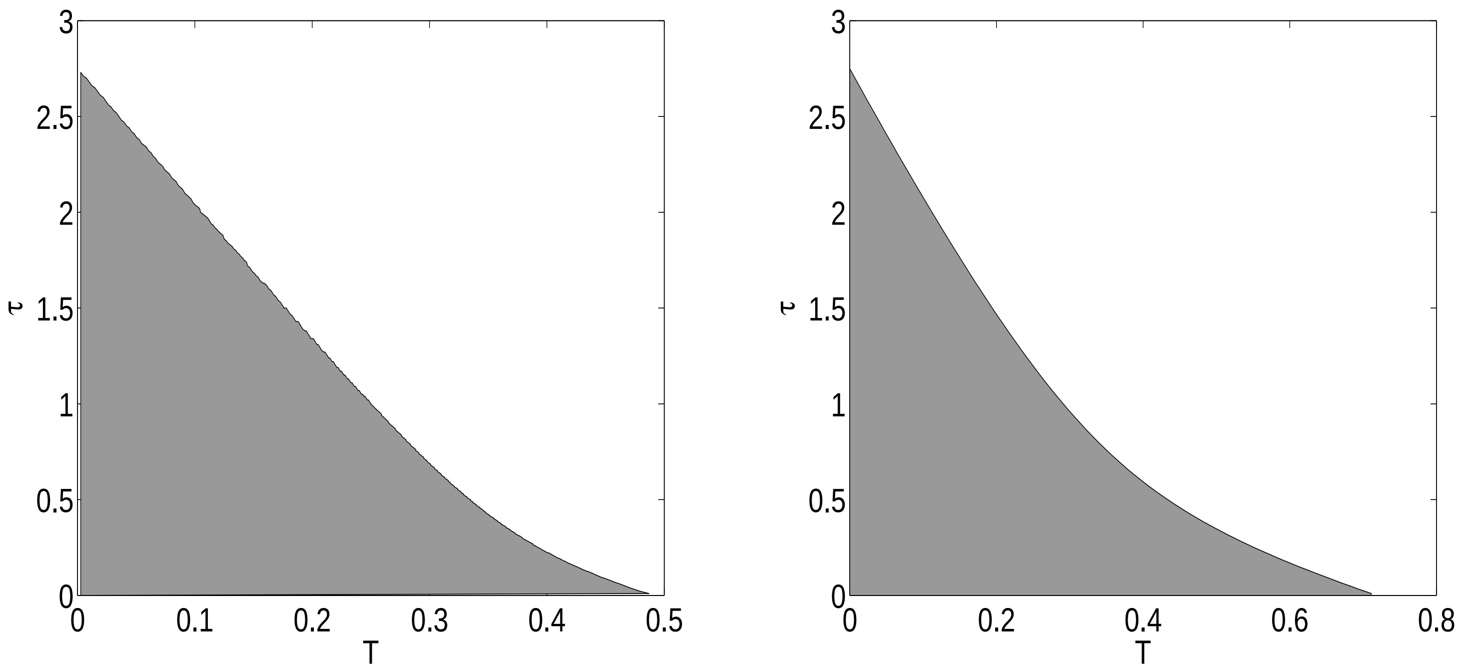
Numerics: HB threshold τ_H vs T on $|x| \leq L$ for $L = 1$, $L = 2$ and $L = \infty$. Exponent set $(3, 2, 3, 0)$



Caption: $v(0, t)$ vs. t for $L = 2$, $T = 0.05$, $\epsilon = 0.05$, computed by MOL and *dde23* of MATLAB: $\tau = 1.0$ (left), $\tau = 1.3$ (middle), and $\tau = 2.0$ (right). Theory yields $\tau_H \simeq 1.23$

Case I and Case II Comparison

For $(p, q, m, s) = (2, 1, 2, 0)$ compare stability region in the τ versus T plane when only inhibitor delay (Case I) and when both activator and inhibitor delay (Case II).



Caption: Case I (left); Case II (right)

Key Conclusion: The stability regions are qualitatively similar for the two cases. With delayed inhibitor kinetics (regardless of activator), a one-pulse solution is more unstable.

