The Existence and Stability of Spike Equilibria in the One-Dimensional Gray-Scott Model on a Finite Domain

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Abstract

Results for the existence, stability, and pulse-splitting behavior of spike patterns in the one-dimensional Gray-Scott model on a finite domain in the semi-strong spike-interaction regime are summarized. Conditions on the parameters for the existence of competition instabilities, synchronous oscillatory instabilities, or pulse-splitting behavior of spike patterns are given.

Keywords: spike patterns, nonlocal eigenvalue problem, oscillatory instability, pulse-splitting.

1 Introduction

In this letter we summarize some results of [1] and [2] for the existence and stability of spike patterns in the one-dimensional Gray-Scott (GS) reaction-diffusion model. The GS model, first introduced in [3], and studied numerically in [4], can be written in dimensionless form as

$$v_t = \varepsilon^2 v_{xx} - v + Auv^2 \,, \qquad -1 < x < 1 \,, \quad t > 0 \,; \qquad v_x = 0 \,, \quad x = -1, 1 \,, \tag{1.1a}$$

$$\tau u_t = D u_{xx} + (1-u) - uv^2 \qquad -1 < x < 1 \,, \quad t > 0 \,; \qquad u_x = 0 \,, \quad x = -1, 1 \,. \tag{1.1b}$$

Here $A > 0$, $D > 0$, $\tau > 1$, and $\varepsilon \ll 1$. This form of the GS model was first introduced in [5].

We consider (1.1) in the semi-strong spike interaction regime defined by $D = O(1)$ and $\varepsilon \ll 1$. In this limit, singular perturbation techniques can be used to partially decouple u and v. From a mathematical viewpoint, our analysis provides a careful case study for the long-term goal of classifying generic mechanisms leading to instabilities of spike-type solutions in reaction-diffusion systems. For general reaction-diffusion systems in the weakly nonlinear regime, a similar classification based on amplitude equations and normal forms has been undertaken over the past several decades. As a step

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towards a classification for instabilities of spike solutions, our results (cf. [1], §2) show that certain instabilities in the GS model have a direct spectral equivalence with those in the Gierer-Meinhardt model of [6] derived in [7] in the semi-strong interaction limit.

For the infinite domain problem, the existence and stability of a one-spike solution for the GS model in the semi-strong interaction regime was studied analytically in [8], [9], [10], and [5]. Periodic solutions were also constructed in [8]. A numerical example of pulse-splitting with a qualitative analysis of the dynamical mechanism of pulse-splitting is given in [11].

The weak spike interaction regime is defined by $D = O(\varepsilon^2)$, and $\varepsilon \ll 1$. In this latter regime, where u and v are both localized near certain points in the domain, the pioneering works of [12] and [13] showed that both pulse-splitting behavior and spatio-temporal chaos is possible. In terms of numerically computed bifurcation diagrams, a clear mechanism for the occurrence of pulse-splitting was formulated. General criteria for pulse-splitting are given in [14]. For symmetric initial data, the pulse-splitting that occurs in the weak interaction regime exhibits edge-splitting, whereas in the semi-strong interaction regime pulses split roughly simultaneously.

In the semi-strong regime where $D = O(1)$, and $\varepsilon \ll 1$, there are three main parameter regimes for A where different behaviors occur. In §2, the results of [1] are summarized for the *low-feed rate* regime, $A = O(\varepsilon^{1/2})$, where both competition and oscillatory instabilities can occur. These results are new. Some new results of [2] for the pulse-splitting regime, $A = O(1)$, verifying the criteria of [14] and predicting the number of pulse-splitting events, are given in §3. In the intermediate regime $O(\varepsilon^{1/2}) \ll A \ll 1$, our results are rather closely related to those in [8], [9], and [10]. For this regime, we refer to §3.4 and §5 of [1] for a detailed comparison to these previous works.

