Exponential Asymptotics, Boundary Layer Resonance, and Dynamic Metastability

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To Julian Cole—Gentleman, Scholar, and Pioneer in Applied Asymptotics

Abstract

This paper considers the linear convection–diffusion equation $u_t = \epsilon u_{xx} - x u_x$, and certain natural generalizations, on fixed bounded spatial domains including the turning point $x = 0$. For constant boundary values and consistent initial values, asymptotic solutions as $\epsilon \to 0^+$ converge to steady states over asymptotically exponentially-long time intervals. The occurrence of an asymptotically exponentially-small eigenvalue is the reason for such metastability, as well as for the sensitivity of the long-time behavior to small perturbations. The utility of these asymptotic results is illustrated through numerical computations with only moderately small ϵ values.

1 Introduction

There has been much recent work on the asymptotic solution of exponentially ill-conditioned boundary value problems for nonlinear singularly perturbed parabolic partial differential equations. Examples of such problems include Burgers equation and the Ginzburg-Landau equation on bounded spatial domains (cf. [2], [4], [8], [10], [16] and the references therein). The solutions to these problems typically involve an extremely sluggish approach to a steady state (i.e., *dynamic metastability*). Moreover, the steady state itself is often extremely *sensitive* to perturbations of the boundary values and the coefficients in the differential operator. Both phenomena relate to the appropriate linearized problem having an asymptotically exponentially-small eigenvalue. The significance of such eigenvalues was already recognized for certain much-studied linear two-point problems involving boundary layer resonance (cf. [1], [5], [9], [11], [14], [20], among much additional literature). Some of these linear problems arise in the asymptotic determination of exit times for stochastic differential equations in one or more space dimensions (cf. [12], [13], [15], and [17]).

In this paper, we examine a class of linear time-dependent convection-diffusion equations exhibiting the phenomenon of dynamic metastability. The corresponding equilibrium problem is exponentially ill-conditioned and is associated with boundary layer resonance. In §2, we use a straightforward eigenfunction expansion to study a simple example of such a problem in detail. In §3, we generalize the analysis to treat a class of convection-diffusion problems and examine the extreme sensitivity of their solutions to some small

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perturbations. Finally, in §4, we compare the results of the asymptotic analysis with numerical computations for specific examples.

2 A Simple Example

We begin by considering the following initial-boundary value problem for $u = u(x,t)$:

$$u_t = \epsilon u_{xx} - x u_x\,, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{1}$$

$$u(-1,t) = \alpha\,, \qquad u(1,t) = \beta\,, \tag{2}$$

$$u(x,0) = u_0(x)\,. \tag{3}$$

Here $\epsilon \to 0^+$, α and β are constants, and $u_0(x)$ is smooth with $u_0(-1) = \alpha$ and $u_0(1) = \beta$. The exact steady-state solution $U(x;\epsilon)$ corresponding to (1)-(3) is given by

$$U(x;\epsilon) = \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\beta-\alpha)\left[\frac{\int_0^x e^{s^2/2\epsilon}ds}{\int_0^1 e^{s^2/2\epsilon}ds}\right]\,. \tag{4}$$

The dominant contributions to the integrals in (4) arise from s values near the upper endpoints, i.e. $\int_0^x e^{s^2/2\epsilon}ds \sim \epsilon x^{-1} e^{x^2/2\epsilon}$ provided that $x^2 \gg \epsilon$. This implies that the ratio of the two integrals behaves like $x^{-1}e^{(x^2-1)/2\epsilon}$ away from $x = 0$. Thus, for $\epsilon \to 0$,

$$U(x;\epsilon) \sim \frac{1}{2}(\alpha+\beta) + \frac{1}{2}(\alpha-\beta)\left(e^{-(1+x)/\epsilon} - e^{-(1-x)/\epsilon}\right)\,, \tag{5}$$

i. e., the steady state tends to the average $(\alpha+\beta)/2$, except in $O(\epsilon)$ boundary layer regions near each endpoint. Nothing special happens near the turning point $x = 0$.

The solution to (1)-(3) can be represented in terms of an eigenfunction expansion as

$$u(x,t) = U(x;\epsilon) + \sum_{k=0}^{\infty} c_k \phi_k(x) e^{-\lambda_k t} \tag{6}$$

(cf. [6] which also takes this approach). Here, the coefficients $c_k(\epsilon)$ are uniquely given by

$$c_k = \int_{-1}^{1} [u_0(x) - U(x;\epsilon)]\phi_k w\, dx \Big/ \int_{-1}^{1} \phi_k^2 w\, dx\,, \tag{7}$$

for the weight $w(x) \equiv e^{-x^2/2\epsilon}$, while the eigenpairs $(\lambda_k(\epsilon), \phi_k(x;\epsilon))$ satisfy the eigenvalue problem

$$\epsilon\phi'' - x\phi' + \lambda\phi = 0\,, \qquad -1 < x < 1\,, \qquad \phi(\pm 1) = 0\,, \tag{8}$$

for real λ_k's which increase with k. To solve (8), we invoke the classical Liouville transformation $\hat{\phi} = e^{-x^2/4\epsilon}\phi$ and find that $\hat{\phi}$ must satisfy Weber's equation, which is solvable in terms of Whittaker's form of the *parabolic cylinder function* $D_\nu(z)$ (cf. [21]). Thus, the eigenfunctions ϕ must have the form

$$\phi(x) = e^{x^2/4\epsilon}[b_1 D_{-1-\lambda}(ix/\sqrt{\epsilon}) + b_2 D_\lambda(-x/\sqrt{\epsilon})]\,, \tag{9}$$

for some (ϵ-dependent) constants b_1 and b_2. Since $\phi(\pm 1) = 0$, the ratio b_1/b_2 becomes specified and the eigenfunctions are proportional to

$$\phi(x) = e^{x^2/4\epsilon}[D_{-1-\lambda}(ix/\sqrt{\epsilon})D_\lambda(-1/\sqrt{\epsilon}) - D_\lambda(-x/\sqrt{\epsilon})D_{-1-\lambda}(i/\sqrt{\epsilon})]\,, \tag{10}$$

while the eigenvalues λ must satisfy the transcendental eigenvalue relation

$$D_\lambda(-1/\sqrt{\epsilon}) = D_\lambda(1/\sqrt{\epsilon})\frac{D_{-1-\lambda}(i/\sqrt{\epsilon})}{D_{-1-\lambda}(-i/\sqrt{\epsilon})}\,. \tag{11}$$

