Topics in Singular Perturbations and Hybrid Asymptotic-Numerical Methods

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Abstract

Hybrid asymptotic-numerical methods are used to study various singular perturbation problems where infinite logarithmic asymptotic expansions occur. In particular, they are used to sum the infinite logarithmic expansions that arise for a low Reynolds number flow problem and for certain two-dimensional eigenvalue problems in perforated domains. Asymptotic and numerical methods are also used to analyze the exponentially slow internal layer motion that occurs for certain phase separation models and for other nonlinear diffusive problems with exponentially ill-conditioned linearizations. The study of this dynamical metastability phenomena involves exponential asymptotics.

0 Introduction

The method of matched asymptotic expansions is a powerful systematic analytical method for asymptotically calculating solutions to singularly perturbed problems. It has been successfully used in a wide variety of applications (see [H], [KC], [LA], [OR], [V]). However, there are certain special classes of problems where this method has some apparent limitations.

In particular, for problems involving infinite logarithmic series in powers of $\nu = -1/\log\varepsilon$, where ε is a small positive parameter, it is well-known that this method may be of only limited practical use in approximating the exact solution accurately. This difficulty stems from the fact that $\nu \to 0$ very slowly as ε decreases. Therefore, unless many coefficients in the infinite logarithmic series can be obtained analytically, the resulting low order truncation of this series will typically not be very accurate unless ε is very small. Singular perturbation problems involving infinite logarithmic expansions arise in many areas of application including; low Reynolds number fluid flow past cylindrical bodies and related model problems [HS], [HM73], [HM82], [HTB], [K], [KHC], [KW], [KWK], [LA], [PP], [S], [V]; vibration properties of membranes containing small holes or concentrated masses [LW94], [LS], [O], [SS], [SW], [WHK], [WK93]; thermal runaway behavior for cylindrical chemical reactors containing cooling rods [LW91], [WK91], [WHK], [LW94]; two-dimensional diffusion problems in singularly perturbed regions [TG], [TW].

In §1-2 we will summarize and extend some recent work of [KWK], [KW], [TW], and [WHK] where hybrid asymptotic-numerical methods are used to sum infinite logarithmic expansions in various contexts. Examples of such problems that will be considered include the calculation of the drag coefficient for slow viscous flow past a cylinder and the determination of eigenvalue parameters for some linear and nonlinear eigenvalue problems in perforated domains.

Another class of problems where difficulties with the method of matched asymptotic expansions arise is for problems where exponentially small terms of the form $O\left(e^{-c/\varepsilon}\right)$, for $\varepsilon \to 0$ with $c > 0$, need to be captured analytically in order to characterize the phenomena under consideration. For such problems in exponential asymptotics, it is typically difficult to apply the method of matched asymptotic expansions in a systematic way to resolve these terms. In many cases, a failure to resolve these exponentially small terms leads to a non-uniqueness in the matched asymptotic expansion solution, which is not present in the original problem. This type of spurious behavior is usually associated with an exponential ill-conditioning inherent in the underlying singularly perturbed problem. By exponential ill-conditioning we mean that the spectrum associated with the linearized version of the underlying problem contains exponentially small eigenvalues. Examples of singularly perturbed problems of this type include; boundary layer resonance phenomena [AM], [D], [GM], [KR], [LG], [LEW], [M]; autonomous boundary value problems with internal layer behavior [KKM], [L], [W92]; a nonlinear turning point problem [ML]; dynamical metastability phenomena for various phase separation models with applications to material science [ABF], [AF94a], [AF94b], [BX], [BH], [CP], [EF], [F], [FH], [G], [N], [RW94], [RW95b], [RSK], [W94], [W95a]; certain viscous shock problems [KK], [L094], [L095], [RW95a], [WR], [W0]; activator-inhibitor systems [KEM]; chaotic pulse solutions to Fitzhugh-Nagumo type equations [BIS], [EMS]; spike-layer solutions in multi-dimensional domains [NT91], [NT93], [W95b].

In §3-4 we will discuss and give some recent results for some of these exponentially ill-conditioned problems. In particular, we will show how these problems can be treated asymptotically by supplementing the method of matched asymptotic expansions with certain spectral information associated with the linearized problem.

1 Logarithmic Expansions in Bounded Domains

We now give various examples where infinite logarithmic expansions arise for partial differential equation problems in bounded two-dimensional domains and we show how hybrid asymptotic-numerical methods can be used to sum these expansions.

1.1 Perturbed Linear Eigenvalue Problems

We first consider the perturbed linear eigenvalue problem

$$\begin{aligned} \triangle u + [\lambda p(x) - V(x)]\, u &= 0\,, \qquad x \in D \setminus \cup_{i=1}^{N} D_\varepsilon^i\,, && (1.1)\\ \partial_n\, u + b\, u &= 0\,, \qquad x \in \partial D\,, && (1.2)\\ \varepsilon\, \partial_n\, u + \kappa_i\, u &= 0\,, \qquad x \in \partial D_\varepsilon^i\,, \quad i = 1, .., N\,, && (1.3)\\ \int_{D \setminus \cup_{i=1}^{N} D_\varepsilon^i} p(x) u^2\, dx &= 1\,, && (1.4) \end{aligned}$$

where $b > 0$ and $\kappa_i > 0$ are constants and $p(x)$ and $V(x)$ are smooth. Here and below we assume that D_ε^i is a small region of 'radius' $O(\varepsilon)$ that is centered at some $x_0^i \in D$. We assume that D_ε^i is obtained from a fixed domain D_0^i by shrinking the distance from every point of ∂D_0^i to x_0^i by the factor ε. Thus, we write $D_\varepsilon^i = \varepsilon D_0^i$. We shall focus on calculating the principal eigenpair $\lambda_0(\varepsilon)$, $u_0(x;\varepsilon)$ for this problem as $\varepsilon \to 0$.

Let λ_0^0, $u_0^0(x)$ denote the principal eigenpair of the unperturbed problem in the absence of any holes. Then, when $\kappa_i = \kappa > 0$ for all i and when each D_ε^i is a circular hole of radius ε, a two-term expansion for $\lambda_0(\varepsilon)$ is

$$\lambda_0(\varepsilon) = \lambda_0^0 + 2\pi \left(\frac{-1}{\log [\varepsilon d]} \right) \sum_{i=1}^{N} \left[u_0^0(x_0^i) \right]^2 + O\left[(-1/\log \varepsilon)^2 \right]\,, \qquad (1.5)$$

where $d = e^{-1/\kappa}$. This result, which gives the first two terms in an infinite logarithmic expansion for $\lambda_0(\varepsilon)$, was given in [O] for the Laplacian with $\kappa = \infty$ ($d = 1$) (see also [SW], [WK93] and [LW94] for related work).

In [WHK] a hybrid method was formulated and implemented numerically to sum the infinite logarithmic expansion for $\lambda_0(\varepsilon)$. The hybrid formulation is to find the first eigenpair λ_0^*, u_0^* of the related problem

$$\begin{aligned} \triangle u_0^* + [\lambda_0^* p(x) - V(x)]\, u_0^* &= 0\,, \qquad x \in D\,, && (1.6)\\ \partial_n u_0^* + b u_0^* &= 0\,, \qquad x \in \partial D\,, && (1.7) \end{aligned}$$

$$u_0^* = A_i + \nu_i A_i \log|x - x_0^i| \quad + \quad o(1)\,, \qquad x \to x_0^i\,, \quad i = 1,..,N\,, \tag{1.8}$$

$$\int_D p(x)\,(u_0^*)^2\, dx \quad = \quad 1\,, \tag{1.9}$$

where $u_0^* = u_0^*(x;\nu_1,..,\nu_N)$, $\lambda_0^* = \lambda_0^*(\nu_1,..,\nu_N)$ and $A_i(\nu_1,..,\nu_N)$ for $i = 1,..,N$ are to be determined. Here $\nu_i \equiv -1/\log z_i$, $z_i \equiv \varepsilon d^i$, and $d^i = d^i(\kappa_i)$ is to be determined from the solution to the canonical inner problem

$$\triangle_y v_c^i \quad = \quad 0\,, \qquad y \notin D_0^i\,, \tag{1.10}$$

$$\partial_n v_c^i + \kappa_i v_c^i \quad = \quad 0\,, \qquad y \in \partial D_0^i\,, \tag{1.11}$$

$$v_c^i \quad \sim \quad \log|y| - \log d^i + p^i \cdot y/|y|^2 + \cdots\,, \qquad y \to \infty\,. \tag{1.12}$$

When $\kappa = \infty$, d^i is the logarithmic capacitance. The asymptotic analysis of [WHK] shows that

$$u_0(x;\varepsilon) \quad = \quad u_0^*(x;\nu_1,..,\nu_N) + O\left[\varepsilon/\log\varepsilon\right]\,, \qquad |x - x_0^i| \gg O(\varepsilon)\,, \tag{1.13}$$

$$\lambda_0(\varepsilon) \quad = \quad \lambda_0^*(\nu_1,..,\nu_N) + O\left[\varepsilon/\log\varepsilon\right]\,. \tag{1.14}$$

The justification for this method is based on examining the form of the infinite logarithmic expansions for each of the inner solutions near the holes (see [WHK]).

