Oscillatory and Competition Instabilities: Dynamics of Spikes for the Gray-Scott Model

Wentao Sun (Mitacs Postdoc, U. Calgary)

Jens Rademacher (CWI, Amsterdam)

Chen Wan, Michael Ward (UBC)

ward@math.ubc.ca

Singularly Perturbed RD Models

Spatially localized solutions occur for RD models of the form:

$$v_t = \varepsilon^2 \triangle v + g(u,v)\,; \quad \tau u_t = D \triangle u + f(u,v)\,, \quad \partial_n u = \partial_n v = 0 \quad \boldsymbol{x} \in \partial\Omega\,.$$

Since $\varepsilon \ll 1$, then v is localized as a spike (1-D) or a spot (2-D). There are two well-known choices:

- **Classic Gierer-Meinhardt Model** (1972):

$$g(u,v) = -v + v^2/u \qquad f(u,v) = -u + v^2\,.$$

  Simplest in a hierarchy of more complicated models (morphogenesis, patterns on sea-shells etc.)

- **Gray-Scott Model** (1988):

$$g(u,v) = -v + Auv^2\,, \qquad f(u,v) = (1-u) - uv^2\,.$$

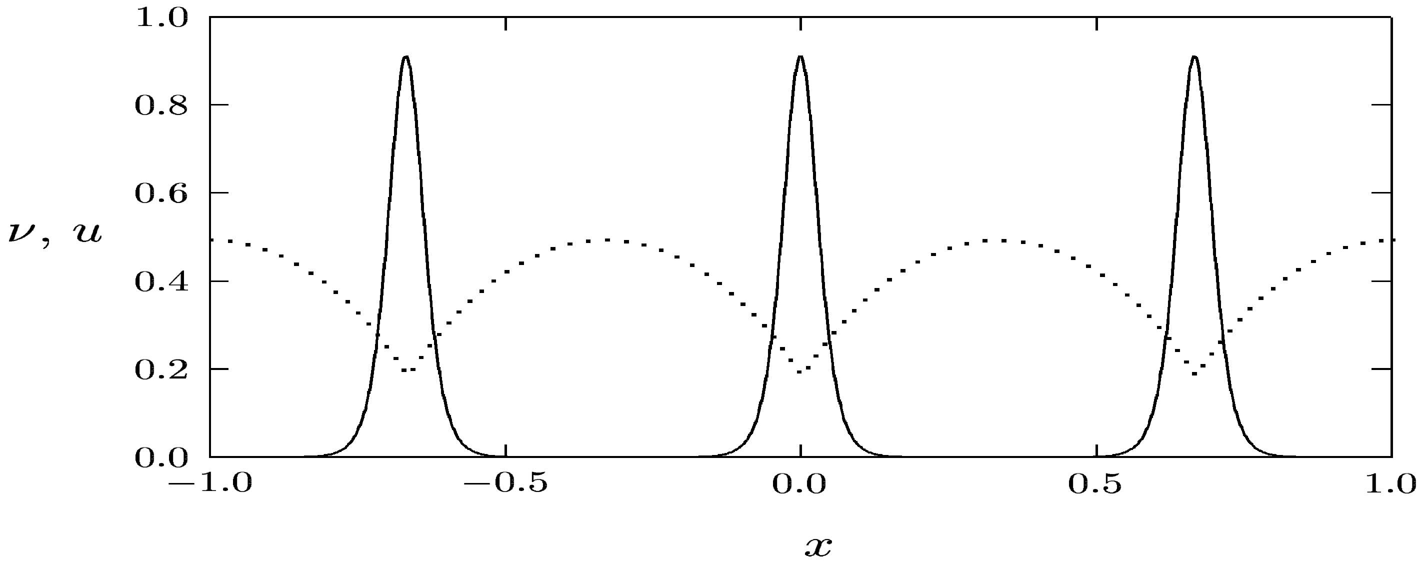
  Chemical patterns in a continuously fed reactor. Intricate patterns depending on D and A (Pearson 1993, Swinney et al. 1993, Nishiura et al., Doleman et al., Muratov and Osipov, KWW).

The Gray-Scott Model

Let $A = \varepsilon^{1/2}\mathcal{A}$. Then, on $|x| < 1$ with $v_x = u_x = 0$ at $x = \pm 1$:

$$v_t = \varepsilon^2 v_{xx} - v + \mathcal{A} u v^2 \,, \qquad \tau u_t = D u_{xx} - u + 1 - \frac{1}{\varepsilon} u v^2 \,.$$

(Nishiura-Ueyama, Doelman et al, Pearson-Reynolds, Muratov-Osipov, KWW). We consider the semi-strong regime $D = O(1)$ with $\varepsilon \ll 1$



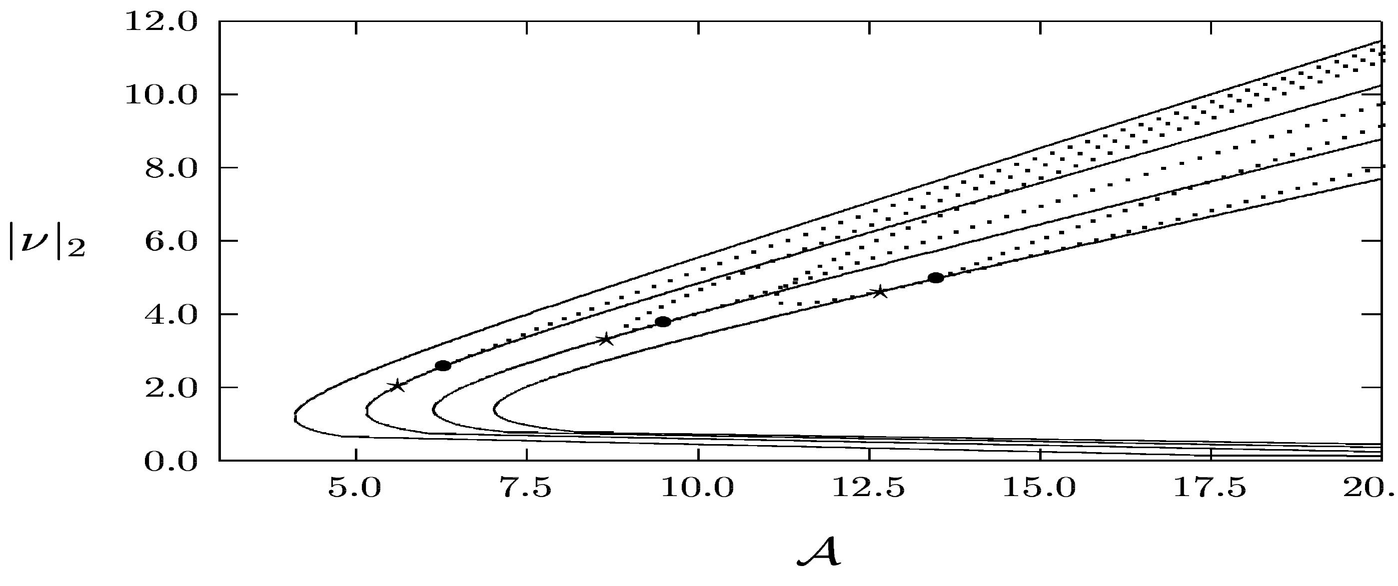
GS Model: 3-Spike equilibrium solution when $D = O(1)$

Gray-Scott Model: Different Regimes I

Low Feed-Rate Regime: $\mathcal{A} = O(1)$ as $\varepsilon \to 0$.

