Mean First Passage Time, Narrow Escape, and Fekete Points

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Topics of the Lectures

Lecture I: Mean First Passage Time, Narrow Escape, and Fekete Points

Lecture II: The Dynamics and Stability of Localized Spot Patterns in the 2-D Gray-Scott Model

Lecture III:

- A) **Optimization of the Persistence Threshold in the Diffusive Logistic Model with Environmental Heterogeneity (Mathematical Ecology)**
- B) **Concentration Behavior in Nonlinear Biharmonic Eigenvalues of MEMS**

Outline of Lecture I

KEY CONCEPTS AND THEMES:

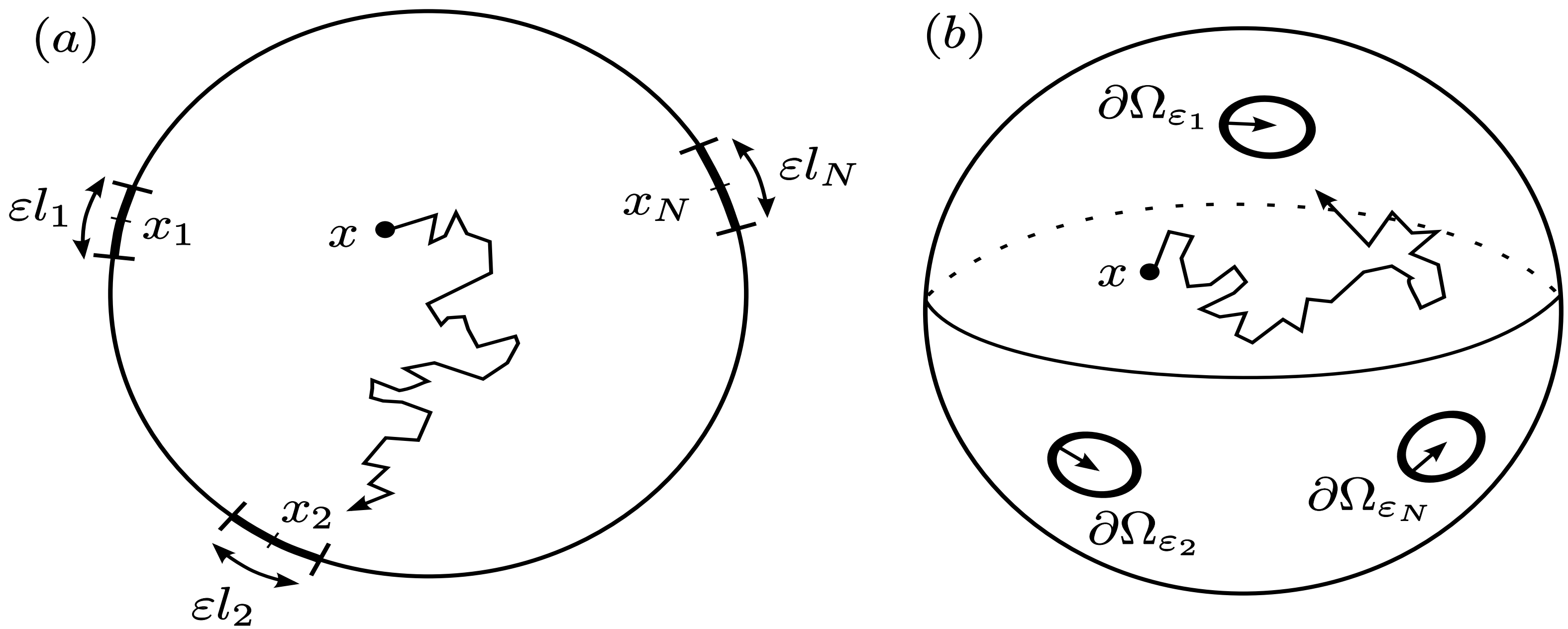
1. Eigenvalue problems in domains with perforated boundaries.
2. The optimization of the principal eigenvalue for these problems leads to certain discrete variational problems.
3. A generalization of the Fekete point problem on the sphere.
4. Central role of the Neumann Green function.

THREE SPECIFIC PROBLEMS CONSIDERED:

1. **Part I:** Diffusion on the Surface of the Sphere with Localized Traps (2-D).
2. **Part II:** The Mean First Passage Time for Escape from a Sphere (3-D)
3. **Part I:** The Mean First Passage Time for Escape from a 2-D Domain

Narrow Escape: Background I

Narrow Escape: Brownian motion with diffusivity D in Ω with $\partial\Omega$ insulated except for an (multi-connected) absorbing patch $\partial\Omega_a$ of measure $O(\varepsilon)$. Let $\partial\Omega_a \to x_j$ as $\varepsilon \to 0$ and $X(0) = x \in \Omega$ be initial point for Brownian motion.



The MFPT $v(x) = E\left[\tau | X(0) = x\right]$ satisfies (Z. Schuss (1980))

$$\Delta v = -\frac{1}{D}\,, \quad x \in \Omega\,,$$

$$\partial_n v = 0 \quad x \in \partial\Omega_r\,; \quad v = 0\,, \quad x \in \partial\Omega_a = \cup_{j=1}^{N} \partial\Omega_{\varepsilon_j}\,.$$

Narrow Escape: Background II

KEY GENERAL REFERENCES:

- Z. Schuss, A. Singer, D. Holcman, *The Narrow Escape Problem for Diffusion in Cellular Microdomains*, PNAS, **104**, No. 41, (2007), pp. 16098-16103.
- O. Bénichou, R. Voituriez, *Narrow Escape Time Problem: Time Needed for a Particle to Exit a Confining Domain Through a Small Window*, Phys. Rev. Lett, **100**, (2008), 168105.
- S. Condamin, et al., Nature, **450**, 77, (2007)
- S. Condamin, O. Bénichou, M. Moreau, Phys. Rev. E., **75**, (2007).

RELEVANCE OF NARROW ESCAPE TIME PROBLEM IN BIOLOGY:

- time needed for a reactive particle released from a specific site to activate a given protein on the cell membrane
- biochemical reactions in cellular microdomains (dendritic spines, synapses, microvesicles), consisting of a small number of particles that must exit the domain to initiate a biological function.
- determines reaction rate in Markov model of chemical reactions

Narrow Escape (3-D Domain): Intro I

In 3-D, let λ_1 be the principal eigenvalue when $\partial\Omega$ is perforated:

$$\Delta u + \lambda u = 0\,, \quad x \in \Omega\,; \quad \int_\Omega u^2\, dx = 1\,,$$

$$\partial_n u = 0 \quad x \in \partial\Omega_r\,, \quad u = 0\,, \quad x \in \partial\Omega_a = \cup_{j=1}^N \partial\Omega_{\varepsilon_j}\,.$$

The corresponding MFPT v satisfies

$$\Delta v = -\frac{1}{D}\,, \quad x \in \Omega\,; \qquad \partial_n v = 0 \quad x \in \partial\Omega_r$$

$$v = 0\,, \quad x \in \partial\Omega_a = \cup_{j=1}^N \partial\Omega_{\varepsilon_j}\,, \qquad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v\, dx \sim \frac{1}{D\lambda_1}\,.$$

OLD AND RECENT RESULTS IN 3-D:

- The principal eigenvalue is $\lambda_1 \sim \frac{2\pi\varepsilon}{|\Omega|}\sum_{j=1}^N C_j$ (MJW, Keller, SIAP, 1993) where C_j is the capacitance of the electrified disk problem:

$$\Delta_y w = 0\,, \quad y_3 \geq 0, \quad -\infty < y_1, y_2 < \infty\,,$$

$$w = 1\,, \quad y_3 = 0\,, \, (y_1, y_2) \in \partial\Omega_j\,; \; \partial_{y_3} w = 0\,, \quad y_3 = 0\,, \, (y_1, y_2) \notin \partial\Omega_j\,;$$

$$w \sim C_j/|y|\,, \quad |y| \to \infty\,.$$

Narrow Escape (3-D Domain): Intro II

- Analysis of $v(x)$ for two traps on the unit sphere (with undetermined $O(1)$ terms fit through Brownian particle simulations). **Ref:** D. Holcman et al., J. of Phys. A: Math Theor., **41**, (2008), 155001.
- For one circular trap of radius ε on the unit sphere Ω with $|\Omega| = 4\pi/3$,

$$\bar{v} \sim \frac{|\Omega|}{4\varepsilon D}\left[1 - \frac{\varepsilon}{\pi}\log\varepsilon + O(\varepsilon)\right].$$

  Ref: A. Singer et al. J. Stat. Phys., **122**, No. 3, (2006).
- For arbitrary Ω with smooth $\partial\Omega$ and one circular trap at $x_0 \in \partial\Omega$

$$\bar{v} \sim \frac{|\Omega|}{4\varepsilon D}\left[1 - \frac{\varepsilon}{\pi}H(x_0)\log\varepsilon + O(\varepsilon)\right].$$

  Here $H(x_0)$ is the mean curvature of $\partial\Omega$ at $x_0 \in \partial\Omega$. **Ref:** A. Singer, Z. Schuss, D. Holcman, Phys. Rev. E., **78**, No. 5, 051111, (2009).

Part II: Main Goal: Calculate a higher-order expansion for $v(x)$ and $\bar{v}$ as $\varepsilon \to 0$ in 3-D to determine the significant effect on $\bar{v}$ of the spatial configuration $\{x_1, \cdots, x_N\}$ of multiple absorbing boundary traps for a fixed area fraction of traps. Minimize $\bar{v}$ with respect to $\{x_1, \cdots, x_N\}$.

The MFPT on the Surface of a Sphere: Intro I

Consider Brownian motion on the surface of a sphere. The MFPT satisfies

$$\triangle_s v = -\frac{1}{D}\,, \qquad x \in \Omega_\varepsilon \equiv \Omega \backslash \cup_{j=1}^N \Omega_{\varepsilon_j}\,,$$

$$v = 0\,, \quad x \in \partial\Omega_{\varepsilon_j}\,; \qquad \bar{v} \sim \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} v\, ds\,.$$

Here Ω is the unit sphere, Ω_{ε_j} are localized non-overlapping circular traps of radius $O(\varepsilon)$ on Ω centered at x_j with $|x_j| = 1$ for $j = 1, \ldots, N$.

Eigenvalue Problem: The corresponding eigenvalue problem is

$$\triangle_s \psi + \lambda\psi = 0\,, \quad x \in \Omega_\varepsilon \equiv \Omega \backslash \cup_{j=1}^N \Omega_{\varepsilon_j}\,,$$

$$\psi = 0\,, \quad x \in \partial\Omega_{\varepsilon_j}\,; \quad \int_{\Omega_\varepsilon} \psi^2\, ds = 1\,.$$

- **Goal:** Calculate the principal eigenvalue λ_1 in the limit $\varepsilon \to 0$. Note that $\bar{v} \sim 1/(D\lambda_1)$ as $\varepsilon \to 0$. What is the effect of the trap locations?
- **Reference: [CSW]** D. Coombs, R. Straube, MJW, *"Diffusion on a Sphere with Traps..."*, SIAM J. Appl. Math., Vol. 70, No. 1, (2009), pp. 302–332.