To study the asymptotic behavior of the eigenpairs as $\epsilon \to 0$, we shall use the leading-order approximations

$$D_\nu(z) = \begin{cases} e^{-z^2/4}z^\nu[1+O(z^{-2})]\,, & |\arg z| < 3\pi/4 \\ e^{-z^2/4}z^\nu[1+O(z^{-2})] - \frac{\sqrt{2\pi}}{\Gamma(-\nu)}\frac{e^{z^2/4-\nu\pi i}}{z^{\nu+1}}[1+O(z^{-2})]\,, & \frac{\pi}{4} < \arg z < \frac{5\pi}{4} \end{cases} \tag{12}$$

as $|z| \to \infty$ for complex arguments z. It is critical to note that the decaying nature of $D_\nu(z)$ for large positive z becomes growing for large negative values of z, unless ν happens to be (near) zero or a positive integer, since then $1/\Gamma(-\nu)$ is (near) zero. (This reflects the Stokes phenomenon for Weber's equation and the fact that $D_\nu(z)$ for a non-negative integer ν reduces to the product of a Hermite polynomial and an exponential function which decays as $z \to \pm\infty$.) A careful use of the approximations (12) in the eigenvalue relation (11) reduces it to the limiting form

$$\frac{\epsilon^\lambda}{\Gamma(-\lambda)} \sim -\sqrt{\frac{2}{\pi\epsilon}}e^{-1/2\epsilon}\,. \tag{13}$$

Thus, for $\epsilon \to 0$, the principal eigenvalue λ_0 satisfies

$$\lambda_0 \sim \sqrt{\frac{2}{\pi\epsilon}}e^{-1/2\epsilon}\,, \tag{14}$$

while the remaining sequence of eigenvalues satisfies

$$\lambda_k \sim k \qquad \text{for} \quad k = 1, 2, 3, \ldots\,. \tag{15}$$

The (un-normalized) eigenfunction corresponding to λ_0 is given asymptotically by

$$\phi_0(x) \sim 1 - e^{-(1+x)/\epsilon} - e^{-(1-x)/\epsilon}\,, \tag{16}$$

while

$$\phi_k(x) \sim x^k - (-1)^k e^{-(1+x)/\epsilon} - e^{-(1-x)/\epsilon} \qquad \text{for} \quad k = 1, 2, 3, \ldots\,. \tag{17}$$

Thus, for large t, the higher terms in (6) are insignificant and the *quasi-steady state*

$$u(x,t) \sim U(x;\epsilon) + c_0\phi_0(x)e^{-\lambda_0 t} \tag{18}$$

is attained. The coefficient c_0 is obtained asymptotically by substituting (16) into (7) with $k = 0$. Since the weighting function w is localized near $x = 0$, the integrals in (7) can be evaluated for $\epsilon \to 0$ (cf. [23]) to yield $c_0 \sim u_0(0) - U(0;\epsilon)$. Thus, (18) reduces to

$$u(x,t) \sim A_0(t) + (\alpha - A_0(t))e^{-(1+x)/\epsilon} + (\beta - A_0(t))e^{-(1-x)/\epsilon} \tag{19}$$

where the level

$$A_0(t) \equiv \frac{1}{2}(\alpha+\beta) + [u_0(0) - \frac{1}{2}(\alpha+\beta)]e^{-\lambda_0 t} \tag{20}$$

is the outer limit within $-1 < x < 1$ and where λ_0 is the negligible principal eigenvalue given by (14). This significant result describes the exponentially-slow evolution of $u(x,t)$

toward the steady state $U(x;\epsilon)$. Higher-order approximations than (19) would result from using higher-order approximations than (12) in (6) and (11).

To illustrate the big effect a small perturbation can produce, consider the equation

$$u_t = \epsilon u_{xx} - x u_x - \delta u \,, \tag{21}$$

where δ is a small, but fixed, positive constant, subject to the boundary and initial conditions (2) and (3). We can again obtain a solution of the form (6), with the eigenpairs now satisfying

$$\epsilon\phi'' - x\phi' + (\lambda - \delta)\phi = 0 \,, \quad \phi(\pm 1) = 0 \,. \tag{22}$$

Thus, the resulting limiting eigenvalue condition $\frac{\epsilon^{\lambda-\delta}}{\Gamma(\delta-\lambda)} \sim -\sqrt{\frac{2}{\pi\epsilon}} e^{-1/2\epsilon}$ implies a sequence

$$\lambda_k \sim \delta + k \tag{23}$$

of positive eigenvalues for $k = 0, 1, 2, \ldots$. Because no eigenvalue remains asymptotically exponentially-negligible, solutions of the perturbed initial-boundary value problem will decay to the steady state in finite time (i.e., the earlier metastability is eliminated). Moreover, corresponding to the original notion of a resonant equilibrium problem having the trivial limit within $-1 < x < 1$ except for nonpositive integer values of δ, the steady state will now be trivial except in endpoint boundary layer regions. Indeed, the solution u will uniformly satisfy

$$u(x,t) \sim \alpha e^{-(1+x)/\epsilon} + \beta e^{-(1-x)/\epsilon} + O(e^{-\delta t}) \tag{24}$$

as $\epsilon \to 0$ (thereby agreeing in form with the long-time limit (19) for $\delta = 0$, but with $A_0(t) \equiv 0$). This big change between the solution with $\delta = 0$ and with any small $\delta > 0$ was called *super-sensitivity* in [10]. If we, instead, allowed $\delta(\epsilon)$ to be of the same asymptotically exponentially-small order as the eigenvalue λ_0 for $\delta = 0$, the solution u will again have a nontrivial outer limit, with metastable decay to a nontrivial steady state over an exponentially-long time interval. On the other hand, if we let δ be negative, the equilibrium solution will lose its stability.

2.1 An Alternative Metastability Analysis

The preceding analysis relied heavily on explicit analytical expressions for the eigenpairs (λ_k, ϕ_k) of (8) and their representations in terms of parabolic cylinder functions. We now present a more direct asymptotic method to study the metastable behavior for (1)-(3) in a manner which does not require such an explicit knowledge of the spectrum.

To begin, recall that the traditional method of matched asymptotic expansions fails to uniquely determine the asymptotic solution to the equilibrium problem

$$L_\epsilon U \equiv \epsilon U_{xx} - x U_x = 0 \,, \qquad U(-1;\epsilon) = \alpha \,, \quad U(1;\epsilon) = \beta \,. \tag{25}$$

For $\epsilon \to 0$, a straightforward leading-order boundary layer analysis shows that

$$U(x;\epsilon) \sim \tilde{u}^\epsilon[x; A_{0e}] \equiv A_{0e} + (\alpha - A_{0e})\, e^{-(1+x)/\epsilon} + (\beta - A_{0e})\, e^{-(1-x)/\epsilon} \,, \tag{26}$$

uniformly in $-1 \le x \le 1$. Note that the form of $\tilde{u}^\epsilon$ is that already found for the long-time limit in (19). Here A_{0e} is a constant to be determined, which we naturally identify as the limit of $U(0;\epsilon)$. Note that the limiting equilibrium profile $\tilde{u}^\epsilon$ satisfies the boundary conditions of (25) to within asymptotically exponentially-small terms. In addition,

$$L_\epsilon \tilde{u}^\epsilon = \frac{1}{\epsilon}[(1+x)e^{-(1+x)/\epsilon}(\alpha - A_{0e}) + (1-x)e^{-(1-x)/\epsilon}(\beta - A_{0e})] \tag{27}$$

is asymptotically negligible away from the boundary layer regions near $x = \pm 1$ for any choice of A_{0e}. This indeterminacy in the matched asymptotic expansion persists even after one attempts to construct higher order boundary layer approximations near the endpoints. Yet, it is clear by symmetry that the correct value to select is $A_{0e} = (\alpha+\beta)/2$. The variational principle of [7] provides one analytical method for obtaining this value.