- Spike equilibria have a saddle-node bifurcation structure in $\mathcal{A}$. For the equilibrium problem there are oscillatory and competition instabilities [KWW, Studies 2004].
- The dynamics and instability mechanisms of quasi-equilibria: dynamic competition (click here), and oscillatory instabilities (click here) can occur [SWR, SIADS 2005].
- **Main Issue:** Give a precise analysis of the dynamics and the onset of instabilities for quasi-equilibrium 2-spike patterns.

Equilibrium bifurcation diagram:



The Gray-Scott Model: Different Regimes II

Intermediate Regime: $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$.

- Dynamics and NLEP stability of 2-spike quasi-equilibria on unbounded domains (Doelman et al. SIADS 2003)
- For N-spike patterns on a bounded domain, static oscillatory profile instabilities for $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/3})$ with $\tau_H = O(\mathcal{A}^4)$ are analyzed from a universal one-multiplier NLEP. (W. Chen, MJW)
- On a bounded domain, for $O(\varepsilon^{-1/3}) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$ oscillatory drift instabilities dominate since $\tau_{TW} = O(\varepsilon^{-2}\mathcal{A}^{-2}) \ll \tau_H$. (Doelman et al, Muratov-Osipov, KWW). Large-scale oscillatory spike motion from time-dependent heat equation (W. Chen, MJW).

High-Feed Regime: $\mathcal{A} = O(\varepsilon^{-1/2})$.

- A "core problem" determines the spike profile (Doelman et al, Muratov-Osipov, KWW). Intricate bifurcation structure (DKP, 2006)
- Instability mechanism is oscillatory drift instability on a finite domain when $\tau = \tau_{TW} = O(\varepsilon^{-1})$ [KWW, Physica D 2004).
- Simulataneous pulse-splitting can occur. Core problem coupled to a time-dependent PDE when $\tau = O(\varepsilon^{-1})$.

Comparison of Two Slow Processes: I

Dynamics of Quasi-Equilibria: Cahn-Hilliard, Allen-Cahn:

$$u_t = \varepsilon^2 u_{xx} + u - u^3 \,, \quad \text{(AC)}\,; \qquad u_t = -(\varepsilon^2 u_{xx} + u - u^3)_{xx} \,, \quad \text{(CH)}\,.$$

- **Metastable dynamics** for widely-spaced heteroclinic layers. The evolution occurs over exponentially long time intervals in 1-D.
- **Coarsening Process:** Collapse events punctuate the metastable dynamics in 1-D. K-layer solutions cascade to $K-2$ layer solutions from pairwise collapse of nearest neighbours. The collapse process is local in space and time. The quasi-equilibrium profile for widely spaced layers is unconditionally stable.
- **Variational Structure and Gradient Flow:** the final equilibrium state of no interfaces (Allen-Cahn), or one interface from mass conservation (Cahn-Hilliard) is a **minimum energy solution.**

Weakly Interacting (Metastable) Pulses: Tail interactions of exponentially localized pulses determine the dynamics (Ei, Sandstede...).

Comparison of Two Slow Processes: II

Dynamics of Quasi-Equilibrium Spike Patterns: GS Model Low Feed

- **No Variational Structure**: Below thresholds on $\mathcal{A}$ and τ depending on D and k, all equilibrium solutions with $\leq k$ spikes are stable.
- **Algebraically Slow Motion**: Slow dynamics with speed $O(\varepsilon^2)$ determined by the global u variable. Slow dynamics occur only when a profile stability condition wrt the large eigenvalues is satisfied. Stability thresholds depend on instantaneous spike locations.
- **Dynamic Instabilities (or Bifurcations)** occur on a bounded domain if stability boundaries are crossed as the spike locations approach their equilibrium values. There are two types: a dynamic oscillatory instability due to a Hopf bifurcation or a dynamic competition instability due to the creation of a positive real eigenvalue. Static competition and oscillatory instabilities as those that arise immediately at $t = 0$ due to the parameters and spike configuration being initially in the unstable zone.
- **Spike Collapses** often result from these instabilities leading to a "coarsening" process for k-spike patterns.

GS Model: Two-Spike Evolution: Low-Feed

Principal Result [SWR, SIADS 2005]: Consider a symmetric two-spike quasi-equilibrium solution for the GS model on $-1 < x < 1$ with spikes at $\alpha \equiv x_1 = -x_0 > 0$. Suppose that $\mathcal{A} > \mathcal{A}_{2e}$, where $\mathcal{A}_{2e} = \mathcal{A}_{2e}(\alpha)$ is the existence threshold given by

$$\mathcal{A}_{2e} = \sqrt{\frac{12\theta_0}{\sinh\theta_0}} \left(\cosh\theta_0 + \cosh\left[2\theta_0\left(\alpha - 1/2\right)\right]\right)^{1/2}, \qquad \theta_0 \equiv D^{-1/2}.$$

Then, for $0 < \varepsilon \ll 1$ and $\tau = 0(1)$, and when the quasi-equilibrium solution is stable on an $O(1)$ time scale, the spike locations $\alpha \equiv x_1 = -x_0$ satisfy the ODE

$$\frac{d\alpha}{dt} \sim \varepsilon^2 \theta_0 s_g \left[\tanh(\theta_0(1-\alpha)) - \tanh(\theta_0 \alpha)\right], \qquad \theta_0 = D^{-1/2}.$$

The equilibrium is $\alpha = 1/2$. Here $s_g = s_g(\alpha)$ is defined by

$$s_g = 2\left[1 - \sqrt{1 - \left(\frac{\mathcal{A}_{2e}}{\mathcal{A}}\right)^2}\right]^{-1} - 1.$$

GS Model: Two-Spike Stability (Low Feed)

Principal Result: Let α with $0 < \alpha < 1$ be fixed. The stability of the 2-spike quasi-equilibrium profile is determined by the spectrum of the NLEP

$$L_0\Phi - \chi_{gs\pm} w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi \, dy}{\int_{-\infty}^{\infty} w \, dy} \right) = \lambda\Phi \,, \quad \Phi \to 0 \,, \text{ as } |y| \to \infty \,.$$

Let $\theta_\lambda = \theta_0 \sqrt{1+\tau\lambda}$ and $\theta_0 = D^{-1/2}$. The two multipliers $\chi_{gs\pm}$ are

$$\chi_{gs\pm} \equiv 2 s_g \left[s_g + \sqrt{1+\tau\lambda} \left(\frac{\kappa_\pm(\tau\lambda)}{\kappa_+(0)} \right) \right]^{-1} \,.$$

$$\kappa_+ = \frac{\tanh(\theta_\lambda\alpha) + \tanh(\theta_\lambda(1-\alpha))}{\tanh(\theta_0\alpha) + \tanh(\theta_0(1-\alpha))} \,, \quad \kappa_- = \frac{\coth(\theta_\lambda\alpha) + \tanh(\theta_\lambda(1-\alpha))}{\tanh(\theta_0\alpha) + \tanh(\theta_0(1-\alpha))} \,.$$

Equivalence Principle: The NLEP multipliers and ODE dynamics for the low-feed GS model are equivalent to that of a generalized GM model with exponent set $(p,q,m,s) = (2, s_g, 2, s_g)$.

$$v_t = \varepsilon^2 v_{xx} - v + \frac{v^p}{u^q} \,, \quad \tau u_t = D u_{xx} - h + \frac{v^m}{\varepsilon u^s} \,.$$