The solution to the hybrid problem can be written in the form

$$u_0^* = 2\pi \sum_{i=1}^{N} A_i \nu_i G\left(x; x_0^i, \lambda_0^*\right)\,, \tag{1.15}$$

where the A_i satisfy the homogeneous linear system

$$A_i\left[1 - 2\pi\nu_i G_r(x_0^i; x_0^i, \lambda_0^*)\right] - 2\pi \sum_{\substack{j=1 \\ j\neq i}}^{N} \nu_j A_j G(x_0^i; x_0^j, \lambda_0^*) = 0\,, \qquad i = 1,..,N\,. \tag{1.16}$$

Here $G\left(x; x_0^i, \lambda_0^*\right)$ is the Green's function for (1.6), (1.7) with singularity at $x = x_0^i$, and G_r is the regular part of G. The relation $\lambda_0^* = \lambda_0^*(\nu_1,..,\nu_N)$ is determined by the condition that this system has a non-trivial solution $A_1,..,A_N$ with $\lambda_0^* \to \lambda_0^0$ as $\varepsilon \to 0$. The condition (1.9) then determines the A_i uniquely.

This hybrid problem is readily solved numerically as a function of the parameters ν_i, for $i = 1,..,N$. Moreover, the hybrid solution can be used for subdomains of different shapes by simply re-computing the N shape-dependent parameters d^i numerically from the canonical inner problems. This gives a significant reduction in the computational complexity associated with solving the full perturbed problem numerically. In [WHK] numerical results for λ_0^*

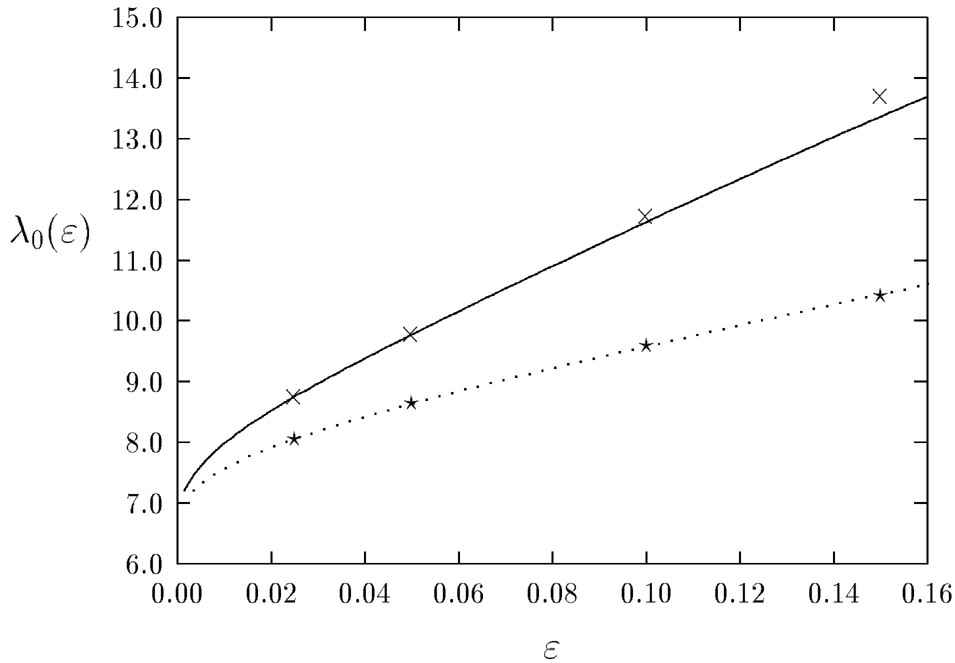

Figure 1: For Example 1, hybrid results for λ_0 (solid and dotted curves) are compared with corresponding full numerical results (discrete points). The solid curve is for $\kappa = \infty$ and the dotted curve is for $\kappa = 1$.

were compared for various examples with the two-term expansion (1.5) and with numerical results computed from the full perturbed problem (1.1)-(1.4) using a finite difference method. We now summarize a few of these results.

Example 1 (One Hole): Let D be a circular cylindrical domain of radius one, with $u = 0$ on D, that contains a small elliptical hole $D_\varepsilon = \varepsilon D_0$ centered at the origin. The semi-axes of D_0 are 2.0 and 0.5. By determining G explicitly in terms of the Bessel functions J_0 and Y_0, we obtain from (1.16) that $\lambda_0^*(z)$ is the solution branch of the transcendental equation

$$\frac{J_0\left(\sqrt{\lambda_0^*}\right)}{Y_0\left(\sqrt{\lambda_0^*}\right)} = \frac{\pi}{2}\left[\log z + \log\left(\sqrt{\lambda_0^*}/2\right) + \gamma\right]^{-1}, \qquad z = \varepsilon d\,, \tag{1.17}$$

for which $\lambda_0^*(z) \to \lambda_0^0$ as $z \to 0$. Here γ is Euler's constant and λ_0^0 is the first zero of $J_0\left(\sqrt{\lambda_0^*}\right)$. In Figure 1 we compare the hybrid results $\lambda_0^*\left[\varepsilon d\right]$ with the

N (no. of Holes)	ε	$\lambda_0(\varepsilon)$ (numerical)	$\lambda_0^*(\varepsilon d)$	(1.18) (2-term)
2	0.025	9.49	9.48	8.697
2	0.050	10.50	10.42	9.372
2	0.100	12.16	11.91	10.452
5	0.025	15.74	15.7	11.406
5	0.050	20.0	19.8	12.707
5	0.100	28.1	27.1	14.792

Table 1: Circular holes of radius ε with $u = 0$ on each hole. For $N = 2$ the holes are centered at $x_0^1 = (0,0)$ and $x_0^2 = (.50, 0.0)$. For $N = 5$ the holes are centered at $x_0^1 = (0,0)$, $x_0^{2,3} = (\pm.50, 0)$, $x_0^{4,5} = (0, \pm.50)$.

corresponding full numerical results for $\kappa = \infty$ and for $\kappa = 1$. The values of d for these two cases are $d(\infty) = 1.25$ and $d(1) = .562$. From this figure we see that the hybrid result provides an accurate determination of $\lambda_0(\varepsilon)$ even when $\varepsilon \approx .15$.

Example 2 (Several Holes): Let D be a circular cylindrical domain of radius one, with $u = 0$ on ∂D, that contains either two or five circular holes. Each hole has radius ε with $u = 0$ on the hole boundary. The locations x_0^i of the hole centers are given in Table 1. For this problem the two-term expansion (1.5) for $\varepsilon \to 0$ becomes

$$\lambda_0(\varepsilon) = \lambda_0^0 + \frac{2\nu}{\left(J_0'(\sqrt{\lambda_0^0})\right)^2} \sum_{i=1}^{N} \left[J_0(\sqrt{\lambda_0^0}|x_0^i|) \right]^2 + \cdots , \tag{1.18}$$

where $\nu = -1/\log\varepsilon$ and λ_0^0 is the first zero of $J_0\left(\sqrt{\lambda_0^0}\right)$. This result does not account for any interactions between the holes. In Table 1 we compare (1.18) with the hybrid and full numerical results for $\lambda_0(\varepsilon)$ computed in [WHK]. This table shows that the two-term result (1.18) is very inaccurate when ε is moderately small.

