I) Optimization of the Persistence Threshold in Diffusive Logistic Model; II) Concentration Behavior in Nonlinear Biharmonic Eigenvalue Problems of MEMS

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Joint With: A. Lindsay (U. Arizona), M.C. Kropinski (SFU): Based on the Ph.D thesis of A. Lindsay, UBC, (2010)

Persistence Problem: Introduction I

Consider the diffusive logistic equation for $u(x,t)$ with $x \in \Omega \in \mathbb{R}^2$

$$u_t = D\Delta u + u\left[m_\varepsilon(x) - c(x)u\right], \quad x \in \Omega; \quad \partial_n u = 0, \quad x \in \partial\Omega.$$

Here D is the constant diffusivity.

We linearize around the zero solution with $u = e^{\mu D t}\phi(x)$ and set $\mu = 0$

$$\Delta\phi + \lambda m_\varepsilon(x)\phi = 0, \quad x \in \Omega; \qquad \partial_n \phi = 0, \quad x \in \partial\Omega.$$

The bifurcation parameter $\lambda \geq 0$ is defined by

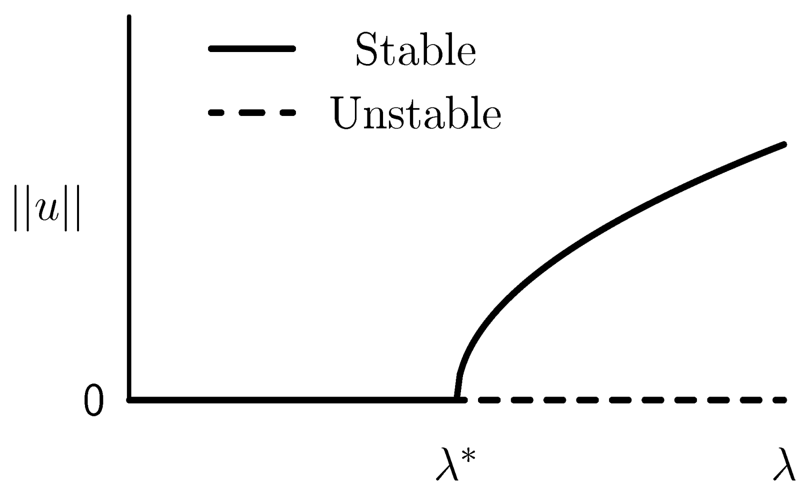
$$\lambda = 1/D.$$

- The extinct solution $u = 0$ exists for all $\lambda \geq 0$. Depending on the form of the spatially depdendent growth rate $m_\varepsilon(x)$, at some critical value of λ there can be a transcritical bifurcation to a spatially dependent solution. This leads to the idea of a persistence threshold.
- **Note:** Growth rate m_ε changes sign $\rightarrow$ indefinite weight eig. problem (no standard oscillation theory, or standard variational characterization of eigenvalues, etc..).

Persistence Problem: Introduction II

Key Previous Result I: Assume that $\int_\Omega m_\varepsilon \, dx < 0$, but that $m_\varepsilon > 0$ on a set of positive measure. Then, there exists a positive principal eigenvalue $\lambda_1 = \lambda^\star$, i.e. the extinction threshold, with corresponding positive eigenfunction ϕ (Brown and Lin, (1980)).

Key Previous Result II: Transcritical bifurcation: $u \to u_\infty(x) \neq 0$ as $t \to \infty$ if $\lambda > \lambda^\star$, while $u \to 0$ as $t \to \infty$ if $0 < \lambda < \lambda^\star$. (many authors; Cantrell, Cosner, Berestycki, etc..)



Persistence Problem: Introduction III

Key Previous Result II: The optimal growth rate $m_\varepsilon(x)$ is of bang-bang type. (Theorem 1.1 of Lou and Yanagida, JJAM, 2006, for 2-D).

Main Goal: Minimize λ_1 wrt $m_\varepsilon(x)$, subject to a fixed $\int_\Omega m_\varepsilon \, dx < 0$: i.e. determine the largest D that can still allow for the persistence of the species. This is a **long-standing open problem** of determining the optimal shape of $m_\varepsilon(x)$ **in a 2-D domain.** (Cantrell and Cosner 1990's, Lou and Yanagida, (2006); Kao, Lou, and Yanagida, (2008); Roques and Stoica, (2007); Berestycki, Hamel, (2005,2006)).

Remark: In a 1-D domain, this problem has been solved (Lou and Yanagida, JJAM (2006)). *The optimal* $m_\varepsilon(x)$ *in 1-D is to concentrate favorable resources near one of the endpoints of the domain, and to have only one favorable patch.*

Persistence Problem: Patch Model I

Patch Model: The eigenvalue problem for the persistence threshold is

$$\Delta\phi + \lambda\, m_\varepsilon(x)\phi = 0\,, \quad x \in \Omega; \quad \partial_n \phi = 0\,, \quad x \in \partial\Omega\,; \quad \int_\Omega \phi^2\, dx = 1\,.$$

Note that the population is confined in the domain (reflecting boundary condition). The piecewise-constant growth rate $m_\varepsilon(x)$ is defined as

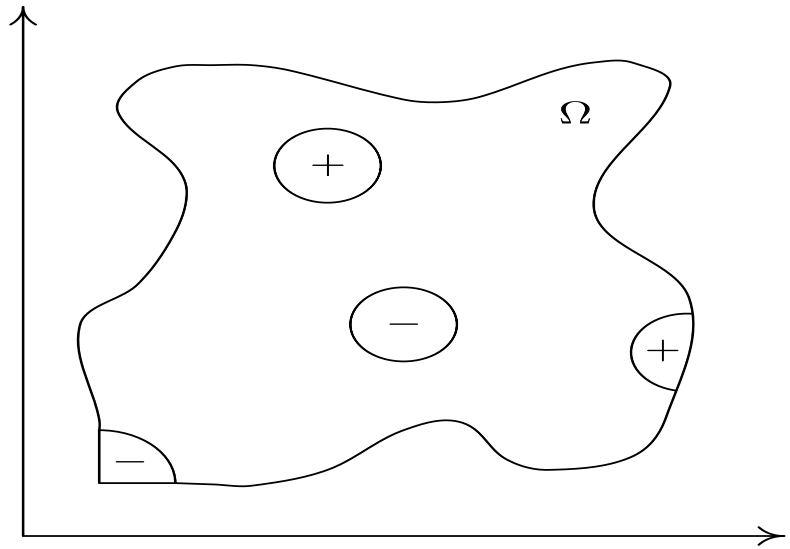
$$m_\varepsilon(x) = \begin{cases} m_j/\varepsilon^2\,, \quad x \in \Omega_{\epsilon_j} \equiv \{x \,|\, |x - x_j| = \varepsilon\rho_j \cap \Omega\}\,, \quad j = 1, \ldots, n\,, \\ \\ -m_b\,, \qquad\qquad x \in \Omega \backslash \bigcup_{j=1}^{n} \Omega_{\varepsilon_j}\,. \end{cases}$$

- Assume that at least one $m_j > 0$, and $\int_\Omega m_\varepsilon\, dx < 0$. Then, there is a positive principal eigenvalue $\lambda_1 > 0$.
- **Biologically:** On the whole the environment is hostile, but there is at least one region that can support growth.

Ref [LW]: A. Lindsay, M. J. Ward, *An Asymptotic Analysis of the Persistence Threshold for the Diffusive Logistic Model in Spatial Environments with Localized Patches*, DCDS-B, **14**(3), (2010), pp. 1139–1179.

Persistence Problem: Patch Model II

Schematic Plot of the Domain in 2-D

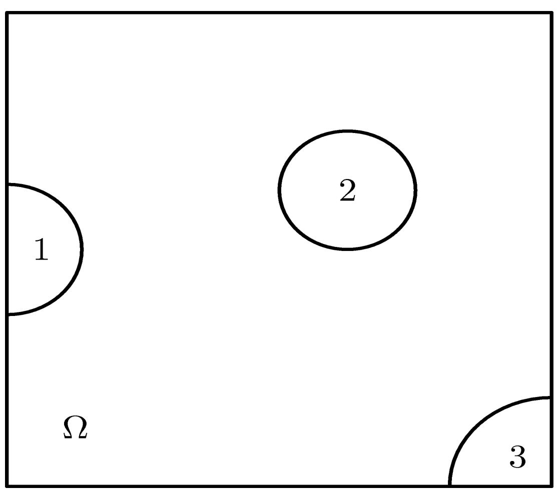


Remarks and Terminology:

- Patches Ω_{ε_j} of radius $O(\varepsilon)$ are portions of small circular disks strictly inside Ω. Circular patches are locally optimal (Hamel, Roques, 2007).
- The constant m_j is the local growth rate of the j^{th} patch, with $m_j > 0$ for a favorable habitat and $m_j < 0$ for a non-favorable habitat.
- The constant m_b is the background bulk decay rate.
- The boundary $\partial\Omega$ is piecewise smooth, with possible corner points.

Persistence Problem: Patch Model III

- Define $\Omega^I \equiv \{x_1, \ldots, x_n\} \cap \Omega$ to be the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \ldots, x_n\} \cap \partial\Omega$ is the set of the centers of the boundary patches. We assume patches are well-separated, i.e. $|x_i - x_j| \gg \mathcal{O}(\varepsilon)$ for $i \neq j$ and that dist$(x_j, \partial\Omega) \gg \mathcal{O}(\varepsilon)$ if $x_j \in \Omega^I$.
- To accommodate a boundary patch, we assign with each x_j for $j = 1, \ldots, n$, an angle $\pi\alpha_j$ representing the angular fraction of a circular patch that is contained within Ω. For example, $\alpha_j = 2$ when $x_j = \Omega^I$, and $\alpha_j = 1$ when $x_j \in \Omega^B$ and x_j is a point where $\partial\Omega$ is smooth, and $\alpha_j = 1/2$ when $x_j \in \partial\Omega$ is at a $\pi/2$ corner of $\partial\Omega$, etc.



Patch	Angle	Radius
1	$\alpha_1 = \pi$	$\epsilon\rho_1$
2	$\alpha_2 = 2\pi$	$\epsilon\rho_2$
3	$\alpha_3 = \pi/2$	$\epsilon\rho_3$

Persistence Problem: Patch Model IV

The condition $\int_\Omega m_\varepsilon \, dx < 0$ is asymptotically equivalent for $\varepsilon \to 0$ to

$$\int_\Omega m_\varepsilon \, dx = -m_b|\Omega| + \frac{\pi}{2}\sum_{j=1}^{n} \alpha_j m_j \rho_j^2 + \mathcal{O}(\varepsilon^2) = C < 0 \, .$$

Assumption I: Assume that this holds and that one m_j is positive. Then, by the Key Previous Result I, there exists a positive principal eigenvalue λ_1.