Next, we shall look for a solution to the time-dependent problem (1)-(3) in the form

$$u(x,t) = \tilde{u}^\epsilon[x; A(t;\epsilon)] + v(x,t)\,. \tag{28}$$

We insist that v remains asymptotically uniformly small for all moderately large $t > 0$, so that the translating profile $\tilde{u}^\epsilon[x; A(t;\epsilon)]$ continues to describe the limiting solution, uniformly in space, as time evolves. We define the outer solution $A(t;\epsilon)$ by

$$u(0,t) = \tilde{u}^\epsilon[0; A(t;\epsilon)]\,, \tag{29}$$

so $v(0,t) = 0$. Substituting (28) into (1)-(2), we obtain

$$L_\epsilon v = f(x,t) + v_t \quad -1 < x < 1\,, \quad t > 0\,, \tag{30}$$

$$v(-1,t) = (A(t;\epsilon) - \alpha)e^{-2/\epsilon}\,, \qquad v(1,t) = (A(t;\epsilon) - \beta)e^{-2/\epsilon}\,. \tag{31}$$

$$v(0,t) = 0\,. \tag{32}$$

Here, we have used the linear operator L_ϵ of (25) and defined $f(x,t)$ by

$$f(x,t) \equiv \frac{\partial \tilde{u}^\epsilon}{\partial A}\frac{dA}{dt} - L_\epsilon \tilde{u}^\epsilon\,, \tag{33}$$

where $L_\epsilon \tilde{u}^\epsilon$ given by (27). Note that (26) implies that $\partial\tilde{u}^\epsilon/\partial t$ is independent of A while (16) implies that $\partial\tilde{u}^\epsilon/\partial A \sim \phi_0$ as $\epsilon \to 0$. Integrating (30) with respect to x and setting $v(0,t) = 0$ shows that v satisfies the integral equation

$$v(x,t) = M\int_0^x e^{r^2/2\epsilon}\,dr + \int_0^x \int_0^r K(r,s)[f(s,t) + v_t(s,t)]\,ds\,dr\,, \tag{34}$$

where the constant of integration $M(t;\epsilon)$ is to be determined and the kernel is $K(r,s) \equiv \epsilon^{-1}e^{(r^2-s^2)/2\epsilon}$. Using symmetry, the boundary conditions (31) at $x = \pm 1$ yield

$$(A-\alpha)e^{-2/\epsilon} = -M\int_0^1 e^{r^2/2\epsilon}\,dr + \int_0^1 \int_0^r K(r,s)[f(-s,t) + v_t(-s,t)]\,ds\,dr\,, \tag{35}$$

and

$$(A-\beta)e^{-2/\epsilon} = M\int_0^1 e^{r^2/2\epsilon}\,dr + \int_0^1 \int_0^r K(r,s)[f(s,t) + v_t(s,t)]\,ds\,dr\,. \tag{36}$$

By adding and subtracting (35) and (36), we can eliminate M to find the relation

$$\begin{aligned} \frac{dA}{dt}&\left(\int_0^1 \int_0^r K(r,s)\frac{\partial \tilde{u}^\epsilon(s;A)}{\partial A}\,ds\,dr\right) \\ &= \left(A - \frac{1}{2}(\alpha+\beta)\right)\left(\epsilon^2 \int_0^1 \int_0^r K(r,s)\frac{\partial}{\partial \epsilon}\left(\frac{\partial \tilde{u}^\epsilon(s;A)}{\partial A}\right)ds\,dr + e^{-2/\epsilon}\right) \\ &\quad + \frac{1}{2}\int_0^1 \int_0^r K(r,s)\left(v_t(s,t) + v_t(-s,t)\right)ds\,dr\,. \end{aligned} \tag{37}$$

We propose solving (34)-(36) (or, equivalently, (34) with M eliminated and (37)) by successive approximations with the first iterate v_0 resulting from setting $v_t \equiv 0$. Then (37) becomes the ordinary differential equation

$$
\begin{aligned}
(38)\qquad & \frac{dA_0}{dt}\left(\int_0^1\int_0^r K(r,s)\frac{\partial \tilde{u}^\epsilon(s;A)}{\partial A}\,ds\,dr\right) \\
&= \left(A_0-\frac{1}{2}(\alpha+\beta)\right)\left(\epsilon^2\int_0^1\int_0^r K(r,s)\frac{\partial}{\partial\epsilon}\left(\frac{\partial \tilde{u}^\epsilon(s;A)}{\partial A}\right)ds\,dr+e^{-2/\epsilon}\right)
\end{aligned}
$$

for A_0, which describes the corresponding motion of the limiting solution $\tilde{u}^\epsilon[x;A_0(t)]$. By asymptotically approximating the integrals in (38), we find that A_0 satisfies the limiting differential equation

$$
(39)\qquad \frac{dA_0}{dt} \sim -\sqrt{\frac{2}{\pi\epsilon}}e^{-1/2\epsilon}\left(A_0-\frac{1}{2}(\alpha+\beta)\right) \quad \text{as } \epsilon\to 0\,.
$$

To determine an appropriate initial condition $A_0(0)$ (which will determine $A_0(t)$ for all $t\geq 0$), note that the solution U^0 of the reduced equation $U_t^0=-xU_x^0$, corresponding to (1), will be constant on its characteristics, defined by a fixed value for xe^{-t}, which spread out from the origin as t increases. Thus, to leading order, it is natural to take $A_0(0)\sim u_0(0)$, the center value of the initial data. The limiting outer solution $A_0(t)$ thereby obtained coincides precisely with the limit (20) previously found for the long time limit, and yields the anticipated steady state $A_{0e}=(\alpha+\beta)/2$. Finding higher-order approximations for the initial value $A_0(0;\epsilon)$ as $\epsilon\to 0$ would, of course, be desirable. Knowing $A_0(t)$ provides us the corresponding constant M_0 and the limiting initial iterate v_0 as a correction to the approximate asymptotic profile $\tilde{u}^\epsilon[x;A_0(t)]$. It is easy to check that v_0 is uniformly asymptotically negligible in $-1\leq x\leq 1$ for $t\geq 0$. For consistency, we can also check that the neglected v_{0t} terms in (34)-(36) are asymptotically much smaller than the terms retained.