GS Competition Instability: 2-Spikes

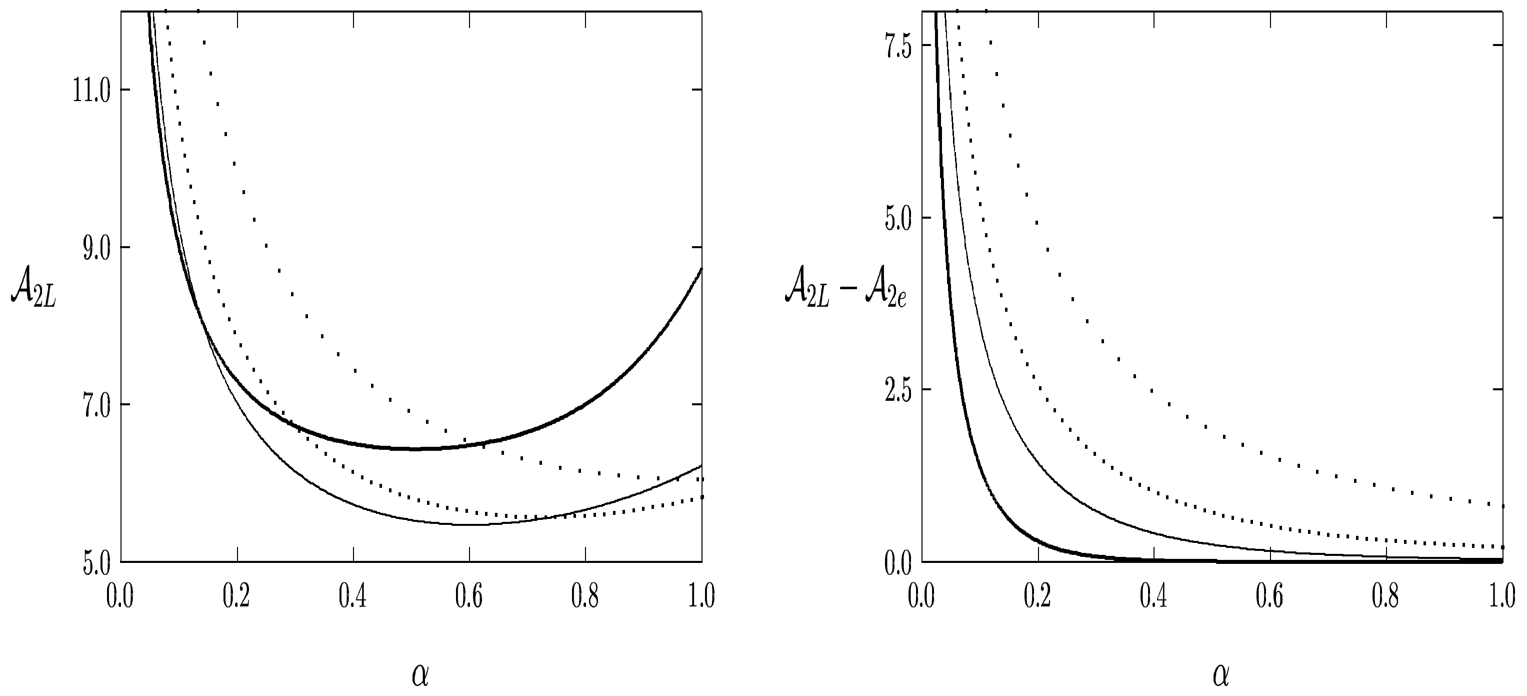
By analyzing the spectrum of the NLEP rigorously:

Proposition: Suppose that $0 \leq \tau < \tau_H$ and that $\mathcal{A}$ satisfies $\mathcal{A}_{2e} < \mathcal{A} < \mathcal{A}_{2L}$, where $\mathcal{A}_{2e}$ is the existence threshold. Then, the quasi-equilibrium solution is unstable as a result of a unique eigenvalue in Re$(\lambda) > 0$ located on the positive real axis. The threshold $\mathcal{A}_{2L}(\alpha)$ is given by

$$\mathcal{A}_{2L} \equiv \mathcal{A}_{2e} \frac{[1 + \coth(\theta_0)\coth(\theta_0\alpha)]}{2\sqrt{\coth(\theta_0)\coth(\theta_0\alpha)}} \,.$$

- Alternatively, for $0 < \tau < \tau_H$, the solution is stable on an $O(1)$ time-scale when $\mathcal{A} > \mathcal{A}_{2L}$.
- Suppose that the initial spike location $\alpha(0)$ satisfies $1/2 < \alpha(0) < 1$ and that $D > D_{2gs} \approx 2.3063$. Suppose that $\mathcal{A}$ satisfies $\mathcal{A}_{2L}(\alpha(0)) < \mathcal{A} < \mathcal{A}_{2L}(1/2)$. Then, there is a dynamic competition instability before the spikes reach their stable equilibria at $\alpha = 1/2$.

GS Existence and Competition Thresholds

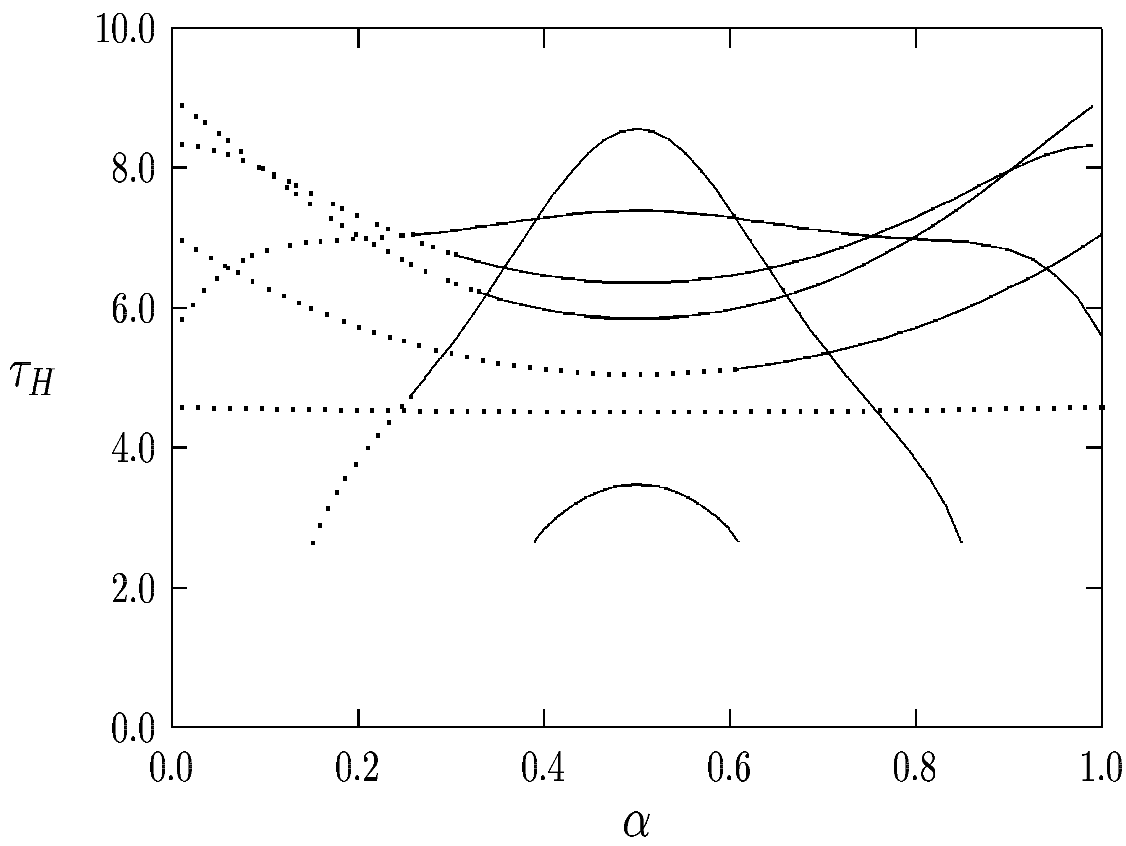


Left: GS competition instability threshold $\mathcal{A}_{2L}$ for $D = 0.1$ (heavy solid), $D = 0.5$ (solid), $D = 1.0$ (dotted), $D = 2.306$ (widely spaced dots). Right: The difference $\mathcal{A}_{2L} - \mathcal{A}_{2e}$ (same labels for D)