Fekete Points: Two Conjectures

- **Part I: 2-D (Elliptic Fekete Points):** minimum point of the logarithmic energy $\mathcal{H}_L$ on the unit sphere

$$\mathcal{H}_L(x_1, \ldots, x_N) = -\sum_{j=1}^{N} \sum_{k>j}^{N} \log |x_j - x_k| \,, \quad |x_j| = 1 \,.$$

(References: Smale and Schub, Saff, Sloane, Kuijlaars, D. Boal, P. Palffy-Muhoray,...) Conjecture I: Are these points related to minimizing the average MFPT $\bar{v}$ for diffusion on the sphere?

- **Part II: 3-D (Fekete Points):** Let Ω be the unit sphere with N-circular holes on $\partial\Omega$ of a common radius. Conjecture II: Is minimizing $\bar{v}$ equivalent to minimizing the Coulomb energy on the sphere? This energy is

$$\mathcal{H}_C(x_1, \ldots, x_N) = \sum_{j=1}^{N} \sum_{k>j}^{N} \frac{1}{|x_j - x_k|} \,, \quad |x_j| = 1 \,.$$

Such Fekete points give the minimal energy configuration of "electrons" on a sphere (References: J.J. Thomson, E. Saff, N. Sloane, A. Kuijlaars, etc..)

The MFPT on the Surface of a Sphere: I

Previous Results for MFPT: For one trap at the north pole we get an ODE problem for $v(\theta)$:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\partial_\theta v\right) = -\frac{1}{D}\,, \quad \theta_c < \theta < \pi\,; \;\; v(\theta_c) = 0\,, \;\; v'(\pi) = 0\,.$$

The solution when $\theta_c = \varepsilon \ll 1$ is

$$v \sim \frac{1}{D}\left[-2\log\left(\frac{\varepsilon}{2}\right) + \log(1-\cos\theta)\right]\,, \quad \bar{v} \sim \frac{1}{D}\left[-2\log\left(\frac{\varepsilon}{2}\right) - 1\right]\,.$$

Ref: Lindeman, et al., Biophys. (1986); Singer et al. J. Stat. Phys. (2006).

Previous Results for λ_1: For one trap at the north pole $\theta_c = \varepsilon \ll 1$,

$$\partial_{\theta\theta}\psi + \cot(\theta)\partial_\theta\psi + \lambda\psi = 0\,, \quad \theta_c < \theta < \pi\,; \quad \psi(\theta_c) = 0\,, \;\; \psi'(\pi) = 0\,.$$

An explicit solution (Weaver (1983), Chao et. al. (1981), Biophys. J.) gives

$$\lambda_1 \sim \frac{\mu}{2} + \mu^2\left(-\frac{\log 2}{2} + \frac{1}{4}\right)\,; \quad \mu = -\frac{1}{\log\varepsilon}$$

Two-Term Asymptotic Result for the MFPT: I

A matched asymptotic expansion yields (**Ref: [CSW]**):

Principal Result: *Consider N perfectly absorbing circular traps of a common radius $\varepsilon \ll 1$ centered at x_j, for $j = 1, \ldots, N$ on S. Then, the asymptotics for the MFPT v in the "outer" region $|x - x_j| \gg O(\varepsilon)$ for $j = 1, \ldots, N$ is*

$$v(x) = -2\pi \sum_{j=1}^{N} A_j G(x; x_j) + \chi \,, \qquad \chi \equiv \frac{1}{4\pi} \int_S v \, ds \,,$$

where A_j for $j = 1, \ldots, N$, with $\mu \equiv -1/\log \varepsilon$ is

$$A_j = \frac{2}{ND} \left[1 + \mu \sum_{\substack{i=1 \\ i \neq j}}^{N} \log |x_i - x_j| - \frac{2\mu}{N} p(x_1, \ldots, x_N) + O(\mu^2) \right] .$$

The averaged MFPT $\bar{v} = \chi$ is given asymptotically by

$$\bar{v} \sim \chi = \frac{2}{ND\mu} + \frac{1}{D} \left[(2 \log 2 - 1) + \frac{4}{N^2} p(x_1, \ldots, x_N) \right] + O(\mu) \,.$$

Two-Term Asymptotic Result for the MFPT: II

Here the discrete energy $p(x_1, \ldots, x_N)$ *is the logarithmic energy*

$$p(x_1, \ldots, x_N) \equiv -\sum_{i=1}^{N} \sum_{j>i}^{N} \log |x_i - x_j| \, .$$

The Neumann Green function $G(x; x_0)$ that appears satisfies

$$\triangle_s G = \frac{1}{4\pi} - \delta(x - x_0) \, , \quad x \in S \, ; \quad \int_S G \, ds = 0$$

G is 2π periodic in ϕ and smooth at $\theta = 0, \pi$.

It is given analytically by

$$G(x; x_0) = -\frac{1}{2\pi} \log |x - x_0| + R \, , \qquad R \equiv \frac{1}{4\pi} \left[2 \log 2 - 1 \right] \, .$$

Remark: G appears in various studies of the motion of fluid vortices on the surface S of a sphere (P. Newton, S. Boatto, etc..).

Two-Term Asymptotic Result for the MFPT: III

Principal Result: *For N identical perfectly absorbing traps of a common radius ε centered at x_j, for $j = 1, \ldots, N$, on S, the principal eigenvalue has asymptotics, with* $\mu \equiv -1/\log \varepsilon$

$$\lambda(\varepsilon) \sim \frac{\mu N}{2} + \mu^2 \left[-\frac{N^2}{4} (2\log 2 - 1) - p(x_1, \ldots, x_N) \right] + O(\mu^3) \, .$$

- **Key Point:** $\lambda(\varepsilon)$ is maximized and $\bar{v}$ minimized at the minumum point of p, i.e. at the elliptic Fekete points for the sphere. Conjecture I holds.
- Can readily adapt the analysis to treat the case of N partially absorbing traps of different radii (see **[CSW,2009]**).
- For $N = 1$, v and λ_1 can be found from ODE problems, and we reproduce old results of Weaver (1983), Chao et. al. (1981), Biophys. J., and Lindeman and Laufenburger, Biophys, J. (1986)). In particular,

$$\lambda(\varepsilon) \sim \frac{\mu}{2} + \frac{\mu^2}{4} (1 - 2\log 2) \ .$$

Summing the Logarithmic Expansion

Can formulate a problem involving the Helmholtz Green function on the sphere that sums the infinite logarithmic expansion for $\lambda(\varepsilon)$. It reads as:

Principal Result: *Consider N perfectly absorbing traps of a common radius ε for $j = 1, \ldots, N$. Let $\nu(\varepsilon)$ be the smallest root of the transcendental equation*

$$\text{Det}\,(I + 2\pi\mu\mathcal{G}_h) = 0\,, \qquad \mu = -\frac{1}{\log\varepsilon}\,.$$

Here $\mathcal{G}_h$ is the Helmholtz Green function matrix with matrix entries

$$\mathcal{G}_{hjj} = R_h(\nu)\,; \qquad \mathcal{G}_{hij} = -\frac{1}{4\sin(\pi\nu)} P_\nu\left(\frac{|x_j - x_i|^2}{2} - 1\right)\,, \quad i \neq j\,.$$

Then, with an error of order $O(\varepsilon)$, we have $\lambda(\varepsilon) \sim \nu(\nu+1)$.

- $P_\nu(z)$ is the Legendre function of the first kind, with regular part

$$R_h(\nu) \equiv -\frac{1}{4\pi}\left[-2\log 2 + 2\gamma + 2\psi(\nu+1) + \pi\cot(\pi\nu)\right]\,,$$

where γ is Euler's constant and ψ is the Di-Gamma function.

Comparison of Asymptotics and Full Numerics

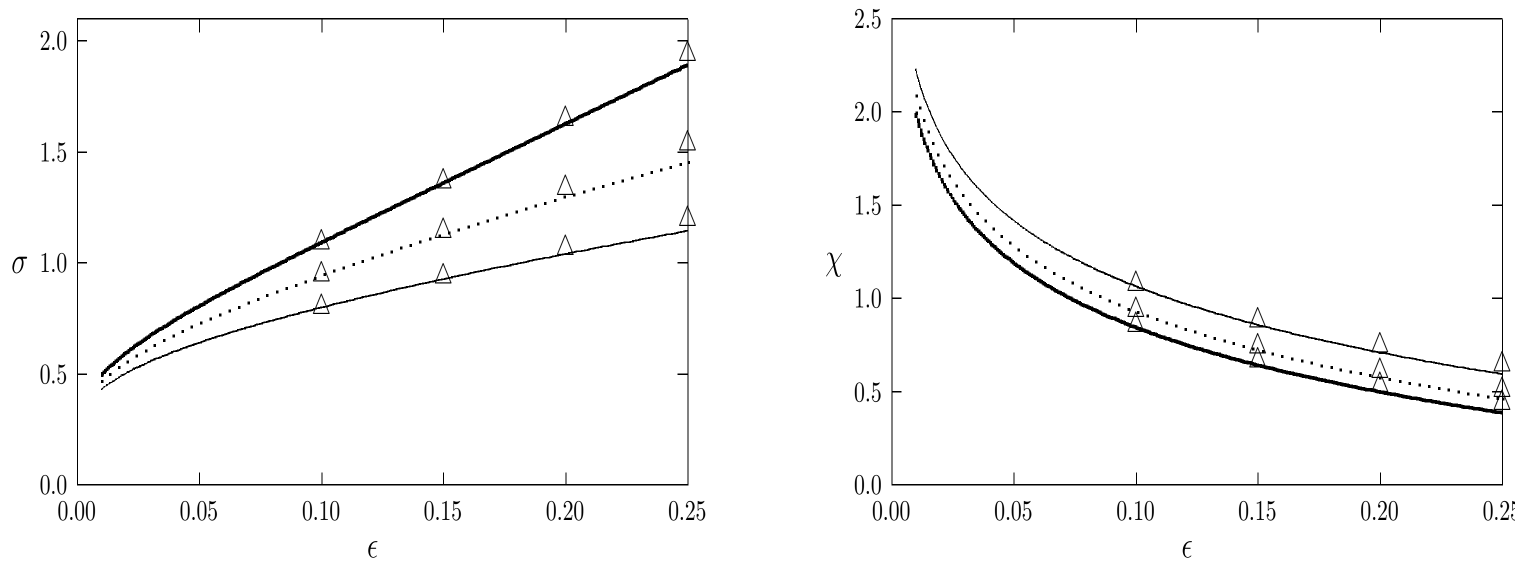
Table 1: Principal eigenvalue $\lambda(\varepsilon)$ for the 2- and 5-trap configurations. For the 2-trap case the traps are at $(\theta_1, \phi_1) = (\pi/4, 0)$ and $(\theta_2, \phi_2) = (3\pi/4, 0)$. Here, λ is the numerical solution found by COMSOL; λ^* corresponds to summing the log expansion; λ_2 is calculated from the two-term expansion.

	5 traps			2 traps		
ε	λ	λ^*	λ_2	λ	λ^*	λ_2
0.02	0.7918	0.7894	0.7701	0.2458	0.2451	0.2530
0.05	1.1003	1.0991	1.0581	0.3124	0.3121	0.3294
0.1	1.5501	1.5452	1.4641	0.3913	0.3903	0.4268
0.2	2.5380	2.4779	2.3278	0.5177	0.5110	0.6060

Note: For $\varepsilon = 0.2$ and $N = 5$, we get 5% trap area fraction. The agreement is still very good: 2.4% error (summing logs) and 8.3% error (2-term).

Effect on $\bar{v}$ of Locations of Traps on Sphere

EFFECT OF SPATIAL ARRANGEMENT OF $N = 4$ IDENTICAL TRAPS:



Note: $\varepsilon = 0.1$ corresponds to 1% trap surface area fraction.