2.2 A Related Shock Layer Problem

We now contrast the metastable behavior of the solution to (1)-(3) with that of the modified problem

$$
(40)\qquad u_t=\epsilon u_{xx}+xu_x\,,\qquad -1<x<1\,,\quad t>0\,,
$$

together with the boundary and initial conditions (2) and (3). The only difference between (40) and (1) is the sign of the xu_x term. The equilibrium solution to this problem is

$$
(41)\qquad U(x;\epsilon)=\frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\beta-\alpha)\left[\frac{\int_0^x e^{-s^2/2\epsilon}ds}{\int_0^1 e^{-s^2/2\epsilon}ds}\right]\,.
$$

For $\epsilon\to 0$, the dominant contributions to the integrals in (41) occur near $s=0$. By evaluating these integrals asymptotically, we readily observe that the ratio of the two integrals switches from the asymptotic limit -1 for $x<0$ to $+1$ for $x>0$ in an $O(\sqrt{\epsilon})$ neighborhood of the *turning point* $x=0$. Thus, as $\epsilon\to 0$, we have

$$
(42)\qquad U(x;\epsilon)\to\begin{cases}\alpha & \text{for } x<0\\ \frac{1}{2}(\alpha+\beta) & \text{for } x=0\\ \beta & \text{for } x>0\end{cases}\,,
$$

with nonuniform convergence in the thin shock or transition layer about $x = 0$.

To solve the time-dependent equation (40), we expand $u(x,t)$ in an eigenfunction expansion as in (6). In place of (8), the relevant eigenvalue problem is

$$\epsilon\phi'' + x\phi' + \lambda\phi = 0\,, \qquad -1 < x < 1\,; \qquad \phi(\pm 1) = 0\,. \tag{43}$$

The solution ϕ to the differential equation with $\phi(1) = 0$ is proportional to

$$\phi(x) = e^{-x^2/4\epsilon}[D_{-\lambda}(ix/\sqrt{\epsilon})D_{\lambda-1}(-1/\sqrt{\epsilon}) - D_{\lambda-1}(-x/\sqrt{\epsilon})D_{-\lambda}(i/\sqrt{\epsilon})]\,. \tag{44}$$

By enforcing $\phi(-1) = 0$, we find that the eigenvalues λ must satisfy

$$D_{\lambda-1}(-1/\sqrt{\epsilon}) = D_{\lambda-1}(1/\sqrt{\epsilon})\frac{D_{-\lambda}(i/\sqrt{\epsilon})}{D_{-\lambda}(-i/\sqrt{\epsilon})}\,. \tag{45}$$

By using the asymptotic approximations (12), we reduce (45) when $\epsilon \to 0$ to

$$\frac{\epsilon^{\lambda-1}}{\Gamma(1-\lambda)} \sim -\sqrt{\frac{2}{\pi\epsilon}}e^{-1/2\epsilon}\,. \tag{46}$$

Thus, for $\epsilon \to 0$, the (increasing) eigenvalues λ_k satisfy

$$\lambda_k \sim k + 1 + O\left(\epsilon^{-k-1/2}e^{-1/2\epsilon}\right) \quad \text{for} \quad k = 0, 1, 2, \ldots\,. \tag{47}$$

From (47) we see that the principal eigenvalue λ_0 for this modified problem is *not exponentially small.* Since λ_0 is positive and bounded away from zero, the solution to the modified problem (40)-(2)-(3) decays to the equilibrium shock-layer solution (41) on an $O(1)$ time scale. Thus, in contrast to the solution of (1)-(3), this convection-diffusion equation has a shock layer solution which does not exhibit dynamic metastability.

However, as was shown in [8], [10], and [16], metastable behavior can occur for certain *nonlinear* convection-diffusion equations with shock-layer solutions. To illustrate qualitatively how this can occur, consider Burgers equation

$$u_t = \epsilon u_{xx} - uu_x\,, \qquad -1 < x < 1\,, \quad t > 0 \tag{48}$$

with

$$u(-1,t) = 1\,, \qquad u(1,t) = -1\,, \quad \text{and} \quad u(x,0) = u_0(x)\,. \tag{49}$$

The unique equilibrium solution $U(x;\epsilon)$ for this problem is given asymptotically by $U(x;\epsilon) \sim -\tanh[x/2\epsilon]$ for $\epsilon \to 0$.

We shall determine the stability of this equilibrium solution by linearizing (48) about U. Substituting $u(x,t) = U(x;\epsilon) + \nu e^{-\lambda t}\Phi(x)$, where $\nu \ll 1$, into (48) and (49), we collect terms of $O(\nu)$ to find that Φ satisfies the eigenvalue problem

$$\epsilon\Phi_{xx} - (U\Phi)_x + \lambda\Phi = 0\,, \qquad -1 < x < 1\,; \qquad \Phi(\pm 1) = 0\,. \tag{50}$$

Equivalently, $\phi \equiv \exp[-\epsilon^{-1}\int_0^x U(s;\epsilon)ds]\Phi$ satisfies

$$\epsilon\phi_{xx} + U\phi_x + \lambda\phi = 0\,, \qquad -1 < x < 1\,; \qquad \phi(\pm 1) = 0\,. \tag{51}$$

Because U is monotonically decreasing in x and zero at the turning point $x = 0$, the nature of the turning point for (51) is very closely related to that for the eigenvalue problem (8), which has an exponentially small eigenvalue. In [8], it was shown that (51) has an asymptotically exponentially-small principal eigenvalue and an asymptotic formula for this eigenvalue was obtained in [10] and [16] to determine the limiting metastable behavior of the Burgers solution.

3 A More General Convection-Diffusion Equation

In the limit $\epsilon \to 0^+$, we now consider the following convection-diffusion equation for $u = u(x,t)$:

$$(52) \qquad u_t = \epsilon u_{xx} - x^{2m+1} p(x) u_x + \epsilon^\nu g(x) e^{-a/\epsilon} u\,, \quad -1 < x < b\,, \quad t > 0\,,$$

$$(53) \qquad u(-1,t) = \alpha \qquad u(b,t) = \beta\,,$$

$$(54) \qquad u(x,0) = u_0(x)\,.$$

Here a and b are fixed positive constants, m is a non-negative integer, and α and β are constants with $u_0(-1) = \alpha$ and $u_0(b) = \beta$. In addition, $p(x) > 0$, $g(x)$, and $u_0(x)$ are smooth functions. Under these conditions, we again anticipate having boundary layer behavior at both endpoints. The final asymptotically negligible perturbation term on the right side of (52) is added to determine its effect on long-time behavior.

For $\epsilon \to 0$, a leading order boundary layer approximation for the equilibrium solution $U(x;\epsilon)$ corresponding to (52)-(53) has the form

$$(55) \qquad U(x;\epsilon) \sim \tilde{u}^\epsilon[x; A_{0e}] \equiv A_{0e} + (\alpha - A_{0e})\, e^{-\xi_l (x+1)/\epsilon} + (\beta - A_{0e})\, e^{-\xi_r (b-x)/\epsilon}\,.$$

Here the outer limit $A_{0e} = A_{0e}(\epsilon)$ is a constant to be determined and the decay constants are $\xi_l \equiv p(-1)$ and $\xi_r \equiv b^{2m+1} p(b)$. In the region away from the endpoint boundary layers, $L_\epsilon \tilde{u}^\epsilon \equiv \epsilon \tilde{u}^\epsilon_{xx} - x^{2m+1} p(x) \tilde{u}^\epsilon_x$ is asymptotically exponentially-small for any choice of A_{0e}. Thus, as for the simple problem of §2, A_{0e} can only be determined by somehow incorporating the effect of asymptotically exponentially-small terms into the analysis. Various methods to calculate A_{0e} are given in [7], [11], and [19].