Case III: Activator Delay Only

Consider a 1-spike solution on the infinite line for

$$v_t = \epsilon^2 v_{xx} - v + v_T^2/u\,, \quad \tau u_t = u_{xx} - u + \epsilon^{-1} v_T^2\,,$$

By linearizing, $v = v_e + e^{\lambda t}\Phi(y)$, with $y = x/\epsilon$, we get the NLEP

$$L_\mu \Phi - \chi(\tau\lambda)\mu w^2 \frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w^2\, dy} = \lambda\Phi\,; \quad \chi(\tau\lambda) \equiv \frac{2}{\sqrt{1+\tau\lambda}}\,,$$

with $\Phi \to 0$ as $|y| \to \infty$. The delayed local operator, L_μ, is

$$L_\mu \Phi \equiv \Phi'' - \Phi + 2w\mu\Phi\,, \qquad \mu \equiv e^{-\lambda T}\,,$$

The eigenvalues of the NLEP are the roots of $g(\lambda) = 0$, where

$$g(\lambda) \equiv \frac{1}{\mu\chi(\tau\lambda)} - \mathcal{F}_\mu(\lambda)\,; \qquad \mathcal{F}_\mu(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\left[(L_\mu - \lambda)^{-1} w^2\right]\, dy}{\int_{-\infty}^{\infty} w^2\, dy}\,.$$

Step 1: Need results for delayed local eigenvalue problem (DLEP):

$$L_\mu \Phi = \lambda\Phi\,, \qquad \Phi \in H^1(\mathbb{R})\,.$$

The Delayed Local Eigenvalue Problem I

Main Result:[FWW2] *Any eigenvalue λ of the DLEP must be a root of one of the transcendental equations $\mathcal{K}_l(\lambda)=0$, for $l=0,1,2,\ldots$, defined by*

$$\mathcal{K}_l(\lambda) \equiv 4\sqrt{1+\lambda}+1-\sqrt{1+48\mu}+2l=0\,, \quad l=0,1,2,\ldots\,; \qquad \mu\equiv e^{-\lambda T}\,.$$

The translation mode $\lambda=0$, $\Phi=w'$, must be an eigenpair for all $T\geq 0$. This corresponds to $l=1$. The continuous spectrum is $\lambda<-1$ with λ real.

Proof: With $\mu=e^{-\lambda T}$ and $w(y)=\frac{3}{2}\text{sech}^2\,(y/2)$, $\Phi(y)$ satisfies

$$\Phi''-\Phi+2\mu w\Phi=\lambda\Phi\,, \qquad \Phi\in H^1(\mathbb{R}^1)\,.$$

Put $\gamma\equiv\sqrt{1+\lambda}$ and $\Phi(y)=w^{\gamma}\mathcal{Y}(z)$, with $z=z(y)$ defined $1-1$ from $0<z<1$ to $-\infty<y<\infty$ by $2z=\left(1-\frac{w'}{w}\right)$. This yields

$$z(1-z)\mathcal{Y}''+\left(c-(a+b+1)z\right)\mathcal{Y}'-ab\mathcal{Y}=0\,,$$

where the coefficients a, b, c are

$$a+b+1=4\gamma+2\,, \quad ab=4\gamma(\gamma-1)+6\gamma-12\mu\,, \quad c=1+2\gamma\,.$$

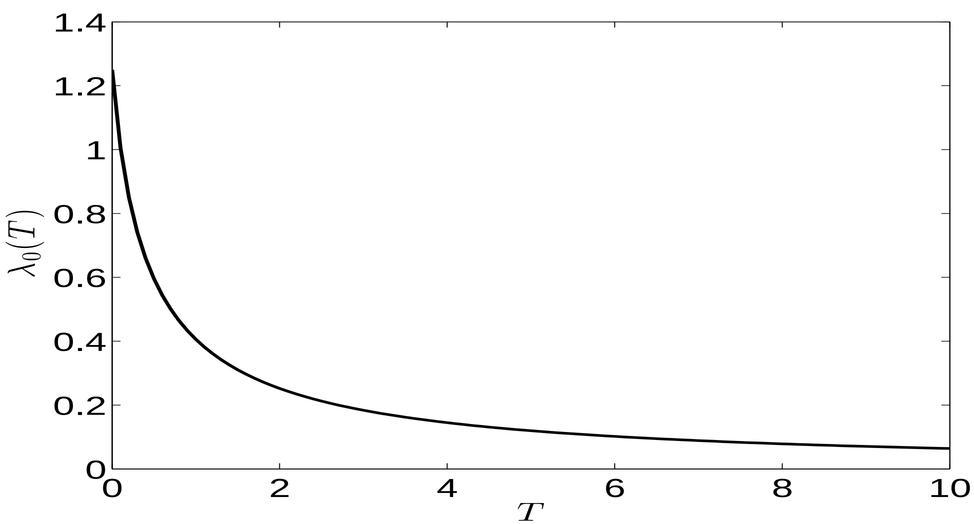
Finally, choose a, b, and c so that $\mathcal{Y}(z)$ is bounded as $z\to 0,1$.