1.3 Perturbed Nonlinear Eigenvalue Problems

Next we consider the following nonlinear eigenvalue problem that arises in steady-state combustion theory:

$$\triangle u + \lambda F(u) = 0 , \qquad x \in D \backslash D_\varepsilon , \tag{1.19}$$

$$\partial_n u + b\, u = 0 , \qquad x \in \partial D , \tag{1.20}$$

$$\varepsilon\, \partial_n u + \kappa u = 0\,, \qquad x \in \partial D_\varepsilon\,, \tag{1.21}$$

$$\alpha = \int_{D \setminus D_\varepsilon} u^2\, dx\,. \tag{1.22}$$

Here $\kappa > 0$, $u = u(x, \alpha; \varepsilon)$, $\lambda = \lambda(\alpha, \varepsilon)$, $D_\varepsilon = \varepsilon D_0$ and $F(u)$ is the Arrhenius reaction term $F(u) \equiv \exp\left[u/(1 + \beta u)\right]$, where $\beta > 0$. With this form of $F(u)$, (1.19)-(1.22) typically has multiple solutions.

Let $u_0^0(x, \alpha)$ and $\lambda_0^0(\alpha)$ denote the solution branch to the unperturbed problem (1.19), (1.20) and (1.22) in all of D for which $\lambda_0^0 \to 0$ as $\alpha \to 0$. We assume that the graph of α versus λ_0^0 is S-shaped so that the unperturbed problem admits multiple solutions for some range of λ_0^0. Let λ_{c0} denote the value of λ_0^0 at the location of the lower fold point which connects the lower and middle branches of this S-shaped curve. In the combustion context, λ_{c0} is the thermal ignition point and as λ_0^0 increases past λ_{c0} a thermal explosion is initiated. The perturbed problem (1.19)-(1.22) models the effect on thermal runaway of inserting a cooling rod of small cross-sectional area inside the cylinder (see [LW91], [WK91], [WHK], [LW94]).

Let $\lambda_c(\varepsilon)$ denote the lower fold point for the perturbed problem. For $\varepsilon \to 0$, it was shown in [WHK] that the expansion of $\lambda_c(\varepsilon) - \lambda_{c0}$ starts with an infinite logarithmic series. Specifically, it was found that

$$\lambda_c(\varepsilon) = \lambda_c^*\left[\varepsilon d\right] + O\left(\varepsilon / \log \varepsilon\right)\,, \tag{1.23}$$

where $\lambda_c^*(z)$ is the lower fold point for the hybrid problem

$$\triangle u^* + \lambda^* F(u^*) = 0\,, \qquad x \in D\,, \tag{1.24}$$

$$\partial_n u^* + b\, u^* = 0\,, \qquad x \in \partial D\,, \tag{1.25}$$

$$u^* = A + \nu A \log|x - x_0| + o(1)\,, \qquad x \to x_0\,, \tag{1.26}$$

$$\alpha = \int_D (u^*)^2\, dx\,. \tag{1.27}$$

Here $\nu = -1/\log z$, $z = \varepsilon d$ and d is computed from the canonical inner problem (1.10)-(1.12), where D_0^i and d^i are replaced by D_0 and d, respectively.

For instance, let $F = e^u$, $b = \infty$ and suppose that D is a circular cylinder of radius one containing a hole D_ε centered at the origin. For this specific example, it was shown in [WHK] that

$$\lambda_c^*(z) = 8\left(e^{-A_c/2} - e^{-A_c}\right)\left(1 - \frac{A_c}{2 \log z}\right)^2\,, \tag{1.28}$$

ε	κ	$\lambda_c(\varepsilon)$ (numerical)	$\lambda_c^*(\varepsilon d)$	(1.30) (2-term)
0.025	∞	3.15	3.1542	2.836
0.050	∞	3.57	3.5808	3.057
0.10	∞	4.35	4.4406	3.436
0.025	1	2.782	2.782	2.691
0.050	1	2.965	2.964	2.835
0.10	1	3.259	3.252	3.056
0.15	1	3.532	3.509	3.248

Table 2: Comparison of hybrid, two-term and full numerical results for $\lambda_c(\varepsilon)$ for an elliptical hole centered at the origin of a cylinder of radius one when $F = e^u$ and $b = \infty$.

where $z = \varepsilon d$ and $A_c = A_c(z)$ is the unique solution to the transcendental equation

$$\frac{1}{\log z} = \left[\frac{2 - 2\, e^{A_c/2}}{e^{A_c/2} - 2} + \frac{A_c}{2} \right]^{-1} . \tag{1.29}$$

For other forms of $F(u)$, $\lambda_c^*(z)$ can be computed numerically by solving a two-point boundary value problem (see [WHK]). A close asymptotic approximation to $\lambda_c(\varepsilon)$ is then obtained by substituting (1.28) into (1.23). This result is analogous to that given in (1.17) for the linear eigenvalue problem. From [LW91] and [WK91], a two-term logarithmic expansion for $\lambda_c(\varepsilon)$ is

$$\lambda_c(\varepsilon) \sim 2 + 4 \log 2 \left(-1/ \log [\varepsilon d] \right) + \cdots . \tag{1.30}$$

In Table 2 we compare $\lambda_c^*(\varepsilon d)$, when $\kappa = \infty$ and $\kappa = 1$, with the corresponding two-term and full numerical results for $\lambda_c(\varepsilon)$ in the case when D_0 is an ellipse with semi-axes 2.5 and 0.4. The values of d are $d(\infty) = 1.45$ and $d(1) = .724$. From this table it is clear that the hybrid result provides a significantly better determination of $\lambda_c(\varepsilon)$ than does the two-term expansion.

2 Logarithmic Expansions in Unbounded Domains

We now consider two problems in unbounded domains where infinite logarithmic expansions arise.

2.1 Convective Heat Transfer

We first consider two-dimensional convective heat transfer past N small cylindrical bodies D_ε^i, which are centered at $x = \xi^i$ for $i = 1,..,N$. For simplicity, we assume that there is a uniform flow in the positive x_1 direction. Thus, the temperature $u(x;\varepsilon)$, where $x = (x_1, x_2)$, satisfies

$$\begin{aligned} \triangle u - u_{x_1} &= 0\,, \qquad x \in \mathcal{R}^2 \setminus \cup_{i=1}^N D_\varepsilon^i\,, && (2.1)\\ \varepsilon\, \partial_n u + \kappa_i\,(u - \alpha_i) &= 0 \qquad x \in \partial D_\varepsilon^i\,,\quad i = 1,..,N\,, && (2.2)\\ u &\sim 1\,, \qquad (x_1^2 + x_2^2)^{1/2} \to \infty\,. && (2.3) \end{aligned}$$

Here $\kappa_i > 0$ and α_i for $i = 1,..,N$ are constants, and $D_\varepsilon^i = \varepsilon D_0^i$.

A hybrid method for summing the logarithmic expansions associated with this problem and with some generalizations of it is given in [TW]. We now briefly summarize some of the main results. The outer expansion for u, valid in the region away from the small bodies, is

$$u(x;\varepsilon) = u_0(x;\nu_1,..,\nu_N) + \varepsilon\, u_1(x;\nu_1,..,\nu_N) + \cdots\,, \qquad (2.4)$$

where $\nu_i \equiv -1/\log\left[\varepsilon d^i\right]$ and $d^i = d^i(\kappa_i)$ is determined from the canonical inner problem (1.10)-(1.12). The hybrid problem for u_0 is

$$\begin{aligned} \triangle u_0 - u_{0x_1} &= 0\,, && (2.5)\\ u_0 &\sim 1\,, \qquad (x_1^2 + x_2^2)^{1/2} \to \infty\,, && (2.6)\\ u_0 = \alpha_i - A_i - \nu_i A_i \log|x - \xi^i| &+ o(1)\,, \quad x \to \xi^i\,,\quad i = 1,..,N\,, && (2.7) \end{aligned}$$

where $A_i = A_i(\nu_1,..,\nu_N)$ for $i = 1,..,N$ are functions to be determined. The solution to the hybrid problem is

$$u_0(x;\nu_1,..,\nu_N) = 1 + \sum_{i=1}^N \nu_i A_i e^{x_1 - \xi_1^i}\, K_0\left(|x - \xi^i|/2\right)\,, \qquad (2.8)$$

where the A_i, for $i = 1,..N$, satisfy the linear system

$$A_i\left[1 - \nu_i(\log 4 - \gamma)\right] + \sum_{\substack{j=1\\ j\neq i}}^N \nu_j A_j e^{\xi_1^i - \xi_1^j} K_0\left(|\xi^i - \xi^j|/2\right) = \alpha_i - 1\,. \qquad (2.9)$$

Here γ is Euler's constant, ξ_1^i is the x_1 coordinate of ξ^i and $K_0(z)$ is the modified Bessel function.