Fig: Results for $\lambda(\varepsilon)$ (left) and $\chi(\varepsilon)$ (right) for three different 4-trap patterns with perfectly absorbing traps and a common radius ε. **Heavy solid:** $(\theta_1, \phi_1) = (0,0)$, $(\theta_2, \phi_2) = (\pi, 0)$, $(\theta_3, \phi_3) = (\pi/2, 0)$, $(\theta_4, \phi_4) = (\pi/2, \pi)$; **Solid:** $(\theta_1, \phi_1) = (0,0)$, $(\theta_2, \phi_2) = (\pi/3, 0)$, $(\theta_3, \phi_3) = (2\pi/3, 0)$, $(\theta_4, \phi_4) = (\pi, 0)$; **Dotted:** $(\theta_1, \phi_1) = (0,0)$, $(\theta_2, \phi_2) = (2\pi/3, 0)$, $(\theta_3, \phi_3) = (\pi/2, \pi)$, $(\theta_4, \phi_4) = (\pi/3, \pi/2)$. The marked points are computed from finite element package COMSOL.

Optimal Trap Configurations: Minimizing $\bar{v}$

For $N \to \infty$, the optimal energy for the discrete variational problem associated with elliptic Fekete points gives

$$\max\left[-p(x_1,\ldots,x_N)\right] \sim \frac{1}{4}\log\left(\frac{4}{e}\right)N^2 + \frac{1}{4}N\log N + l_1 N + l_2\,, \quad N \to \infty\,,$$

with $l_1 = 0.02642$ and $l_2 = 0.1382$.

Ref: E. A. Rakhmanov, E. B. Saff, Y. M. Zhou, (1994); B. Bergersen, D. Boal, P. Palffy-Muhoray, J. Phys. A: Math Gen., **27**, No. 7, (1994).

This yields a key scaling law for the minimum of the averaged MFPT:

Principal Result: *For $N \gg 1$, and N circular disks of common radius ε, and with small trap area fraction $N(\pi\varepsilon^2) \ll 1$ with $|S| = 4\pi$, then*

$$\min \bar{v} \sim \frac{1}{ND}\left[-\log\left(\frac{\sum_{j=1}^{N}|\Omega_{\varepsilon_j}|}{|S|}\right) - 4l_1 - \log 4 + O(N^{-1})\right].$$

Effect of Trap Fragmentation: Biophysics

Application: Estimate the averaged MFPT T for a surface-bound molecule to reach a molecular cluster on a spherical cell.

Physical Parameters: The diffusion coefficient of a typical surface molecule (e.g. LAT) is $D \approx 0.25\mu$m^2/s. Take $N = 100$ (traps) of common radius 10nm on a cell of radius 5μm. This gives a 1% trap area fraction:

$$\varepsilon = 0.002\,, \qquad N\pi\varepsilon^2/(4\pi) = 0.01\,.$$

Scaling Law: The scaling law gives an asymptotic lower bound on the averaged MFPT. For $N = 100$ traps, the bound is 7.7s, achieved at the elliptic Fekete points.

One Big Trap: As a comparison, for one big trap of the same area the averaged MFPT is 360s, which is very different.

Bounds: Therefore, for any other arrangement, $7.7\text{s} < T < 360\text{s}$.

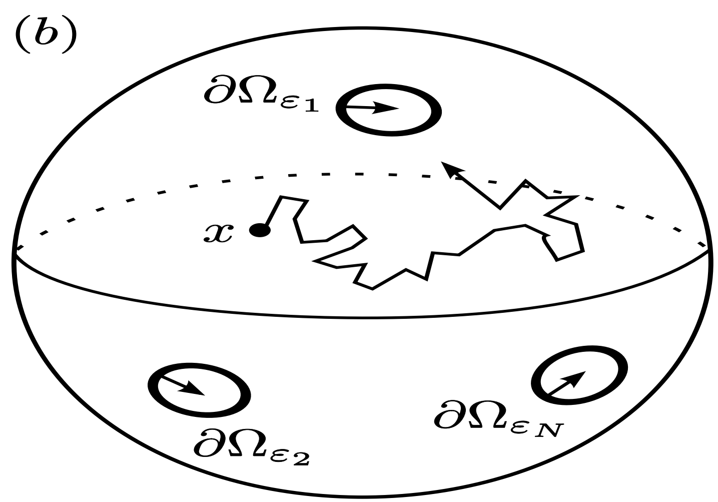
Conclusion: Both the Spatial Distribution and Fragmentation Effect of Localized Traps are Rather Significant even at Small Trap Area Fraction

Narrow Escape From a Sphere

Narrow Escape Problem for MFPT $v(x)$ and averaged MFPT $\bar{v}$:

$$\Delta v = -\frac{1}{D}\,, \quad x \in \Omega\,,$$

$$\partial_n v = 0 \quad x \in \partial\Omega_r\,; \quad v = 0\,, \quad x \in \partial\Omega_a = \cup_{j=1}^{N} \partial\Omega_{\varepsilon_j}\,.$$



Key Question: What is effect of spatial arrangement of traps on the unit sphere? Relation to Fekete Points? Need high order asymptotics.

Ref: [CWS] *An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems: SIAM J. Multiscale Modeling and Simulation: Part II: The Sphere (A. Cheviakov, M.J. Ward, R. Straube) (2010)*

The Surface Neumann G-Function for a Sphere

The surface Neumann G-function, G_s, is central to the analysis:

$$\triangle G_s = \frac{1}{|\Omega|}\,, \quad x \in \Omega\,; \qquad \partial_r G_s = \delta(\cos\theta - \cos\theta_j)\delta(\phi - \phi_j)\,, \quad x \in \partial\Omega\,,$$

Lemma: *Let* $\cos\gamma = x \cdot x_j$ *and* $\int_\Omega G_s\, dx = 0\,$. *Then* $G_s = G_s(x;x_j)$ *is*

$$G_s = \frac{1}{2\pi|x-x_j|} + \frac{1}{8\pi}(|x|^2+1) + \frac{1}{4\pi}\log\left[\frac{2}{1-|x|\cos\gamma + |x-x_j|}\right] - \frac{7}{10\pi}\,.$$

Define the matrix $\mathcal{G}_s$ using $R = -\frac{9}{20\pi}$ and $G_{sij} \equiv G_s(x_i;x_j)$ as

$$\mathcal{G}_s \equiv \begin{pmatrix} R & G_{s12} & \cdots & G_{s1N} \\ G_{s21} & R & \cdots & G_{s2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{sN1} & \cdots & G_{sN,N-1} & R \end{pmatrix},$$

Remark: As $x \to x_j$, G_s has a subdominant logarithmic singularity:

$$G_s(x;x_j) \sim \frac{1}{2\pi|x-x_j|} - \frac{1}{4\pi}\log|x-x_j| + R + o(1)\,.$$

Main Result for the MFPT: I

Principal Result: *For $\varepsilon \to 0$, and for N circular traps of radii εa_j centered at x_j, for $j = 1, \ldots, N$, the averaged MFPT $\bar{v}$ satisfies*

$$\bar{v} = \frac{|\Omega|}{2\pi\varepsilon D N\bar{c}} \left[1 + \varepsilon \log\left(\frac{2}{\varepsilon}\right) \frac{\sum_{j=1}^{N} c_j^2}{2N\bar{c}} + \frac{2\pi\varepsilon}{N\bar{c}} p_c(x_1, \ldots, x_N) - \frac{\varepsilon}{N\bar{c}} \sum_{j=1}^{N} c_j \kappa_j + O(\varepsilon^2 \log \varepsilon) \right] .$$

Here $c_j = 2a_j/\pi$ is the capacitance of the j^{th} circular absorbing window of radius εa_j, $\bar{c} \equiv N^{-1}(c_1 + \ldots + c_N)$, $|\Omega| = 4\pi/3$, and κ_j is defined by

$$\kappa_j = \frac{c_j}{2} \left[2\log 2 - \frac{3}{2} + \log a_j \right] .$$

Moreover, $p_c(x_1, \ldots, x_N)$ is a quadratic form in terms of $\mathcal{C} = (c_1, \ldots, c_N)^T$

$$p_c(x_1, \ldots, x_N) \equiv \mathcal{C}^T \mathcal{G}_s \mathcal{C} .$$

Remarks: 1) A similar result holds for non-circular traps. 2) The logarithmic term in ε arises from the subdominant singularity in G_s.

Main Result for the MFPT: II

- **One Trap:** Let $N = 1$, $c_1 = 2/\pi$, $a_1 = 1$, (compare with Holcman et al)

$$\bar{v} = \frac{|\Omega|}{4\varepsilon D}\left[1 + \frac{\varepsilon}{\pi}\log\left(\frac{2}{\varepsilon}\right) + \frac{\varepsilon}{\pi}\left(-\frac{9}{5} - 2\log 2 + \frac{3}{2}\right) + \mathcal{O}(\varepsilon^2\log\varepsilon)\right].$$

- **N Identical Circular Traps** of a common radius ε:

$$\bar{v} = \frac{|\Omega|}{4\varepsilon DN}\left[1 + \frac{\varepsilon}{\pi}\log\left(\frac{2}{\varepsilon}\right) + \frac{\varepsilon}{\pi}\left(-\frac{9N}{5} + 2(N-2)\log 2 + \frac{3}{2} + \frac{4}{N}\mathcal{H}(x_1,\ldots,x_N)\right) + \mathcal{O}(\varepsilon^2\log\varepsilon)\right],$$

with discrete energy $\mathcal{H}(x_1,\ldots,x_N)$ given by

$$\mathcal{H}(x_1,\ldots,x_N) = \sum_{i=1}^{N}\sum_{k>i}^{N}\left(\frac{1}{|x_i - x_k|} - \frac{1}{2}\log|x_i - x_k| - \frac{1}{2}\log\left(2 + |x_i - x_k|\right)\right).$$

- **Key point:** Minimizing $\bar{v}$ corresponds to minimizing $\mathcal{H}$. This discrete energy is a generalization of the purely Coulombic or logarithmic energies associated with Fekete points. So Conjecture II is false.

Key Steps in Derivation of Main Result: I

- Asymptotic expansion of global (outer) solution and local (inner solutions near each trap.
- Tangential-normal coordinate system used near each trap.
- The Neumann G-function has a subdominant logarithmic singularity on the boundary (related to surface diffusion). This fact requires adding "logarithmic switchback terms in ε" in the outer expansion (ubiquitous in Low Reynolds number flow problems)
- The leading-order local solution is the tangent plane approximation and yields electrified disk problem in a half-space, with capacitance c_j.
- **Key:** Need corrections to the tangent plane approximation in the inner region, i.e. near the trap. This higher order correction term in the inner expansion satisfies a Poission type problem. The far-field behavior of this inhomogeneous problem is a monopole term and determines κ_j.
- Asymptotic matching and solvability conditions (Divergence theorem) determine v and $\bar{v}$

Key Steps in Derivation of Main Result: II

The outer expansion has the form

$$v \sim \varepsilon^{-1} v_0 + \log(\varepsilon/2)\, v_{1/2} + v_1 + \varepsilon \log\left(\frac{\varepsilon}{2}\right) v_2 + \varepsilon v_3 + \cdots .$$

Here v_0 is an unknown constant, while v_1, v_2, and v_3 are functions to be determined. The unknown constant $v_{1/2}$ is a logarithmic switchback term.