GS Dynamic Oscillatory Instabilities: 2-Spikes

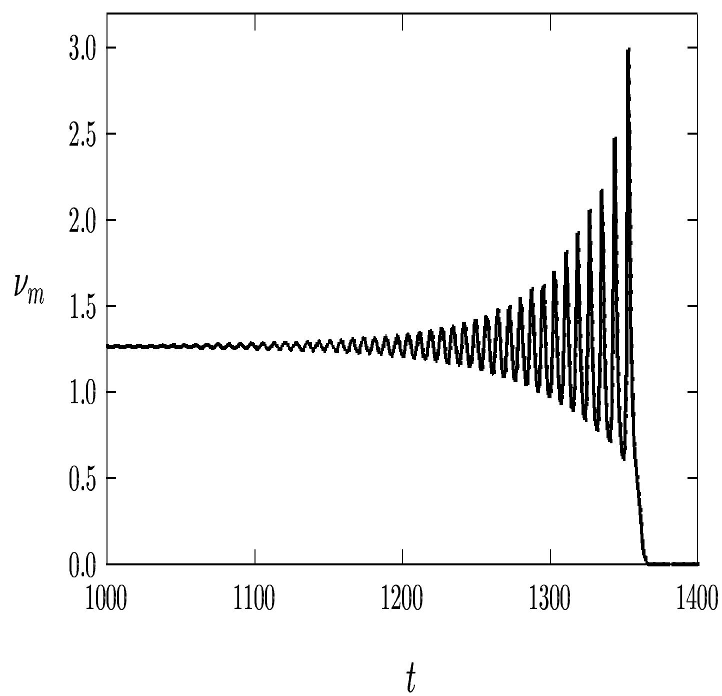
We calculate the spectrum of the NLEP numerically and use Newton's method to determine the Hopf bifurcation value.

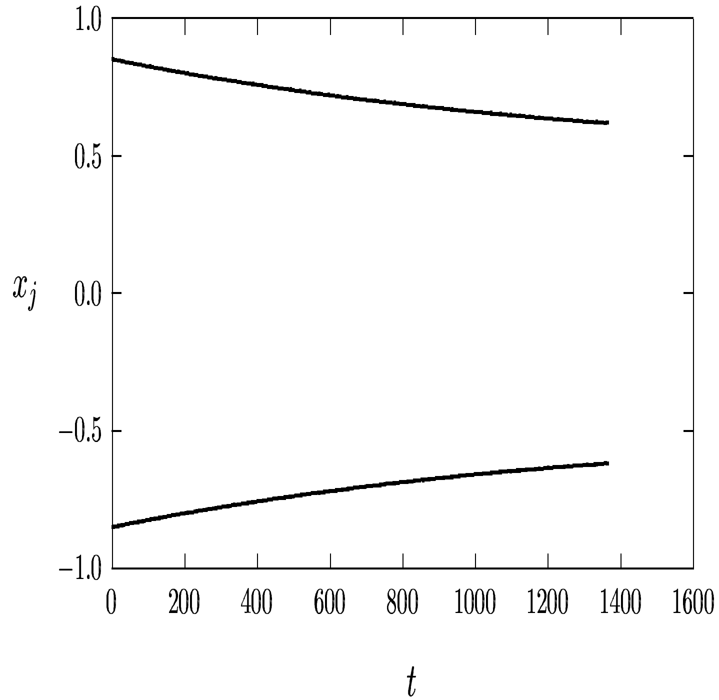
GS Model: $\tau_H(\alpha)$ when $\mathcal{A} = 6.5$ labeled by $(\tau_H(1/2), D)$: $(3.47, 0.1)$, $(8.56, 0.2)$, $(7.39, 0.5)$, $(6.36, 0.75)$, $(5.84, 1.0)$, $(5.04, 2.25)$, and $(4.5, 50)$. For the dotted segments there is a competition instability



Two-Spike GS: Dynamic Oscillatory

Experiment: We take $\varepsilon = 0.015$, $D = 2.25$, $\mathcal{A} = 6.5$, $\tau = 5.3$, $\alpha(0) = x_1(0) = -x_2(0) = 0.85$. The theory predicts $\tau_H \approx 5.2$ when $\alpha = 0.7$ and $\tau_H < 5.3$ on $0.5 < \alpha < 0.7$. The theory predicts that there is a dynamic oscillatory instability that is triggered when $\alpha < 0.7$.

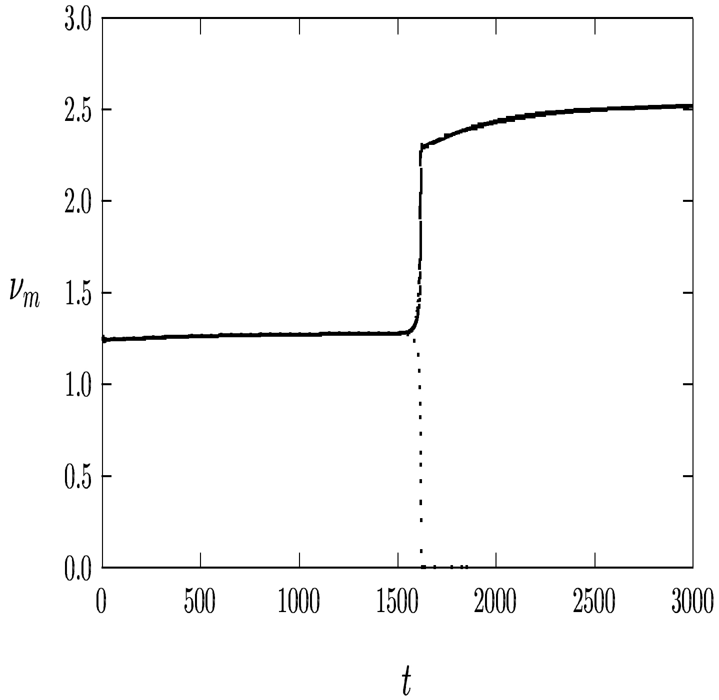


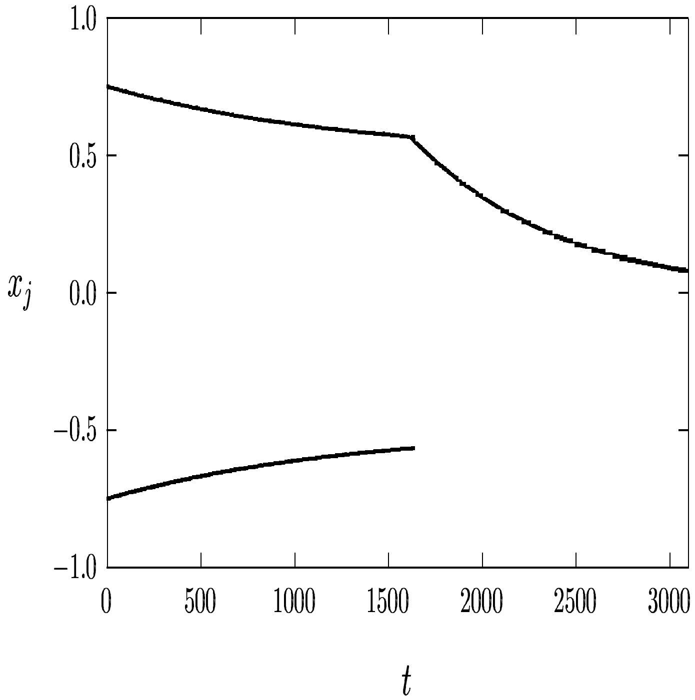


Left: Synchronous oscillation in the amplitudes (subcritical bifurcation?).
Right: Spike locations versus t. For the movie click here.