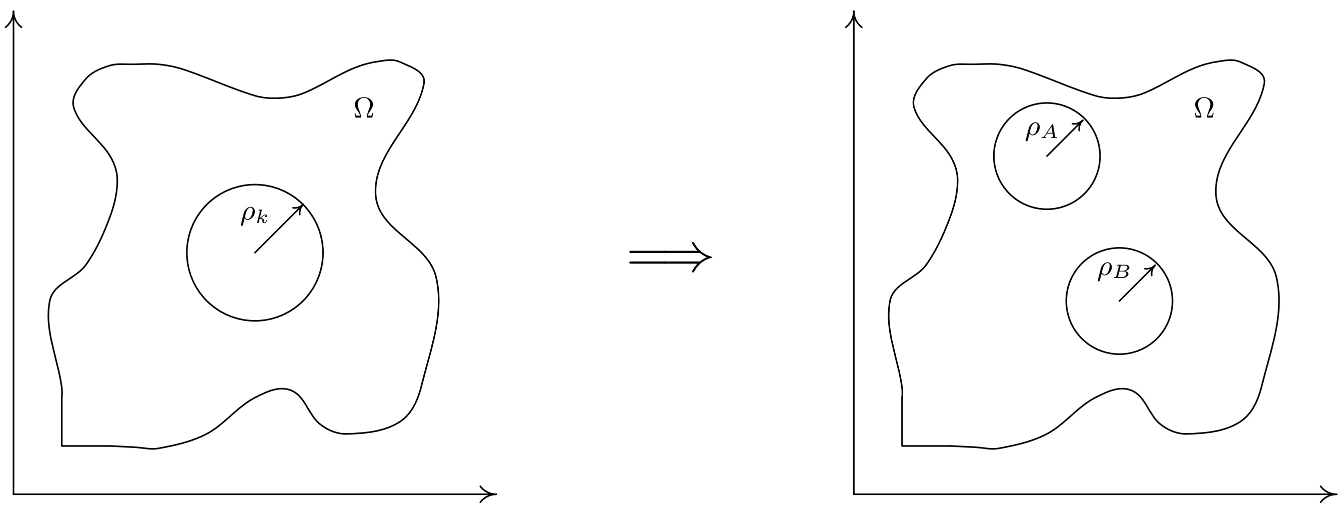
Main Goal: Calculate λ_1 as $\varepsilon \to 0$ by using an approach based on strong localized perturbation theory. Then, minimize it for a fixed $\int_\Omega m_\varepsilon \, dx < 0$. The parameter set is $\{m_1, \ldots, m_n\}$, $\{\rho_1, \ldots, \rho_n\}$, $\{x_1, \ldots, x_n\}$, and $\{\alpha_1, \ldots, \alpha_n\}$.

Persistence Problem: Qualitative Questions

After calculating λ_1 as $\varepsilon \to 0$ we then address several interesting qualitative questions:

Q1: What is the effect on λ_1 of resource location. Are boundary habitats preferable to interior habitats with regards to decreasing the extinction threshold?

Q2: What is the effect of resource fragementation? Does fragmentation lead to larger persistence thresholds?. To maintain the value of $\int_\Omega m_\varepsilon \, dx$, we need that $m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2$.



Persistence Problem: Green Functions

In the analysis, the following Green functions play an important role:

Modified G-Function: Define the modified G-function G_m by

$$G_m(x;x_j) \equiv G(x;x_j)\,, \quad x_j \in \Omega\,; \quad G_m(x;x_j) \equiv G_s(x;x_j)\,, \quad x_j \in \partial\Omega\,.$$

Here $G(x;x_j)$ is the unique Neumann Green function satisfying

$$\Delta G = \frac{1}{|\Omega|} - \delta(x-x_j)\,, \quad x \in \Omega\,; \quad \partial_n G = 0\,, \quad x \in \partial\Omega\,; \quad \int_\Omega G\,dx = 0\,,$$

$$G(x;x_j) \sim -\frac{1}{2\pi}\log|x-x_j| + R(x_j;x_j)\,, \quad \text{as} \quad x \to x_j\,,$$

while $G_s(x;x_j)$ is the unique surface Neumann Green function

$$\triangle G_s = \frac{1}{|\Omega|}\,, \quad x \in \Omega\,; \quad \partial_n G_s = 0\,, \quad x \in \partial\Omega \backslash \{x_j\}\,; \quad \int_\Omega G_s\,dx = 0\,,$$

$$G_s(x;x_j) \sim -\frac{1}{\alpha_j \pi}\log|x-x_j| + R_s(x_j;x_j)\,, \quad \text{as} \;\; x \to x_j \in \partial\Omega\,.$$

Persistence Problem: Main Result I

Principal Result 4.3: *In the limit $\varepsilon \to 0$, the positive principal eigenvalue λ_1 has the following two-term asymptotic expansion*

$$\lambda_1 = \mu_0 \nu - \mu_0 \nu^2 \left(\frac{\kappa^T \left(\pi \mathcal{G}_m - \mathcal{P} \right) \kappa}{\kappa^T \kappa} + \frac{1}{4} \right) + \mathcal{O}(\nu^3) \,, \quad \nu = -1/\log \varepsilon \,.$$

Here $\kappa = (\kappa_1, \ldots, \kappa_n)^T$ and $\mu_0 > 0$ is the first positive root of $\mathcal{B}(\mu_0) = 0$

$$\mathcal{B}(\mu_0) \equiv -m_b |\Omega| + \pi \sum_{j=1}^{n} \sqrt{\alpha_j} \kappa_j \,, \qquad \kappa_j \equiv \frac{\sqrt{\alpha_j} m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0} \,.$$

Finally, the $n \times n$ matrix $\mathcal{G}_m$ and diagonal matrix $\mathcal{P}$ are defined by

$$\mathcal{G}_{mij} = \sqrt{\alpha_i \alpha_j} G_{mij} \,, \quad i \neq j \,; \quad \mathcal{G}_{mjj} = \alpha_j R_{mjj} \,; \quad \mathcal{P}_{jj} = \log \rho_j \,.$$

Remarks:

- The coefficient μ_0 is independent of the precise relative locations of the patches within the domain.
- The coefficient of order $\mathcal{O}(\nu^2)$ has this spatial information through the Green interaction matrix.

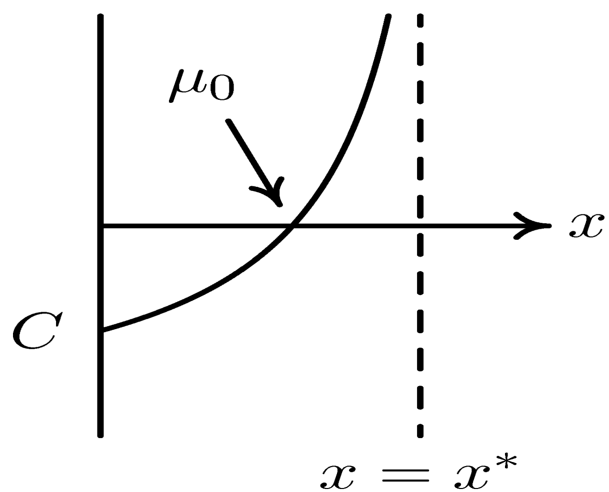
Persistence Problem: Main Result II

Principal Result: *There exists a unique root* μ_0 *to* $\mathcal{B}(x) = 0$ *on the range* $0 < x < \mu_{0u} \equiv 2/(m_J \rho_J^2)$*, where* $m_J \rho_J^2 = \max_{m_j > 0}\{m_j \rho_j^2 \,|\, j = 1, \ldots, n\}$*. The corresponding eigenfunction has one sign.*

Proof: $\mathcal{B}(0) = \int_\Omega m_\varepsilon(x)\, dx \sim C < 0$ by Assumption I. In addition, $\mathcal{B}(x) \to +\infty$ as $x \to \mu_{0u}^-$, and

$$\mathcal{B}'(x) = \sum_{j=1}^{n} \frac{\alpha_j m_j^2 \rho_j^4}{(2 - m_j \rho_j^2 x)^2} > 0 \,, \qquad 0 < x < \mu_{0u} \,.$$

Notice also that μ_{0u} is the smallest vertical asymptote of $\mathcal{B}(x)$. Hence, there exists a unique root $\mu_0 > 0$. With $x^\star = \mu_{0u}$ we plot:



Persistence Problem: Derivation of μ_0 I

We now sketch the derivation of the leading-order term in the Principal Result.

We expand the positive principal eigenvalue λ_1 as

$$\lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \cdots \,, \quad \nu = -1/\log \varepsilon \,,$$

for some unknown μ_0 and μ_1 to be found. In the outer region, defined away from an $\mathcal{O}(\varepsilon)$ neighborhoods of x_j, we expand the corresponding eigenfunction as

$$\phi \sim \phi_0 + \nu \phi_1 + \nu^2 \phi_2 + \cdots \,.$$

We obtain that $\phi_0 = |\Omega|^{-1/2}$ is a constant, and that ϕ_1 satisfies

$$\Delta \phi_1 = \mu_0 m_b \phi_0 \,, \quad x \in \Omega \backslash \Omega^I \,;$$

$$\partial_n \phi_1 = 0 \,, \quad x \in \partial\Omega \backslash \Omega^B \,; \quad \int_\Omega \phi_1 \, dx = 0 \,.$$

Here $\Omega^I \equiv \{x_1, \ldots, x_n\} \cap \Omega$ is the set of the centers of the interior patches, while $\Omega^B \equiv \{x_1, \ldots, x_n\} \cap \partial\Omega$ is the set of the centers of the boundary patches.

Persistence Problem: Derivation of μ_0 II

In the inner region, near the j^{th} patch we introduce $y = \varepsilon^{-1}(x - x_j)$ and $\psi(y) = \phi(x_j + \varepsilon y)$, and expand

$$\psi \sim \psi_{0j} + \nu \psi_{1j} + \nu^2 \psi_{2j} + \cdots \,,$$

where ψ_{0j} is a constant to be determined. For an interior patch with $x_j \in \Omega^I$, we obtain that ψ_{1j} satisfies

$$\Delta \psi_{1j} = \begin{cases} \mathcal{F}_{1j} \,, & |y| \le \rho_j \,, \\ 0 \,, & |y| \ge \rho_j \,, \end{cases}$$

where $\mathcal{F}_{1j} = -\mu_0 m_j \psi_{0j}$. The solution for ψ_{1j}, with $\rho = |y|$, is

$$\psi_{1j} = \begin{cases} A_{1j} \left(\frac{\rho^2}{2\rho_j^2} \right) + \bar{\psi}_{1j} \,, & 0 \le \rho \le \rho_j \,, \\ A_{1j} \log \left(\frac{\rho}{\rho_j} \right) + \frac{A_{1j}}{2} + \bar{\psi}_{1j} \,, & \rho \ge \rho_j \,, \end{cases}$$

where $\bar{\psi}_{1j}$ is an unknown constant.

Persistence Problem: Derivation of μ_0 III

The divergence theorem yields A_{1j} as

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j} \,,$$

for both boundary and interior patches.

The matching condition between the outer solution as $x \to x_j$ and the inner solution as $|y| = \varepsilon^{-1}|x - x_j| \to \infty$ is

$$\phi_0 + \nu\phi_1 + \cdots \sim$$

$$\psi_{0j} + A_{1j} + \nu\left(A_{1j}\log|x - x_j| - A_{1j}\log\rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j}\right) + \cdots \,.$$

The leading-order matching condition (blue terms) yields

$$\phi_0 = \psi_{0j} + A_{1j} \,, \qquad j = 1, \ldots, n \,.$$

Persistence Problem: Derivation of μ_0 IV

Recall that the problem for ϕ_1 is

$$\Delta\phi_1 = \mu_0 m_b \phi_0\,, \quad x \in \Omega \backslash \Omega^I\,;$$

$$\partial_n \phi_1 = 0\,, \quad x \in \partial\Omega \backslash \Omega^B\,; \quad \int_\Omega \phi_1 \, dx = 0\,.$$

From the $\mathcal{O}(\nu)$ red terms in the matching condition we obtain that ϕ_1 has the following singular behavior as $x \to x_j$

$$\phi_1 \sim A_{1j} \log|x - x_j| - A_{1j} \log \rho_j + \frac{A_{1j}}{2} + \bar{\psi}_{1j} + A_{2j}\,, \quad \text{as} \quad x \to x_j\,.$$

Next, by using the divergence theorem on the solution ϕ_1 to we get

$$\mu_0 m_b |\Omega| \phi_0 = -\pi \sum_{j=1}^{n} \alpha_j A_{1j}\,.$$

Persistence Problem: Derivation of μ_0 V

Then, by recalling that

$$A_{1j} = -\frac{\mu_0}{2} m_j \rho_j^2 \psi_{0j} \,, \qquad \phi_0 = \psi_{0j} + A_{1j} \,,$$

we get that

$$\psi_{0j} = \frac{2\phi_0}{2 - m_j \rho_j^2 \mu_0} \,, \qquad A_{1j} = -\frac{m_j \rho_j^2 \mu_0 \phi_0}{2 - m_j \rho_j^2 \mu_0} \,, \qquad j = 1, \ldots, n \,.$$

From the equation above we obtain that the leading-order eigenvalue correction μ_0 is a root of the nonlinear algebraic equation

$$\frac{m_b |\Omega|}{\pi} = \sum_{j=1}^{n} \frac{\alpha_j m_j \rho_j^2}{2 - m_j \rho_j^2 \mu_0} \,.$$

This yields the nonlinear algebraic equation for the leading-order term μ_0 in the expansion of the eigenvalue, as given in the Principal Result.