The Delayed Local Eigenvalue Problem II

Real Eigenvalues: First characterize any non-zero real-valued eigenvalue satisfying $\lambda > -1$ that exists for all $T \geq 0$.

- **l=0 branch:** $\mathcal{K}_0(0) < 0$, $\mathcal{K}_0(\lambda) \to +\infty$ as $\lambda \to +\infty$, and $\mathcal{K}_0'(\lambda) < 0$ on $\lambda > 0$. Thus, $\exists$ a unique $\lambda_0 = \lambda_0(T) > 0$ for any $T \geq 0$. We calculate $\lambda_0(0) = 5/4$, $\lambda_0 \sim \log(2)/T$ for $T \gg 1$, and $\lambda_0'(T) > 0$.
- **l=2 branch:** We get $\mathcal{K}_2(-1) < 0$, $\mathcal{K}_2(0) = 2 > 0$, and $\mathcal{K}_2'(\lambda) > 0$. Thus $\exists$ a unique root $\lambda_2(T)$ in $-1 < \lambda_2(T) < 0$ for any $T \geq 0$. We get $\lambda_2(0) = -3/4$, $\lambda_2 \sim \log\left(3/5\right)/T$ for $T \gg 1$.

Plot of $\lambda_0(T)$:



The Delayed Local Eigenvalue Problem III

Question: Can we get edge bifurcations as T increases from $\lambda = -1$ of the continuous spectrum? What about complex-valued spectra?

Edge Bifurcations: For $l \geq 3$, a real eigenvalue bifurcates from the edge of the continuous spectrum $\lambda \leq -1$ when T exceeds $T^l_{\text{edge}} \geq 0$, given by

$$T^l_{\text{edge}} \equiv \log\left(\frac{l^2 + l}{12}\right), \quad l = 3, 4, \ldots ; \quad \text{with} \quad T^{l+1}_{\text{edge}} > T^l_{\text{edge}} .$$

Notice that $T^3_{\text{edge}} = 0$.

Proof: set $\mathcal{K}_l(-1) = 0$ and solve for T.

Since $\mathcal{K}_l(0) > 0$, $\mathcal{K}_l(-1) < 0$ when $T > T^l_{\text{edge}}$, $\exists$ a unique $\lambda_l(T)$ in $-1 < \lambda < 0$ when $T > T^l_{\text{edge}}$. For $T \gg 1$,

$$\lambda_l(T) \sim c_l/T \,, \qquad c_l \equiv \log\left(\frac{12}{(l+3)(l+2)}\right) < 0 \,, \quad \text{as} \quad T \to \infty \,.$$

Key: Edge bifurcations occur, but only produce stable spectra, as they remain in Re$(\lambda) < 0$ for all T.

The Delayed Local Eigenvalue Problem IV

Complex Spectra: Can only occur for $l = 0$ mode. Set $\lambda = i\lambda_I$ for $l = 0$. Such a pure imaginary eigenvalue occurs when $T = T_{\mathsf{H}}^n$, given by

$$T = T_{\mathsf{H}}^n \equiv \frac{(\theta_0 + 2\pi n)}{\lambda_I}\,, \qquad n = 1, 2, 3, \ldots\,.$$

The corresponding eigenvalue λ_I (independent of n) is

$$\lambda_I = \frac{1}{8}\mathrm{Re}\left(-1 + \sqrt{1 + 48\mu}\right)\mathrm{Im}\left(\sqrt{1 + 48\mu}\right)\,, \qquad \mu \equiv \cos\theta_0 - i\sin\theta_0\,.$$

Here θ_0 in $-\pi/3 \leq \theta \leq 0$ is unique root of $\mathcal{H}(\theta) = 0$:

$$\mathcal{H}(\theta) \equiv (24\cos\theta - 7)^2\left(12\cos^2\theta - 8\cos\theta + 1\right) - 12\sin^2\theta\,.$$