Two-Spike GS: Dynamic Competition

Experiment: We take $\varepsilon = 0.015$, $D = 2.25$, $\mathcal{A} = 6.5$, $\tau = 4.0$, $\alpha(0) = x_1(0) = -x_2(0) = 0.75$. We have $\tau < \tau_H(\alpha)$ for $0.5 < \alpha < 0.7$, and so there is no oscillatory instability. However, $\mathcal{A}_{2L} \geq 6.5$ for $\alpha = \alpha_c \approx 0.6$, and so there will be a dynamic competition instability triggered at this point.

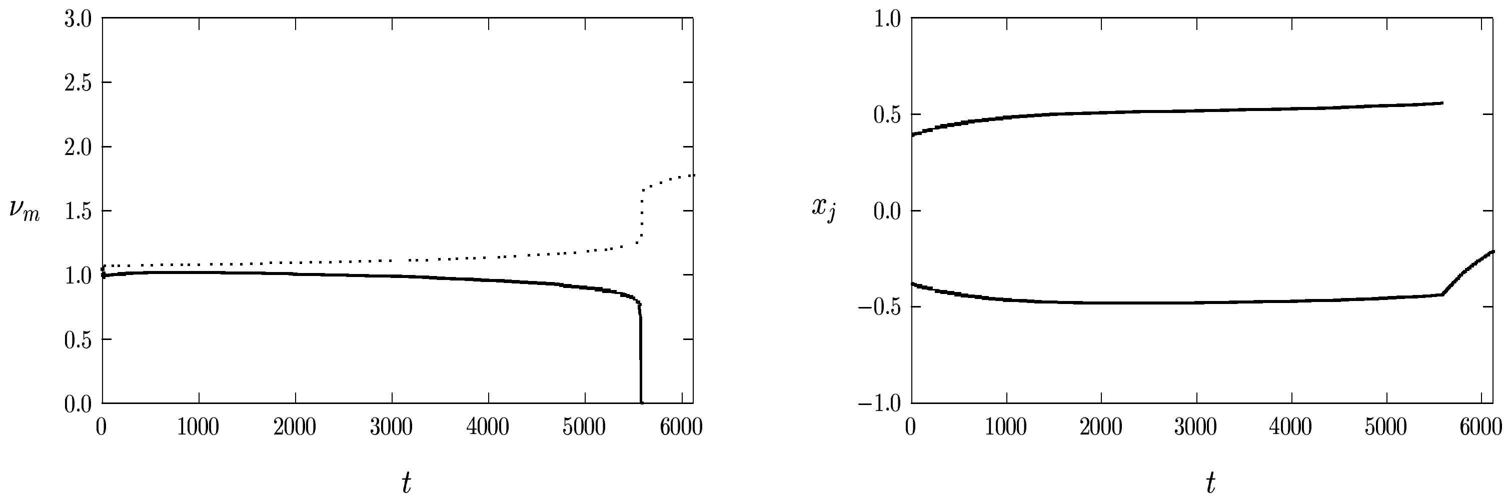




Left: A competition instability kills one of the spikes Right: Spike locations versus t. For the movie click here.

2-Spike GS: Unstable Small Eigenvalues

Experiment: We take $\varepsilon = 0.015$, $D = 0.75$, $\mathcal{A} = 6.1$, $\tau = 4.1$, with slightly asymmetric initial spike locations $x_1(0) = 0.39$, $x_2(0) = -0.38$. For symmetric data, $x_1(0) = -x_2(0) < 0.39$, the NLEP predicts no instabilities.



Left: Spike amplitudes. Right: Spike locations. Movie click here.

Explanation (Saddle Behavior of Small Eigenvalues): The equilibrium solution with $x_1 = -x_0 = 1/2$ is unstable wrt small $O(\varepsilon^2)$ eigenvalues when [KWW, Int. Free. Boun. 2006]

$$\mathcal{A}_{2e} < \mathcal{A} < \mathcal{A}_{2S} < \mathcal{A}_{2L}\,, \quad \mathcal{A}_{2S} = \mathcal{A}_{2e}\coth(D^{-1/2}) = 6.296\,,$$

For $\mathcal{A}_{2e} < \mathcal{A} < \mathcal{A}_{2L}$, there is one positive and one negative eigenvalue. The stable direction occurs for a symmetric approach to $x_1 = -x_0 = 1/2$.

GS Model: Infinite-Domain (Low Feed)

Principal Result: Let $\varepsilon \ll 1$ and and consider a quasi-equilibrium two-spike solution for the GS model with spikes located at $\alpha_i \equiv x_1 = -x_0 > 0$. Suppose that $\mathcal{A}_i > \mathcal{A}_{2e}^{\infty}$, where

$$\mathcal{A}_{2e}^{\infty} = \sqrt{12}\left(1 + e^{-2\alpha_i}\right)^{1/2} ,$$

and that this solution is stable on an $O(1)$ time-scale. Then,

$$\frac{d\alpha_i}{dt} \sim \frac{2\varepsilon_i^2 s_g e^{-2\alpha_i}}{1 + e^{-2\alpha_i}} , \qquad s_g = 2\left[1 - \sqrt{1 - \left(\frac{\mathcal{A}_{2e}^{\infty}}{\mathcal{A}_i}\right)^2}\,\right]^{-1} - 1 .$$

The stability of this solution is determined by the NLEP

$$L_0\Phi - \chi_{gs\pm}^{\infty} w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi\, dy}{\int_{-\infty}^{\infty} w\, dy}\right) = \lambda\Phi , \quad \Phi \to 0 , \text{ as } |y| \to \infty .$$

$$\chi_{gs\pm}^{\infty} \equiv 2s_g \left[s_g + \sqrt{1+\tau\lambda}\left(\frac{1 + e^{-2\alpha_i}}{1 \pm e^{-2\alpha_i\sqrt{1+\tau\lambda}}}\right)\right]^{-1} .$$

GS Model: Infinite-Domain (Low Feed)

Proposition: Suppose that $\mathcal{A}_{2e}^{\infty} < \mathcal{A}_i < \mathcal{A}_{2L}^{\infty}$, where

$$\mathcal{A}_{2L}^{\infty} \equiv \mathcal{A}_{2e}^{\infty} \frac{[1 + \coth(\alpha_i)]}{2\sqrt{\coth(\alpha_i)}} \, .$$

Then, for $0 \leq \tau < \tau_H$, the q.e. is unstable from a unique positive real eigenvalue. Alternatively, for $0 < \tau < \tau_H$, the solution is stable when $\mathcal{A}_i > \mathcal{A}_{2L}^{\infty}$. A Hopf Bifurcation occurs at $\tau = \tau_H$.