Persistence Problem: Derivation of μ_0 VI

Remarks:

- The calculation of the higher-order term of order $\mathcal{O}(\nu^2)$, is more involved and is given **[LW]**. This second-order term has the spatial information on the location of the traps.
- Note that $\psi_{0j} > 0$ if $\mu_0 < \mu_{0u}$. This is the positivity property of the principal eigenfunction.
- We emphasize, that in contrast to the Laplacian eigenvalue problems for the MFPT, the equation for μ_0 does contain some key qualitative information, which we now illustrate.

By optimizing the leading-order coefficient μ_0 subject to $\int_\Omega m_\varepsilon dx < 0$, we can obtain key qualititative results regarding the optimal resource distribution.

Persistence Problem: Implications I

The following very simple Lemma is needed:

Lemma: Consider two smooth functions $C_{\text{old}}(\zeta)$ and $C_{\text{new}}(\zeta)$ defined on $0 \leq \zeta < \mu_{\text{m}}^{\text{old}}$ and $0 \leq \zeta < \mu_{\text{m}}^{\text{new}}$, respectively, with $C_{\text{old}}(0) = C_{\text{new}}(0) < 0$, and $C_{\text{old}}(\zeta) \to +\infty$ as $\zeta \to \mu_{\text{m}}^{\text{old}}$ from below, and $C_{\text{new}}(\zeta) \to +\infty$ as $\zeta \to \mu_{\text{m}}^{\text{new}}$ from below. Suppose further that there exist unique roots $\zeta = \mu_0^{\text{old}}$ and $\zeta = \mu_0^{\text{new}}$ to $C_{\text{old}}(\zeta) = 0$ and $C_{\text{new}}(\zeta) = 0$ on the intervals $0 < \zeta < \mu_{\text{m}}^{\text{old}}$ and $0 < \zeta < \mu_{\text{m}}^{\text{new}}$, respectively. Then,

- Case I: If $\mu_{\text{m}}^{\text{new}} \leq \mu_{\text{m}}^{\text{old}}$ and $C_{\text{new}}(\zeta) > C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{m}}^{\text{new}}$, then $\mu_0^{\text{new}} < \mu_0^{\text{old}}$.
- Case II: If $\mu_{\text{m}}^{\text{new}} \geq \mu_{\text{m}}^{\text{old}}$ and $C_{\text{new}}(\zeta) < C_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{m}}^{\text{old}}$, then $\mu_0^{\text{new}} > \mu_0^{\text{old}}$.

Persistence Problem: Implications II

Qualitative Result I: *The movement of a single favorable habitat to the boundary of the domain is advantageous for species persistence.*

Proof: Move the j^{th} interior favorable patch with $m_j > 0$ of radius $\varepsilon\rho_j$ and angle 2π (i.e. $\alpha_j = 2$) to an unoccupied boundary location with patch radius $\varepsilon\rho_k$, "mass" $m_k > 0$, and angle $\pi\alpha_k$, with $\alpha_k < 2$. To maintain $\int_\Omega m_\varepsilon \, dx$, we need $m_j\rho_j^2 = \alpha_k m_k \rho_k^2$, which implies $m_k\rho_k^2 > m_j\rho_j^2$. Then,

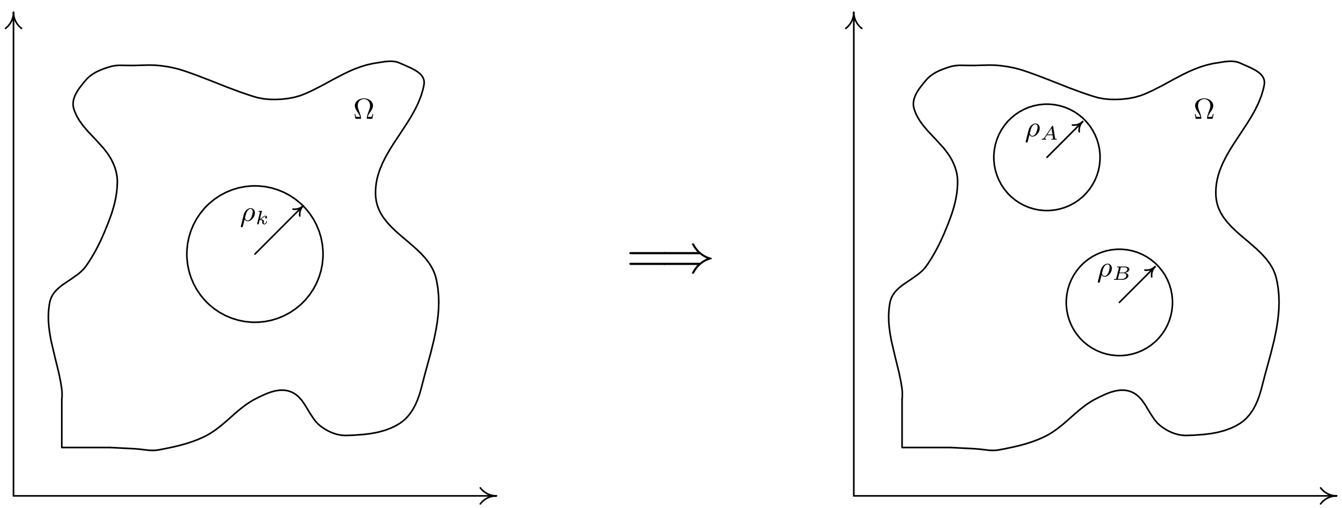
$$\mathcal{B}_{\text{new}}(\zeta) - \mathcal{B}_{\text{old}}(\zeta) = \frac{\pi\alpha_k m_k \rho_k^2}{2 - \zeta m_k \rho_k^2} - \frac{2\pi m_j \rho_j^2}{2 - \zeta m_j \rho_j^2}$$
$$= \pi \left(\frac{2}{\alpha_k}\right) \frac{m_j^2 \rho_j^4 \zeta}{\left(2 - \zeta m_j \rho_j^2\right)\left(2 - \zeta m_k \rho_k^2\right)} (2 - \alpha_k) \; .$$

Recall that $\mathcal{B}_{\text{old}}(\zeta) = 0$ has a root ζ on $0 < \zeta < \mu_{\text{m}}^{\text{old}} \equiv 2/(m_J\rho_J^2)$, where $m_J\rho_J^2 = \max_j m_j\rho_j^2$. Since $m_k\rho_k^2 > m_j\rho_j^2$, the first vertical asymptote for $\mathcal{B}_{\text{new}}(\zeta)$ cannot be larger than that of $\mathcal{B}_{\text{old}}(\zeta)$. Thus, there is a unique root $\zeta = \mu_0^{\text{new}}$ to $\mathcal{B}_{\text{new}}(\zeta) = 0$ on $0 < \zeta < \mu_{\text{m}}^{\text{new}} \equiv 2/(m_K\rho_K^2)$, where $m_K\rho_K^2 \equiv \max\{m_J\rho_J^2, m_k\rho_k^2\}$. Since $\mu_{\text{m}}^{\text{new}} \le \mu_{\text{m}}^{\text{old}}$, and $\mathcal{B}_{\text{new}}(\zeta) > \mathcal{B}_{\text{old}}(\zeta)$ for $0 < \zeta < \mu_{\text{m}}^{\text{new}}$, Case I of the Lemma yields $\mu_0^{\text{new}} < \mu_0^{\text{old}}$. ■

Persistence Problem: Implications III

Qualitative Result II: *The fragmentation of one favorable interior habitat into two separate favorable interior habitats is not advantageous for species persistence. Similarly, the fragmentation of a favorable boundary habitat into two favorable boundary habitats, with each centered at a smooth point of $\partial\Omega$, is not advantageous.*

Proof: Split k^{th} patch of radius ρ_k into two patches of radius ρ_A and ρ_B.



The constraint that $\int_\Omega m_\varepsilon(x)\,\mathrm{d}x$ is unchanged requires:

$$m_k \rho_k^2 = m_A \rho_A^2 + m_B \rho_B^2.$$

Persistence Problem: Implications IV

We prove this result for $\alpha_A = \alpha_B = \alpha_k$ as follows.

Suppose that we are fragmenting one favorable habitat into two smaller favorable habitats. Then, $m_A > 0$, $m_B > 0$, and $m_k > 0$.

For the original patch distribution, $\mathcal{B}_{\text{old}}(\zeta) = 0$ has a positive root $\zeta = \mu_0^{\text{old}}$ on $0 < \zeta < \mu_{\text{m}}^{\text{old}} \equiv 2/(m_J \rho_J^2)$, where $m_J \rho_J^2 = \max_{m_j > 0}(m_j \rho_j^2)$.

Since, clearly, the first vertical asymptote for $\mathcal{B}_{\text{new}}(\zeta)$ cannot be smaller than that of $\mathcal{B}_{\text{old}}(\zeta)$ under this fragmentation, it follows that $\mathcal{B}_{\text{new}}(\zeta) = 0$ has a positive root $\zeta = \mu_0^{\text{new}}$ on $0 < \zeta < \mu_{\text{m}}^{\text{new}}$ with $\mu_{\text{m}}^{\text{new}} \geq \mu_{\text{m}}^{\text{old}}$.

Persistence Problem: Implications V

To maintain $\int_\Omega m_\varepsilon \, dx$ we require that $m_k\rho_k^2 = m_A\rho_A^2 + m_B\rho_A^2$. The change in $\mathcal{B}(\zeta)$ induced by this fragmentation action is

$$\begin{aligned}\mathcal{B}_{\text{new}}(x) - \mathcal{B}_{\text{old}}(x) &= \frac{\alpha_k m_A\rho_A^2}{2 - m_A\rho_A^2 x} + \frac{\alpha_k m_B\rho_B^2}{2 - m_B\rho_B^2 x} - \frac{\alpha_k m_k\rho_k^2}{2 - m_k\rho_k^2 x} \\ &= \frac{-x\alpha_k(m_A\rho_A^2 m_B\rho_B^2)[(2 - m_A\rho_A^2 x) + (2 - m_B\rho_B^2 x)]}{(2 - m_A\rho_A^2 x)(2 - m_B\rho_B^2 x)(2 - m_k\rho_k^2 x)} \,.\end{aligned}$$

Hence, we have that $\mathcal{B}_{\text{new}}(\zeta) < \mathcal{B}_{\text{old}}(\zeta)$ on $0 < \zeta < \mu_{\text{m}}^{\text{old}} \equiv 2/(m_J\rho_J^2)$. Since, in addition $\mu_{\text{m}}^{\text{new}} \geq \mu_{\text{m}}^{\text{old}}$, it follows from Case II of the Lemma that $\mu_0^{\text{new}} > \mu_0^{\text{old}}$. This proves the Qualitative Result II. ■

Key: Fragmentation of an interior favorable habitat into two separate favorable interior habitats is deleterious to survival of the species.