The difficulty in determining A_{0e} using standard asymptotic methods results from the fact that the equilibrium problem is exponentially ill-conditioned as $\epsilon \to 0$ (see [5], [9], and [11]). More specifically, as shown in [11], the principal eigenvalue λ_0 for the eigenvalue problem

$$(56) \qquad L_\epsilon \phi \equiv \epsilon \phi_{xx} - x^{2m+1} p(x) \phi_x = -\lambda \phi\,, \quad -1 < x < b\,; \qquad \phi(-1) = \phi(b) = 0$$

is positive, but exponentially small as $\epsilon \to 0$ (see (76) below for the precise estimate). The corresponding (un-normalized) eigenfunction ϕ_0 is in the boundary layer form

$$(57) \qquad \phi_0 \sim 1 + B_l[(1+x)/\epsilon; \epsilon]\, e^{-\xi_l (1+x)/\epsilon} + B_r[(b-x)/\epsilon; \epsilon]\, e^{-\xi_r (b-x)/\epsilon}\,,$$

where $B_l(z;\epsilon)$ and $B_r(z;\epsilon)$ behave like polynomials in z. The exponentially small eigenvalue implies that the equilibrium solution for (52)-(54) will be very sensitive to the exponentially small perturbation term $\epsilon^\nu g(x) e^{-a/\epsilon} u$. Such sensitivity of the equilibrium solution to changes in either a or the endpoint location b was studied in [11], [18], [19], and [22].

Since $\lambda_0 > 0$, it follows that if a is sufficiently large, the equilibrium solution for (52) – (54) will remain asymptotically stable. However, because λ_0 is exponentially small, the approach to the equilibrium will occur over an asymptotically exponentially-long time scale. We shall study this slow motion asymptotically by using the projection method developed in [16] to treat related nonlinear problems. This method, which yields a higher order asymptotic theory than that given in §2.1, relies to a significant extent on the equilibrium theory of [11].

3.1 The Metastability Analysis

Following (28), we seek a solution to (52)-(54) in the form

$$u(x,t) = \tilde{u}^\epsilon[x; A(t;\epsilon)] + v(x,t)\,, \tag{58}$$

where $\tilde{u}^\epsilon$ is defined in (55). Using the projection method, we will, as in (39), derive a differential equation for A and obtain its steady-state limit A_e, provided a is large enough. For large values of t, the differential equation will capture the metastable dynamics of $u(x,t)$, since then $v \ll \tilde{u}^\epsilon$. The projection method differs somewhat from the method of §2.1 in that we exploit the occurrence of the exponentially small eigenvalue to directly enforce a limiting solvability condition on the correction term v, rather than explicitly obtaining an integral equation for v like (34).

Substituting (58) into (52)-(54), we obtain

$$L_\epsilon v = \tilde{u}_t^\epsilon + v_t - L_\epsilon \tilde{u}^\epsilon - \epsilon^\nu g(x) e^{-a/\epsilon}\left(\tilde{u}^\epsilon + v\right)\,, \quad -1 < x < b\,, \quad t > 0\,, \tag{59}$$

$$v(-1,t) = \alpha - \tilde{u}^\epsilon[-1; A(t;\epsilon)]\,, \qquad v(b,t) = \beta - \tilde{u}^\epsilon[b; A(t;\epsilon)]\,, \tag{60}$$

$$v(x,0) = u_0(x) - \tilde{u}^\epsilon[x; A(0;\epsilon)]\,. \tag{61}$$

Now let $\phi_k(x)$ and λ_k, for $k = 0, 1, ...$, be the normalized eigenfunctions and eigenvalues of (56). The λ_k are real and the ϕ_k satisfy the orthogonality relations

$$\left(\phi_j, \phi_k\right)_w \equiv \int_{-1}^{b} \phi_j \phi_k w\, dx = \delta_{jk} \qquad \text{for} \qquad w \equiv \exp[-\epsilon^{-1} \int_0^x t^{2m+1} p(t)\, dt]\,. \tag{62}$$

We then expand $v(x,t)$ in terms of the ϕ_k as

$$v(x,t) = \sum_{k=0}^{\infty} h_k(t;\epsilon) \phi_k(x)\,. \tag{63}$$

Substituting (63) into (59)-(61), orthogonality implies that the h_k will satisfy the differential equation

$$h_k' + \lambda_k h_k = -\epsilon w v \phi_{kx}\Big|_{-1}^{b} - \left(\phi_k, \tilde{u}_t^\epsilon\right)_w + \left(\phi_k, L_\epsilon \tilde{u}^\epsilon\right)_w + \epsilon^\nu e^{-a/\epsilon}\left(g\phi_k, \tilde{u}^\epsilon + v\right)_w\,, \tag{64}$$

together with the initial value

$$h_k(0;\epsilon) = \int_{-1}^{b} \left(u_0(x) - \tilde{u}^\epsilon[x; A(0;\epsilon)]\right) \phi_k w\, dx\,. \tag{65}$$

Since λ_0 is asymptotically exponentially small and the λ_k for $k \geq 1$ are bounded away from zero, to ensure that $v \ll \tilde{u}^\epsilon$ over exponentially long time intervals requires $h_0(t) \equiv 0$. Thus, the right sides of (64) and (65) must vanish when $k = 0$. Then, using $v \ll \tilde{u}^\epsilon$ to simplify the last term on the right of (64), we obtain

$$\left(\phi_0, \tilde{u}_t^\epsilon\right)_w \sim -\epsilon w v \phi_{0x}\Big|_{-1}^{b} + \left(\phi_0, L_\epsilon \tilde{u}^\epsilon\right)_w + \epsilon^\nu e^{-a/\epsilon}\left(g\phi_0, \tilde{u}^\epsilon\right)_w \tag{66}$$

and

$$\int_{-1}^{b} \tilde{u}^\epsilon[x; A(0;\epsilon)] \phi_0 w\, dx = \int_{-1}^{b} u_0(x) \phi_0 w\, dx\,. \tag{67}$$

Equation (66) will provide a differential equation for A and (67) will determine its initial value.