- **Static Competition Instability:** occurs at $t = 0$ when $\mathcal{A}_{2e}^{\infty} < \mathcal{A}_i < \mathcal{A}_{2L}^{\infty}$ for the initial $\alpha_i(0)$. Setting $\mathcal{A}_i = \mathcal{A}_{2L}^{\infty}$ we calculate

$$\alpha_{ic} \equiv \frac{1}{2} \log \left(\frac{s_g + 1}{s_g - 1} \right) \, , \quad s_g = 2 \left[1 - \sqrt{1 - \left(\frac{\mathcal{A}_{2e}^{\infty}}{\mathcal{A}_i} \right)^2} \right]^{-1} - 1 \, .$$

  A static competition instability occurs when $0 < \alpha_i(0) < \alpha_{ic}$. However, since $\alpha_i^{'}(t) > 0$ and since $\mathcal{A}_{2L}^{\infty}$ is monotone decreasing in α_i, there are no dynamic competition instabilities.

- **Static Oscillatory Instability:** Occurs when $\alpha_i(0) > \alpha_{ic}$ and $\tau > \tau_H$. Since $\tau_H^{'}(\alpha_i) > 0$ there are no dynamic oscillatory instabilities.

GS Intermediate Regime: Scaling Law I

Consider the subregime $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/3})$ of the intermediate regime. In this regime, n-spike quasi-equilibria are de-stabililized first by a Hopf bifurcation in the spike amplitudes as τ is increased.

Principal Result:[WanW, 2006]: Let $\varepsilon \ll 1$, $\tau = O(\mathcal{A}^4)$, and let $\sigma = \varepsilon^2 \mathcal{A}^2 t$ be the slow time-scale. Near the j^{th} spike

$$v \sim \mathcal{A}\gamma_j w\left[\varepsilon^{-1}(x - x_j)\right], \qquad u \sim \frac{1}{\mathcal{A}^2 \gamma_j}.$$

The amplitudes $\gamma_j(\sigma)$ and locations $x_j(\sigma)$ satisfy the tridiagonal DAE system:

$$\frac{d\boldsymbol{x}}{d\sigma} \sim -\theta_0 \mathcal{U}\mathcal{P}_b \boldsymbol{e}, \qquad \mathcal{B}\boldsymbol{e} \sim 6\theta_0 \boldsymbol{u}, \qquad \theta_0 = D^{-1/2}.$$

Here $\boldsymbol{x} \equiv (x_1, ..., x_n)^t$, $\boldsymbol{u} \equiv (\gamma_1, .., \gamma_n)^t$, and $\mathcal{U}_{ij} = \gamma_i \delta_{ij}$.

GS Intermediate Regime: Scaling Law II

The tridiagonal matrices $\mathcal{B}$ and $\mathcal{P}_b$ are defined by

$$\mathcal{B}_{ii} = c_i\,, \qquad \mathcal{B}_{i,i+1} = \mathcal{B}_{i+1,i} = d_i\,,$$
$$\mathcal{P}_{b\,ii} = g_i\,, \qquad \mathcal{P}_{b\,i,i+1} = f_i\,, \qquad \mathcal{P}_{b\,i+1,i} = -f_i$$

The matrix coefficients depending on $x_1, \ldots, x_n$ and θ_0 by

$$\begin{aligned}
c_1 &= \coth\left[\theta_0(x_2 - x_1)\right] + \tanh\left[\theta_0(1 + x_1)\right]\,,\\
c_n &= \coth\left[\theta_0(x_n - x_{n-1})\right] + \tanh\left[\theta_0(1 - x_n)\right]\,,\\
c_j &= \coth\left[\theta_0(x_{j+1} - x_j)\right] + \coth\left[\theta_0(x_j - x_{j-1})\right]\,,\\
d_j &= -\text{csch}\left[\theta_0(x_{j+1} - x_j)\right]\,, \qquad j = 1, .., n-1\,.\\
g_1 &= \tanh\left[\theta_0(1 + x_1)\right] - \coth\left[\theta_0(x_2 - x_1)\right]\,,\\
g_n &= \coth\left[\theta_0(x_n - x_{n-1})\right] - \tanh\left[\theta_0(1 - x_n)\right]\,,\\
g_j &= \coth\left[\theta_0(x_j - x_{j-1})\right] - \coth\left[\theta_0(x_{j+1} - x_j)\right]\,,\\
f_j &= \text{csch}\left[\theta_0(x_{j+1} - x_j)\right]\,, \qquad j = 1, .., n-1\,.
\end{aligned}$$

GS Intermediate Regime: Scaling Law III

For spikes patterns with $O(1)$ spike separation, the stability of the n-spike quasi-equilibrium is determined by the spectrum of a universal NLEP.

Principal Result: [WanW, 2006]: Let $\varepsilon \ll 1$ and define τ_j for $j = 1, \ldots, N$ by

$$\tau_j \equiv \frac{D\mathcal{A}^4}{144} \left[(\mathcal{B}e)_j \right]^4 \tau_H \,, \quad j = 1, \ldots, N \,,$$

where $\tau_H = 1.748$ is the value of $\tilde{\tau}$ for which the following NLEP problem has a complex conjugate pair of eigenvalues on the imaginary axis:

$$L_0 \Phi - \frac{2}{1 + \sqrt{\tilde{\tau} \lambda}} w^2 \left(\frac{\int_{-\infty}^{\infty} w \Phi \, dy}{\int_{-\infty}^{\infty} w \, dy} \right) = \lambda \Phi \,, \quad \Phi \to 0 \,, \text{ as } |y| \to \infty \,.$$

Then, the quasi-equilibrium pattern is stable for a given configuration $x_1, \ldots, x_n$ iff $\tau < \min_{j=1,\ldots,n} \tau_j$. (This NLEP first appeared in Doelman et al. 1998, then in Muratov-Osipov and KWW).

In contrast to low-feed rate regime, the are no dynamic instabilities in this regime. If the pattern is stable at $t = 0$ then it is stable for $t > 0$.

GS Intermediate Regime: Oscillatory Drift I

In the sub-regime $O(\varepsilon^{-1/3}) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$ of the intermediate regime for the GS model, equilibrium spikes are de-stabilized first by a drift instability (rather than an NLEP profile instability) as τ is increased.

Principal Result:[WanW, 2006]: Let $\varepsilon \ll 1$ and $\tau_0 = \varepsilon^2 \mathcal{A}^2 \tau = O(1)$. Let $\sigma = \varepsilon^2 \mathcal{A}^2 t$ be the slow time-scale. Near the j^{th} spike

$$v \sim \mathcal{A} \gamma_j w \left[\varepsilon^{-1} (x - x_j) \right] , \qquad u \sim \frac{1}{\mathcal{A}^2 \gamma_j} .$$

The amplitudes $\gamma_j(\sigma)$ and locations $x_j(\sigma)$ satisfy the ODE-PDE system:

$$\tau_0 u_\sigma = D u_{xx} + (1 - u) - 6 \sum_{j=1}^{N} \gamma_j \delta(x - x_j) , \quad |x| < 1 ,$$

$$u_x(\pm 1, \sigma) = 0 ,$$

$$\frac{dx_j}{d\sigma} = \gamma_j \left(u_x(x_j^+, \sigma) + u_x(x_j^-, \sigma) \right) ; \quad u(x_j, \sigma) = 0 .$$

GS Intermediate Regime: Oscillatory Drift II

Qualitative: The slow-component u has "memory" and hence, for 1-spike, the dynamics can be written qualitatively in the form