For $k = 1, 2, 3$, v_k satisfies

$$\triangle v_k = -\frac{1}{D}\delta_{k1}\,, \quad x \in \Omega\,; \quad \partial_n v_k = 0\,, \quad x \in \partial\Omega \backslash \{x_1, \ldots, x_N\}\,,$$

where $\delta_{k1} = 1$ if $k = 1$ and $\delta_{k1} = 0$ for $k > 1$, and x_j for $j = 1, \ldots, N$ are centers of the trap locations on $\partial\Omega$.

Singularity behaviors for each v_k as $x \to x_j$ in terms of an unknown constant will be derived upon matching to the inner solution. The solvability condition (i.e. divergence theorem), will determine these unknown constants.

Key Steps in Derivation of Main Result: III

In the inner region we expand

$$v \sim \varepsilon^{-1} w_0 + \log\left(\frac{\varepsilon}{2}\right) w_1 + w_2 + \cdots \,,$$

in terms of a local orthogonal coordinate system (s_1, s_2, η). The trap Ω_j (in the stretched local variable) is

$$\Omega_j = \{(s_1, s_2) \,|\, s_1^2 + s_2^2 \leq a_j^2\}$$

We obtain that w_k for $k = 0, 1, 2$ satisfies

$$\mathcal{L} w_k \equiv w_{k\eta\eta} + w_{k s_1 s_1} + w_{k s_2 s_2} = \delta_{k2}\, \mathcal{F}_2 \,, \quad \eta \geq 0 \,, \quad -\infty < s_1, s_2 < \infty \,,$$

$$\partial_\eta w_k = 0 \,, \quad \text{on} \quad \eta = 0 \,, \;\; (s_1, s_2) \notin \Omega_j \,; \quad w_k = 0 \,, \quad \text{on} \quad \eta = 0 \,, \;\; (s_1, s_2) \in \Omega_j \,,$$

where $\delta_{22} = 1$ and $\delta_{k2} = 0$ if $k = 0, 1$. Here $\mathcal{F}_2$, is defined by

$$\mathcal{F}_2 \equiv 2\left(\eta w_{0\eta\eta} + w_{0\eta}\right) - \cot\theta_j \left(w_{0 s_2} - 2 s_2 w_{0 s_1 s_1}\right) \,.$$

Note: w_2 satisfies a Poisson-type problem (correction to tangent plane approximation of w_0 and w_1.)

Key Steps in Derivation of Main Result: IV

The leading order matching condition is that $w_0 \sim v_0$ as $\rho \equiv (\eta^2 + s_1^2 + s_2^2)^{1/2} \to \infty$.

Therefore, $w_0 = v_0\,(1 - w_c)$, where v_0 is an unknown constant and w_c satisfies

$$\mathcal{L} w_c = 0 \,, \qquad \eta \geq 0 \,, \quad -\infty < s_1, s_2 < \infty \,,$$

$$\partial_\eta w_c = 0 \,, \quad \eta = 0 \,, \;\; (s_1, s_2) \notin \Omega_j \,; \qquad w_c = 1 \,, \quad \text{on} \quad \eta = 0 \,, \;\; (s_1, s_2) \in \Omega_j \,.$$

With $\sigma \equiv (s_1^2 + s_2^2)^{1/2}$, the solution to this electrified disk problem is

$$w_c = \frac{2}{\pi} \sin^{-1}\left(\frac{a_j}{L}\right) \,,$$

$$L(\eta,\sigma) \equiv \frac{1}{2}\left(\left[(\sigma + a_j)^2 + \eta^2\right]^{1/2} + \left[(\sigma - a_j)^2 + \eta^2\right]^{1/2}\right) \,.$$

Thus, the far-field behavior is $w_c \sim c_j/\rho + \mathcal{O}(\rho^{-3})$ as $\rho \to \infty$, where $c_j = 2a_j/\pi$. This gives the far-field behavior

$$w_0 \sim -c_j v_0/\rho \,, \quad \rho \to \infty \,.$$

Key Steps in Derivation of Main Result: V

The matching condition then yields that $v_1 \sim -v_0 c_j / |x - x_j|$ as $x \to x_j$ for $j = 1, \ldots, N$. Thus, v_1 satisfies

$$\triangle v_1 = -\frac{1}{D}\,, \quad x \in \Omega\,; \quad \partial_r v_1|_{r=1} = -2\pi v_0 \sum_{j=1}^{N} \frac{c_j}{\sin\theta_j} \delta(\theta - \theta_j)\delta(\phi - \phi_j)\,.$$

The divergence theorem then determines v_0 as

$$v_0 = \frac{|\Omega|}{2\pi D N \bar{c}}\,, \qquad \bar{c} \equiv \frac{1}{N}\sum_{j=1}^{N} c_j\,, \qquad c_j = \frac{2a_j}{\pi}\,,$$

which yields the leading-order term in the expansion of $\bar{v}$.

The solution for v_1 up to an arbitrary constant χ_1 is

$$v_1 = -2\pi v_0 \sum_{i=1}^{N} c_i G_s(x; x_i) + \chi_1\,, \qquad \chi_1 \equiv |\Omega|^{-1} \int_\Omega v_1 \, dx\,.$$

We expand v_1 as $x \to x_j$ in terms of the local behavior of G_s. Note: G_s has a subdominant logarithmic singularity.

Key Steps in Derivation of Main Result: VI

At next order we get that v_2 solves

$$\triangle v_2 = 0\,, \quad x \in \Omega\,,$$

$$\partial_r v_2|_{r=1} = -2\pi \sum_{j=1}^{N} c_j \left(\frac{v_0 c_j}{2} + v_{1/2} \right) \frac{\delta(\theta - \theta_j)\delta(\phi - \phi_j)}{\sin\theta_j}\,.$$

By using the divergence theorem, we calculate $v_{1/2}$ as

$$v_{1/2} = -\frac{v_0}{2N\bar{c}} \sum_{j=1}^{N} c_j^2\,,$$

which specifies the second term in the MFPT.

Then, we solve for v_2 in terms of an arbitrary constant χ_2 as

$$v_2 = -2\pi \sum_{i=1}^{N} c_i \left(\frac{v_0 c_i}{2} + v_{1/2} \right) G_s(x; x_i) + \chi_2\,,$$

Key Steps in Derivation of Main Result: VII

The second inner correction w_2 satisfies a Poisson-type problem. The explicit solution to this problem yields that

$$w_2 \sim (B_j + \chi_1)\left(1 - \frac{c_j}{\rho}\right) + \frac{v_0 c_j}{2}\left[\log(\eta+\rho) - \frac{\eta}{\rho^3}(s_1^2 + s_2^2) + \frac{s_1^2 s_2}{\rho^3}\cot\theta_j - \frac{2\kappa_j}{\rho} + \mathcal{O}(\rho^{-2})\right], \quad \text{as} \quad \rho \to \infty\,.$$

Here χ_1 is unknown as yet, and B_j is given by

$$B_j = -2\pi v_0 \left(-\frac{9}{20\pi}c_j + \sum_{\substack{i=1\\ i\neq j}}^{N} c_i G_{sji}\right), \quad G_{sji} \equiv G_s(x_j; x_i)\,.$$

Many of these terms match identically with the outer expansion.
The unmatched monopole terms give rise to a singularity behavior for the outer correction v_3 of the form

$$v_3 \sim -\frac{c_j\left(B_j + \chi_1 + v_0\kappa_j\right)}{|x - x_j|} \quad \text{as} \quad x \to x_j\,.$$

Key Steps in Derivation of Main Result: VIII

Therefore, the problem for v_3 is

$$\triangle v_3 = 0\,, \quad x \in \Omega\,,$$

$$\partial_r v_3|_{r=1} = -2\pi \sum_{j=1}^{N} c_j \left(B_j + \chi_1 + v_0 \kappa_j\right) \frac{\delta(\theta-\theta_j)\delta(\phi-\phi_j)}{\sin\theta_j}\,.$$

The divergence theorem yields that $\chi_1 = -\frac{1}{N\bar{c}} \sum_{j=1}^{N} c_j \left[B_j + v_0 \kappa_j\right]$, which can be written in terms of $\mathcal{C}^T \equiv (c_1, \ldots, c_N)$ and the Green matrix as

$$\chi_1 = \frac{2\pi v_0}{N\bar{c}} p_c(x_1, \ldots, x_N) - \frac{v_0}{N\bar{c}} \sum_{j=1}^{N} c_j \kappa_j\,, \quad p_c(x_1, \ldots, x_N) \equiv \mathcal{C}^T \mathcal{G}_s \mathcal{C}\,.$$

This yields the trap-location dependent third term in $\bar{v}$. The final result is

$$\bar{v} = \frac{|\Omega|}{2\pi\varepsilon D N\bar{c}} \left[1 + \varepsilon \log\left(\frac{2}{\varepsilon}\right) \frac{\sum_{j=1}^{N} c_j^2}{2N\bar{c}} + \frac{2\pi\varepsilon}{N\bar{c}} p_c(x_1, \ldots, x_N) \right.$$

$$\left. - \frac{\varepsilon}{N\bar{c}} \sum_{j=1}^{N} c_j \kappa_j + O(\varepsilon^2 \log\varepsilon) \right]\,.$$

Key Steps in Derivation of Main Result: IX

Remark: The Principal Result still holds for arbitrary-shaped traps Ω_j with two changes: First, c_j is the **capacitance** of the electrified disk problem;

$$w_{c\eta\eta} + w_{cs_1s_1} + w_{cs_2s_2} = 0\,, \quad \eta \geq 0\,, \quad -\infty < s_1, s_2 < \infty\,,$$

$$\partial_\eta w_c = 0\,, \quad \text{on} \quad \eta = 0\,,\ (s_1,s_2) \notin \Omega_j\,; \quad w_c = 1\,, \quad \text{on} \quad \eta = 0\,,\ (s_1,s_2) \in \Omega_j\,,$$

$$w_c \sim c_j/\rho\,, \quad \text{as} \quad \rho \to \infty\,.$$

Secondly, the constant κ_j is now found from a modified electrified disk problem with non-constant potential on the disk:

$$w_{2h\eta\eta} + w_{2hs_1s_1} + w_{2hs_2s_2} = 0\,, \qquad \eta \geq 0\,, \quad -\infty < s_1, s_2 < \infty\,,$$

$$\partial_\eta w_{2h} = 0\,, \quad \eta = 0\,,\ (s_1,s_2) \notin \Omega_j\,; \quad w_{2h} = -\mathcal{K}(s_1,s_2)\,, \quad \eta = 0\,,\ (s_1,s_2) \in \Omega_j\,,$$

$$w_{2h} \sim -\kappa_j c_j/\rho\,, \quad \text{as} \quad \rho = (\eta^2 + s_1^2 + s_2^2)^{1/2} \to \infty\,,$$

where $\mathcal{K}(s_1,s_2)$ is defined from

$$\mathcal{K}(s_1,s_2) = -\frac{1}{4\pi}\int_{\Omega_j} \log|\tilde{s} - s|\, w_{c\eta}|_{\eta=0}\, ds\,.$$