To obtain an *explicit* differential equation for A, we evaluate the terms of (66), as in [11]. Upon integrating by parts, we can show that

$$(\phi_0, L_\epsilon \tilde{u}^\epsilon)_w \sim \epsilon\,(\alpha - A_0)\,w(-1)\phi_{0x}(-1) - \epsilon\,(\beta - A_0)\,w(b)\phi_{0x}(b)\,. \tag{68}$$

Since v is exponentially small at the endpoints, (68) dominates the first term on the right side of (66). From (56), the identity

$$-\,\lambda_0\,(1,\phi_0)_w = \epsilon w \phi_{0x}\,\Big|_{-1}^{b} \tag{69}$$

follows. Next, (55) implies

$$(\phi_0, \tilde{u}^\epsilon_t)_w \sim \frac{dA}{dt}\,(\phi_0, 1)_w \qquad \text{and} \qquad (g\phi_0, \tilde{u}^\epsilon)_w \sim A\,(g\phi_0,1)_w\,. \tag{70}$$

Substituting (69) and (70) into (66) and neglecting the insignificant term yields

$$\frac{dA}{dt} \sim \left(-\lambda_0 + \epsilon^\nu e^{-a/\epsilon}\frac{(g\phi_0,1)_w}{(\phi_0,1)_w}\right) A + \frac{\epsilon}{(\phi_0,1)_w}\left(\alpha w(-1)\phi_{0x}(-1) - \beta w(b)\phi_{0x}(b)\right)\,. \tag{71}$$

For convenience, we re-normalize ϕ_0 so $\phi_0(0) = 1$. Then, using a boundary layer analysis to calculate the terms B_l and B_r in (57) as in [11], we obtain

$$\phi_{0x}(b) \;=\; -\frac{1}{\epsilon}\xi_r\gamma_r(\epsilon) \quad \text{for} \quad \gamma_r(\epsilon) = 1 - \frac{\epsilon}{\xi_r}\left(\frac{p'(b)}{p(b)} + \frac{(2m+1)}{b}\right) + O(\epsilon^2)\,, \tag{72}$$

$$\phi_{0x}(-1) \;=\; \frac{1}{\epsilon}\xi_l\gamma_l(\epsilon) \quad \text{for} \quad \gamma_l(\epsilon) = 1 + \frac{\epsilon}{\xi_l}\left(\frac{p'(-1)}{p(-1)} - (2m+1)\right) + O(\epsilon^2)\,. \tag{73}$$

Higher order coefficients in γ_l and γ_r can also be obtained. Next, since the difference $\phi_0 - 1$ is exponentially small near $x = 0$, we can calculate $(g\phi_0,1)_w$ and $(\phi_0,1)_w$ asymptotically by Laplace's method, as in [11]. This yields

$$(\phi_0,1)_w \;=\; \epsilon^{1/(2m+2)}\theta_\epsilon \quad \text{for} \quad \theta_\epsilon = \frac{r^{1/(2m+2)}}{(m+1)}\Gamma\left(\frac{1}{2m+2}\right) + O(\epsilon^{1/(m+1)})\,, \tag{74}$$

$$(g\phi_0,1)_w \;=\; \epsilon^{1/(2m+2)}g_\epsilon \quad \text{for} \quad g_\epsilon = \frac{r^{1/(2m+2)}}{(m+1)}g(0)\Gamma\left(\frac{1}{2m+2}\right) + O(\epsilon^{1/(m+1)})\,, \tag{75}$$

where $r \equiv 2(m+1)/p(0)$. Explicit formulas for some higher-order correction terms in θ_ϵ and g_ϵ are given in (3.9b) and (3.17b) of [11]. Substituting (71)–(73) into (69), we obtain the explicit asymptotic estimate

$$\lambda_0 \sim \epsilon^{-1/(2m+2)}\,\theta_\epsilon^{-1}\,[b^{2m+1}p(b)\gamma_r(\epsilon)e^{-\omega_r/\epsilon} + p(-1)\gamma_l(\epsilon)e^{-\omega_l/\epsilon}]\,, \tag{76}$$

where $\omega_l \equiv \int_0^{-1} t^{2m+1}p(t)\,dt$ and $\omega_r \equiv \int_0^b t^{2m+1}p(t)\,dt$.

Finally, by substituting (72)-(75) into (71), we obtain the explicit differential equation

$$\frac{dA}{dt} \sim \left(-\lambda_0 + \epsilon^\nu e^{-a/\epsilon}\frac{g_\epsilon}{\theta_\epsilon}\right) A + \frac{1}{\theta_\epsilon}\epsilon^{-1/(2m+2)}\,\left(\alpha\xi_l\gamma_l(\epsilon)e^{-\omega_l/\epsilon} + \beta\xi_r\gamma_r(\epsilon)e^{-\omega_r/\epsilon}\right)\,. \tag{77}$$

By using Laplace's method to evaluate (67), we obtain $A(0;\epsilon)\,(\phi_0,1)_w \sim (\phi_0 u_0, 1)_w$, so

$$A(0;\epsilon) \sim \frac{1}{\theta_\epsilon}\epsilon^{-1/(2m+2)}\,(\phi_0 u_0,1)_w\,. \tag{78}$$

TABLE 1

Comparison of numerical and asymptotic values for the principal eigenvalue for Example 1 at different values of ϵ.

ϵ	λ_0 (numerical)	λ_0 (1-term) (14)	λ_0 (2-term) (82)
0.100	0.15304×10^{-1}	0.17001×10^{-1}	0.15301×10^{-1}
0.075	0.33891×10^{-2}	0.37078×10^{-2}	0.34297×10^{-2}
0.050	0.15280×10^{-3}	0.16200×10^{-3}	0.15390×10^{-3}
0.025	0.10126×10^{-7}	0.10401×10^{-7}	0.10141×10^{-7}
0.020	0.76709×10^{-10}	0.78354×10^{-10}	0.76787×10^{-10}
0.0175	0.227×10^{-11}	0.23551×10^{-11}	0.23139×10^{-11}

The asymptotic evaluation of the integral $(\phi_0 u_0, 1)_w$ results from replacing $g(x)$ with $u_0(x)$ in (75). To leading order, we have $A(0;\epsilon) = u_0(0) + O(\epsilon^{1/(m+1)})$. Higher order correction terms for $A(0;\epsilon)$ are obtained explicitly in §4 for a specific example.

In summary, the main result of this section is an explicit asymptotic description of the metastable dynamics for (52)-(54), valid away from an initial time layer, namely

$$(79)\quad u(x,t) \sim \tilde{u}^{\epsilon}[x; A(t;\epsilon)] \equiv A(t;\epsilon) + [\alpha - A(t;\epsilon)]e^{-\xi_l(x+1)/\epsilon} + [\beta - A(t;\epsilon)]e^{-\xi_r(b-x)/\epsilon}$$

where A satisfies (77) and the initial value (78). If a is large enough, it is clear from (77) that $A(t;\epsilon)$ tends to its equilibrium value A_e, which is defined by setting the right side of (77) to zero. This equilibrium result was obtained in [11].

4 Some Examples of the Theory

Example 1: Consider the perturbed problem

$$(80)\qquad u_t = \epsilon u_{xx} - x u_x + s\sqrt{\frac{2}{\pi\epsilon}}\left(1 + x^2 + x^4\right)e^{-1/2\epsilon}u\,, \qquad -1 < x < 1\,,$$

$$(81)\qquad u(-1,t) = 2\,, \qquad u(1,t) = 1/2\,, \qquad u(x,0) = 1/2 + 3(x-1)^2/8\,,$$

where s is a constant.