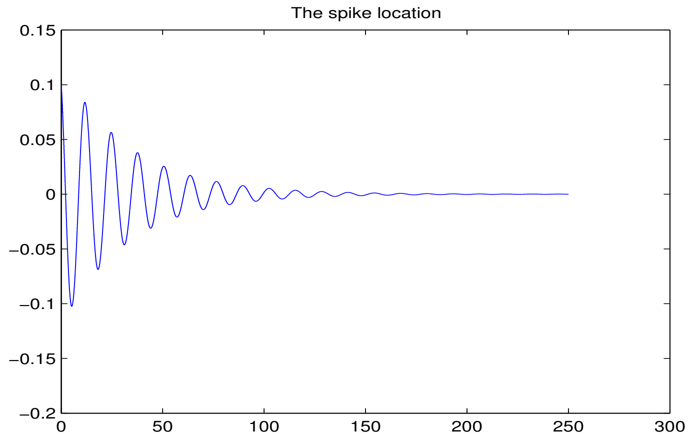
$$\frac{dx_1}{d\sigma} = \gamma_1(\sigma) \int_0^t K(x_0(s), x_0(\sigma), \sigma, s, \gamma_1(s))\, ds + \ldots.$$

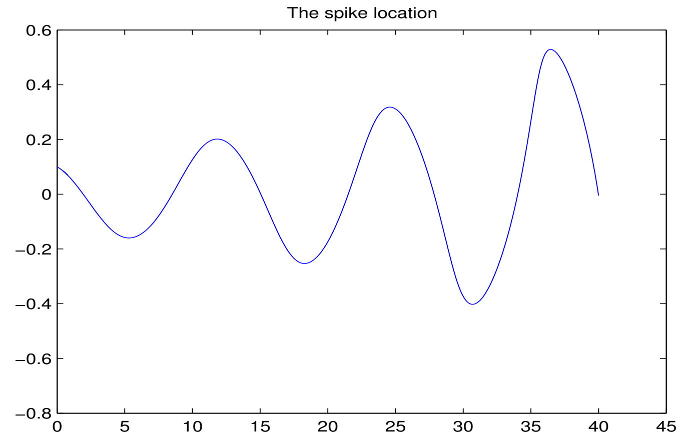
This is a non-linear Volterra integro-differential equation. Oscillations are possible if the "delay" τ_0 is large enough.

- **VIDE:** long history in delay systems in biology (J. Wu, K. Kuang...)
- **Numerically:** Moving source problems require careful discretization of the delta function (Leveque 1991, Enquist-Tornberg (2002, 2005,2006),
- **Related Problems:** By moving a concentrated e^u source term at a sufficiently large speed, blowup can be prevented (W. Olmstead 1994-1999). A similar VIDE occurs in analyzing the route to extinction of a flame-front in non-adiabatic solid flames. In certain cases this leads to chaotic time-dependent behavior of the flame-front (Park, Bayliss, Matkowsky, et al. SIAM J. Appl. Math. (2006)).

GS Intermediate Regime: Oscillatory Drift III

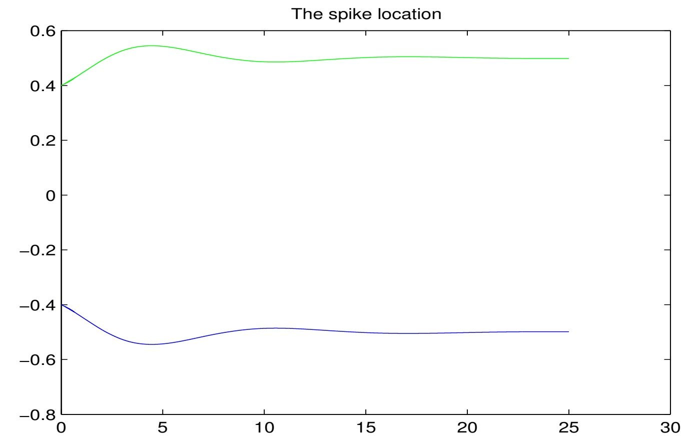
One-Spike Dynamics

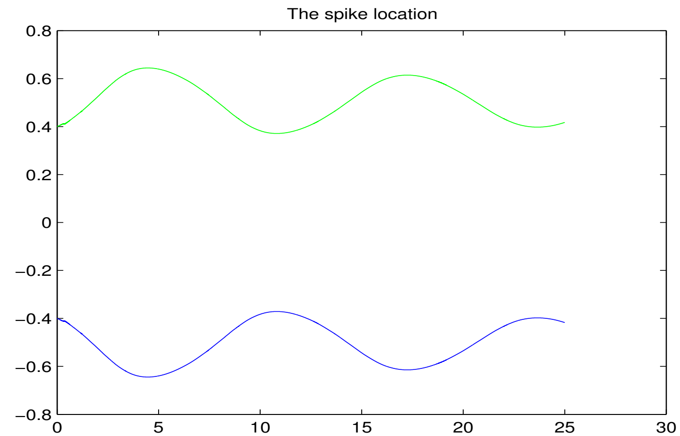




Caption: $x_1(0) = 0.1$. Left: $\tau_0 = 4$. Right: $\tau_0 = 4.5$. HB is $\tau \approx 4.22$ and agrees with equilibrium theory of [KWW, Int. Free BVP 2006].

Two-Spike Dynamics





Caption: $x_2(0) = -x_1(0) = 0.4$ Left: $\tau = 10.0$. Right: $\tau = 15.0$

Main Question: Does chaotic behavior occur for τ sufficiently large?

GS High-Feed Regime: Equilibria I

On $|x| < 1$ with $A = O(1)$, or $\mathcal{A} = O(\varepsilon^{-1/2})$, we have

$$v_t = \varepsilon^2 v_{xx} - v + A u v^2 \,, \qquad \tau u_t = D u_{xx} + (1-u) - u v^2 \,.$$

Principal Result:[KWW, 2004]: The inner solution for a k-spike equilibria

$$v \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} V\left[\varepsilon^{-1}(x - x_j)\right] \,, \quad u \sim \frac{\varepsilon}{A\sqrt{D}} \sum_{j=1}^{k} U\left[\varepsilon^{-1}(x - x_j)\right] \,.$$

Here $U(y)$ and $V(y)$ satisfy the core problem on $0 < y < \infty$:

$$V'' - V + V^2 U = 0 \,, \qquad U'' = U V^2 \,,$$

$$V'(0) = U'(0) = 0 \,; \quad V \to 0 \,, \quad U \sim B y \,, \quad \text{as} \quad y \to \infty \,.$$