Persistence Problem: Implications VI

Q3: What about a partial fragmentation scenario, whereby an interior favorable habitat is fragmented into a boundary habitat and a smaller interior favorable habitat?

Qualitative Result III: *The fragmentation of one favorable interior habitat into a new smaller interior favorable habitat together with a favorable boundary habitat, is advantageous for species persistence when the boundary habitat is sufficiently strong in the sense that*

$$m_k \rho_k^2 > \frac{4}{2 - \alpha_k} m_j \rho_j^2 > 0 \,.$$

Such a fragmentation of a favorable interior habitat is not advantageous when the new boundary habitat is too weak in the sense that

$$0 < m_k \rho_k^2 < m_j \rho_j^2 \,.$$

Finally, the clumping of a favorable boundary habitat and an unfavorable interior habitat into one single interior habitat is not advantageous for species persistence when the resulting interior habitat is still unfavorable.

Persistence Problem: Examples I

Example 1: We illustrate Qualitative Result III inside the unit disk for the case $m_b = 2$: Fragment a single interior patch of radius ε centered at the origin into a favorable boundary patch of radius $\varepsilon\rho_0$ and a smaller favorable interior patch of radius $\varepsilon\rho_1$. Assume that $m_j = 1$ for each patch. To maintain the constraint $\int_\Omega m_\varepsilon \, dx = -\pi$, we require that ρ_0 and ρ_1, with $0 < \rho_1 < 1$, satisfy

$$1 = \rho_1^2 + \frac{1}{2}\rho_0^2 \, .$$

For the new configuration, the equation for μ_0 is simply

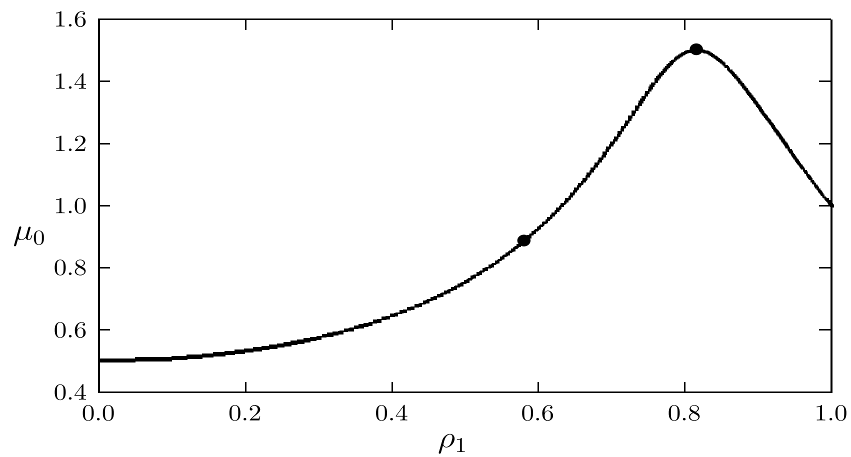
$$\mathcal{B}(\mu_0) \equiv -2\pi + \pi \left(\frac{\rho_1^2}{2 - \rho_1^2 \mu_0} - \frac{\rho_0^2/2}{2 - \rho_0^2 \mu_0} \right) = 0$$

For this two-patch problem, μ_0 satisfies the quadratic equation

$$\mu_0^2 \rho_1^2 \left(1 - \rho_1^2\right) + \mu_0 \left(-2 + \frac{5}{2}\rho_1^2 - \frac{3}{2}\rho_1^4 \right) + 1 = 0 \, .$$

Note: $\mu_0 = 1$ when $\rho_1 = 1$ (original configuration of one interior patch); Also $\mu_0 = 1/2$ when $\rho_1 = 0$ (only a boundary patch).

Persistence Problem: Examples II



The (sufficient condition) bounds in Qualitative Result III state that:

- fragmentation of an interior patch into a boundary patch is undesirable when $\rho_1 > \rho_0$, which yields $\rho_1 > \sqrt{2/3}$.
- such a fragmentation is advantageous when $\rho_1 < 1/\sqrt{3}$.

For this simple two-patch case, we obtain that $\mu_0 = 1$ when $\rho_1 = \sqrt{2/5}$, or equivalently $\rho_0 = \sqrt{6/5}$. Thus, fragmentation is advantageous when $\rho_1 < \sqrt{2/5}$, or equivalently $\rho_0 > \sqrt{6/5}$.

Persistence Problem: Examples III

Example 2: Illustrate Qualitative Result III for a unit disk with $m_b = 3$ that has one pre-existing favorable interior patch of radius ε and growth rate $m_+ = 1$, together with one pre-existing unfavorable interior patch of radius ε and growth rate $m_- = -1$.

We introduce an additional favorable resource with $m_0 = 1$ that can occupy an area $\varepsilon^2 A_0$ if it is separated from the other two patches.

We then compare three different options for using this additional favorable resource, subject to the constraint that $\int_\Omega m_\varepsilon \, dx = -3\pi + A_0$ remains fixed.

Persistence Problem: Examples IV

- **Case I:** If we concentrate the additional favorable resource at a smooth point on the boundary, then μ_0 satisfies

$$-3 + 2\left(\frac{1}{2-\mu_0} - \frac{1}{2+\mu_0}\right) + \frac{A_0/\pi}{1-\mu_0 A_0/\pi} = 0\,.$$

- **Case II:** If the additional favorable resource is used to strengthen the pre-existing favorable interior patch, then μ_0 satisfies

$$-3 + \frac{2\rho_+^2}{2-\rho_+^2\mu_0} - \frac{2}{2+\mu_0} = 0\,, \qquad \rho_+^2 = 1 + A_0/\pi\,.$$

- **Case III:** Finally, if the additional favorable resource is used to diminish the strength of the unfavorable pre-existing interior patch, then μ_0 satisfies

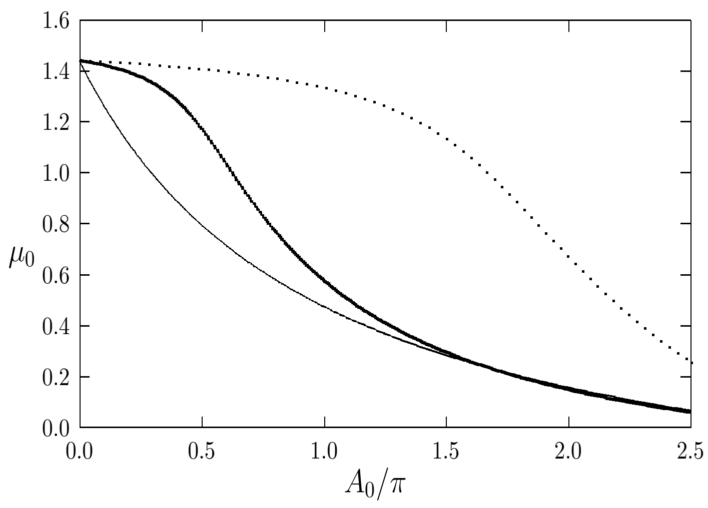
$$-3 + \frac{2}{2-\mu_0} + \frac{m_-}{2-m_-\mu_0} = 0\,, \qquad m_- = -1 + A_0/\pi\,.$$

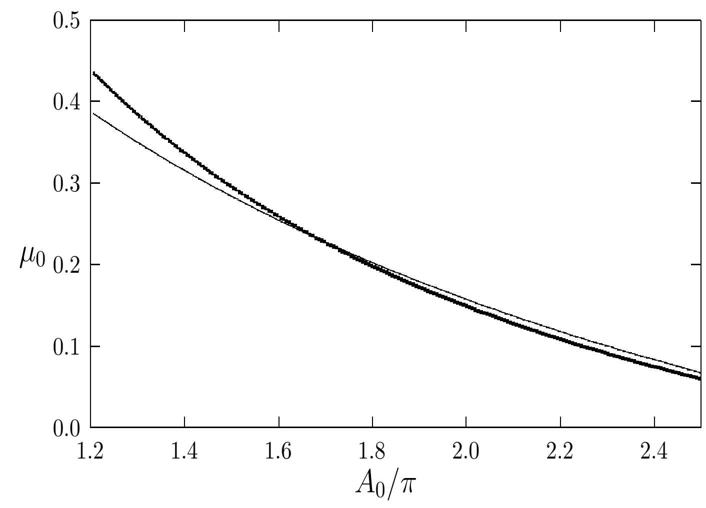
Persistence Problem: Examples V

We conclude that:

- inserting a favorable boundary patch is preferable only when it has a sufficiently large size.
- if only a limited amount of an additional favorable resource is available, it is preferable to re-enforce the pre-existing favorable habitat.
- It is never optimal for any range of A_0/π to use the additional favorable resource to mitigate the effect of the unfavorable interior patch.

Plot: μ_0 versus A_0/π: Heavy solid (new boundary patch); Solid curve (re-inforce favorable interior patch); dashed curve (weaken unfavorable patch). Right figure is a zoom of left.





Persistence Problem: Main Comment

Key Remark: These qualitative results show that, given some fixed amount of favorable resources to distribute, the optimal strategy is to clump them all together at a point on the boundary of the domain, and more specifically at the corner point of the boundary (if any are present) with the smallest angle $\leq 90^{\circ}$. This strategy will minimize μ_0, thereby maximizing the chance for the persistence of the species.

Remark: Starting with the algebraic equation for μ_0 as derived by formal asymptotics, these qualitative results regarding fragmentation are rigorous results based on manipulating the formula for μ_0. A key issue then is to give a rigorous proof of the expression for μ_0.

2nd-Order Optimization I

The persistence threshold has a two-term asymptotic expansion

$$\lambda_1 \sim \mu_0 \nu + \mu_1 \nu^2 + \cdots \,, \qquad \nu \equiv = -1/\log \varepsilon \,.$$

Remark: The minimization of λ_1 is typically accomplished by optimizing μ_0. However, in certain degenerate cases, the problem of optimizing λ_1 requires the examination of the μ_1 term.

In particular, suppose that the boundary $\partial\Omega$ is smooth, and that there is one favorable patch. Then, to optimize μ_0, we must put this patch on the boundary. To determine which boundary point to center the patch is optimal, we must optimize μ_1. For a boundary patch of radius ρ_1, then the Principal Result becomes

$$\mu_1 = -\mu_0 \left(\frac{1}{4} + \pi R_s(x_0; x_0) - \log \rho_1 \right)$$

$$\mu_0 \equiv \frac{2}{m_+ \rho_1^2} \left[1 - \frac{\alpha_1 \pi m_+ \rho_1^2}{2|\Omega| m_b} \right] > 0 \,.$$

2nd-Order Optimization II

Principal Result: *For a single boundary patch centered at x_0 on a smooth boundary $\partial\Omega$, then λ_1 is minimized at the global maximum of the regular part $R_s(x_0; x_0)$ of the surface Neumann Green function.*

Recall that $R_s(x_0; x_0)$ is defined via

$$\triangle G_s = \frac{1}{|\Omega|}\,, \quad x \in \Omega\,; \quad \partial_n G_s = 0\,, \qquad x \in \partial\Omega \backslash \{x_0\}\,; \quad \int_\Omega G_s\, dx = 0\,,$$

$$G_s(x; x_0) \sim -\frac{1}{\alpha_0 \pi} \log |x - x_0| + R_s(x_0; x_0)\,, \quad \text{as } x \to x_0 \in \partial\Omega\,.$$

Question: For $\partial\Omega$ smooth, is the global maximum of $R_s(x_1; x_1)$ obtained at the global maximum of the boundary curvature κ? (No; we can find a counterexample for smooth perturbations of the unit disk, by deriving a perturbation formula for R_s)

Remark: Given a pre-existing patch distribution, finding the optimal location of a new favorable habitat may also require optimizing the $\mathcal{O}(\nu^2)$ term.