We first obtain a high order estimate for the principal eigenvalue λ_0 of (56). Since $p(x) = 1$, $m = 0$ and $b = 1$ in (56), (72)-(74) imply that $\gamma_l(\epsilon) = \gamma_r(\epsilon) = 1 - \epsilon + O(\epsilon^2)$ and $\theta_\epsilon = \sqrt{2\pi} + o(\epsilon^k)$ for any $k > 0$. Thus, (76) yields the estimate:

$$(82)\qquad \lambda_0 = \sqrt{\frac{2}{\pi\epsilon}}\,[1 - \epsilon + O(\epsilon^2)]\,e^{-1/2\epsilon}\,.$$

Here we give two terms in the pre-exponential factor rather than only one term as in (14). To compare with this asymptotic result, we compute λ_0 numerically for various values of ϵ using the boundary value solver COLSYS [3]. From Table 1, it is clear that the two-term result (82) provides a significantly better determination of λ_0 than the one-term result.

Now let $s = 0$ in (80) so the asymptotically negligible term vanishes. Then, the ODE (77) for $A(t;\epsilon)$ reduces to $dA/dt \sim -\lambda_0(A - 5/4)$. Substituting $u_0(x)$ into (78) and evaluating the resulting integral asymptotically gives the two-term expansion $A(0;\epsilon) \sim 7/8 + 3\epsilon/8$ for the initial condition. Thus, for $t > 0$

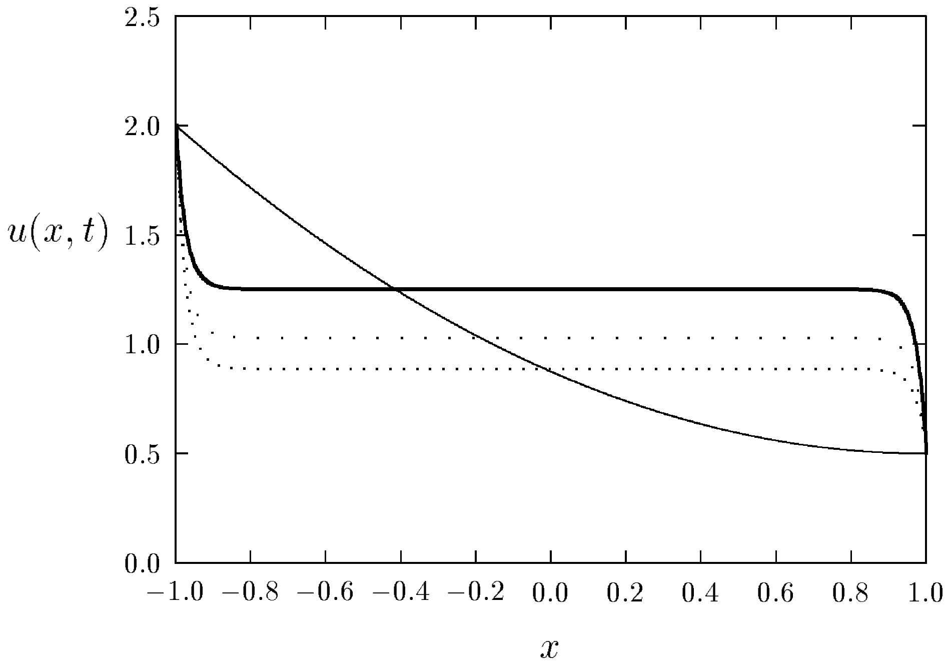

FIG. 1. *Plot of $u(x,t)$ versus x for Example 1 at the four different times, $t_0 = 0.0$ (light solid curve), $t_1 = 20.2$ (closely spaced dots), $t_2 = 4.90 \times 10^8$ (widely spaced dots) and $t_3 = 8.64 \times 10^9$ (heavy solid line). The parameter values are $s = 0$ and $\epsilon = 0.025$.*

$$A(t;\epsilon) \sim 5/4 + (-3/8 + 3\epsilon/8)\, e^{-\lambda_0 t}\,. \tag{83}$$

To compare with (83), we use the routine D03PAF of the NAG software library to numerically compute the solution $u(x,t)$ to (80)-(81) when $s = 0$. From this numerical solution, we output the value of $u(0,t)$, which gives the numerical prediction for $A(t;\epsilon)$. In Fig. 1 we plot the numerical solution $u(x,t)$ versus x at different times for the moderately small $\epsilon = 0.025$. Notice that the numerical solution has the boundary layer form predicted by (79). In Fig. 2 we plot $\log_{10}(t)$ versus A for the asymptotic result (83) and the numerically-computed result. Note that the two curves are virtually indistinguishable and that $A(t;\epsilon)$ remains constant over a time interval $t \approx 10^7$.

Now suppose that $s \neq 0$ in (80). Then, from (72)-(77), $A(t;\epsilon)$ satisfies

$$\frac{dA}{dt} \sim \lambda_0\,(5/4 - A) + s\sqrt{\frac{2}{\pi\epsilon}}\, e^{-1/2\epsilon}\,(1+\epsilon)\,A\,. \tag{84}$$

Note that when s is a constant independent of ϵ, the perturbing term in (84) has the same asymptotic order as λ_0 when $\epsilon \to 0$. The initial condition for (84) is $A(0;\epsilon) \sim 7/8 + 3\epsilon/8$ and the equilibrium value A_e for A, obtained by setting dA/dt to zero, is

$$A_e = \frac{5}{4}\left(\frac{1-\epsilon}{(1-s) - \epsilon(s+1)}\right)\,. \tag{85}$$

This result clearly shows the *super-sensitivity* of the equilibrium solution to the asymptotically exponentially small term in (80) since by varying s on the range $0 < s < 1$, A_e changes by $O(1)$. In Fig. 3, we show the very close agreement between the asymptotic result (84)

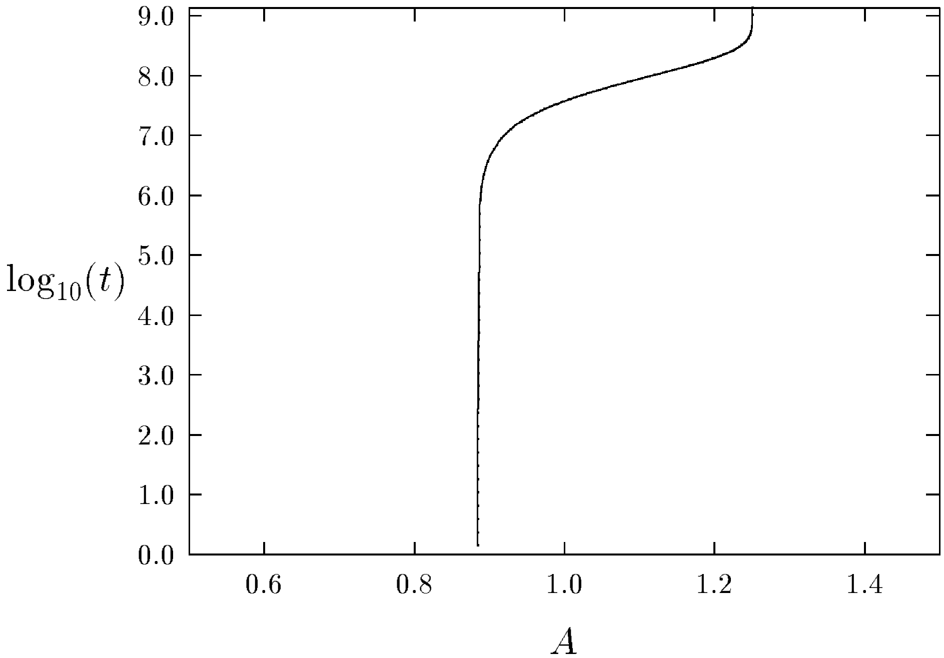

FIG. 2. *Plots of* $\log_{10}(t)$ *versus* A *for Example 1 from the full numerical solution (solid line) and from the asymptotic approximation (dotted line) when* $s = 0$ *and* $\epsilon = 0.025$.