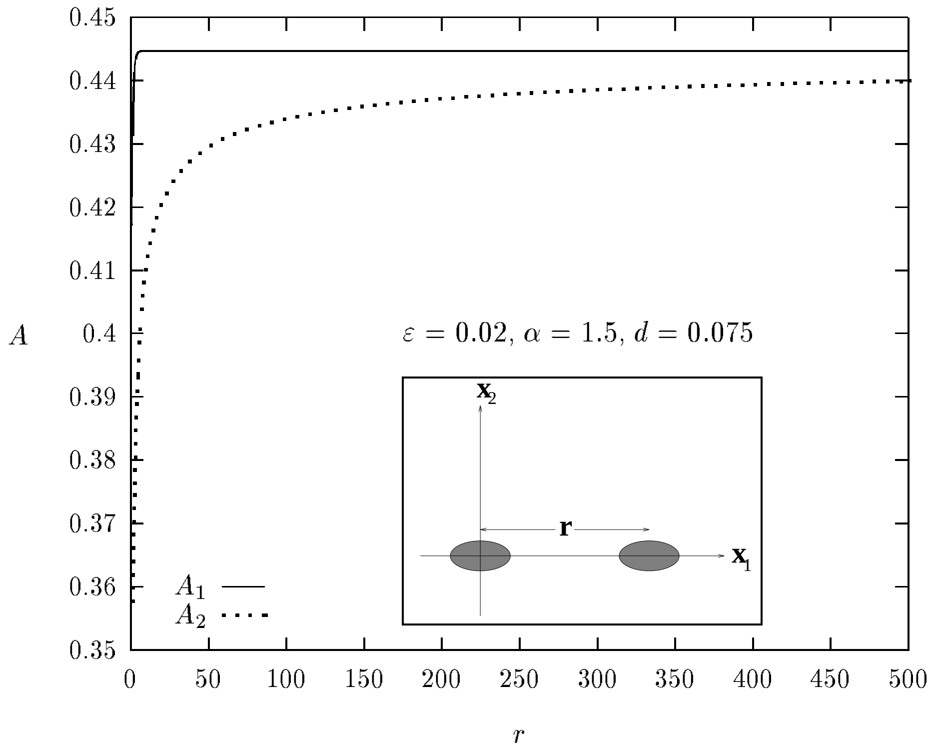

Figure 2: Plots of A_1 and A_2 versus the separation distance r are shown for two identical elliptical bodies aligned with the flow.

The asymmetry of the flow is reflected by the off-diagonal terms $e^{\xi_1^i - \xi_1^j}$ in (2.9). To leading order as $\nu_i \to 0$ we have that $A_i \sim \alpha_i - 1$. Thus, to this order, the A_i are independent of the locations of the small bodies and are not influenced by the asymmetry of the flow.

We now test the range of validity of this no-interaction limit for a specific example. Consider two identical elliptical bodies with $d \equiv d^1 = d^2 = 0.075$, $\alpha \equiv \alpha_1 = \alpha_2 = 1.5$ and with $\kappa_1 = \kappa_2 = \infty$. The ellipses are aligned with the flow and their centers are located at $(0,0)$ and $(r,0)$ with $r > 0$ (see the insert in Figure 2). We fix $\varepsilon = 0.02$ and we determine A_1 and A_2 as a function of the separation distance r by solving (2.9) numerically. For values of r where the interaction between the bodies is negligible it follows from (2.9) that $A_1 \approx A_2$. From Figure 2, where we plot A_1 and A_2 as a function of r, it is clear that even at the large separation value $r = 300$, the first body has an appreciable effect on the second body when $\varepsilon = 0.02$. This suggests that the no-interaction limit, which is predicted by the first term in the infinite logarithmic expansion for the A_i, can have a very limited range of validity.

2.2 Slow Viscous Flow Past a Cylinder

Next we consider the steady two-dimensional flow of a viscous incompressible fluid around an infinitely long straight cylinder whose cross-section is symmetric about the direction of the oncoming stream. In terms of polar coordinates centered inside the body, the dimensionless stream function $\psi(r,\theta;\varepsilon)$ satisfies

$$\begin{aligned} \triangle^2\psi + \varepsilon J\left(\psi, \triangle\psi\right) &= 0\,, \qquad r > r_b(\theta)\,, &(2.10)\\ \psi = 0\,, \qquad \psi_n &= 0\,, \qquad r = r_b(\theta)\,, &(2.11)\\ \psi &\sim r\sin\theta\,, \qquad r \to \infty\,. &(2.12)\end{aligned}$$

Here $\varepsilon \ll 1$ is the Reynolds number based on the radius of the cross-section, $J(u,v) \equiv r^{-1}\left(u_r v_\theta - u_\theta v_r\right)$ is the Jacobian, and $r = r_b(\theta)$, with $r_b(\theta) = r_b(-\theta)$, is the (dimensionless) boundary of the cross-section of the cylinder.

For $\varepsilon \to 0$, this problem was studied using the method of matched asymptotic expansions in the classical works of [K] and [PP]. The methods and results are summarized in [V]. The analysis of [K] for $\varepsilon \to 0$ showed that the expansion of the drag coefficient C_D for this problem starts with $C_D \sim 4\pi\varepsilon^{-1}A(\varepsilon d)$, where $A(z)$ is an infinite logarithmic series. Here d is an 'effective' radius of the cross-section of the cylinder. The three-term (re-normalized) result of [K] is

$$C_D \sim 4\pi\varepsilon^{-1}\nu(\varepsilon d)\left[1 - 0.8669\,\nu^2(\varepsilon d)\right]\,, \qquad \nu(z) \equiv \left[\log\left(3.7027/z\right)\right]^{-1}\,. \quad (2.13)$$

This result agrees rather poorly with corresponding experimental results when ε is moderately small. To date, no other terms in $A(z)$ have been obtained analytically.

We now summarize the hybrid method of [KWK], improved in [KW], for summing the logarithmic expansion for C_D. In the inner (Stokes) region we expand $\psi(r,\theta;\varepsilon)$ as

$$\psi(r,\theta;\varepsilon) \sim \psi_0(r,\theta;\nu) + \varepsilon\psi_1(r,\theta;\nu) + \cdots\,. \quad (2.14)$$

Here $\nu \equiv \left(\log 4 - \gamma - \log z + 1/2\right)^{-1} = \left[\log\left(3.7026/z\right)\right]^{-1}$, where γ is Euler's constant, $z \equiv \varepsilon d$ and d is specified below. Substituting (2.14) into (2.10) and (2.11), and allowing ψ_0 to grow like $r\log r$ at infinity, we obtain that ψ_0 is a multiple of a canonical inner Stokes solution $\psi_c(r,\theta)$, which satisfies

$$\begin{aligned} \triangle^2\psi_c &= 0\,, \qquad r > r_b(\theta)\,; \quad \psi_c(r,\theta) = -\psi_c(r,-\theta)\,, &(2.15)\\ \psi &= 0\,, \qquad \psi_n = 0\,, \qquad r = r_b(\theta)\,, &(2.16)\\ \psi_c &\sim \left(r\log r - r\log\left[de^{1/2}\right]\right)\sin\theta\,, \qquad r \to \infty\,. &(2.17)\end{aligned}$$

The shape-dependent parameter d is uniquely determined by this problem.