2 The Low-Feed Rate Regime: $\mathcal{A} = O(\varepsilon^{1/2})$

In the low feed-rate regime, we define a new $O(1)$ variable $\mathcal{A}$ by $A = \varepsilon^{1/2}\mathcal{A}$. The first result of [1] for this regime characterizes symmetric k-spike equilibrium patterns, characterized by spikes in v that have a common amplitude. For each $k = 1, 2\ldots$, and for $\mathcal{A} > \mathcal{A}_{ke}$, there are two branches of such solutions for (1.1) that meet at the saddle-node bifurcation value $\mathcal{A}_{ke}$ given by

$$\mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh\left(\theta_0/k\right)}}\,, \qquad \theta_0 \equiv D^{-1/2}\,. \tag{2.1}$$

These solution branches are conveniently parameterized in terms of s, with $0 < s < \infty$, where

$$s \equiv \frac{1-U_\pm}{U_\pm}\,, \qquad U_\pm = \frac{1}{2}\left[1 \pm \sqrt{1-\frac{\mathcal{A}_{ke}^2}{\mathcal{A}^2}}\,\right]\,. \tag{2.2}$$

We refer to the range $0 < s < 1$ as the *large solution branch* ($U = U_+$ is large), while the range $s > 1$ is called the *small solution branch* ($U = U_-$ is small). The precise result is as follows (see Proposition 2.1 of [1]):

Proposition 1: *Let $\varepsilon \to 0$, with $\mathcal{A} = O(1)$ and $D = O(1)$. Then, when $\mathcal{A} > \mathcal{A}_{ke}$, the small and large k-spike symmetric equilibrium solutions for (1.1), denoted by u_-, v_- and u_+, v_+, respectively, are given asymptotically by*

$$v_\pm(x) \sim \frac{1}{\sqrt{\varepsilon}\mathcal{A}U_\pm} \sum_{j=1}^{k} w\left[\varepsilon^{-1}(x - x_j)\right] , \qquad u_\pm(x) \sim 1 - \frac{(1 - U_\pm)}{a_g} \sum_{j=1}^{k} G(x; x_j) . \tag{2.3}$$

Here $x_j = -1 + \frac{(2j-1)}{k}$, $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, $w(y) = \frac{3}{2}sech^2(y/2)$ satisfies $w'' - w + w^2 = 0$, and $G(x; x_j)$ is the Green's function satisfying $DG_{xx} - G = -\delta(x - x_j)$ with $G_x(\pm 1; x_j) = 0$.

To analyze the stability of these equilibrium solutions on an $O(1)$ time-scale, we let $u(x,t) = u_\pm(x) + e^{\lambda t}\eta(x)$, and $v(x,t) = v_\pm(x) + e^{\lambda t}\phi(x)$, where ϕ is a localized eigenfunction of the form

$$\phi(x) \sim \sum_{j=1}^{k} c_j \Phi\left[\varepsilon^{-1}(x - x_j)\right] , \tag{2.4}$$

for some c_j to be found. We consider eigenfunctions with $\int_{-\infty}^{\infty} w(y)\Phi(y)\, dy \neq 0$. In §3.1 of [1], the following spectral problem for $\Phi(y)$ is derived (see Proposition 3.2 of [1]):

Proposition 2: *Assume that $0 < \varepsilon \ll 1$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues determining the stability of k-spike equilibria satisfy the nonlocal eigenvalue problem*

$$L_0\Phi - \chi_{gs} w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy} \right) = \lambda\Phi , \qquad -\infty < y < \infty ; \qquad \Phi \to 0 , \quad \text{as} \quad |y| \to \infty . \tag{2.5}$$