for $A(t;\epsilon)$ and the corresponding numerically-computed result when $s = 0.5$ and $\epsilon = 0.025$. For these parameter values, we find $A_e \approx 2.635$.

Example 2: Next, consider the perturbed problem

$$u_t = \epsilon u_{xx} - x^3 u_x + s\epsilon^{-1/4}\left(1 + x^2 + x^4\right)e^{-1/4\epsilon}u\,, \qquad -1 < x < b\,, \tag{86}$$

$$u(-1,t) = 2\,, \qquad u(b,t) = \frac{1}{2}\,, \qquad u(x,0) = \frac{1}{2} + \frac{3}{2}\left(\frac{x-b}{1+b}\right)^2\,, \tag{87}$$

where $b > 0$ and s is a constant. The main difference between this and the previous example is that we now have a higher-order turning point and the interval is $-1 < x < b$.

We first obtain a higher-order estimate for λ_0. From (72)-(74), $\gamma_l(\epsilon) \sim 1 - 3\epsilon$, $\gamma_r(\epsilon) \sim 1 - 3\epsilon/b^4$ and $\theta_\epsilon = \Gamma(1/4)/\sqrt{2} + o(\epsilon^k)$ for any $k > 0$. Thus, from (76), we obtain

$$\lambda_0 \sim \left(\frac{1}{4\epsilon}\right)^{1/4}\frac{2}{\Gamma(1/4)}\left[b^3\left(1 - 3\epsilon/b^4\right)e^{-b^4/4\epsilon} + (1 - 3\epsilon)\,e^{-1/4\epsilon}\right]\,. \tag{88}$$

For this example, (72)-(77) imply the differential equation

$$\begin{aligned}\frac{dA}{dt} \sim\ & \left[-\lambda_0 + s\epsilon^{-1/4}\left(1 + 2\epsilon^{1/2}\Gamma(3/4)/\Gamma(1/4)\right)\,e^{-1/4\epsilon}\right] \\ & + \left(\frac{1}{4\epsilon}\right)^{1/4}\frac{2}{\Gamma(1/4)}\left[\frac{b^3}{2}\left(1 - 3\epsilon/b^4\right)e^{-b^4/4\epsilon} + 2\,(1 - 3\epsilon)\,e^{-1/4\epsilon}\right]\,.\end{aligned} \tag{89}$$

From (78), the initial condition is

$$A(0;\epsilon) \sim \frac{1}{2}\left[1 + 3\frac{b^2}{(1+b)^2}\right] + \frac{3\epsilon^{1/2}}{(1+b)^2}\frac{\Gamma(3/4)}{\Gamma(1/4)}\,, \tag{90}$$

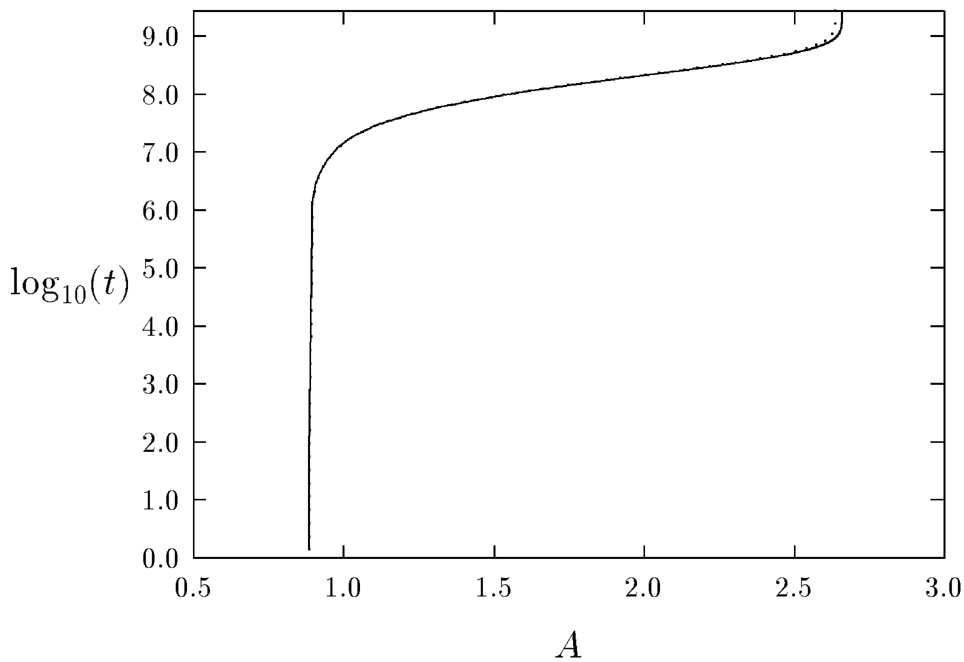

FIG. 3. *Plots of* $\log_{10}(t)$ *versus* A *for Example 1 from the full numerical solution (solid line) and from the asymptotic approximation (dotted line) when* $s = 0.5$ *and* $\epsilon = 0.025$.

when $\epsilon \to 0$. Comparing (89) with corresponding numerical results yields similar agreement as for Example 1.

Finally, we illustrate the *super-sensitivity* of the solution with respect to changes in the endpoint location b, by determining the equilibrium value A_e when $b > 1$ and $b = 1$. A simple calculation shows that

$$A_e = \frac{5}{4}\left(1 - \frac{s\sqrt{2}\mu\,\Gamma(1/4)}{4(1-3\epsilon)}\right)^{-1} \quad \text{for} \quad b = 1\,, \tag{91}$$

$$A_e = 2\left(1 - \frac{s\sqrt{2}\mu\,\Gamma(1/4)}{2(1-3\epsilon)}\right)^{-1} \quad \text{for} \quad b > 1\,, \tag{92}$$

with $\mu \equiv 1 + 2\epsilon^{1/2}\Gamma(3/4)/\Gamma(1/4)$. From (89) it is clear that A_e varies by an $O(1)$ amount as b is varied in an $O(\epsilon)$ region near $b = 1$.

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