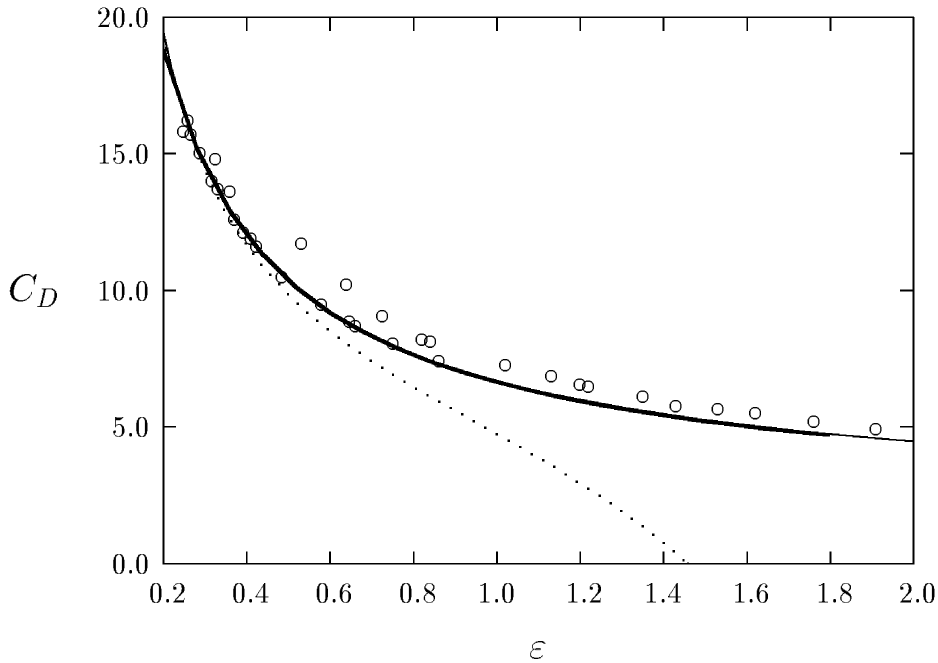

Figure 3: The hybrid result (2.22) for C_D (solid line) is compared with the experimental results of Tritton (discrete points), with full numerical results (heavy solid line) and with the three-term expansion (2.13) (dotted line).

We then take $\psi_0(r,\theta;\nu) = A\psi_c(r,\theta)$, where $A = A(\nu)$ is to be determined. In the outer (Stokes) region we set $\Psi(R,\theta;\varepsilon) = \varepsilon\psi(\varepsilon^{-1}R,\theta;\varepsilon)$ and we expand

$$\Psi(R,\theta;\varepsilon) \sim \Psi_0(R,\theta;\nu) + \varepsilon\Psi_1(R,\theta;\nu) + \cdots . \tag{2.18}$$

Substituting (2.18) into (2.10) and (2.12), and matching Ψ_0 to the far field form of the inner solution, we obtain that $\Psi_0(R,\theta;\nu)$ and $A(\nu)$ satisfy

$$\begin{aligned} \triangle^2\Psi_0 &+ J\left(\Psi_0, \triangle\Psi_0\right) = 0 \,, \qquad R > 0 \,, & (2.19)\\ \Psi_0 &\sim R\sin\theta \,, \qquad R \to \infty \,, & (2.20)\\ \Psi_0 &\sim AR\log R\sin\theta - AR\log\left[ze^{1/2}\right]\sin\theta + o(R) \,, \quad R \to 0 \,. & (2.21) \end{aligned}$$

Since we are specifying the regular part of Ψ_0 in (2.21), the singularity form (2.21) provides an implicit equation for A in terms of z.

The solution to (2.19), (2.20) with $\Psi_0 \sim AR \log R \sin\theta$ as $R \to 0$ can be computed numerically as a function of A, at least when A is sufficiently small. After computing Ψ_0 numerically, we subtract off the singular part to obtain $\Psi_0 - AR\log R \sin\theta \sim CR\sin\theta$ as $R \to 0$, where $C = C(A)$. The function $C(A)$ is computed numerically in terms of the behavior of the Fourier sine coefficient of Ψ_0 as $R \to 0$. For details on the computational procedures used see [KWK] and [KW]. Since C must agree with the coefficient of $O(R\sin\theta)$ in (2.21) we have that $C(A) = -A\log\left(ze^{1/2}\right)$, which determines z as a function of A.

The curve $A = A(z)$ can be used for cylindrical cross-sections of arbitrary shape and this universality behavior is known as Kaplun's equivalence principle. To obtain A as a function of ε for a specific shape, we compute d numerically from (2.15)-(2.17) and then set $z = \varepsilon d$. Finally, to within all logarithmic correction terms, the drag coefficient is given asymptotically by

$$C_D \sim 4\pi\varepsilon^{-1}A\left(\varepsilon d\right) . \tag{2.22}$$

In Figure 3, we show some results for C_D computed in [KWK], [KW] for a circular cylinder of radius one ($d = 1$). In this figure, we compare the hybrid result (2.22) for C_D with the full numerical result for C_D, with the experimental results of Tritton and with the three-term result (2.13). Notice that the hybrid results and the full numerical results for C_D are virtually indistinguishable on the range $0 < \varepsilon < 1.8$. Similar results for various cross-sectional shapes are given in [KWK]. Finally, the effect of transcendentally small terms on C_D and on the solution were analyzed in [S] and [KW] for a circular cylinder.

3 Dynamic Metastability in One Dimension

We now consider some problems exhibiting dynamical metastability phenomena in one spatial dimension.

3.1 Allen-Cahn Equation

The analysis of [CP] and [FH], justifying the formal asymptotic analysis of [N], proved that dynamical metastability occurs for the Allen-Cahn equation modeling slow phase separation

$$u_t = \varepsilon^2 u_{xx} + Q(u)\,, \quad -1 < x < 1\,; \qquad u_x(1,t) = u_x(-1,t) = 0\,. \tag{3.1}$$

Here $u(x,0) = u_0(x)$, $\varepsilon \ll 1$ and $Q(u) = -V'(u)$, where $V(u) \equiv -\int_{s_-}^{u} Q(\eta)\,d\eta$ is a double-well potential with wells of equal depth (i. e. $V(s_+) = 0$) located at

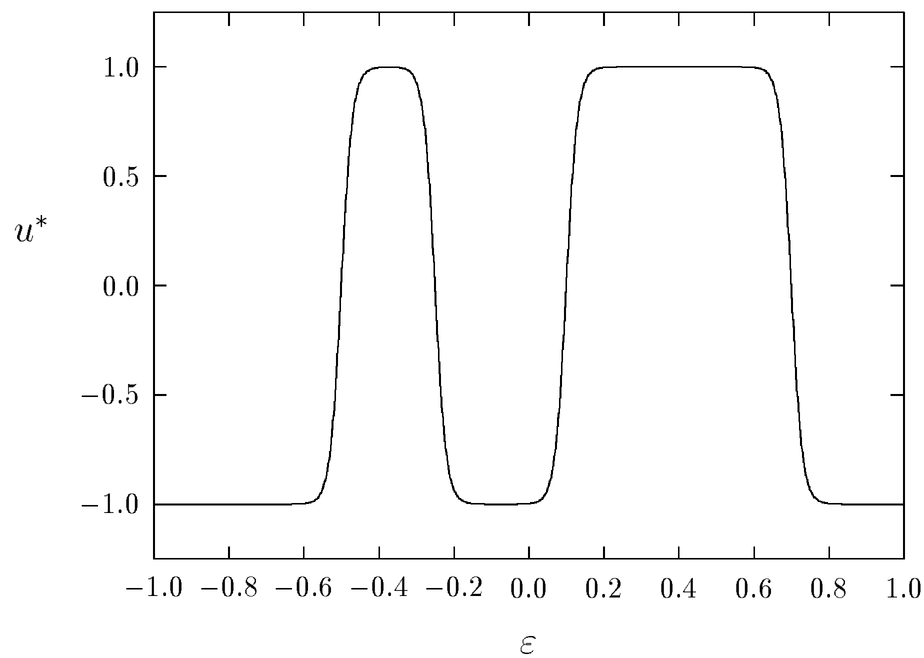

Figure 4: Schematic plot of the form of a metastable pattern of internal layers.

the two preferred phases $s_+ > 0$ and $s_- < 0$. Prototypical is $Q(u) = 2(u - u^3)$.