2nd-Order Optimization III

Pre-Existing Patch Distribution Formulation:

- Suppose that $\partial\Omega$ is smooth. Let x_j for $j = 1, \ldots, n$ be the centers of n pre-exisiting circular patches within Ω with local growth rates m_j for $j = 1, \ldots, n$.
- Add a new favorable habitat, centered at x_0, and assume that μ_0 is smallest when it is located on $\partial\Omega$ rather than inside Ω.
- To determine the point on $\partial\Omega$ that minimizes λ_1, we must optimize

$$\mu_1 = \mu_0 \left(-\frac{1}{4} + \frac{\kappa^T \left(\mathcal{P} - \pi \mathcal{G}_m\right) \kappa + \kappa_0^2 \log \rho_0 - \pi p(x_0)}{\kappa^T \kappa + \kappa_0^2} \right) .$$

where $\kappa = (\kappa_1, \ldots, \kappa_n)^T > 0$, $\mathcal{P}$, and $\mathcal{G}_m$ refer to the pre-existing patches. To minimize μ_1, we must maximize $p(x_0)$ defined by

$$p(x_0) = \kappa_0^2 R_m(x_0; x_0) + 2 \sum_{j=1}^{n} \sqrt{2} \kappa_0 \kappa_j G_m(x_j; x_0) \, , \qquad \kappa_0 \equiv \frac{m_0 \rho_0^2}{2 - \mu_0 m_0 \rho_0^2} \, ,$$

The scalar represents the interaction of the additional favorable boundary patch with the fixed patch distribution.

2nd-Order Optimization IV

Example: Let Ω be the unit disk with n pre-existing favorable resources of a common radius ε and growth rate $m_j = m_c > 0$ that are equally-spaced on a concentric ring of radius r with $0 < r < 1$ at $x_j = \exp(2\pi i j/n)$ for $j = 0, \ldots, n-1$. Then, $\kappa_j = \kappa_c$ for $j = 1, \ldots, n$. We add an extra favorable resource on the boundary at angle θ_0. Since the G-functions are known for the disk, we calculate (after some lengthy algebra) that

$$p(\theta_0) = \frac{\kappa_0^2}{8\pi} + \frac{\kappa_0}{2\pi}\left(r^2 - \frac{1}{2}\right)(m_b - \kappa_0) - \frac{\sqrt{2}\kappa_0\kappa_c}{\pi}\chi(\theta_0)\,,$$

where $\chi(\theta_0) = \log\left[(r^n - \cos(n\theta_0))^2 + \sin^2(n\theta_0)\right]$.

The local minima of $\chi(\theta_0)$ (local maxima of $p(\theta_0)$) are at $\theta_0 = \frac{2\pi j}{n}$ for $j = 1, \ldots, n-1$.

Thus, for a ring of n pre-exisiting equally-spaced favorable patches, the optimal boundary locations for one additional favorable boundary patch is to put it at the shortest distance to any of the n favorable habits on the ring.

Persistence Problem: Open Problems

Open I: Give a rigorous proof of the formula for μ_0.

Open II: Consider the effect of a predator v, modeled in Ω by

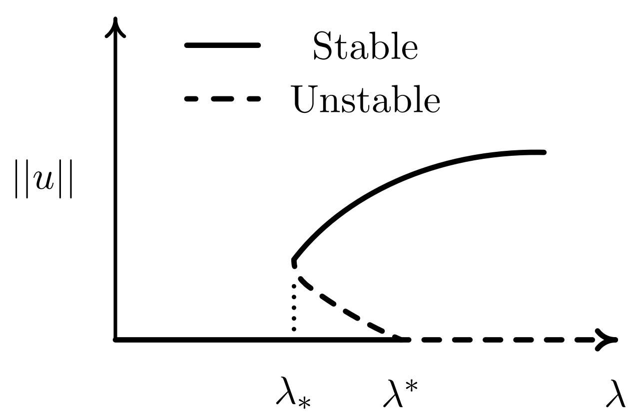
$$u_t = D\Delta u + m_\varepsilon(x)u(1-u) - \beta uv\,,$$
$$v_t = \Delta v - \sigma v + \mu + \beta uv\,,$$

with $\partial_n u = \partial_n v = 0$ for $x \in \partial\Omega$. One might guess that a predator has an advantage when its prey is concentrated. Does the optimal strategy for the prey still remain the same as for the basic problem?

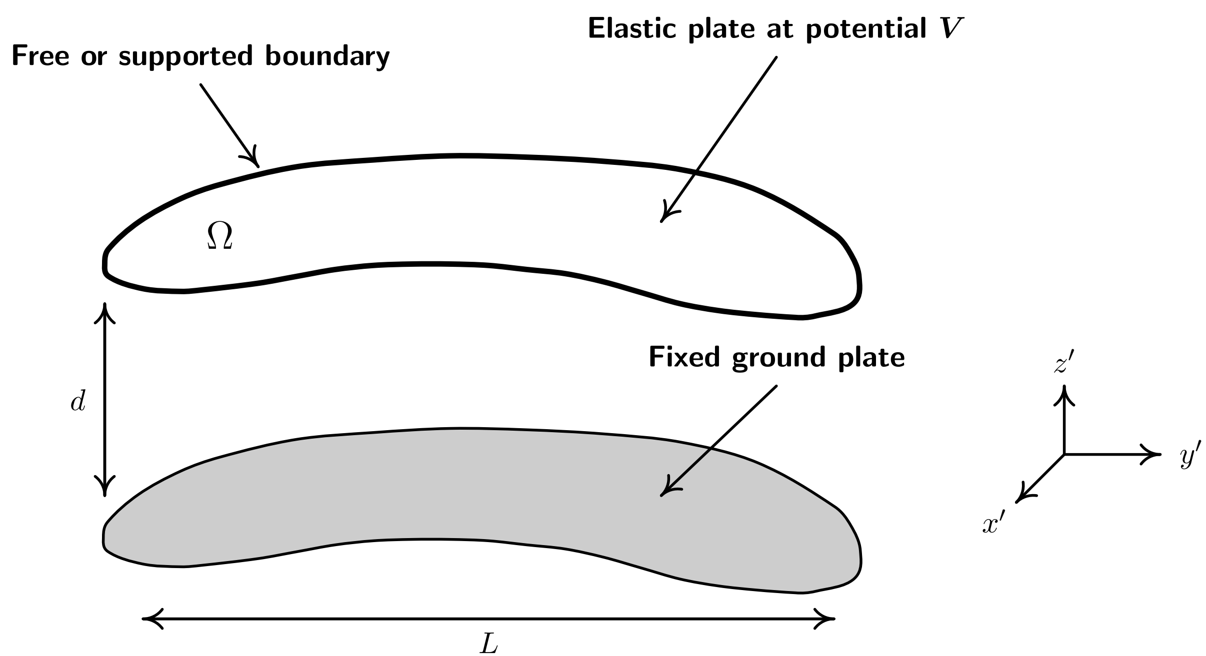
Open III: Consider including the weak Allee effect for which

$$\Delta u + \lambda m_\varepsilon(x)u(1-u)(a+u) = 0\,, \quad x \in \Omega\,; \quad \partial_n u = 0 \quad x \in \partial\Omega\,.$$

The extinction threshold is now a saddle node bifurcation point.



Nonlinear Biharmonic Problems of MEMS I



- Plate will deflect in the presence of an electric field
- Top plate can make contact with the lower plate (i.e. touchdown) when $V > V^*$ at some quenching time $t = T$.
- Device can act as a switch, valve, or capacitor.
- If $V > V^*$ then no stable steady-state solutions. The threshold V^* is called the pull-in voltage threshold.

Nonlinear Biharmonic Problems of MEMS II

For small aspect ratio, the plate deflection u in $\mathbb{R}^2$ satisfies

$$u_t = -\delta\Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}\,, \quad x \in \Omega\,; \quad u = u_n = 0\,, \quad x \in \partial\Omega\,.$$

- We focus on clamped boundary conditions.
- The singular nonlinearity represents a Coulomb attractive force; λ is proportional to V^2; δ represents bending rigidity.
- Model originally derived by Pelesko (2000) in the narrow gap limit. It neglects self-stretching term, fringing-field effect, nonlocal term due to external circuit, etc...

Main Questions:

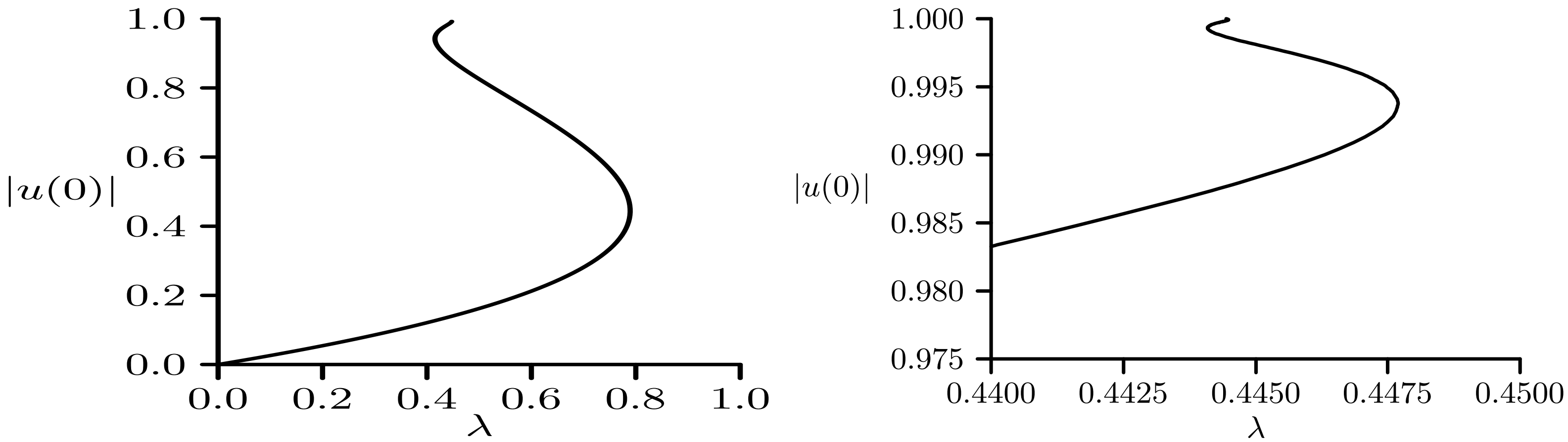
- **Pull-in Threshold:** Of importance for applications is the location of the saddle-node point at the end of the minimal solution branch for $|u|_\infty$ versus λ. This sets the stable operating range of the device.
- **Solution Multiplicity:** An interesting theoretical question is how does the global bifurcation diagram depend on δ?

Nonlinear Biharmonic Problems of MEMS III

Pelesko (SIAM J. App. M., (2000)) considered the membrane problem

$$\Delta u = \frac{\lambda}{(1+u)^2}\,, \quad x \in \Omega\,; \quad u = 0\,, \quad x \in \partial\Omega\,.$$

For the unit disk the numerically computed bifurcation diagram is:



Left: Bifurcation diagram **Right:** Zoom of left figure.