Compare Asymptotic and Full Numerics for $\bar{v}$: I

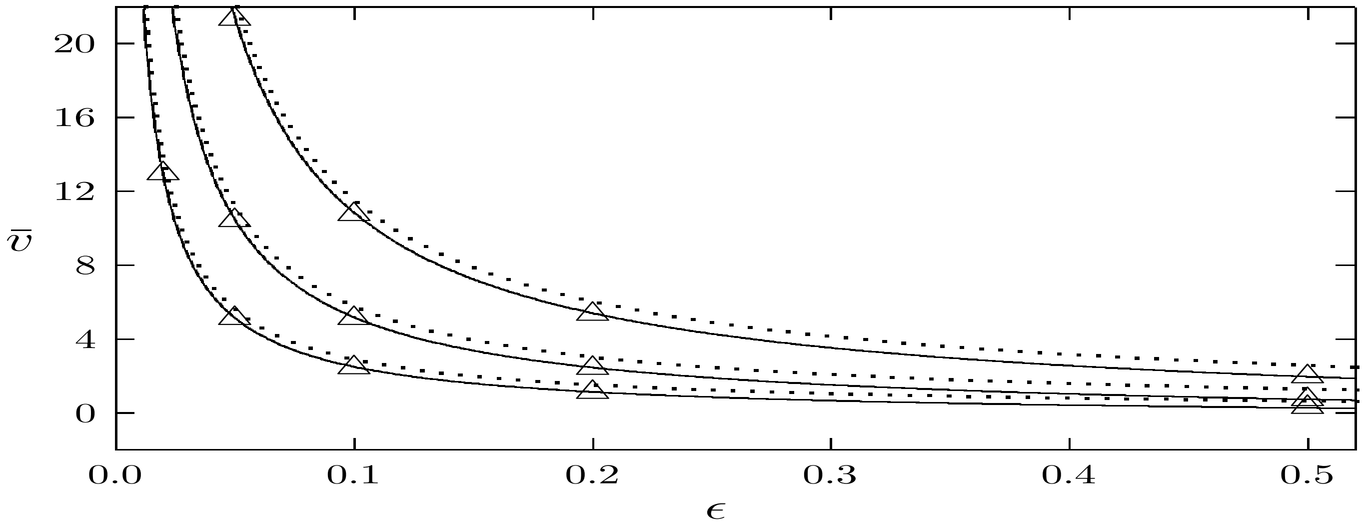


Fig: $\bar{v}$ vs. ε with $D = 1$ and either $N = 1, 2, 4$ equidistantly spaced circular windows of radius ε. **Solid:** 3-term expansion. **Dotted:** 2-term expansion. **Discrete:** COMSOL. **Top:** $N = 1$. **Middle:** $N = 2$. **Bottom:** $N = 4$.

	$N=1$			$N=4$		
ε	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_n$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_n$
0.02	53.89	53.33	52.81	13.47	13.11	12.99
0.05	22.17	21.61	21.35	5.54	5.18	5.12
0.10	11.47	10.91	10.78	2.87	2.51	2.47
0.20	6.00	5.44	5.36	1.50	1.14	1.13
0.50	2.56	1.99	1.97	0.64	0.28	0.30

Compare Asymptotic and Full Numerics for $\bar{v}$: II

Remark: For $\varepsilon = 0.5$ and $N = 4$, traps occupy $\approx 20\%$ of the surface. Yet, the 3-term asymptotics for $\bar{v}$ differs from COMSOL by only $\approx 7.5\%$.

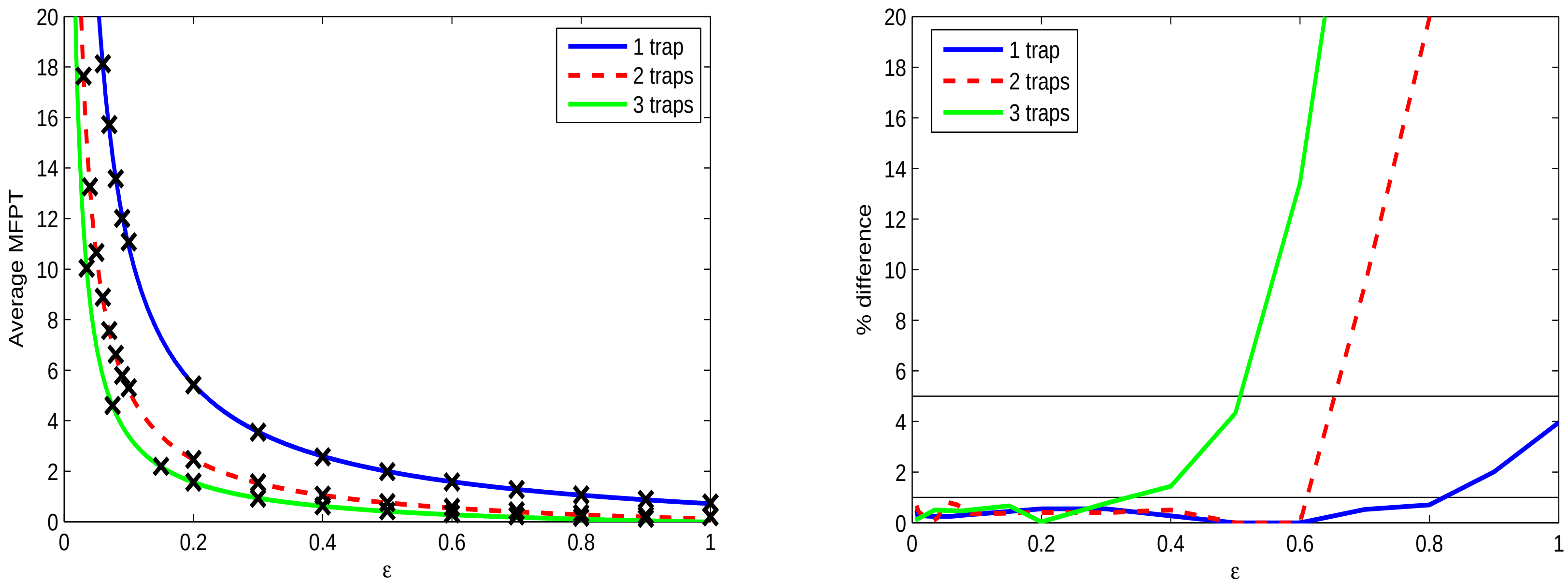
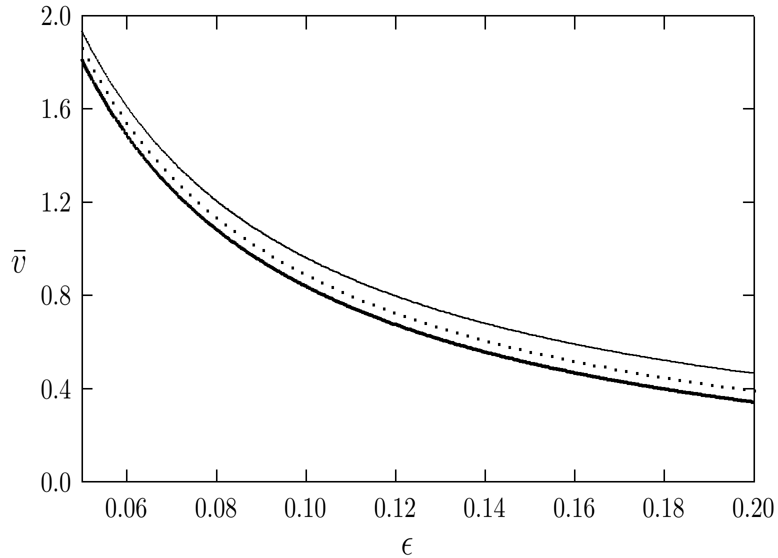
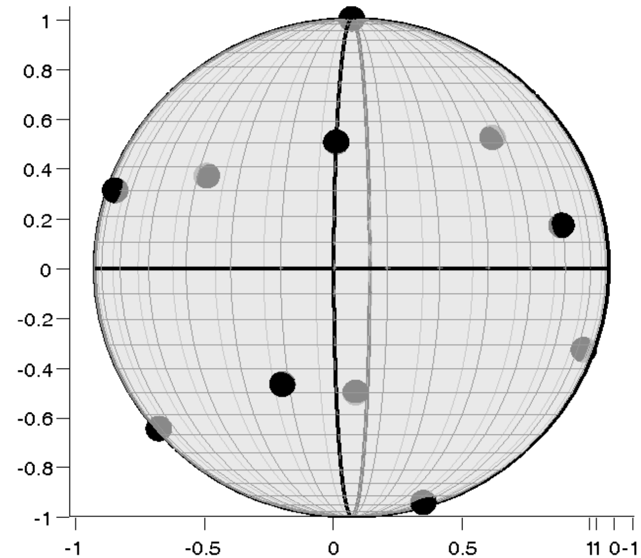


Fig: $\bar{v}$ vs. trap radius ε for $D = 1$ for one, two, and three traps equally spaced on the equator: curves (asymptotics), crosses (full numerics).

- For one trap, we get only 1% error for a trap of radius $\varepsilon \lesssim 0.8$, i.e. $\varepsilon^2/4 \times 100 = 16\%$ percent surface trap area fraction.
- For 3 traps, 1% error when $\varepsilon \lesssim 0.3$, which is 6.8% percent surface trap area fraction.

Effect of Location of Traps on Sphere





Plot: $\bar{v}(\varepsilon)$ for $D = 1$, $N = 11$, and three trap configurations. **Heavy:** global minimum of $\mathcal{H}$ (right figure). **Solid:** equidistant points on equator. **Dotted:** random.

- The effect of trap location is still rather significant.
- For $\varepsilon = 0.1907$, $N = 11$ traps occupy $\approx 10\%$ of surface area; The optimal arrangement gives $\bar{v} \approx 0.368$. For a single large trap with a 10% surface area, $\bar{v} \approx 1.48$; a result 3 times larger. Thus, trap fragmentation effects are important.

Trap Locations that Minimize $\bar{v}$: I

Numerical Computations: to compare optimal energies and point arrangements of $\mathcal{H}$, given by

$$\mathcal{H}(x_1, \ldots, x_N) = \sum_{i=1}^{N} \sum_{k>i}^{N} \left(\frac{1}{|x_i - x_k|} - \frac{1}{2} \log |x_i - x_k| - \frac{1}{2} \log \left(2 + |x_i - x_k|\right) \right) ,$$

with those of classic Coulomb or Logarithmic energies

$$\mathcal{H}_{\mathrm{C}} = \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{1}{|x_i - x_j|} , \quad \mathcal{H}_{\mathrm{L}} = -\sum_{i=1}^{N} \sum_{j>i}^{N} \log |x_i - x_j| .$$

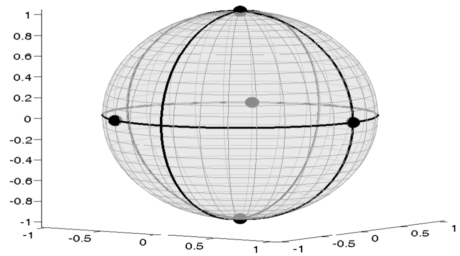
(A. Cheviakov, R. Spiteri, MJW).

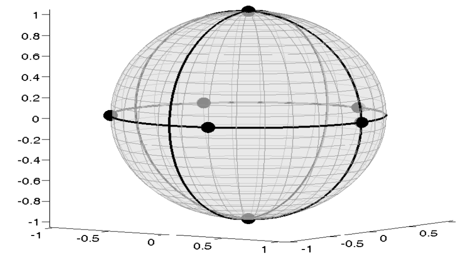
Numerical Methods:

- **Extended Cutting Angle method (ECAM).** (cf. G. Beliakov, Optimization Methods and Software, **19** (2), (2004), pp. 137-151).
- **Dynamical systems – based optimization (DSO).** (cf. M.A. Mammadov, A. Rubinov, and J. Yearwood, (2005)).
- Our computational results obtained by using the open software library GANSO where both the ECAM and DSO methods are implemented.