Here the operator L_0 is $L_0\Phi \equiv \Phi'' - \Phi + 2w\Phi$, and the multiplier $\chi_{gs} = \chi_{gs}(z; j)$ is defined by

$$\chi_{gs} \equiv 2s \left(s + \frac{\sqrt{1+z}}{\tanh(\theta_0/k)} \left[\tanh(\theta_\lambda/k) + \frac{(1 - \cos[\pi(j-1)/k])}{\sinh(2\theta_\lambda/k)} \right] \right)^{-1} , \tag{2.6}$$

where $z \equiv \tau\lambda$, $\theta_\lambda \equiv \theta_0\sqrt{1+z}$, and $\theta_0 \equiv D^{-1/2}$. Here s is given in (2.2). The coefficients $\mathbf{c}^t = (c_1, \ldots, c_k)$ in (2.4) are given by

$$\mathbf{c}_1^t = \frac{1}{\sqrt{k}}(1, \ldots, 1) \; ; \quad c_{l,j} = \sqrt{\frac{2}{k}} \cos\left(\frac{\pi(j-1)}{k}(l - 1/2) \right) , \quad j = 2, \ldots, k . \tag{2.7}$$

There is an equivalent formulation of (2.5) which states that the eigenvalues of (2.5) with $\int_{-\infty}^{\infty} w\Phi\, dy \neq 0$ are the union of the zeros of the functions $g_j(\lambda) = 0$ for $j = 1, .., k$, where

$$g_j(\lambda) \equiv C_j(\lambda) - f(\lambda)\,, \qquad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\,(L_0 - \lambda)^{-1}\, w^2\, dy}{\int_{-\infty}^{\infty} w^2\, dy}\,, \qquad C_j(\lambda) \equiv [\chi_{gs}]^{-1}\,. \tag{2.8}$$

By analyzing the zeroes of $g_j(\lambda)$ in the right half-plane $\mathrm{Re}(\lambda) \geq 0$, the main stability result for multi-spike solutions is as follows (cf. Proposition 3.10 and 3.13 of [1]):

Proposition 3: *The large solution u_+, v_+ is unstable for any $0 < s < 1$, $k \geq 1$, and $D > 0$. Next, let $k > 1$, and consider the multi-spike small solution u_-, v_-, where $s > 1$. For $\mathcal{A} > \mathcal{A}_{kL}$, the solution will be stable on an $O(1)$ time-scale when $0 < \tau < \tau_{hL}$. Alternatively, on the range $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$, the small solution is unstable for any $\tau > 0$. The threshold $\mathcal{A}_{kL}$ is given analytically by*

$$\mathcal{A}_{kL} = \frac{\mathcal{A}_{ke}\left((\gamma_k/2) + 2\sinh^2(\theta_0/k)\right)}{\left(\left[(\gamma_k/2) + 2\sinh^2(\theta_0/k)\right]^2 - (\gamma_k/2)^2\right)^{1/2}}\,, \qquad \gamma_k \equiv 1 + \cos\left(\frac{\pi}{k}\right)\,. \tag{2.9}$$

Let $\mathcal{A}$ satisfy $\mathcal{A} > \mathcal{A}_{kL}$. Then, as τ increases beyond τ_{hL}, a Hopf bifurcation in the spike amplitudes was computed numerically in [1]. The threshold τ_{hL} is given by the *minimum value* of the set τ_j, $j = 1, .., k$, for which $g_j(\lambda) = 0$, $j = 1, .., k$, has complex conjugate roots on the imaginary axis. Let $\lambda = \pm i\lambda_h$ be the corresponding value of λ. Then, as was shown in [1], the unstable eigenfunction generically has the form of a *synchronous oscillatory instability* with

$$v = v_- + \delta e^{i\lambda_h t}\phi + \text{c.c}\,, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right]\,, \quad c_n = 1\,, \quad n = 1, .., k\,. \tag{2.10}$$

Here c.c denotes complex conjugate and $\delta << 1$. Alternatively, suppose that $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$. Then, for any $\tau > 0$, the dominant initial instability was shown in [1] to have the form

$$v = v_- + \delta e^{\lambda_{Rk} t}\phi\,, \quad \phi(x) = \sum_{n=1}^{k} c_n \Phi\left[\varepsilon^{-1}(x - x_n)\right]\,, \quad c_n = \cos\left(\frac{\pi(k-1)}{k}(n - 1/2)\right)\,, \quad n = 1, .., k\,. \tag{2.11a}$$