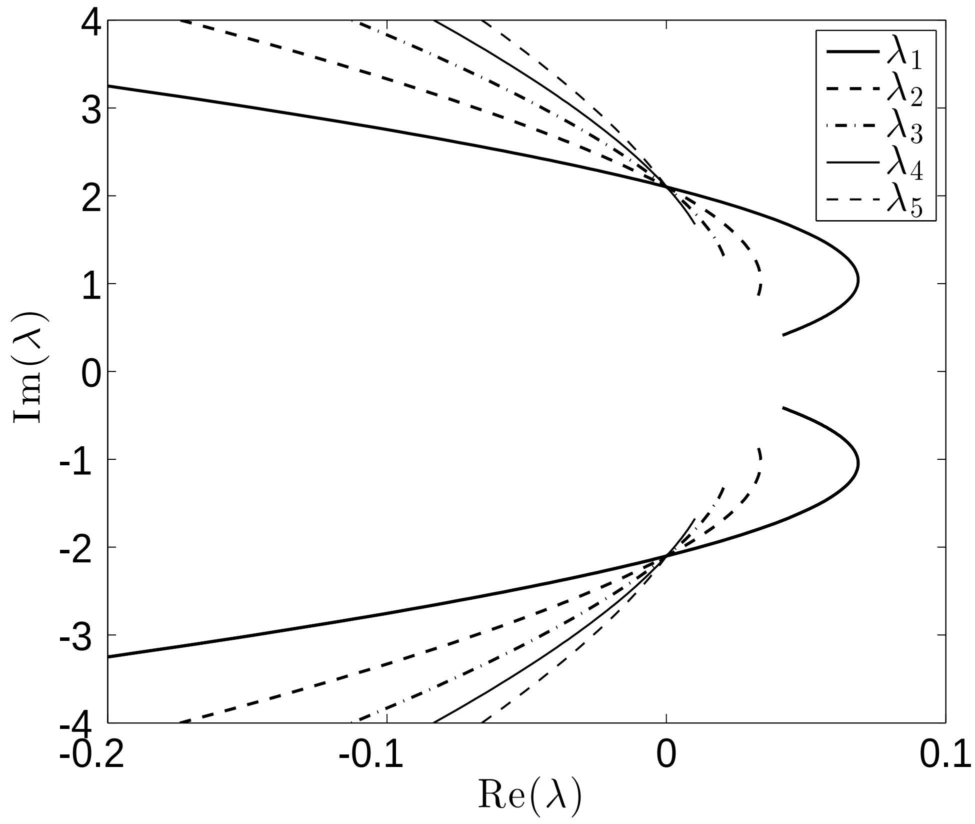
Note: Uniqueness follows since $\mathcal{H}(0) > 0$, $\mathcal{H}(-\pi/3) < 0$, and $\mathcal{H}(\theta)$ monotonic on $-\pi/3 < \theta < 0$. We compute

$$\theta_0 \approx -0.99046\,, \qquad \lambda_I \approx 2.1015\,,$$

and

$$T_{\mathsf{H}}^1 \approx 2.5185\,,\ \ T_{\mathsf{H}}^2 \approx 5.5084\,,\ \ T_{\mathsf{H}}^3 \approx 8.4982\,,\ \ T_{\mathsf{H}}^4 \approx 11.488\,.$$