In an $O(1)$ time interval a pattern of internal layers is formed that separates the two phases (see Figure 4). The term dynamic metastability refers to the subsequent exponentially slow motion of this pattern of layers. Each of these internal layers is closely approximated by a dilation of the standing wave solution $u_c(z)$, which satisfies

$$u_c''(z) + Q\left[u_c(z)\right] = 0\,, \quad -\infty < z < \infty\,, \qquad u_c(0) = 0\,, \tag{3.2}$$

$$u_c(z) \sim s_\pm \mp a_\pm e^{\mp\nu_\pm z}\,, \quad z \to \pm\infty\,. \tag{3.3}$$

Here $\nu_\pm = \left[-Q'(s_\pm)\right]^{1/2}$ and $a_\pm$ are positive constants. A metastable pattern of N widely separated layers is then represented by the composite expansion

$$u \sim u^*\left(x; x_0, .., x_{N-1}\right) \equiv \sum_{i=1}^{N} u_{ci} + \text{ constant}\,, \qquad u_{ci} \equiv u_c\left[\varepsilon^{-1}\xi_i(x - x_i)\right]\,, \tag{3.4}$$

where $x_i = x_i(t)$, $x_{i+1} > x_i$, $\xi_i = (-1)^i \xi_0$ and $\xi_0 = \pm 1$. By widely separated we mean that $x_{i+1} - x_i = O(1)$ as $\varepsilon \to 0$. The difficulty with this problem is that a conventional singular perturbation analysis fails to determine the $x_i = x_i(t)$ unless the exponentially weak interactions between neighboring layers is accounted for.

We now outline a formal asymptotic method, referred to in [W94] as the projection method, for deriving equations of motion for the $x_i(t)$. It is closely related to the method of [EMS].

We first linearize u about u^*. Let $w(x,t)$ denote the solution to the linearized problem and assume that $w_t \ll u_t^*$. Then w satisfies

$$L_\varepsilon w \equiv \varepsilon^2 w_{xx} + Q'(u^*) w = E + u_t^* \, ; \quad E \equiv \sum_{i=0}^{N-1} Q(u_{ci}) - Q(u^*) \tag{3.5}$$

$$w_x(-1,t) = u_x^*(-1, x_0, .., x_{N-1}) \qquad w_x(1,t) = u_x^*(1, x_0, .., x_{N-1}) \, . \tag{3.6}$$

Here E represents the exponentially weak layer interactions. We then expand w in terms of the normalized eigenpairs ϕ_j, λ_j of L_ε to get

$$w = \sum_{j=0}^{\infty} \frac{A_j}{\lambda_j} \phi_j \, , \qquad A_j = (\phi_j, E) + (\phi_j, u_t^*) - \varepsilon^2 \phi_j u_x^* \Big|_{-1}^{1} \, , \tag{3.7}$$

where $(u,v) \equiv \int_{-1}^{1} uv \, dx$. As a result of the slight break in the translation degeneracy, the first N eigenvalues of L_ε are exponentially small as $\varepsilon \to 0$ and the corresponding eigenfunctions are $\phi_j \sim u_c' \left[\varepsilon^{-1} \xi_j (x - x_j)\right]$ for $j = 0, .., N-1$. Thus the solution w must satisfy the N limiting solvability conditions that $A_j \to 0$ as $\varepsilon \to 0$ for $j = 0, .., N-1$. Setting $A_j = 0$ for $j = 0, .., N-1$ then yields an ODE system for the $x_j(t)$. An *explicit* ODE system is obtained by evaluating the inner products in it asymptotically (see [W94]). The equilibrium internal layer locations correspond to the equilibria of this ODE system. This type of analysis is valid only when the layers are widely separated. In [W94] a hybrid asymptotic-numerical approach was used to study layer collapse events for (3.1) and some related problems. In this way, the coarsening process that occurs for (3.1) was characterized quantitatively.

A similar projection approach, which supplements the method of matched asymptotic expansions, can be used to treat many related problems. Some of these problems are described below in §3.2, §3.3 and §4. This approach has also been used in [LEW] to study boundary layer resonance phenomena. The main feature that is needed in order to apply this method is that the linearized problem about an appropriate approximate solution having undetermined parameters is exponentially ill-conditioned.

Method 2 (Nonlinear WKB): We now summarize another method for studying metastable behavior in (3.1). This method, introduced in [RW94], is based on an exact re-formulation of (3.1) in terms of the new variables τ, $v(x,\tau)$ and $\phi(x,\tau)$ defined by $\tau=\varepsilon t$ and $u=u_c(\varepsilon^{-1}v)$ and $v_x=\tanh(\varepsilon^{-1}\phi)$. Here $u_c(z)$ satisfies (3.2) and (3.3). This change of variables, which is akin to a nonlinear WKB-type transformation, transforms (3.1) to the first order system

$$\cosh^2(\varepsilon^{-1}\phi)v_\tau = \phi_x + b(\varepsilon^{-1}v)\,, \qquad \phi(1,\tau)=\phi(-1,\tau)=0\,, \tag{3.8}$$

$$v_x = \tanh(\varepsilon^{-1}\phi)\,. \tag{3.9}$$

In (3.8) the function $b(z)$ is defined by

$$b(z)\equiv \frac{Q\left[u_c(z)\right]}{u_c'(z)}\,, \qquad \text{with} \qquad b(z)\sim \pm\nu_\pm\,, \quad z\to\pm\infty\,, \tag{3.10}$$

where $\nu_\pm=\left[-Q'(s_\pm)\right]^{1/2}$. Since $u_c(0)=0$, the internal layer locations for u and the zeroes of v coincide.

Four main advantages in studying (3.8), (3.9) instead of (3.1) were discussed in [RW94]. They are that: exponential asymptotics are no longer needed to construct equilibrium solutions; the exponentially long time scale for metastable motion appears in front of the v_τ term; numerical results for metastable motion are readily computed accurately; the extreme sensitivity of solutions to (3.1) to certain exponentially small perturbations can be easily studied both asymptotically and numerically.

The first advantage is illustrated by constructing a two-layer equilibrium solution for u of the form

$$u\sim u^*(x;x_0,x_1)\equiv u_c\left[\varepsilon^{-1}(x-x_0)\right]+u_c\left[\varepsilon^{-1}(x_1-x)\right]-s_+\,. \tag{3.11}$$

As outlined in §3.1, the equilibrium locations x_0 and x_1 can be determined from (3.1) by using the projection method, which requires exponential asymptotic estimates. However, in terms of the re-formulated problem (3.8) and (3.9), x_0 and x_1 can be determined from a conventional singular perturbation analysis. For an equilibrium solution of the form (3.11), the equilibrium v and ϕ for $\varepsilon\to 0$ are piecewise linear and have the form

$$v\sim\begin{cases} x-x_0 & \text{if } -1<x<x_0^* \\ x_1-x & \text{if } x_0^*<x<1 \end{cases}\,, \qquad \phi\sim\begin{cases} \nu_-(x+1) & \text{if } -1<x<x_0 \\ -\nu_+(x-x_0^*) & \text{if } x_0<x<x_1 \\ \nu_-(x-1) & \text{if } x_1<x<1 \end{cases}\,. \tag{3.12}$$

Here $x_0^* = (x_0 + x_1)/2$. The $O(1)$ terms in x_0 and x_1 are obtained by the simple conditions that ϕ is continuous at x_0 and at x_1. The $O(\varepsilon)$ terms in x_0 and x_1 can be calculated by inserting corner layers for ϕ near x_0 and x_1 as was done in [RW94]. A simple corner layer analysis for $\varepsilon \to 0$ yields

$$x_1 = -x_0 \sim \frac{\nu_-}{\nu_+ + \nu_-} + \frac{\varepsilon}{(\nu_+ + \nu_-)} \log\left(\frac{a_+\nu_+}{a_-\nu_-}\right) . \tag{3.13}$$

Likewise, for the time-dependent problem, a conventional singular perturbation analysis of (3.8) and (3.9) can be done to derive the ODE's for $x_0(t)$ and $x_1(t)$ corresponding to the metastable dynamics (see [RW94]).

To illustrate the second advantage, we note that when x_0 and x_1 are near their equilibrium values, the two exponentially small eigenvalues associated with the pattern (3.11) are proportional to $\operatorname{sech}^2(\varepsilon^{-1}\phi_i)$ for $i = 1, 2$, where ϕ_i is one of the two interior extrema of the function ϕ written in (3.12). This observation allows us to use an accurate time-stepping strategy for computing numerical solutions to (3.8) and (3.9). The numerical method and the numerical results are described in [RW94] (see also [WR]).