Here $\delta \ll 1$, and $\lambda_{Rk} > 0$ is the unique root of $g_k(\lambda_R) = 0$. Since $\sum_{n=1}^{k} c_n = 0$, this instability locally conserves the sum of the heights of the spikes. Hence, it is referred to as a *competition instability*. The numerical experiments in §3.3 of [1] show that this instability leads to a spike competition process whereby certain spikes in a spike sequence are ultimately annihilated. Finally, in Proposition 3.3 of [1] it was shown there there is a spectral equivalence principle between (2.5) and a corresponding nonlocal eigenvalue problem derived in Proposition 2.3 of [7] for the Gierer-Meinhardt (GM) model.

3 The Pulse-Splitting Regime: $A = O(1)$

In this regime, where $A = O(1)$, it was shown in [2] that equilibrium k-spike solutions can be constructed in terms of the solutions $V(y) > 0$ and $U(y) > 0$ to a certain *core problem* defined by

$$V'' - V + V^2 U = 0\,, \qquad U'' = UV^2\,, \qquad 0 < y < \infty\,, \tag{3.1a}$$

$$V'(0) = U'(0) = 0\,; \quad V \to 0\,, \quad U \sim By\,, \quad \text{as} \quad y \to \infty\,; \quad B \equiv A \tanh\left(\theta_0/k\right)\,. \tag{3.1b}$$

Here $\theta_0 \equiv D^{-1/2}$. This core problem, without the term $B \equiv A \tanh(\theta_0/k)$ associated with the finite domain, was first derived in [5] for the infinite-line problem. The equilibrium result given in Proposition 3.1 of [2] is as follows:

Proposition 4: *Let $\varepsilon \to 0$, $A = O(1)$, $\varepsilon A/\sqrt{D} \ll 1$, and suppose that (3.1) has a solution. Then, the v-component for a k-spike equilibrium solution to (1.1) is given by*

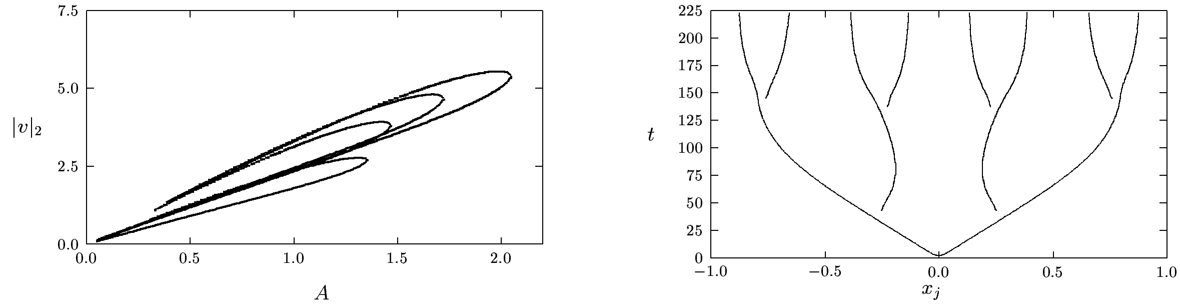
$$v \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} \left(V\left[\varepsilon^{-1}(x - x_j)\right] + O\left(\frac{\varepsilon A}{\sqrt{D}}\right)\right)\,. \tag{3.2}$$

In the j^{th} inner region, where $|x - x_j| = O(\varepsilon)$, u satisfies $u \sim \frac{\varepsilon}{A\sqrt{D}} U\left[\varepsilon^{-1}(x - x_j)\right]$. The outer solution for u, valid for $|x - x_j| \gg O(\varepsilon)$, and $j = 1, \ldots, k$, has the form $u \sim 1 - \frac{1}{a_g}\sum_{j=1}^{k} G(x; x_j)$, where a_g, x_j, and $G(x; x_j)$ are as defined in Proposition 1.