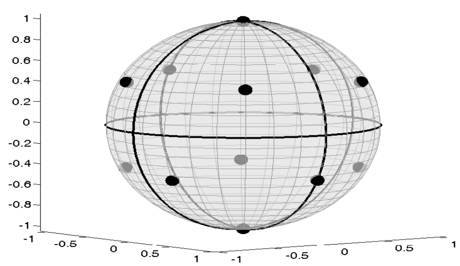
Trap Locations that Minimize $\bar{v}$: II

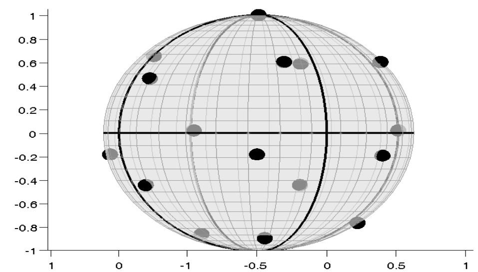
Left: **N=5:** Right: **N=11**.





Left: **N=12:** Right: **N=16:**





Computational Results and Questions:

- For $N = 2, \ldots, 20$ optimal point arrangments of the three energies coincide (proof?) **Q1:** Does this still occur for $N > 20$?.
- **Q2:** Derive a rigorous scaling law for the optimal energy of $\mathcal{H}$ when $N \gg 1$.
- **Q3:** Does the limiting result from $\mathcal{H}$ approach a homogenization theory result in the dilute trap area limit?

Trap Locations that Minimize $\bar{v}$: III

OPTIMAL ENERGIES: (Computations by R. Spiteri and A. Cheviakov)

N	$\mathcal{H}$	$\mathcal{H}_{\rm C}$	$\mathcal{H}_{\rm L}$
3	-1.067345	1.732051	-1.647918
4	-1.667180	3.674234	-2.942488
5	-2.087988	6.474691	-4.420507
6	-2.581006	9.985281	-6.238324
7	-2.763658	14.452978	-8.182476
8	-2.949577	19.675288	-10.428018
9	-2.976434	25.759987	-12.887753
10	-2.835735	32.716950	-15.563123
11	-2.456734	40.596450	-18.420480
12	-2.161284	49.165253	-21.606145
16	1.678405	92.911655	-36.106152
20	8.481790	150.881571	-54.011130
25	21.724913	243.812760	-80.997510
30	40.354439	359.603946	-113.089255
35	64.736711	498.569272	-150.192059
40	94.817831	660.675278	-192.337690
45	130.905316	846.188401	-239.453698
50	173.078675	1055.182315	-291.528601
55	221.463814	1287.772721	-348.541796
60	275.909417	1543.830401	-410.533163
65	336.769710	1823.667960	-477.426426

Trap Locations that Minimize $\bar{v}$: III

For $N \gg 1$, the optimal $\mathcal{H}$ has the scaling law

$$\mathcal{H} \approx \mathcal{F}(N) \equiv \frac{N^2}{2} \log\left(\frac{e}{2}\right) + b_1 N^{3/2} + N\left(b_2 \log N + b_3\right) + b_4 N^{1/2} + b_5 \log N + b_6\,,$$

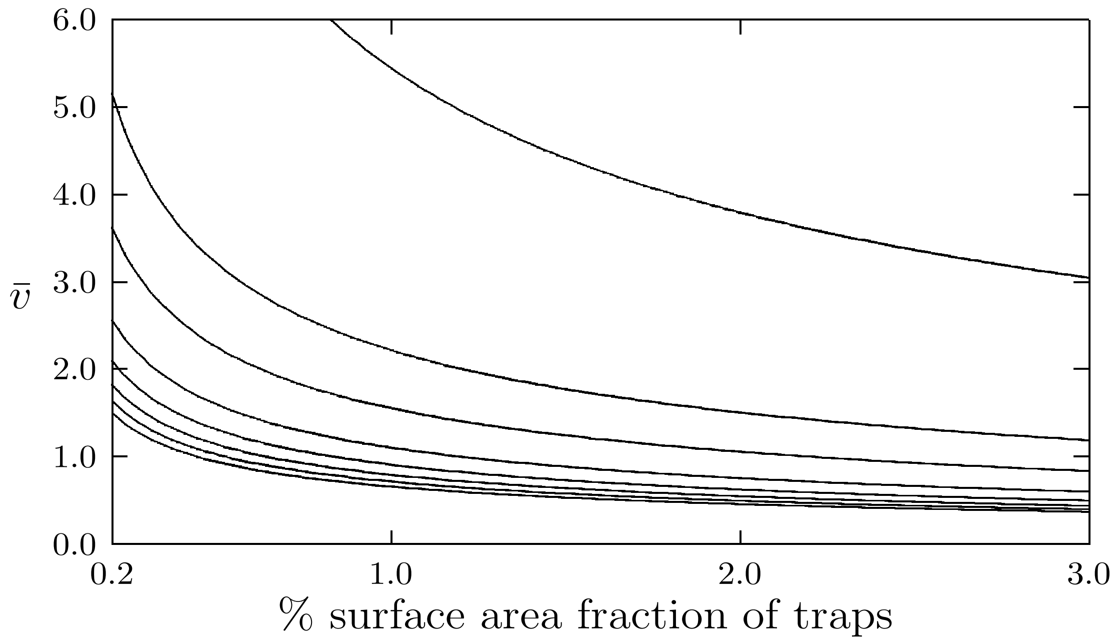
where a least-squares fit to the optimal energy yields

$$b_1 \approx -0.5668\,, \quad b_2 \approx 0.0628\,, \quad b_3 \approx -0.8420\,,$$
$$b_4 \approx 3.8894\,, \quad b_5 \approx -1.3512\,, \quad b_6 \approx -2.4523\,.$$

Scaling Law For $\bar{v}$: For $1 \ll N \ll 1/\varepsilon$, the optimal average MFPT $\bar{v}$, in terms of the trap surface area fraction $f = N\varepsilon^2/4$, satisfies

$$\bar{v} \sim \frac{|\Omega|}{8D\sqrt{fN}}\left[1 - \frac{\sqrt{f/N}}{\pi}\log\left(\frac{4f}{N}\right) + \frac{2\sqrt{fN}}{\pi}\left(\frac{1}{5} + \frac{4b_1}{\sqrt{N}}\right)\right]\,.$$

Effect of Fragmentation of the Trap Set



Plot: averaged MFPT $\bar{v}$ versus $\%$ trap area fraction for $N = 1, 5, 10, 20, 30, 40, 50, 60$ (top to bottom) at optimal trap locations.

- **Fragmentation** effect of traps on the sphere is a significant factor.
- **Only a minimal benefit** by increasing N when N is already large. Does $\bar{v}$ approach a limiting curve (homoegnization limit?)

Narrow Escape in 2-D: Surface G-Function

Consider the narrow escape problem from a 2-D domain. The surface Neumann G-function, G, with $\int_\Omega G\,dx = 0$ is

$$\triangle G = \frac{1}{|\Omega|}\,, \quad x \in \Omega\,; \quad \partial_n G = 0\,, \quad x \in \partial\Omega \backslash \{x_j\}\,,$$

$$G(x;x_j) \sim -\frac{1}{\pi}\log|x - x_j| + R(x_j;x_j)\,, \quad \text{as } \; x \to x_j \in \partial\Omega\,,$$

when x_j is a smooth point of $\partial\Omega$. Define the G-matrix by

$$\mathcal{G} \equiv \begin{pmatrix} R_1 & G_{12} & \cdots & G_{1N} \\ G_{21} & R_2 & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & \cdots & G_{N,N-1} & R_N \end{pmatrix}.$$

The local or inner problem near the j^{th} arc determines a constant d_j

$$w_{0\eta\eta} + w_{0ss} = 0\,, \quad 0 < \eta < \infty\,, \quad -\infty < s < \infty\,,$$

$$\partial_\eta w_0 = 0\,, \quad \text{on } \; |s| > l_j/2\,, \quad \eta = 0\,; \quad w_0 = 0\,, \quad \text{on } \; |s| < l_j/2\,, \quad \eta = 0\,.$$

$$w_0 \sim \log|y| - \log d_j + o(1)\,, \quad \text{as } \quad |y| \to \infty\,, \quad d_j = l_j/4\,.$$

Two-Term Asymptotic Result for the MFPT: I

Principal Result: *Consider N well-separated absorbing arcs of length εl_j for $j=1,\ldots,N$ centered at smooth points $x_j \in \partial\Omega$. Then, in the outer region $|x-x_j| \gg \mathcal{O}(\varepsilon)$ for $j=1,\ldots,N$ the MFPT is*

$$v \sim -\pi \sum_{i=1}^{N} A_i G(x;x_i) + \chi \,, \qquad \chi = \bar{v} = \frac{1}{|\Omega|}\int_\Omega v\, dx \,,$$

where a two-term expansion for A_j and χ (average MFPT) are

$$A_j \sim \frac{|\Omega|\mu_j}{ND\pi\bar{\mu}}\left(1 - \pi\sum_{i=1}^{N}\mu_i \mathcal{G}_{ij} + \frac{\pi}{N\bar{\mu}} p_w(x_1,\ldots,x_N)\right) + \mathcal{O}(|\mu|^2)\,,$$

$$\bar{v} \equiv \chi \sim \frac{|\Omega|}{ND\pi\bar{\mu}} + \frac{|\Omega|}{N^2 D\bar{\mu}^2} p_w(x_1,\ldots,x_N) + \mathcal{O}(|\mu|)\,.$$

Here p_w is a weighted discrete sum in terms of $\mathcal{G}_{ij}$:

$$p_w(x_1,\ldots,x_N) \equiv \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\mu_j\mathcal{G}_{ij}\,, \qquad \mu_j = -\frac{1}{\log(\varepsilon d_j)}\,, \quad d_j = \frac{l_j}{4}\,.$$

Summing the Logarithmic Expansion

Ref [PWPK]: S. Pillay, M. J. Ward, A. Pierce, T. Kolokolnikov, *An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems: Part I: Two-Dimensional Domains*, SIAM Multiscale Modeling and Simulation, (2010).

Remark: In [PWPK] there an analogous result that sums all logarithmic terms for $\bar{v}$ and for A_j for $j = 1, \ldots, N$.

Principal Result: *Define the matrices* $\mathcal{M} = diag(\mu_1, ..., \mu_N)$ *and* $E = ee^T/N$, *where* $e = (1, \ldots, 1)^T$. *Then, with an error of order* $\mathcal{O}(\varepsilon/[-\log\varepsilon])$, $A = (A_1, \ldots, A_N)^T$ *is the solution of the linear system*

$$\left(I + \pi\mathcal{M}(I - \frac{1}{\bar{\mu}}E\mathcal{M})\mathcal{G}\right) A = \frac{|\Omega|}{D\pi N\bar{\mu}}\,,$$

where $\bar{\mu} = (1/N)\sum_{j=1}^{N} \mu_j$. *In addition, the average MFPT* $\bar{v}$ *is*

$$\bar{v} = \frac{|\Omega|}{D\pi N\bar{\mu}} + \frac{\pi}{N\bar{\mu}} e^T \mathcal{M}\mathcal{G}A\,.$$

Remark: If $\mathcal{G}$ is a cyclic matrix, and the traps have the same length, then $A = \beta(1, \ldots, 1)^T$, and the theory simplifies.