The Delayed Local Eigenvalue Problem V



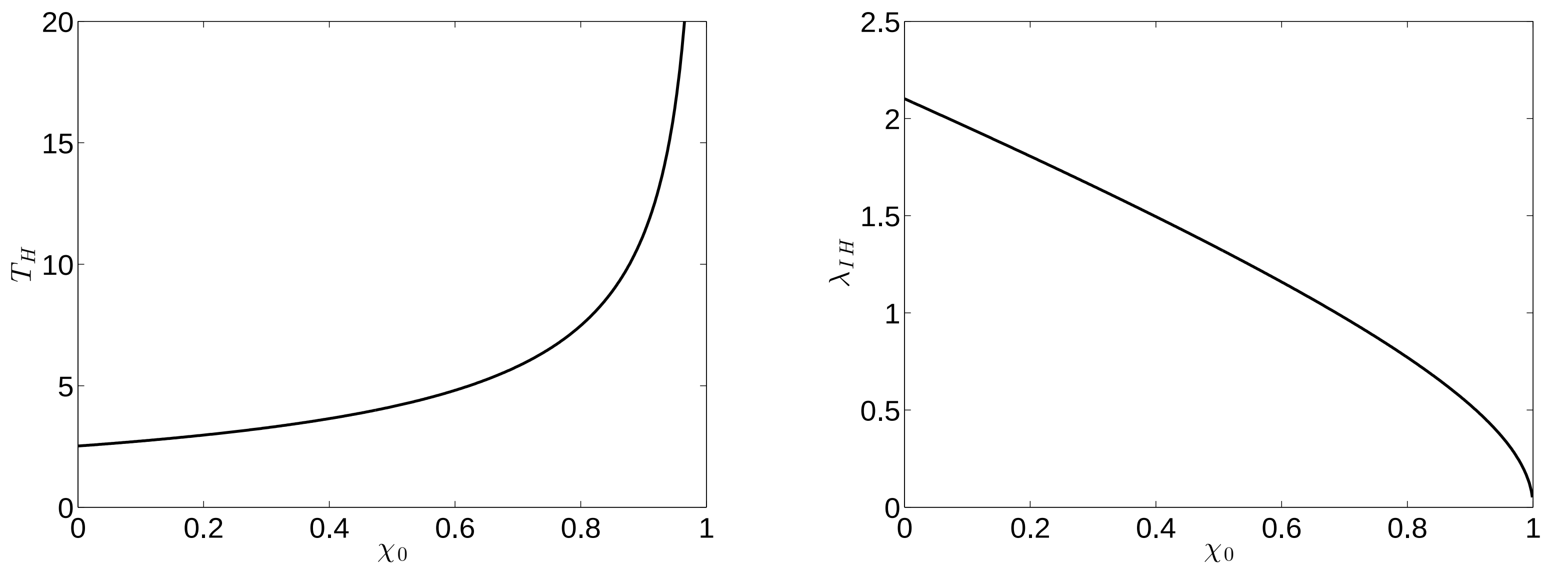
Caption: Complex spectra of L_μ in Re$(\lambda) \geq 0$ for $T_{\mathrm{H}}^1 \leq T \leq T_{\mathrm{H}}^5$. For $T \to \infty$, the paths all tend to the origin but in Re$(\lambda) > 0$. For each path, we also plot its continuation into Re$(\lambda) < 0$ for smaller delays.

Delayed Nonlocal Problem I

With $\chi(\tau\lambda) = 2/\sqrt{1+\tau\lambda}$, the eigenvalues of the NLEP are the roots of $g(\lambda) = 0$:

$$g(\lambda) \equiv \frac{1}{\mu\chi(\tau\lambda)} - \mathcal{F}_\mu(\lambda)\,; \qquad \mathcal{F}_\mu(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\left[(L_\mu - \lambda)^{-1} w^2\right]\, dy}{\int_{-\infty}^{\infty} w^2\, dy}\,;$$

Homotopy in $\chi(0)$: Consider the NLEP with $\tau = 0$ for which $\chi(0) = 2$. By a homotopy in the value $\chi(0)$ from $0 < \chi(0) < 2$, we conclude that no HB is possible for the NLEP when $\tau = 0$ for any fixed $T \geq 0$.



Caption: Left: min value T_H of T vs $\chi(0)$ for a HB. Right: λ_{IH} vs $\chi(0)$.

Delayed Nonlocal Problem II

Implication: HB occurs only on $0 \leq \chi(0) < 1$ with $\lambda_{IH} \to 0^+$ and $T_H \to +\infty$ as $\chi(0) \to 1^-$. For $\chi(0) > 1$ the NLEP does not have any HB as T is increased. Since $\chi(0) = 2$ when $\tau = 0$, we conclude that no HB occurs when $\tau = 0$ as T is increased.

Next, we fix $T > 0$, take $\chi = \chi(\tau\lambda)$ and let $\tau \to +\infty$. The next result shows that $\exists$ at least two positive real eigenvalues to the NLEP when $\tau \gg 1$.

Lemma: *For $\lambda > 0$ real and $\lambda \neq \lambda_0(T)$, we have $\mathcal{F}'_\mu(\lambda) > 0$. Moreover, $\mathcal{F}_1(0) = 1$ and $\mathcal{F}_\mu \to +\infty$ as $\lambda \to \lambda_0(T)^-$.*