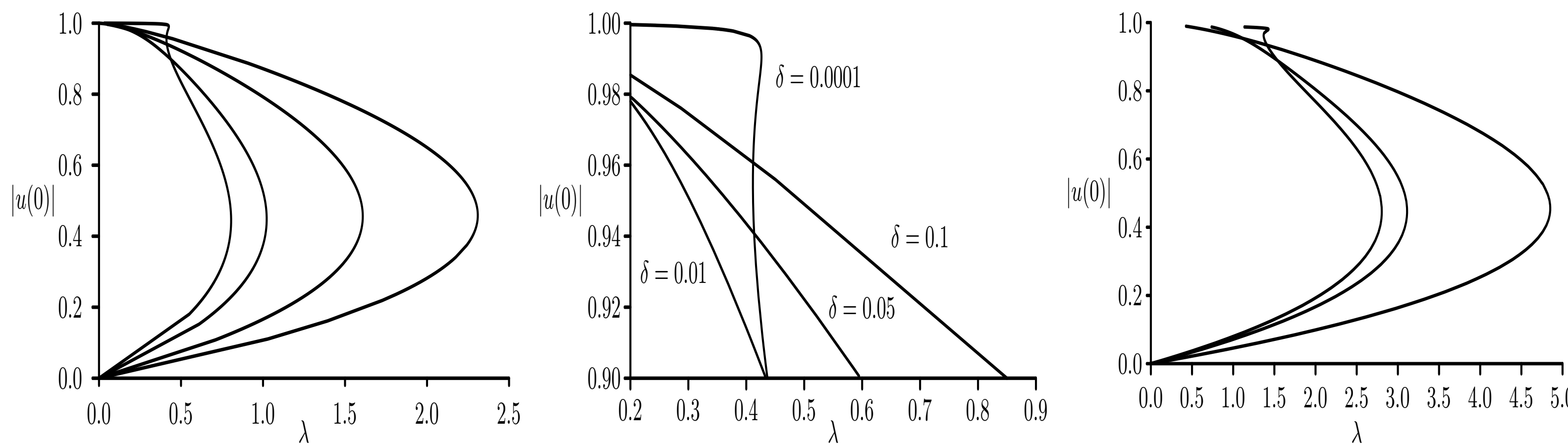
Key Features:

- In the unit disk there are an infinite number of fold points with limiting behavior $\lambda \to 4/9$ as $u(0)+1 \equiv \varepsilon \to 0^+$.
- In contrast, for the unit slab there is either zero, one, or two steady-state solutions.

Perturbation of the Membrane Problem by δ

$$u_t = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}\,, \quad x \in \Omega, ; \quad u = u_n = 0\,, \quad x \in \partial\Omega\,.$$

Numerical computations with either shooting or psuedo-arclength yield:



Left and Middle (Unit Disk): $\delta = 0.0001, 0.01, 0.05, 0.1$. **Right (Unit Square):** $\delta = 0.0001, 0.001, 0.01$.

- **Practical Interest for Engineers:** Derive perturbation results for $\delta \ll 1$ and for $\delta \gg 1$ for the saddle-node point on minimal branch (Lindsay, MJW, MAA (2008)).
- **Theoretical Interest:** Infinite fold point structure destroyed when $\delta > 0$, and there is a maximal solution branch with $\lambda \to 0$ and $|u|_\infty \to 1^-$.

Asymptotics of Maximal Solution Branch

- **In the Unit Ball in** $\mathbb{R}^2$ construct the limiting asymptotics of the maximal solution branch to the pure Biharmonic problem

$$\Delta^2 u = -\lambda/(1+u)^2\,, \quad 0 < r < 1\,; \qquad u(1) = u_r(1) = 0\,,$$

  for which $\lambda \to 0$ as $u(0) + 1 = \varepsilon \to 0^+$.

- **In the Unit Ball in** $\mathbb{R}^2$ perform a similar calculation for the mixed Biharmonic operator

$$\delta\Delta^2 u - \Delta u = -\lambda/(1+u)^2\,, \quad 0 < r < 1\,; \qquad u(1) = u_r(1) = 0\,.$$

- **In a General 2-D Domain** construct the limiting asymptotics of the maximal solution branch to

$$\Delta^2 u = -\lambda/(1+u)^2\,, \quad x \in \Omega\,; \qquad u = \partial_n u = 0\,, \quad x \in \partial\Omega\,.$$

  Where is the concentration point x_0? What is the asymptotics for λ as $u(x_0) + 1 = \varepsilon \to 0^+$?.

Biharmonic Operator and Point Constraints I

Simple Model Problem: For $\varepsilon \to 0$, consider in the 2-D annulus $0 < \varepsilon < r < 1$:

$$\Delta^2 u = 0 \,, \quad 0 < \varepsilon < r < 1 \,,$$
$$u(1) = 1 \,, \quad u_r(1) = 0 \,; \qquad u(\varepsilon) = u_r(\varepsilon) = 0 \,.$$

We first find the exact solution and then expand it as $\varepsilon \to 0$.

Since the radial solutions are linear combinations of $\{r^2, r^2 \log r, \log r, 1\}$, the solution with $u(1) = 1$ and $u_r(1) = 0$ has the form

$$u = A(r^2 - 1) + Br^2 \log r - (2A + B) \log r + 1 \,.$$

Upon satisfying $u(\varepsilon) = u_r(\varepsilon) = 0$, we get two equations for A and B.

By solving these equations in the limit $\varepsilon \to 0$, we get

$$B \sim -2 - 8\varepsilon^2 \left(\log \varepsilon\right)^2 \,, \qquad A \sim 1 + 4\varepsilon^2 \left(\log \varepsilon\right)^2 \,.$$

Biharmonic Operator and Point Constraints II

This gives the two-term outer approximation in $r \gg \mathcal{O}(\varepsilon)$:

$$u \sim u_0(r) + \varepsilon^2 (\log \varepsilon)^2 u_1(r) + \cdots \,,$$

where

$$u_0 = r^2 - 2r^2 \log r \,, \qquad u_1 = 4(r^2 - 1) - 8r^2 \log r \,.$$

- **Key:** In the limit $\varepsilon \to 0$, the outer solution does not tend to the simple solution $u = 1$ that occurs when there is no hole.
- **Note:** u_0 is not C^2. It does satisfy point constraint $u_0(0) = 0$. In fact $u_0 = 1 + cG(r:0)$, where $\Delta^2 G = \delta(x)$ and $c = -1/G(0;0)$.
- **Remarks:** Point constraints are allowed since $G_f = \frac{1}{8\pi} r^2 \log r$ in $\mathbb{R}^2$.
- Satisfying point constraints by using the Biharmonic Green's function is the basis of biharmonic spline interpolation, i.e. writing the interpolant as $f = \sum_{i=1}^{N} \alpha_i G(x; x_i)$ where we have determined the α_i by fitting the data $f = f_j$ at $x = x_j$, for $j = 1, \ldots, N$.

Pure Biharmonic in the Unit Ball: I

Ref: [LW2] A. Lindsay, MJ. Ward, *Asymptotics of Some Nonlinear Eigenvalue Problems Modelling a MEMS Capacitor: Part II: Multiple Solutions and Singular Asymptotics*, EJAM, **22**(2), (2010), pp. 83–123.

Consider $\Delta^2 u = -\lambda/(1+u)^2$, with $u = u_r = 0$ on $r = 1$.

Principal Result: *In the limit $\varepsilon \equiv u(0) + 1 \to 0^+$, the limiting asymptotic behaviour of the maximal solution branch in the outer region, away from $r = 0$, is*

$$u = u_0 + \frac{\varepsilon}{\sigma} \log \sigma \, u_{1/2} + \frac{\varepsilon}{\sigma} u_1 + \varepsilon \log \sigma \, u_{3/2} + \varepsilon u_2 + \mathcal{O}(\varepsilon \sigma \log \sigma) \,,$$

$$\lambda = \frac{\varepsilon}{\sigma} \left[\lambda_0 + \sigma \lambda_1 + \mathcal{O}(\sigma^2) \right] \; ; \quad \lambda_0 = 32 \,, \quad \lambda_1 = 16 \left(\log 2 - \frac{\pi^2}{6} \right) \,,$$

where $\sigma = -1/\log \gamma$ and the boundary layer width γ are determined in terms of ε by $-\gamma^2 \log \gamma = \varepsilon$. The point constraint $u_0(0) = -1$ holds, and

$$u_0 = -1 + r^2 - 2r^2 \log r \,, \quad u_{1/2} = -\frac{\lambda_0}{16} u_0 \,, \quad u_{3/2} = -\frac{\lambda_1}{16} u_0 \,.$$

Here $u_{1/2}$, $u_{3/2}$ are switchback terms proportional to u_0.

Pure Biharmonic in the Unit Ball: II

In addition, u_1 and u_2 are the unique solutions of

$$\Delta^2 u_1 = -\frac{\lambda_0}{(1+u_0)^2}\,, \quad 0 < r < 1\,; \qquad u_1(1) = u_{1r}(1) = 0\,,$$

$$u_1 = \frac{\lambda_0}{16}\log(-\log r) + \frac{\lambda_0}{16} + \mathcal{O}(\log^{-1} r)\,, \quad r \to 0\,,$$

$$\Delta^2 u_2 = -\frac{\lambda_1}{(1+u_0)^2}\,, \quad 0 < r < 1\,; \qquad u_2(1) = u_{2r}(1) = 0\,,$$

$$u_2 = \frac{\lambda_1}{16}\log(-\log r) + \frac{1}{16}(\lambda_0 + \lambda_1) - \log 2 + \frac{\lambda_0}{16}\log r + \mathcal{O}(\log^{-1} r)\,, \quad r \to 0\,.$$

In the inner region with $\rho = r/\gamma$, we have

$$u = -1 + \varepsilon v(\rho)\,, \qquad v(\rho) = v_0 + \sigma v_1 + \sigma^2 v_2 + \cdots\,; \qquad \sigma = \left(\frac{-1}{\log\gamma}\right)\,,$$

where $v_0 = 2\rho^2 + 1$ is the unique solution of

$$\Delta_\rho^2 v_0 = 0\,, \quad 0 < \rho < \infty\,; \qquad v_0(0) = 1\,, \quad v_0'(0) = v_0'''(0) = 0\,;$$

$$v_0 \sim 2\rho^2\,, \quad \text{as} \quad \rho \to \infty\,.$$

Pure Biharmonic in the Unit Ball: III

The higher order (inner) terms v_1 and v_2 are the unique solutions of

$$\Delta_\rho^2 v_1 = -\frac{\lambda_0}{v_0^2} \quad 0 < \rho < \infty\,; \qquad v_1(0) = v_1'(0) = v_1'''(0) = 0\,;$$

$$v_1 \sim -2\rho^2 \log\rho + \rho^2 + \cdots\,, \quad \text{as} \quad \rho \to \infty\,.$$

$$\Delta_\rho^2 v_2 = -\frac{\lambda_1}{v_0^2} + \frac{2\lambda_0}{v_0^3} v_1\,, \quad 0 < \rho < \infty\,; \qquad v_2(0) = v_2'(0) = v_2'''(0) = 0\,,$$

$$v_2 = \mathcal{O}\left[\log^3(\rho)\right]\,, \quad \text{as} \quad \rho \to \infty\,.$$

To calculate λ_0, we use the v_1 equation to obtain

$$\lim_{R\to\infty}\left[\int_0^R \left(\Delta_\rho^2 v_1 + \frac{\lambda_0}{(1+2\rho^2)^2}\right)\rho\,d\rho\right] = \lim_{R\to\infty}\left[\rho\frac{d}{d\rho}\left(\Delta_\rho v_1\right)\Big|_{\rho=R} + \frac{\lambda_0}{4}\right] = 0\,,$$

which yields $\lambda_0 = 32$. Similarly, from the v_2 equation we get

$$\lambda_1 = 8\lambda_0 \int_0^\infty \frac{v_1}{v_0^3}\rho\,d\rho = 16\left(\log 2 - \frac{\pi^2}{6}\right)\,,$$

as obtained by calculating v_1 explicitly, and performing the integration.