Calculating the Surface Neumann G-Function

The key issue to obtain an explicit theory is to calculate the surface Neumann G-function and its regular part R.

- For the unit disk, G and R are

$$G(x;\xi) = -\frac{1}{\pi}\log|x-\xi| + \frac{|x|^2}{4\pi} - \frac{1}{8\pi}\,, \qquad R(\xi;\xi) = \frac{1}{8\pi}\,.$$

- G and R can be calculated explicitly for a rectangle by using Ewald-summation formulae.
- Can calculate G and R for smooth perturbation of unit disk (T. Kolokolnikov).
- For an arbitrary $2D$ domain, G and R are calculated numerically by the regularization of letting $\mu \to 0$ in $\Delta G - \mu G = 0$ with $G \sim -\frac{1}{\pi}\log|x - x_j|$ as $x_j \in \partial\Omega$ (A. Peirce, MJW).

Narrow Escape in 2-D: Simple Results

- **For** $N = 1$ **arc** of length $|\partial\Omega_{\varepsilon_1}| = 2\varepsilon$ (i.e. $d = 1/2$), then

$$v(x) \sim \frac{|\Omega|}{D\pi}\left[-\log\left(\frac{\varepsilon}{2}\right) + \pi\left(R(x_1;x_1) - G(x;x_1)\right)\right] + \mathcal{O}\left(\varepsilon/[-\log\varepsilon]\right)\ ,$$

$$\bar{v} = \chi \sim \frac{|\Omega|}{D\pi}\left[-\log\left(\frac{\varepsilon}{2}\right) + \pi R(x_1;x_1)\right] + \mathcal{O}\left(\varepsilon/[-\log\varepsilon]\right)\ .$$

  Extends work of Singer et al. to arbitrary Ω with smooth $\partial\Omega$.

- For N equidistant arcs on unit disk, i.e. $x_j = e^{2\pi i j/N}$ for $j = 1, \ldots, N$,

$$v(x) \sim \frac{1}{DN}\left[-\log\left(\frac{\varepsilon N}{2}\right) + \frac{N}{8} - \pi\sum_{j=1}^{N} G(x;x_j)\right] + \mathcal{O}\left(\varepsilon/[-\log\varepsilon]\right)\ ,$$

$$\chi \sim \frac{1}{DN}\left[-\log\left(\frac{\varepsilon N}{2}\right) + \frac{N}{8}\right] + \mathcal{O}\left(\varepsilon/[-\log\varepsilon]\right)\ ,$$

,

Compare Asymptotics and Full Numerics: I

For the unit disk, the following four trap configurations were studied:

- a single trap (arc) of arclength ε;
- two oppositely placed traps each of arclength ε;
- seven equally-spaced traps each of arclength ε;
- a three-trap configuration: two traps of length ε centered at $\theta = \pi/2$ and $3\pi/2$, and one larger trap of length 3ε located at $\theta = \pi$.

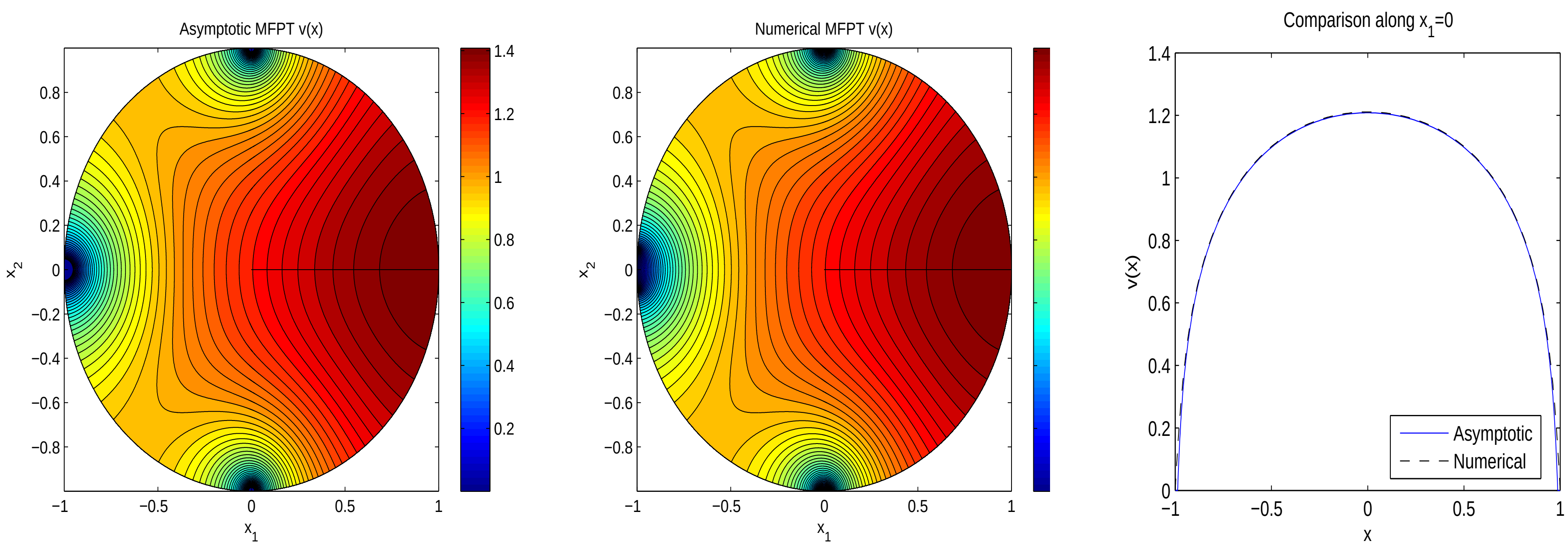
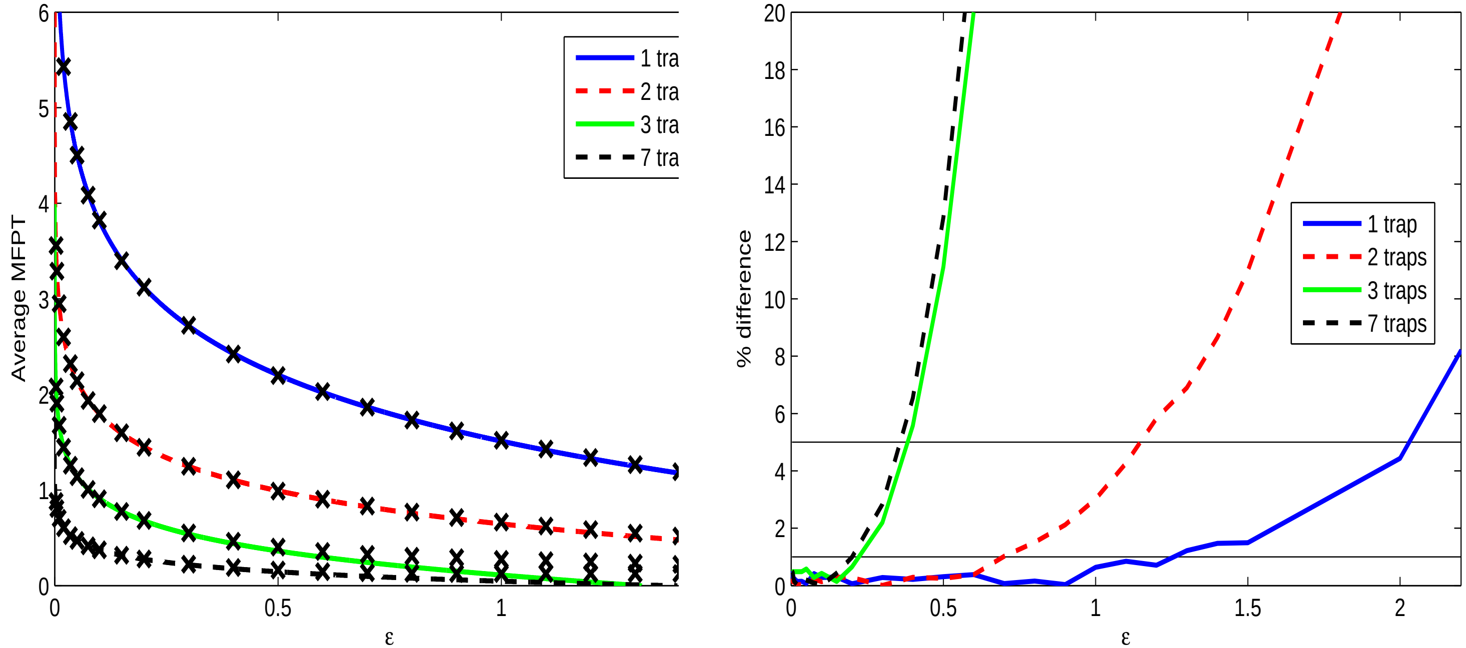


Fig: 3-trap contour plot (left: asymptotics) and (right: full numerics)

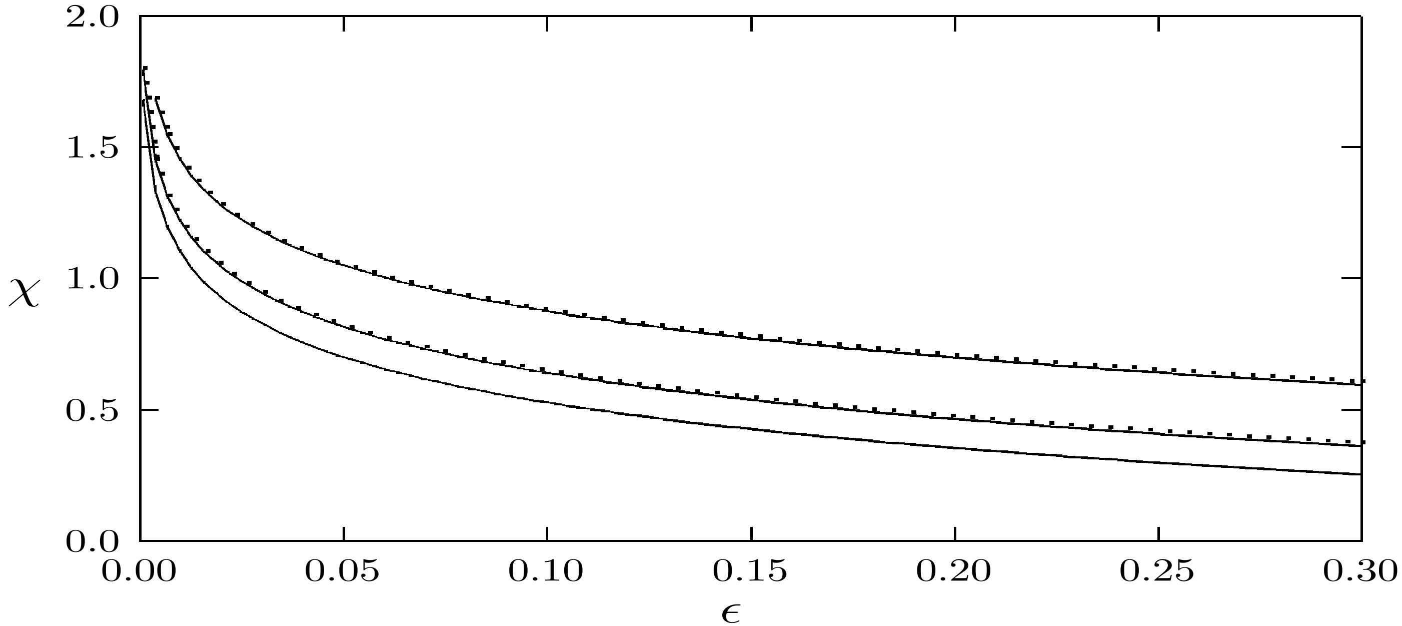
Compare Asymptotics and Full Numerics: II

Fig: $\bar{v}$ vs. trap size ε for $D = 1$ for one-, two-, three-, and seven-trap configurations on unit disk. Curves (asymptotics); crosses (Full numerics).