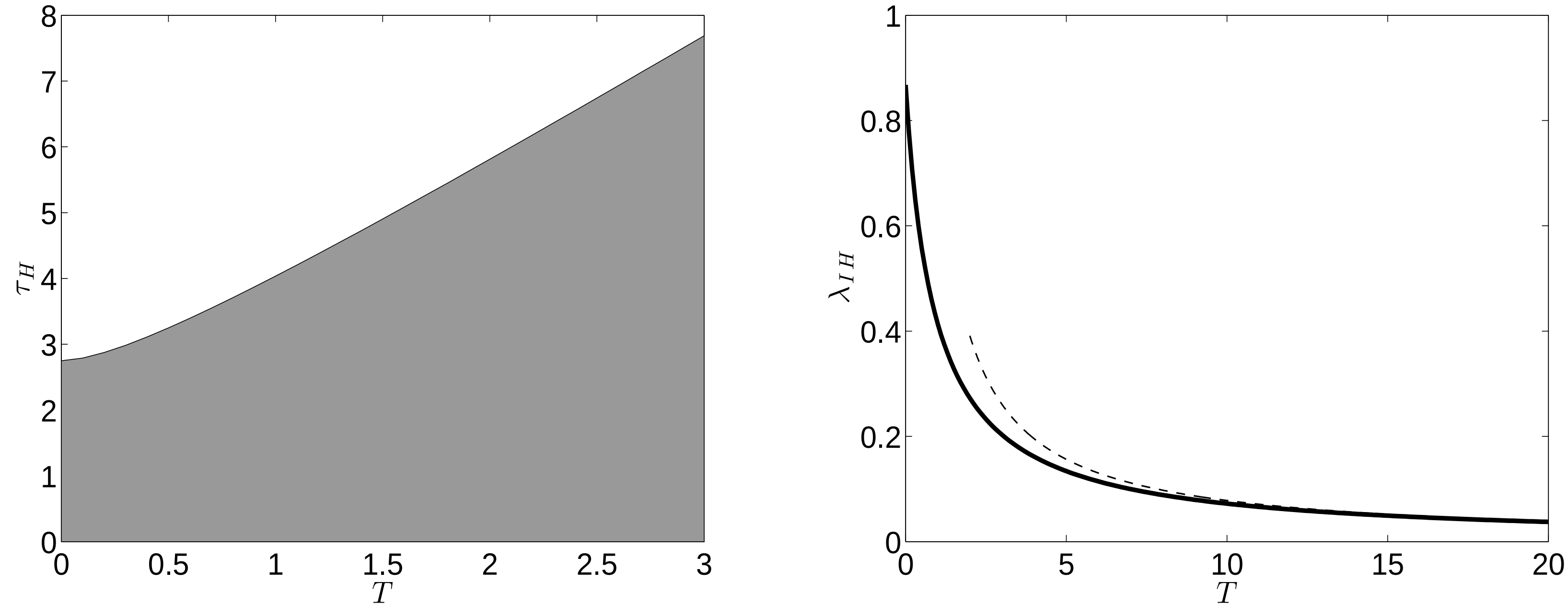
Therefore $1/(\mu\chi) = e^{-\lambda T}\sqrt{1+\tau\lambda}/2$ must intersect $\mathcal{F}_\mu(\lambda)$ at least twice on the positive real axis when τ is large enough.

By continuous dependence of eigenvalue paths in τ for any fixed T we obtain that:

Main Result: *Let $T > 0$ be fixed. There must be a HB at some $\tau = \tau_H > 0$ depending on T.*

Stability Diagram with Activator Delay I

Numerics: To determine the HB threshold set Re $(g(i\lambda_{IH})) = 0$ and Im $(g(i\lambda_{IH})) = 0$ and solve 2×2 nonlinear system for the HB values τ_H and λ_{IH} at a fixed $T \geq 0$. Path-follow in T.



Caption: Left: τ_H vs T. Middle: λ_{IH} vs T. (large T asymptotics (dashed) $\lambda_{IH} \sim 0.78/T$.)

Scaling Law with Activator Delay I

Question: Is there a scaling law for the HB when $T \gg 1$?