Key Points in the Construction: IV

- For $\varepsilon \to 0^+$ and $\lambda \to 0$, then $\lambda/(1+u)^2 \to 0$ except in a narrow zone near $r = 0$, where $u = -1 + \mathcal{O}(\varepsilon)$.
- Leading order term u_0 in the outer region satisfies $\Delta^2 u_0 = 0$ in $0 < r < 1$, with $u_0 = u_{0r} = 0$ on $r = 1$. We must impose the point constraint $u_0(0) = -1$ in order to match to the inner solution. Thus,

$$u_0 = -1 + r^2 - 2r^2 \log r \,.$$

- If we expand $u = u_0 + \nu u_1$ and $\lambda = \nu \lambda_0$, then $\Delta^2 u_1 = -\lambda_0/(1+u_0)^2$, for which $u_{1p} \sim \frac{\lambda_0}{16} \log\left(-\log r\right)$ as $r \to 0$. This divergence of the particular solution as $r \to 0$ requires the inclusion of switchback terms.
- To find the boundary layer width γ, set $\rho = r/\gamma$, with $\gamma \ll 1$, to obtain $u_0 \sim -1 + \left(-\gamma^2 \log \gamma\right)(2\rho^2) + \gamma^2 \left(\rho^2 - 2\rho^2 \log \rho\right)$.
- Thus, in the inner region, we set $u = -1 + \varepsilon \left(v_0(\rho) + \sigma v_1 + \cdots\right)$ with $\sigma = -1/\log \gamma$. Thus, γ is given implicitly by $\varepsilon = -\gamma^2 \log \gamma$.
- The leading order inner problem is $\Delta^2 v_0 = 0$ with $v_0 = 2\rho^2 + 1$. The constant term 1 is then matched by adding an appropriate term in outer expansion.
- Next, we get $\Delta^2 v_1 = -\lambda_0/v_0^2$ with $v_1 \sim -2\rho^2 \log \rho + \rho^2$ as $\rho \to \infty$.

Mixed Biharmonic in the Unit Ball: I

Let $\delta > 0$ be fixed. In the unit ball we construct a solution with $\lambda \to 0$ as $u(0) + 1 = \varepsilon \to 0^+$ for

$$\delta \Delta^2 u - \Delta u = -\lambda/(1+u)^2 \,, \quad 0 < r < 1 \,; \qquad u(1) = u_r(1) = 0 \,.$$

The leading-order outer solution u_0 is the solution to:

$$\delta \Delta^2 u_0 - \Delta u_0 = 0 \,, \quad 0 < r < 1 \,; \quad u_0(1) = u_{0r}(1) = 0 \,, \quad u_0(0) = -1 \,,$$

which is given explicitly by

$$u_0 = \mathcal{A} + \mathcal{B} \log r + \mathcal{C} K_0(\eta r) + \mathcal{D} I_0(\eta r) \,, \qquad \eta \equiv 1/\sqrt{\delta} \,.$$

Here the constants are given by

$$\mathcal{A} = \left[I_0(\eta) \left(1 + \eta K_0'(\eta)\right) - \eta I_0'(\eta) K_0(\eta) \right] \mathcal{G}(\eta) \,,$$
$$\mathcal{B} = \mathcal{C} = \eta I_0'(\eta) \mathcal{G}(\eta) \,, \qquad \mathcal{D} = - \left[1 + \eta K_0'(\eta) \right] \mathcal{G}(\eta) \,,$$

where $\mathcal{G}(\eta)$ is defined by

$$\mathcal{G}(\eta) \equiv \left[\eta I_0'(\eta) \left(K_0(\eta) + \log\left(\eta/2\right) + \gamma_e \right) + \left(1 + \eta K_0'(\eta)\right) \left(1 - I_0(\eta)\right) \right]^{-1} \,,$$

and $\gamma_e \sim 0.5772$ is Euler's constant.

Mixed Biharmonic in the Unit Ball: II

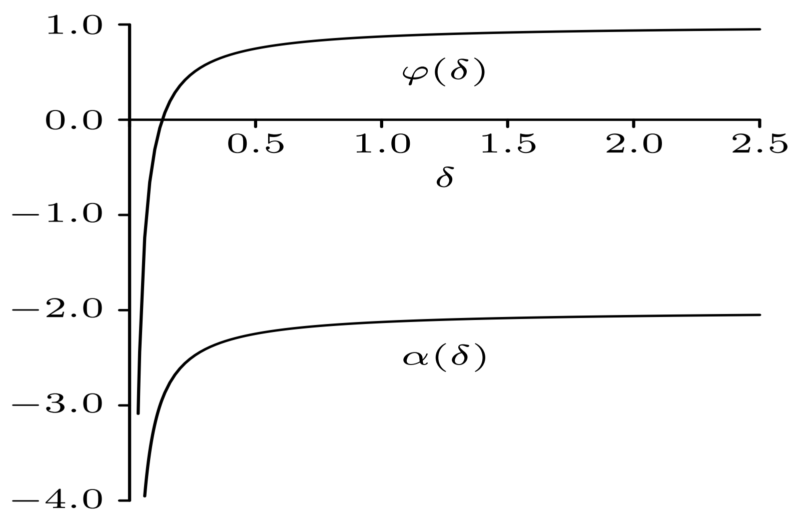
The asymptotics as $r \to 0$ are

$$u_0 = -1 + \alpha r^2 \log r + \varphi r^2 + o(r^2)\,, \quad \text{as} \quad r \to 0\,,$$

$$\alpha = -\left(\frac{\eta^3}{4}\right) I_0'(\eta)\mathcal{G}(\eta)\,,$$

$$\varphi = -\frac{\eta^2}{4}\left[\eta I_0'(\eta)\left(\log\left(\eta/2\right) + \gamma_e - 1\right) + \left(1 + \eta K_0'(\eta)\right)\right]\mathcal{G}(\eta)\,.$$

Plot of $\alpha(\delta)$ and $\varphi(\delta)$ for $0 < \delta < 2.5$ showing $\alpha < 0$ are:



Mixed Biharmonic in the Unit Ball: III

Principal Result: *In the limit* $\varepsilon \equiv u(0) + 1 \to 0^+$*, the limiting asymptotic behaviour of the maximal radially symmetric solution branch is*

$$\lambda = \frac{\delta \varepsilon}{\sigma} \left[\lambda_0 + \sigma \lambda_1 + \mathcal{O}(\sigma^2) \right] \,,$$

where $\sigma = -1/\log \gamma$ *and the boundary layer width* γ *is determined in terms of* ε *by* $-\gamma^2 \log \gamma = \varepsilon$*. Here,*

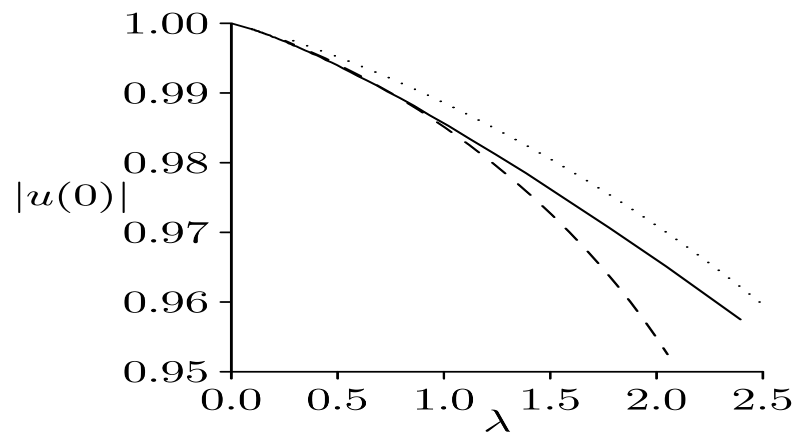
$$\lambda_0 = 8\alpha^2 \,, \qquad \lambda_1 = -\frac{\lambda_0}{2} \left[\frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\varphi}{\alpha} \right) \right] \,,$$

where $\alpha(\delta)$ *and* $\varphi(\delta)$ *are determined by the local behavior of* u_0 *as* $r \to 0$*.*

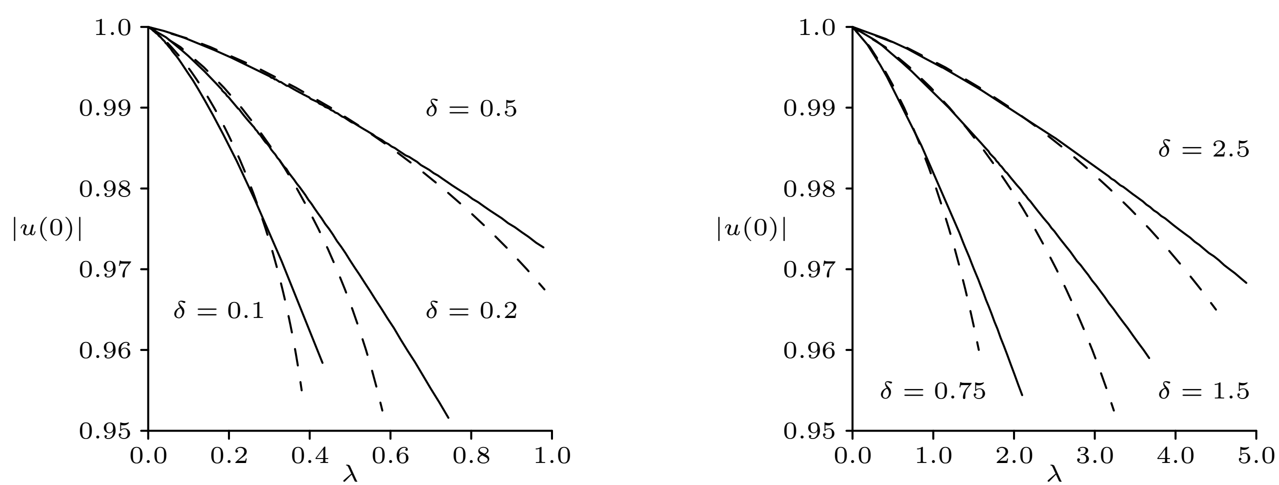
Comparison of Asymptotics and Full Numerics

Pure Biharmonic: Comparison with Full Numerics Comparison of Asympotics

1-term (dotted), 2-term (dashed), and Numerics (solid).



Mixed Biharmonic: Comparison of 2-term Asymptotics with Full Numerics



(a) $|u(0)|$ vs. λ (smaller δ)

(b) $|u(0)|$ vs. λ (larger δ)

Concentration in Arbitrary 2-D Domain I

Ref: [KLW] M. C. Kropinski, A. Lindsay, M. J. Ward, *An Asymptotic Analysis of Localized Solutions to Some Linear and Nonlinear Biharmonic Eigenvalue Problems*, to appear, Stud. Appl. Math, (2011), 63 pages.

Consider $\Delta^2 u = -\lambda/(1+u)^2$ in Ω, with $u = \partial_n u = 0$ on $\partial\Omega$.