- For $N = 1$, we get 5% agreement for a trap length $\varepsilon \lesssim 2$, which is roughly $1/3$ of the perimeter of unit disk.
- For $N = 7$, we get 5% agreement for a trap length $\varepsilon \lesssim 0.35$, which is roughly 40% of perimeter.

Spatial Arrangement of Arcs is Very Significant



Plot: Plot of $\chi = \bar{v}$ versus ε from log-summed result (solid curves) vs. ε for $D = 1$ and for four traps on the boundary of the unit disk.

Trap locations at $x_1 = e^{\pi i/6}$, $x_2 = e^{\pi i/3}$, $x_3 = e^{2\pi i/3}$, $x_4 = e^{5\pi i/6}$ **(top curves)**; $x_1 = (1, 0)$, $x_2 = e^{\pi i/3}$, $x_3 = e^{2\pi i/3}$, $x_4 = (-1, 0)$ **(middle curves)**; $x_1 = e^{\pi i/4}$, $x_2 = e^{3\pi i/4}$, $x_3 = e^{5\pi i/4}$, $x_4 = e^{7\pi i/4}$ **(bottom curves)**.

Considerable effect of location of traps on $\bar{v}$ even at small ε.

Optimization of the MFPT for One Trap: I

Optimization: For one absorbing arc of length 2ε on a smooth boundary,

$$\bar{v} = \chi \sim \frac{|\Omega|}{D\pi} \left[-\log\left(\frac{\varepsilon}{2}\right) + \pi R(x_1; x_1) \right] , \qquad \mu_1 \equiv -\frac{1}{\log[\varepsilon/2]} ,$$

$$\lambda(\varepsilon) \sim \lambda^* \sim \frac{\pi \mu_1}{|\Omega|} - \frac{\pi^2 \mu_1^2}{|\Omega|} R(x_1; x_1) + \mathcal{O}(\mu_1^3) ,$$

Question: For $\partial\Omega$ smooth, is the global maximum of $R(x_1; x_1)$ attained at the global maximum of the boundary curvature κ? In other words, will a boundary trap centered at the maximum of κ minimize the heat loss from the 2-D domain? (i.e. yield the smallest λ_1, and thus the largest $\bar{v}$).

- **Remark:** Related to conjecture of J. Denzler, *Windows of a Given Area with Minimal Heat Diffusion*, Trans. Amer. Math. Soc., 351, (1999).
- **3-D Case:** The conjecture is true in 3-D since for $\varepsilon \to 0$,

$$\bar{v} \sim \frac{|\Omega|}{4\varepsilon D} \left[1 + \frac{(-\varepsilon \log \varepsilon)}{\pi} H(x_1) + \mathcal{O}(\varepsilon) \right] , \qquad \lambda_1 \sim \frac{1}{D\bar{v}} ,$$

where $H(x_1)$ is the mean curvature of $\partial\Omega$ at $x_0 \in \partial\Omega$. **Ref:** D. Holcman et al., Phys. Rev. E. 78, No. 5, 051111, (2009).

Optimization of the MFPT for One Trap: II

Principal Result in 2-D [PWPK]: *The global maximum of $R(x_1, x_1)$ does not necessarily coincide with the global maximum of the curvature κ of the boundary of a smooth perturbation of the unit disk. Consequently, for $\varepsilon \to 0$, $\lambda_1(\varepsilon)$ does not necessarily have a global minimum at the location of the global maximum of the boundary curvature κ.*

Proof: based on the following explicit perturbation formula for R for arbitrary smooth perturbations of the unit disk (T. Kolokolnikov)

Principal Result [PWPK]: *Let Ω be a smooth perturbation of the unit disk with boundary given in terms of polar coordinates by*

$$r = r(\theta) = 1 + \delta\sigma(\theta)\,, \quad \sigma(\theta) = \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)\,, \qquad \delta \ll 1\,,$$

where σ is C^2 smooth. Let $x_1 = x_1(\theta_1) = (r_1 \cos\theta_1, r_1 \sin\theta_1)$ be a point on the boundary where $r_1 = 1 + \delta\sigma(\theta_1)$. Then, for $\delta \ll 1$,

$$\rho'(\theta_1) \equiv \frac{d}{d\theta} R(x;x)|_{x=x_1} = \frac{\delta}{\pi} \sum_{n=1}^{\infty} \left(n^2 + n - 2\right) \left(b_n \cos n\theta_1 - a_n \sin n\theta_1\right) + \mathcal{O}(\delta^2)\,.$$

Optimization of the MFPT for One Trap: III

We construct a counterexample with the choice

$$\sigma(\theta) = \cos(2\theta) + b\sin(3\theta)\,, \quad \text{so that} \quad \kappa \sim 1 + \delta\left[3\cos(2\theta) + 8b\sin(3\theta)\right]\,.$$

We then calculate

$$\kappa'(\theta) = -6\delta\left[\sin(2\theta) - 4b\cos(3\theta)\right]\,, \quad \kappa''(\theta) = -12\delta\left[\cos(2\theta) + 6b\sin(3\theta)\right]\,,$$

$$\rho'(\theta) = -\frac{4\delta}{\pi}\left[\sin(2\theta) - \frac{5b}{2}\cos(3\theta)\right]\,, \quad \rho''(\theta) = -\frac{8\delta}{\pi}\left[\cos(2\theta) + \frac{15b}{4}\sin(3\theta)\right]\,.$$

Note: $\theta = \pi/2$ and $\theta = 3\pi/2$ are the only two critical points shared by κ and ρ. For the range $-4/15 < b < -1/6$, then

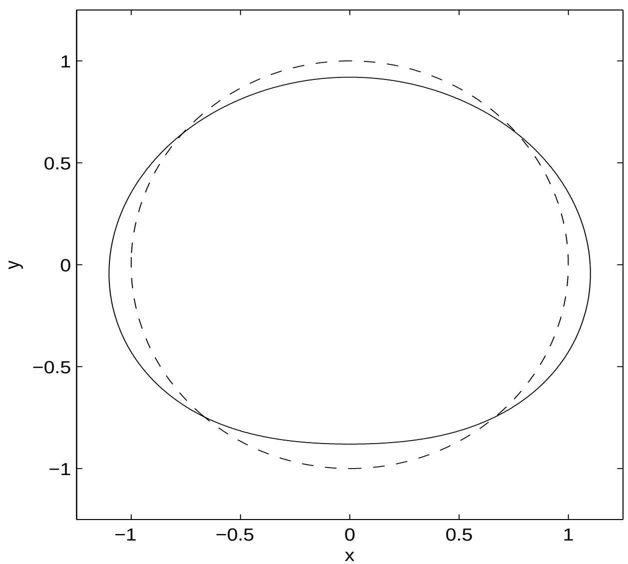
- κ has a local maximum while ρ has a local minimum at $\theta = \pi/2$.
- κ has a local minimum while ρ has a local minimum at $\theta = 3\pi/2$.

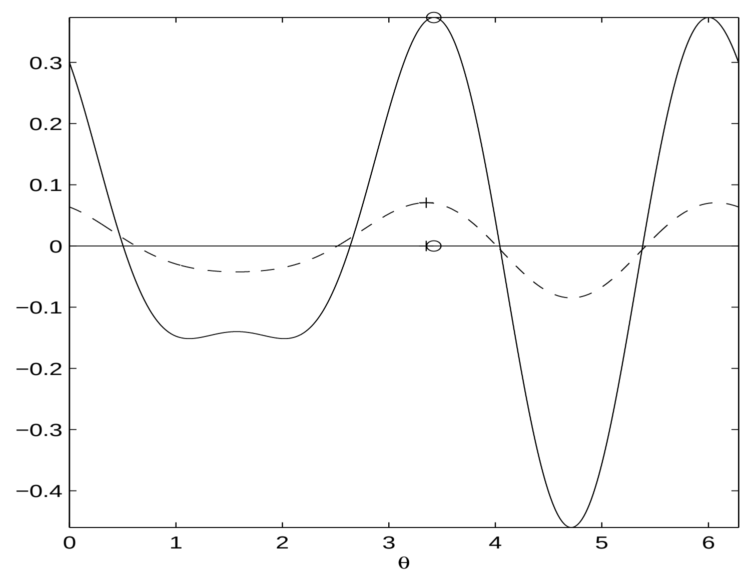
Since the only critical points shared by κ and ρ are local minima of ρ, then the absolute maximum value of ρ occurs at a point where $\kappa'(\theta) \neq 0$. Therefore, the point(s) where the absolute maximum value of ρ is attained do not coincide precisely with the maximum curvature of the boundary.

Optimization of the MFPT for One Trap: IV

Left Fig: Plot of the domain boundary when $\delta = 0.1$ and $b = -1/5$.

Right Fig: Plot of $\rho(\theta) - C$ and $\kappa(\theta) - 1$ for $\delta = 0.1$ and $b = -1/5$ showing that the global maxima of ρ and $\kappa - 1$ occur at different, but nearby, locations.





Further Directions and Open Problems

- Establish relation between the optimal MFPT as $N \to \infty$ with results that can be obtained from dilute fraction limit of homogenization theory.
- Narrow escape problems in arbitrary 3-d domains: require Neumann G-functions in 3-D with boundary singularity. How does one numerically compute the regular part of the singularity?
- Surface diffusion on arbitrary 2-d surfaces: require Neumann G-function and regular part on surface. What is the effect of the mean curvature of the surface on the MFPT.
- Include chemical reactions occurring within each trap, with binding and unbinding events. Can diffusive transport between traps induce synchronous time-dependent oscillations for localized reactions (ode's) valid inside each trap (with Y. Nec and D. Coombs)? Yields a new Steklov-type eigenvalue problem.
- Couple surface diffusion to diffusion processes within the cell.

References

Available at: http://www.math.ubc.ca/ ward/prepr.html

- D. Coombs, R. Straube, M.J. Ward, *Diffusion on a Sphere with Localized Traps: Mean First Passage Time, Eigenvalue Asymptotics, and Fekete Points*, SIAM J. Appl. Math., Vol. 70, No. 1, (2009), pp. 302–332.
- A. Cheviakov, M. J. Ward, R. Straube, *An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems: Part II: The Sphere*, SIAM Multiscale Modeling and Simulation, Vol. 8, No. 3, (2010), pp. 836–870.
- S. Pillay, M. J. Ward, A. Pierce, T. Kolokolnikov, *An Asymptotic Analysis of the Mean First Passage Time for Narrow Escape Problems: Part I: Two-Dimensional Domains*, SIAM Multiscale Modeling and Simulation, Vol. 8, No. 3, (2010), pp. 803–835.
- A. Cheviakov, A. Reimer, M. J. Ward, *A Comparison of Asymptotic and Numerical Results for the Mean First Passage Time for Narrow Escape Problems*, submitted, Phys. Rev. E., Feb. 2011, (21 pages).