Scaling Law: For $T \gg 1$, put $\tau_H \sim \tau_0 T$ and $\lambda \sim i c_0 / T$ for some $c_0 > 0$ and $\tau_0 > 0$. Then, the NLEP becomes

$$\Phi'' - \Phi + 2w \left[e^{-ic_0} + \cdots \right] \Phi - \left[\chi_0 e^{-ic_0} + \cdots \right] w^2 \frac{\int_{-\infty}^{\infty} w \Phi \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} = \left[\frac{ic_0}{T} + \cdots \right] \Phi \,,$$

where $\chi_0 \equiv \chi(ic_0 \tau_0)$. Since $\Phi \sim w$ and $w'' - w + w^2 = 0$, we get

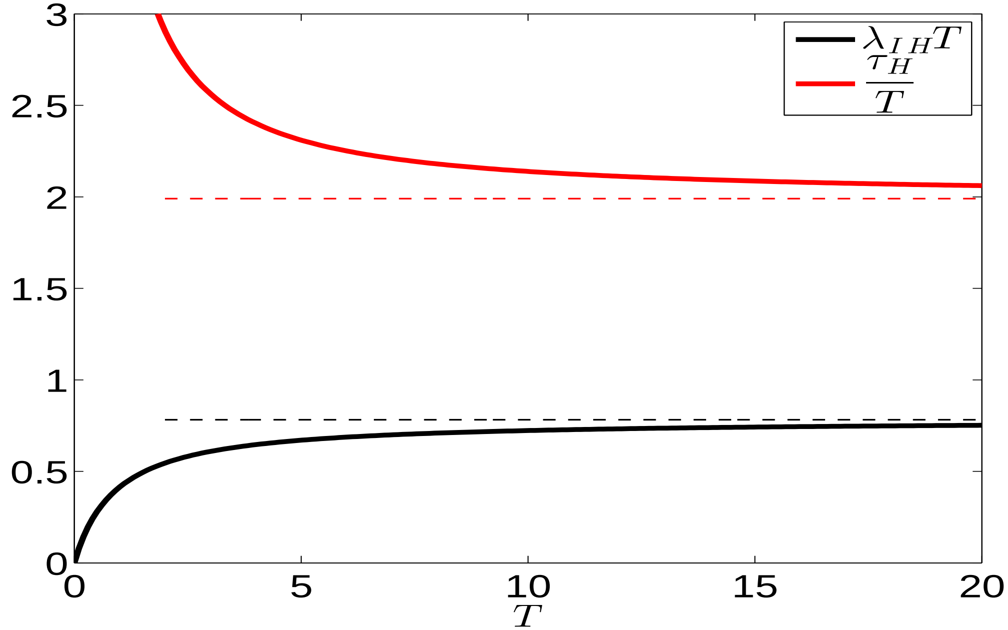
$$e^{ic_0} = 2 - \chi_0 = \frac{2 \left[\sqrt{1 + i\tau_0 c_0} - 1 \right]}{\sqrt{1 + i \tau_0 c_0}} \,.$$

From the real and imaginary parts of this expression:

$$c_0 = \sin^{-1} \left(\frac{\sqrt{2}}{\alpha} \sqrt{\alpha - 1} \right) \approx 0.782106 \,, \qquad \tau_0 = \frac{\sqrt{95 + 32\sqrt{10}}}{9 c_0} \approx 1.9899 \,,$$

where $\alpha = 4(1 + \sqrt{10})/9$. Thus, $\tau_H \sim 1.99 T$ and $\lambda \sim 0.782 i / T$ for $T \gg 1$.