Principal Result: *In the limit $\varepsilon \equiv u(x_0) + 1 \to 0^+$, the limiting asymptotics of the maximal solution branch in the outer region, away from x_0, and λ is*

$$u = u_0 + \mathcal{O}\left(\varepsilon\sigma^{-1}\log\sigma\right) , \qquad \lambda = \frac{\varepsilon}{\sigma}\lambda_0 + \varepsilon\lambda_1 + \mathcal{O}(\varepsilon\sigma) ,$$

where $\sigma = -1/\log\gamma$ and the boundary layer width γ is given implicitly in terms of ε by $-\gamma^2\log\gamma = \varepsilon$. Here

$$u_0 = -\frac{G(x;x_0)}{R(x_0;x_0)} ,$$

with point constraint $u_0(x_0) = -1$, where $G(x;x_0)$ satisfies

$$\Delta^2 G = \delta(x - x_0) , \quad x \in \Omega ; \qquad G = \partial_n G = 0 , \quad x \in \partial\Omega ,$$

$$G(x;x_0) = \frac{1}{8\pi}|x - x_0|^2 \log|x - x_0| + R(x;x_0) .$$

Concentration in Arbitrary 2-D Domain II

To leading order, the concentration point $x_0 \in \Omega$ satisfies

$$\nabla_x R(x;x_0)|_{x=x_0} = 0\,, \quad \text{provided that} \quad R(x_0;x_0) > 0\,,$$

As $x \to x_0$, with $r = |x - x_0|$, we identify α and β by

$$u_0 \sim -1 + \alpha r^2 \log r + r^2(\beta + a_c \cos 2\theta + a_s \sin 2\theta) + \cdots\,,$$

where $\alpha < 0$ by assumption, and β (sign $\pm$) are

$$\alpha \equiv \frac{-1}{8\pi R(x_0;x_0)}\,, \qquad \beta \equiv \frac{-1}{4R(x_0;x_0)} \left[\frac{\partial^2 R}{\partial x_1^2} + \frac{\partial^2 R}{\partial x_2^2}\right]_{x=x_0}\,,$$

Finally, λ_0 and λ_1 are given by

$$\lambda_0 = 8\alpha^2, \qquad \lambda_1 = -\frac{\lambda_0}{2}\left[\frac{\pi^2}{6} - \log(-\alpha) + \left(1 + \frac{2\beta}{\alpha}\right)\right]\,.$$

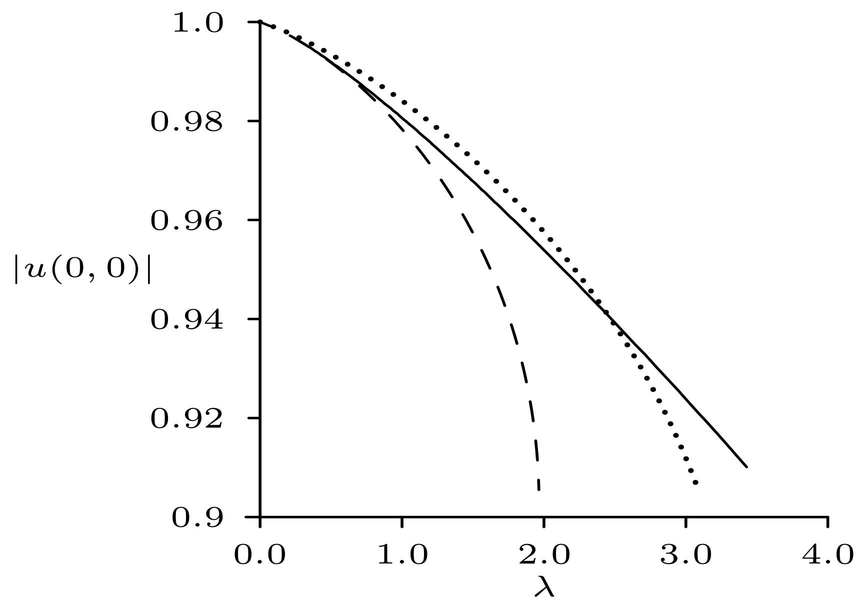
- Thus, 2-term asymptotics of λ are determined by properties of the regular part of the Biharmonic Green function
- **Note:** $R_{00} = R(x_0;x_0)$ and Trace$(\mathcal{R}_{00})$ can be computed by fast multipole methods for Low Reynolds number flow (Kropinski).

Concentration in Arbitrary 2-D Domain III

Comparison of Asymptotics and Numerics in Square Domain: For the square $[-1,1]^2$, then $x_0 = 0$, and to evaluate asymptotic result we computed

$$R_{00} \approx 0.0226\ldots, \qquad \text{Trace}(\mathcal{R}_{00}) \approx -0.0892\ldots.$$

Numerics (solid); 1-term asymptotics (dots); 2-term asymptotics (dashed)



Concentration in Arbitrary 2-D Domain IV

Class of Dumbbell-Shaped Domains

Determine points x_0 for which $\nabla R|_{x_0} = 0$ for a one-parameter family of mappings of the unit disk $\mathcal{B}$:

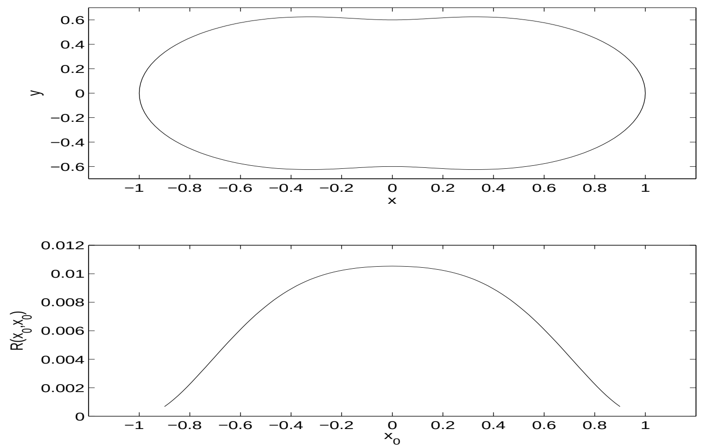
$$w = f(z;b) = \frac{(1-b^2)z}{z^2 - b^2}\,, \quad z \in \mathcal{B}\,.$$

For various values of b, numerical values for $R(x_0;x_0)$ and Trace $(\mathcal{R}_{00})$ at the points $x_0 = (x_0, 0)$ where $dR/dx_0 = 0$.

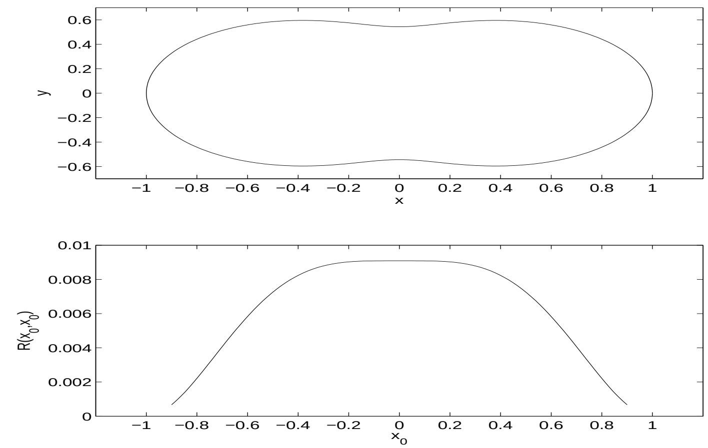
b	x_0	$R(x_0, x_0)$	Trace$(\mathcal{R}_{00})$
2.00000	0.00000	1.05312×10^{-2}	-2.44476×10^{-2}
1.83995	0.00000	9.08917×10^{-3}	-2.44656×10^{-2}
1.50000	-0.39000	6.48716×10^{-3}	1.12095×10^{-2}
	0.00000	5.15298×10^{-3}	4.00099×10^{-2}
	0.39000	6.48716×10^{-3}	1.12095×10^{-2}
1.05000	-0.49450	4.94718×10^{-3}	3.11557×10^{-2}
	0.000000	9.59768×10^{-5}	0.379489×10^{-2}
	0.494500	4.94718×10^{-3}	3.11557×10^{-2}

Concentration in Arbitrary 2-D Domain V

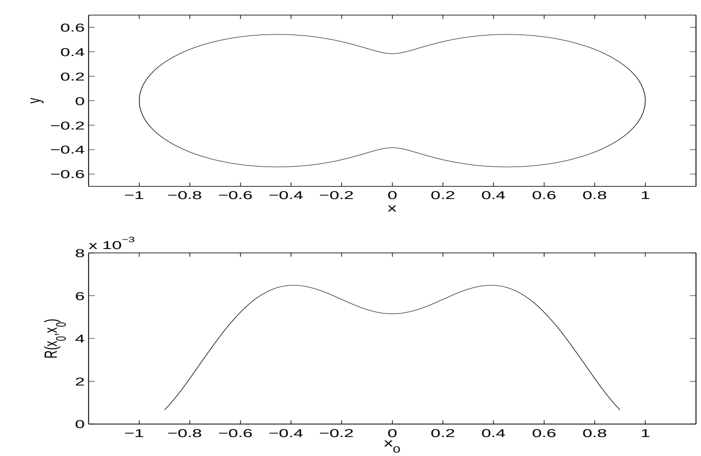
Plot of the domain and R_{00} along the horizontal axis $(x_0, 0)$



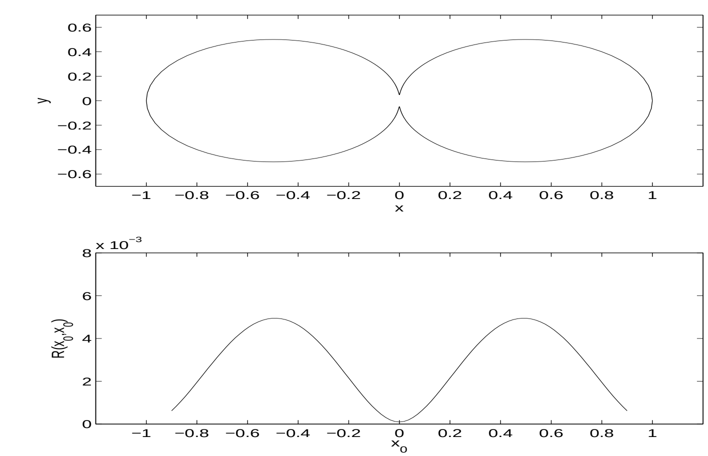
(c) $b = 2.0$



(d) $b = b_c = 1.83995$



(e) $b = 1.5$



(f) $b = 1.05$

Open Questions

Questions:

- Give a rigorous proof of the limiting concentration behavior in a disk and an arbitrary domain.
- Can one find an example with $\nabla R|_{x_0} = 0$, but $R(x_0, x_0) < 0$. Then, concentration at x_0 cannot occur. This might be theoretically possible since the Green function G is not guaranteed to be of one sign.
- Construct solutions in 2-D domains with multiple concentrations.
- Describe in detail the breakup of infinite fold point structure associated with the membrane MEMS problem when $\delta > 0$ but small. In particular, how many fold points are there for δ small but fixed?
- Can we construct limiting asymptotics of some related Bihmarmonic nonlinear eigenvalue problems with other nonlinearities and different dimension N?.

References

The following papers are available on my UBC website:

References:

- M. C. Kropinski, A. Lindsay, M. J. Ward, *An Asymptotic Analysis of Localized Solutions to Some Linear and Nonlinear Biharmonic Eigenvalue Problems*, to appear, Stud. Appl. Math, (2011), 63 pages.
- A. Lindsay, M. J. Ward, *Asymptotics of Some Nonlinear Eigenvalue Problems Modelling a MEMS Capacitor: Part II: Multiple Solutions and Singular Asymptotics*, EJAM, **22**(2), (2010), pp. 83–123.
- A. Lindsay, M.J. Ward, *Asymptotics of Some Nonlinear Eigenvalue Problems for a MEMS Capacitor: Part I: Fold Point Asymptotics*, Meth. Appl. Anal., **15**(3), (2008), pp. 297–326.