The core problem (3.1) was studied qualitatively and numerically in [2] in terms of B. In [2] it was shown that $0 < \gamma < \frac{3}{2}$, where $\gamma \equiv U(0)V(0)$. Numerical solutions to (3.1) were computed for which V has a single maximum at $y = 0$ as $\gamma \to 3/2$ from below, and a resulting curve $B = B(\gamma)$ was computed. This limiting solution for V asymptotically matches onto the solution constructed in the low feed-rate regime in §2. As shown in [2], the curve $B = B(\gamma)$ is double-valued with $B \to 0$ as $\gamma \to 0$ and as $\gamma \to 3/2$, and it has a saddle-node bifurcation point at the maximum value B_c of B given by $B_c = 1.347$ (see also [5]), where $\gamma = \gamma_c = 1.02$. We refer to the range $\gamma_c < \gamma < 3/2$ and $0 < \gamma < \gamma_c$ as the *primary* and *secondary* branches of the $B = B(\gamma)$ bifurcation diagram. Since $B \equiv A \tanh(\theta_0/k)$ from (3.1b), these results of [2] show that equilibrium k-spike solutions exist only when A is small enough. In particular, we have (cf. Proposition 3.2 of [2]):

Proposition 5: *Let $\varepsilon \ll 1$, $A = O(1)$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, there will be no k-spike equilibrium solution to (1.1) that merges onto the low feed-rate regime solution when*

$$A > A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right)\,. \tag{3.3}$$



(a) $|v|_2$ versus A

(b) t versus x_j

Figure 1: Left figure: bifurcation diagram when $A = O(1)$ for k-spike solutions with $\varepsilon = 0.01$, $D = 0.1$, and $k = 1, \ldots, 4$. The saddle-node values A_{pk} increase with k. Right figure: trajectories of the maxima of v showing pulse-splitting behavior when $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and $\tau = 2.0$.

Next, the following three pulse-splitting criteria of [14] were verified in [1]: 1) each k-spike equilibrium branch must have a saddle-node bifurcation that occurs at approximately the same bifurcation value (i.e. the *lining-up property*): 2) for each k, one branch of equilibria is unstable while the other is stable: 3) the linearization around the equilibrium solution at each saddle-node value has a *dimple-shaped* eigenfunction associated with a zero eigenvalue. When these criteria of [14] are satisfied, and when the bifurcation parameter is taken to be only slightly beyond the saddle-node value, pulse-splitting should occur from a single localized initial pulse due to the *ghost* of the dimple eigenfunction. In Fig. 1(a) we plot the norm $|v|_2$, defined by $|v|_2^2 \equiv \int_{-1}^{1} v^2\, dx \sim \varepsilon^{-1} k D \int_{-\infty}^{\infty} V^2\, dy$, versus A when $D = 0.1$, to show that the approximate lining-up condition 1) holds. For $\tau \ll O(\varepsilon^{-1})$ the stability condition 2) of [14] is verified in §5 and §6 of [2] by showing that the primary and secondary branches are stable and unstable, respectively. Finally, the dimple eigenfunction property is verified in §6 of [2]. Therefore, we conjecture:

Conjecture 1: *Let $\varepsilon \ll 1$, $\tau \ll O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Suppose that we have even one-spike initial data centered at the origin. Then, the final equilibrium state is stable, and it has 2^m spikes where, for some integer $m \geq 0$, A is related to A_{pk} by*

$$A_{p2^{m-1}} < A < A_{p2^m} \,. \tag{3.4}$$

To verify this conjecture, we computed numerical solutions to (1.1) starting from an initial pulse in v centered at $x = 0$. In Fig. 1(b) we show pulse-splitting behavior for $\varepsilon = 0.01$, $D = 0.1$, $A = 2.4$, and

$\tau = 2.0$. For these values we calculate $A_{p4} = 2.045 < A < A_{p8} = 3.583$. Thus, (3.4) correctly predicts the eight-spike final state in Fig. 1(b). In addition, in Experiment 5 of [2], Conjecture 1 correctly predicts the final eight-spike solution for the pulse-splitting example in Fig. 4 of [11].

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