3.2 The Viscous Cahn-Hilliard Equation

The viscous Cahn-Hilliard equation, considered in [NO] and [RW95b], is a model of slow phase separation, accounting for viscoelastic effects, that conserves mass. It can be written in the form

$$(1-\alpha)u_t = -\left(\varepsilon^2 u_{xx} + Q(u) - \alpha\kappa u_t\right)_{xx} , \qquad -1 < x < 1 , \tag{3.14}$$

$$u_x(\pm 1, t) = u_{xxx}(\pm 1, t) = 0 . \tag{3.15}$$

Here $u(x,0) = u_0(x)$, $\varepsilon \ll 1$, $\kappa > 0$, $Q(u) = -V'(u)$ where $V(u)$ is a double well potential with $V(s_\pm) = 0$, and α, with $0 \le \alpha \le 1$, is a homotopy parameter. When $\alpha = 0$ and $\alpha = 1$ we obtain the Cahn-Hilliard and the constrained Allen-Cahn equations, respectively.

Dynamical metastability for the Cahn-Hilliard equation has been studied analytically in [ABF], [BH], [G] and [BX]. In [RW95b] an extension of the projection method, as outlined in §3.1, was used to derive a differential-algebraic system for the internal layer locations $x_i = x_i(t)$ corresponding to N-layer metastable patterns for (3.14), (3.15). In [RW95b] a numerical scheme was formulated and implemented to accurately compute metastable solutions on time intervals of the order $t \approx 10^{12}$. For the case of a two-layer evolution of the form given in (3.11), the following asymptotic result was derived in [RW95b]:

Proposition 3.1. (Two-Layers): *Consider a two-layer metastable pattern for (3.14), (3.15) of the form given in (3.11), where $x_0 = x_0(t)$ and $x_1 = x_1(t)$. Define $d_i = d_i(t)$ for $i = 0,.,2$ by $d_0 = x_0 + 1$, $d_1 = x_1 - x_0$ and $d_2 = 1 - x_1$. Then, for $\varepsilon \to 0$, d_1 is a constant and $d_0(t)$ satisfies the asymptotic ODE*

$$\dot{d}_0 \sim \varepsilon\, a_-^2 \nu_-^2 \zeta_\varepsilon^{-1} \left(e^{-2\varepsilon^{-1}\nu_- d_2} - e^{-2\varepsilon^{-1}\nu_- d_0} \right) ; \qquad d_1 \equiv \frac{m_\varepsilon - 2s_-}{s_+ - s_-} . \tag{3.16}$$

Here ζ_ε and m_ε have power series expansions in ε of the form

$$\zeta_\varepsilon = \alpha\kappa\beta + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots , \qquad m_\varepsilon = m_0 + \varepsilon m_1 + \cdots , \tag{3.17}$$

where $\beta \equiv \int_{s_-}^{s_+} [2V(u)]^{1/2}\, du$ and $m_0 = \int_{-1}^{1} u_0(x)\, dx$. The other coefficients m_1, ζ_1 and ζ_2 were obtained explicitly in [RW95b].

3.3 Viscous Shocks for Burger-Type Equations

Metastable internal layer motion also occurs for the viscous shock problem

$$u_t + [f(u)]_x = \varepsilon\, u_{xx} , \qquad -1 < x < 1 , \tag{3.18}$$

$$u(-1,t) = \alpha_- > 0 \qquad u(1,t) = \alpha_+ < 0 , \tag{3.19}$$

where $u(x,0) = u_0(x)$. Here $\varepsilon \ll 1$, $\alpha_\pm$ are constants and the convex $f(u)$ satisfies $f(\alpha_+) = f(\alpha_-)$, $f(0) = f'(0) = 0$ and $uf'(u) > 0$ for $u \neq 0$. The condition $f(\alpha_+) = f(\alpha_-)$ ensures that there exists a stationary shock profile $u_c(z)$ to (3.18), connecting α_- and α_+, which satisfies

$$u_c'(z) = f[u_c(z)] - f(\alpha_+) , \quad -\infty < z < \infty , \qquad u_c(0) = 0 , \tag{3.20}$$

$$u_c(z) \sim \alpha_\pm \mp a_\pm e^{\mp\nu_\pm z} , \quad z \to \pm\infty . \tag{3.21}$$

Here $\nu_\pm \equiv \mp f'(\alpha_\pm)$ and $a_\pm > 0$ are certain explicit constants (see [RW95a]).

For the equilibrium problem we look for a solution to (3.18), (3.19) of the form $u \sim u_c\left[\varepsilon^{-1}(x - x_0)\right]$ for some $x_0 \in (-1,1)$. Since $u_s(z)$ decays exponentially as $z \to \pm\infty$, this form satisfies (3.19) to within exponentially small terms as $\varepsilon \to 0$ for any $x_0 \in (-1,1)$. Thus, determining x_0 is exponentially ill-conditioned.

To quantify this ill-conditioning, we first linearize (3.18), (3.19) around the shock profile $u_c\left[\varepsilon^{-1}(x - x_0)\right]$. The eigenvalue problem associated with this linearization can be transformed into

$$\varepsilon\, \phi_{xx} - q_\varepsilon(x)\phi_x = \lambda\phi , \qquad q_\varepsilon(x) \equiv -f'\left(u_c\left[\varepsilon^{-1}(x - x_0)\right]\right) , \tag{3.22}$$

$$\phi(-1) = 0 , \qquad \phi(1) = 0 . \tag{3.23}$$

This problem has a turning point at $x = x_0$ and is strikingly similar in form to the eigenvalue problem for the exit operator, which is associated with boundary layer resonance phenomena (see [MS], [D], [LEW]). Therefore, it is natural to expect that (3.22), (3.23) has an exponentially small eigenvalue. In fact, for $\varepsilon \to 0$, it was shown in [RW95a] that the principal eigenvalue λ_0 for this problem is exponentially small with the estimate

$$\lambda_0 \sim \frac{-1}{(\alpha_- - \alpha_+)} \left[a_+ \nu_+^2 \, e^{-\nu_+ \varepsilon^{-1}(1-x_0)} + a_- \nu_-^2 \, e^{-\nu_- \varepsilon^{-1}(1+x_0)} \right] . \tag{3.24}$$

As a consequence of the existence of this eigenvalue, the solution to (3.18), (3.19) can be very sensitive to certain exponentially small perturbations. Moreover, since λ_0 is the principal eigenvalue, dynamical metastability behavior is observed for (3.18), (3.19). These aspects, and some related issues, have been studied in detail in [KK], [RW95a], [LO94], [WR] and [LO95].

For the time-dependent problem a shock-layer of width $O(\varepsilon)$ is formed from initial data in an $O(1)$ time. However, the subsequent motion of this internal layer towards the equilibrium location is exponentially slow. By using an extension of the projection method as outlined in §3.1, it was shown in [RW95a] that for $\varepsilon \to 0$ this metastable motion can be described by $u \sim u_c \left[\varepsilon^{-1}(x - x_0)\right]$ where $x_0 = x_0(t)$ satisfies the asymptotic ODE

$$\dot{x}_0 \sim \frac{-1}{(\alpha_- - \alpha_+)} \left[a_+ \nu_+ \, e^{-\nu_+ \varepsilon^{-1}(1-x_0)} - a_- \nu_- \, e^{-\nu_- \varepsilon^{-1}(1+x_0)} \right] . \tag{3.25}$$

Therefore, the stable equilibrium location for x_0 is

$$x_0 = \frac{\nu_+ - \nu_-}{\nu_+ + \nu_-} - \frac{\varepsilon}{\nu_+ + \nu_-} \log \left[\frac{a_+ \nu_+}{a_- \nu_-} \right] . \tag{3.26}$$

4 Internal Layers in Multi-Dimensional Domains

We now give some results for the asymptotic construction of exponentially ill-conditioned localized solutions to certain reaction-diffusion equations in a multi-dimensional setting.