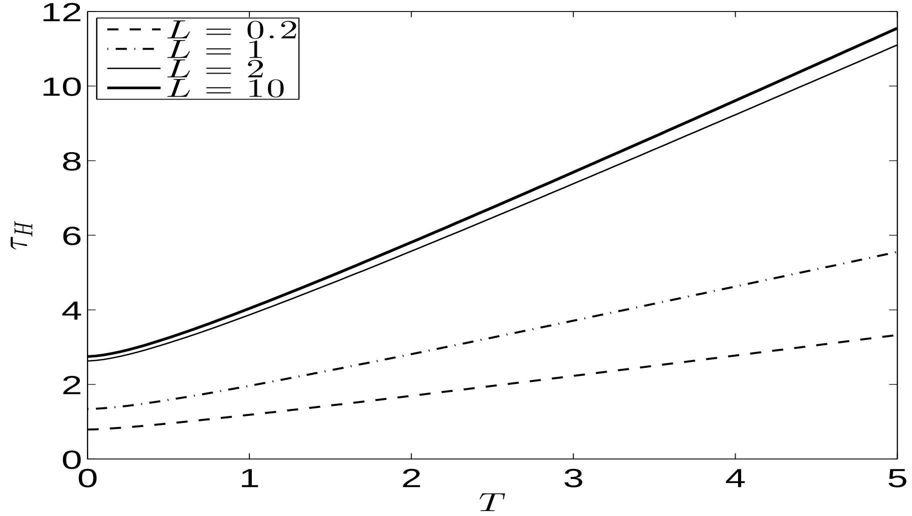
Scaling Law with Activator Delay II



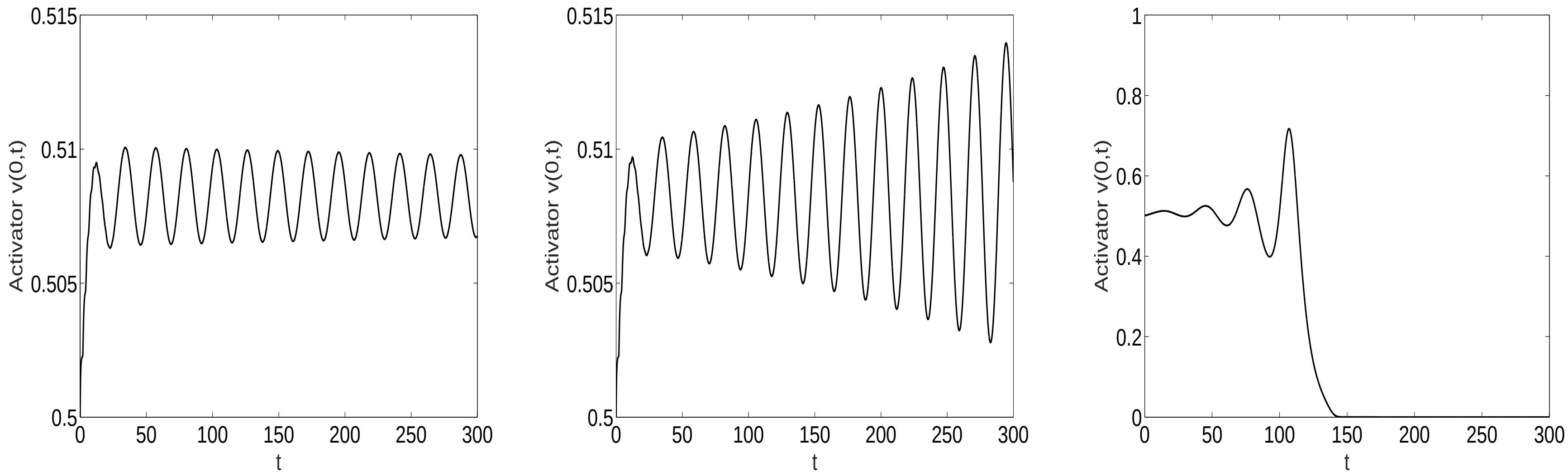
Caption: Right: validation of theoretical asymptotes $\lim_{T\to\infty} \tau_H/T \approx 1.99$ and $\lim_{T\to\infty} \lambda_{IH}T \approx 0.782$.

Key Qualitative Conclusion: A spike is stabilized by the presence of only activator delay in the reaction kinetics. The region in τ where no HB occur is larger with increasing delay.

Numerical Validation on Finite Domain



Numerics: HB threshold τ_H vs T on $|x| \leq L$ for $\epsilon = 0.05$: $L = 0.2$ (dashed), $L = 1$ (dashed-dotted), $L = 2$ (solid), and $L = 10$ (heavy solid).



Caption: $v(0,t)$ vs. t for $L = 2$, $T = 2$, computed by MOL and *dde23* of MATLAB: $\tau = 5.3$ (left), $\tau = 5.6$ (middle), and $\tau = 10$ (right). Theory yields $\tau_H \approx 5.573$.

Conclusion and Outlook

Conclusions:

- When the delay is in the activator kinetics only, a one-pulse solution has better stability properties than with no delay.
- A delay in the inhibitor kinetics is very destabilizing.

Extension to 2-D: A similar analysis can be done for 2-D multi-spot patterns to the GM model in the regime of large inhibitor diffusivity, by analyzing a 2-D NLEP problem.

Open: Oscillations with delayed kinetics appear to be subcritical. With no delay they are supercritical (Veerman, Nonlinearity, 2015). Normal form theory with delayed kinetics?

References:

[FWW1]: N. Fadai, M. Ward, J. Wei, *Delayed Reaction-Kinetics and the Stability of Spikes for the Gierer-Meinhardt Model*, SIAM J. Appl. Math., **77**(2), pp. 664-696

[FWW2]: N. Fadai, M. Ward, J. Wei, *A Time Delay in the Activator Kinetics is Stabilizing*........, under review, DCDS-B, (2017), (23 pages).