4.1 Spike-Layers in Multi-Dimensional Domains

In [W95b] a spike-layer solution is constructed in the limit $\varepsilon \to 0$ for

$$\varepsilon^2 \triangle u + Q(u) = 0 , \qquad x \in D \subset \mathcal{R}^N , \tag{4.1}$$

$$\varepsilon \, \partial_n u + b u = 0 , \qquad x \in \partial D , \tag{4.2}$$

where $b > 0$ and D is a bounded convex domain. Here, $Q(u)$ is assumed to be such that there exists a unique radially symmetric solution $u_c(\rho)$ to (4.1) in all of $\mathcal{R}^N$, where $\rho = \varepsilon^{-1} r$ and $r = |x|$, which has the following far-field behavior:

$$u_c(\rho) \sim a\rho^{(1-N)/2} e^{-\nu\rho}\,, \qquad \rho \to \infty\,; \qquad \nu \equiv \left[-Q'(0)\right]^{1/2}\,. \tag{4.3}$$

Here a is a positive constant. When $b = 0$ it was proved in [NT91] and [NT93] that (4.1), (4.2) has a spike-layer solution of the form $u \sim u_c\left[\varepsilon^{-1}|x - x_0|\right]$, where $x_0 \in \partial D$ is located at the point of maximal mean curvature of ∂D.

In [W95b] a spike-layer solution for (4.1), (4.2) was constructed where the spike-layer location x_0 is *strictly contained* within D. Since $u_c(\rho)$ decays exponentially as $\rho \to \infty$, this problem is exponentially ill-conditioned in the sense that $u_c\left[\varepsilon^{-1}|x - x_0|\right]$ fails to satisfy (4.2) by only exponentially small terms for *any* $x_0 \in D$. Thus, determining the correct x_0 is a problem in exponential asymptotics. In the simpler one-dimensional context, where phase-plane methods are available, ill-conditioned problems of this type have been studied from various viewpoints in [L], [KKM] and [W92]. In the simpler one-dimensional case it is readily clear that a solution with exactly one spike-layer must be such that the spike is centered at the midpoint of the interval.

In [W95b] a very similar geometric criterion determining the spike-layer location for (4.1), (4.2) was obtained by extending the method of [W92] to a multi-dimensional setting. The method used was a projection method that exploits the spectral properties associated with linearizing (4.1), (4.2) around $u_c\left[\varepsilon^{-1}|x - x_0|\right]$ and is closely related to similar methods used to treat dynamical metastability problems. The associated linearized problem was shown to have N exponentially small eigenvalues resulting from a slight break in the translation invariance. Explicit asymptotic estimates for the exponentially small eigenvalues were derived in a similar manner as for the exit problem of [MS]. Solvability conditions that must hold in the limit $\varepsilon \to 0$ were then imposed to derive an equation for x_0, which involves a Laplace-type integral defined over ∂D. Explicit asymptotic results for x_0 were obtained by evaluating this integral using Laplace's method. We now summarize one of the main results of [W95b] obtained for the two-dimensional case $N = 2$.

Suppose that there exists a unique largest inscribed circle $\mathcal{B}$ for D. Let r_{in} and x_{in} denote the radius and center of $\mathcal{B}$, respectively. The following result pertains to the case where $\mathcal{B}$ makes exactly three-point contact with ∂D. A similar result was given in [W95b] for the case of two-point contact.

Proposition 4.1. (Three-Point Contact): *Assume that $\mathcal{B}$ makes exactly* *three-point contact with ∂D at $x(\xi_i) \in \partial D$ for $i = 1, 2, 3$. Suppose that $b \neq \nu$ and that $\kappa_i r_{in} > -1$ for $i = 1, 2, 3$, where $\kappa_i < 0$ is the curvature of ∂D at $x(\xi_i)$. Then, for $\varepsilon \to 0$, $x_0 = x_0(\varepsilon)$ satisfies $x_0(\varepsilon) = x_{in} + \varepsilon x_0^1 + \cdots$, where x_0^1 is the solution to the linear system*

$$(\hat{n}_3 - \hat{n}_1) \cdot x_0^1 = (2\nu)^{-1} \left\{ \log \left(A_1 / A_3 \right) + \log \left(-\hat{n}_1 \cdot \hat{t}_2 / \hat{n}_3 \cdot \hat{t}_2 \right) \right\} , \quad (4.4)$$

$$(\hat{n}_1 - \hat{n}_2) \cdot x_0^1 = (2\nu)^{-1} \left\{ \log \left(A_2 / A_1 \right) + \log \left(-\hat{n}_2 \cdot \hat{t}_3 / \hat{n}_1 \cdot \hat{t}_3 \right) \right\} . \quad (4.5)$$

Here $A_i \equiv (1 + \kappa_i r_{in})^{-1/2}$ and $\hat{n}_i$ and $\hat{t}_i$ are the unit outward normal vector and the unit tangent vector at $x(\xi_i) \in \partial D$, respectively.

For the time-dependent parabolic problem associated with (4.1) and (4.2), the motion of a spike-layer is *not* exponentially slow. This conclusion follows from the fact that the exponentially small eigenvalues are not the principal eigenvalues associated with linearizing (4.1), (4.2) about the canonical spike solution (see [W95b]).

4.2 Exponentially Slow Bubble Motion

The following non-local Allen-Cahn equation for $u = u(x, t; \varepsilon)$ and $\sigma = \sigma(t; \varepsilon)$, introduced in [RS], is the simplest phase separation model that conserves mass:

$$u_t = \varepsilon^2 \triangle u + Q(u) - \sigma , \qquad x \in D \subset \mathcal{R}^N , \quad (4.6)$$

$$\partial_n u = 0 , \qquad x \in \partial D ; \qquad \int_D u(x, t) \, dx = M . \quad (4.7)$$

Here $\varepsilon \ll 1$, D is a bounded convex domain, M is constant, and $Q(u) = -V'(u)$, where $V(u)$ is the double-well potential described following (3.1) with $V(s_\pm) = 0$.

In [W95a] the slow dynamics of a radially symmetric internal layer solution for (4.6)-(4.7), which separates the two preferred phases s_+ and s_-, was studied asymptotically. Such a solution is referred to in [AF94a] and [AF94b] as a bubble solution. In [W95a] the bubble was shown to drift exponentially slowly across the domain, without change of shape, towards the closest point on ∂D. In addition, an explicit ODE for the motion of the center of the bubble was derived. The geometric criterion determining the (unstable) equilibrium location for the center of the bubble was found to be essentially the same as that for the spike-layer problem considered in §4.1. The work in [W95a] was motivated by the rigorous study of [AF94a] and [AF94b] on metastable bubble dynamics for the related fourth-order Cahn-Hilliard model, which also conserves mass.

Let $r = r_b$ denote the radius of the bubble with $u \sim s_-$ inside the bubble and $u \sim s_+$ outside the bubble. This radius is determined by M. Let $x_0 = x_0(t)$ denote the center of the bubble at time t. One of the main results of [W95a] obtained using the projection method is

Proposition 4.2. (Slow Bubble Motion): *Assume that at $t = 0$ the bubble is located inside D with center at $x_0(0) \equiv x_0^0$ and that there exists a unique point $x(\xi_0)$ located on ∂D which is closest to $x_0(0)$. Then, until the bubble collapses against ∂D, the motion of the center of the bubble is in the direction of $x(\xi_0) - x_0^0$ and the distance $r_m(t) = |x(\xi_0) - x_0(t)|$ satisfies the asymptotic ODE*

$$\dot{r}_m \sim -\zeta r_m \left(\frac{\varepsilon}{r_m}\right)^{(N+1)/2} H(r_m)\, e^{-2\nu_+^\varepsilon \varepsilon^{-1}(r_m - r_b)} \,. \tag{4.8}$$

In (4.8), ν_+^ε for $\varepsilon \to 0$ and $H(r_m)$ are defined by

$$\nu_+^\varepsilon \sim \nu_+ \left[1 + \frac{\varepsilon\beta(N-1)\nu_+^{-4}}{2(s_+ - s_-)r_b} Q''(s_+)\right] \; ; \quad \nu_+ = \left[-Q'(s_+)\right]^{1/2} , \tag{4.9}$$

$$H(r_m) = \Pi_{i=1}^{N-1} (1 - r_m/R_i)^{-1/2} \; ; \quad \beta \equiv \sqrt{2} \int_{s_-}^{s_+} [V(u)]^{1/2} \, du \,, \tag{4.10}$$

where $R_i \geq 0$ for $i = 1, .., N-1$ are the principal radii of curvature of ∂D at $x(\xi_0) \in \partial D$. The constant ζ in (4.8) was given explicitly in [W95a] in terms of the dimension N and the nonlinearity $Q(u)$.

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