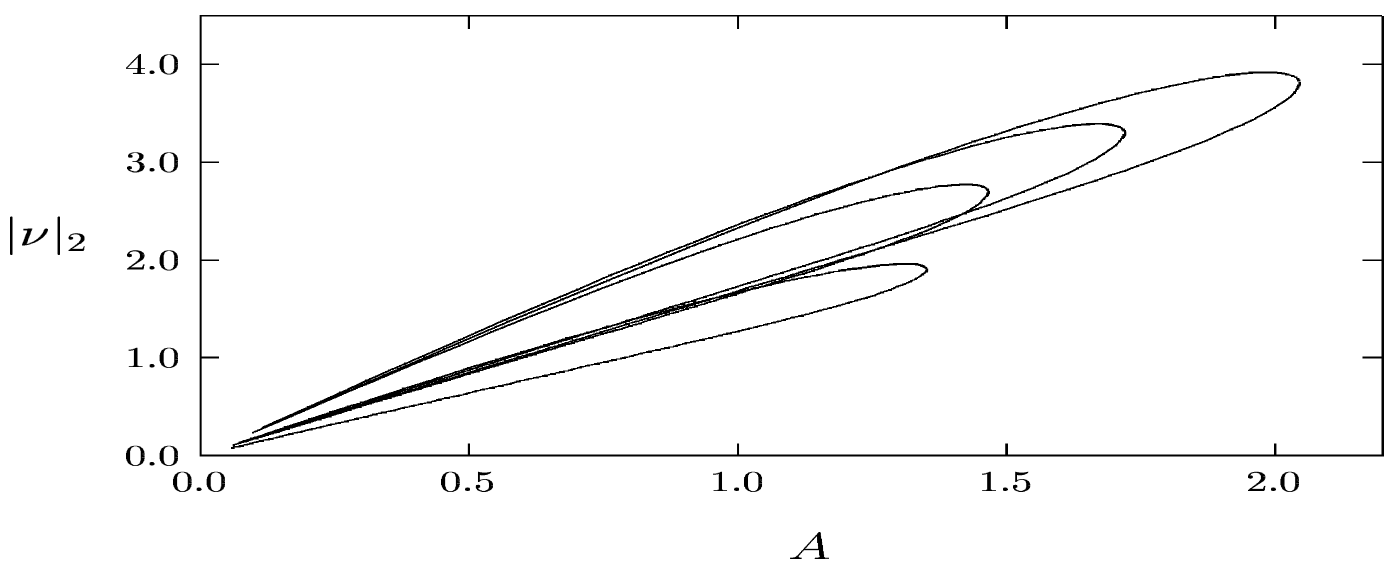
Core problem identified by Doelman et. al (1998), Muratov-Osipov (2000). Matching of core problem to outer solution for u involving Green's functions (KWW, 2004). Detailed properties of bifurcation diagram (DKP, 2006).

GS High-Feed Regime: Equilibria II

The matching condition yields $B \equiv A \tanh\left(\frac{1}{k\sqrt{D}}\right)$. As shown numerically, there is a branch of equilibrium solutions that connect two and one-pulse solutions of the core problem. These solutions exist when $0 < B < 1.347$.

Principal Result: Let $\varepsilon \ll 1$, $A = O(1)$, and assume that $\varepsilon A/\sqrt{D} \ll 1$. Then, there will are no k-spike equilibrium solutions when

$$A > A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right) .$$



GS High-Feed Regime: Drift Instability I

Oscillatory profile instabilities occur when $\tau = O(\varepsilon^{-2})$. Oscillatory drift instabilities occur when $\tau = O(\varepsilon^{-1})$.

Principal Result:[KWW, 2004]: Let $\varepsilon \ll 1$ and $\varepsilon A/\sqrt{D} \ll 1$. Then, the k small eigenvalues λ_j for $j = 1, \ldots, k$ associated with drift instabilities of the k-spike equilibrium solution satisfy

$$\lambda \sim \frac{\varepsilon \alpha A}{\sqrt{D}} \left[\sqrt{1+\tau\lambda} \tanh\left(\frac{\theta_0}{k}\right) z_j - 1 \right], \quad \theta_\lambda \equiv \theta_0 \sqrt{1+\tau\lambda},$$

$$z_j = \coth\left(\frac{2\theta_\lambda}{k}\right) + \text{csch}\left(\frac{2\theta_\lambda}{k}\right) \cos\left(\frac{\pi j}{k}\right), \quad \theta_0 = D^{-1/2}.$$

Here the constant $\alpha > 0$ is given by $\alpha \equiv -\Psi_2^\dagger(\infty)/\int_0^\infty \Psi_1^\dagger V_y \, dy$, where

$$\mathcal{L}^\dagger \Psi^\dagger \equiv \begin{pmatrix} \Psi_{1yy}^\dagger \\ \Psi_{2yy}^\dagger \end{pmatrix} + \begin{pmatrix} -1+2UV & -2UV \\ V^2 & -V^2 \end{pmatrix} \begin{pmatrix} \Psi_1^\dagger \\ \Psi_2^\dagger \end{pmatrix} = 0,$$

and $\Psi_1 \to 0$ and $\Psi_{2y} \to 0$ as $|y| \to \infty$.

GS High-Feed Regime: Drift Instability II

In the inner region near the j^{th} spike, the spatial form of such an instability for $\delta \ll 1$ is

$$v(x,t) \sim \frac{\sqrt{D}}{\varepsilon} \left(V[\varepsilon^{-1}(x - x_j)] + \delta c_j V_y[\varepsilon^{-1}(x - x_j)] e^{\lambda t} \right) .$$

where $\boldsymbol{c}_k^t = \left(1, -1, \ldots, (-1)^{k+1}\right)$. The other $k-1$ possible modes of instabilty satisfy

$$\boldsymbol{c}_{l,j} = \sin\left(\frac{\pi j}{k} (l - 1/2) \right), \quad j = 1, \ldots, k-1 , \quad l = 1, \ldots, k .$$

Our analysis shows that the primary branch is stable when $\tau \ll O(\varepsilon^{-1})$. Instabilities can only occur when $\tau = O(\varepsilon^{-1})$ through oscillatory drift instabilities (Hopf bifurcations) in the spike locations.

GS High-Feed Regime: Drift Instability III

Principal Result: [KWW, 2004]: Let $\varepsilon \ll 1$, $\tau = O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, along the primary branch of the equilibrium core problem, the k-spike equilibrium solution is stable when $0 < \tau < \tau_{TW}$, and is unstable when $\tau > \tau_{TW}$. The stability is lost due to a Hopf bifurcation in the spike locations when $\tau = \tau_{TW}$, where

$$\tau_{TW} \sim \left(\frac{\sqrt{D}}{\varepsilon \alpha A} \right) \tau_{dh} \,, \qquad \tau_{dh} \equiv \min_{j=1,\ldots,k} (\tau_{dj}) \,.$$

Here τ_{dj} for $j = 1, \ldots, k$ is defined by

$$\tau_{dj} \equiv \frac{\xi_{Ij}}{\text{Im}(G_j(i\xi_{Ij}))} \,, \qquad j = 1, \ldots, k \,; \qquad F_j(i\xi_{Ij}) = 0 \,,$$

$$F_j(\xi) \equiv \frac{\xi}{\tau_d} - G_j(\xi) \,, \qquad G_j(\xi) \equiv \sqrt{1+\xi} \tanh\left(\frac{\theta_0}{k}\right) z_j(\xi) - 1 \,,$$

$$z_j(\xi) \equiv \coth\left(\frac{2\theta_0\sqrt{1+\xi}}{k}\right) + \text{csch}\left(\frac{2\theta_0\sqrt{1+\xi}}{k}\right) \cos\left(\frac{\pi j}{k}\right) - 1 \,.$$

GS High-Feed Regime: Drift Instability IV

Conjecture: Let $\varepsilon \ll 1$, and $\varepsilon A/\sqrt{D} \ll 1$. Then, along the primary branch, a k-spike solution first loses its stability to a breathing-type instability at the value $\tau = \tau_{TW} \sim \left(\frac{\sqrt{D}}{\varepsilon \alpha A}\right) \tau_{dk}$. In terms of the v-component, this small-scale oscillatory instability takes the form,

$$v(x,t) \sim \frac{\sqrt{D}}{\varepsilon} \sum_{j=1}^{k} V\left(\varepsilon^{-1}[x - x_j(t)]\right) \,,$$

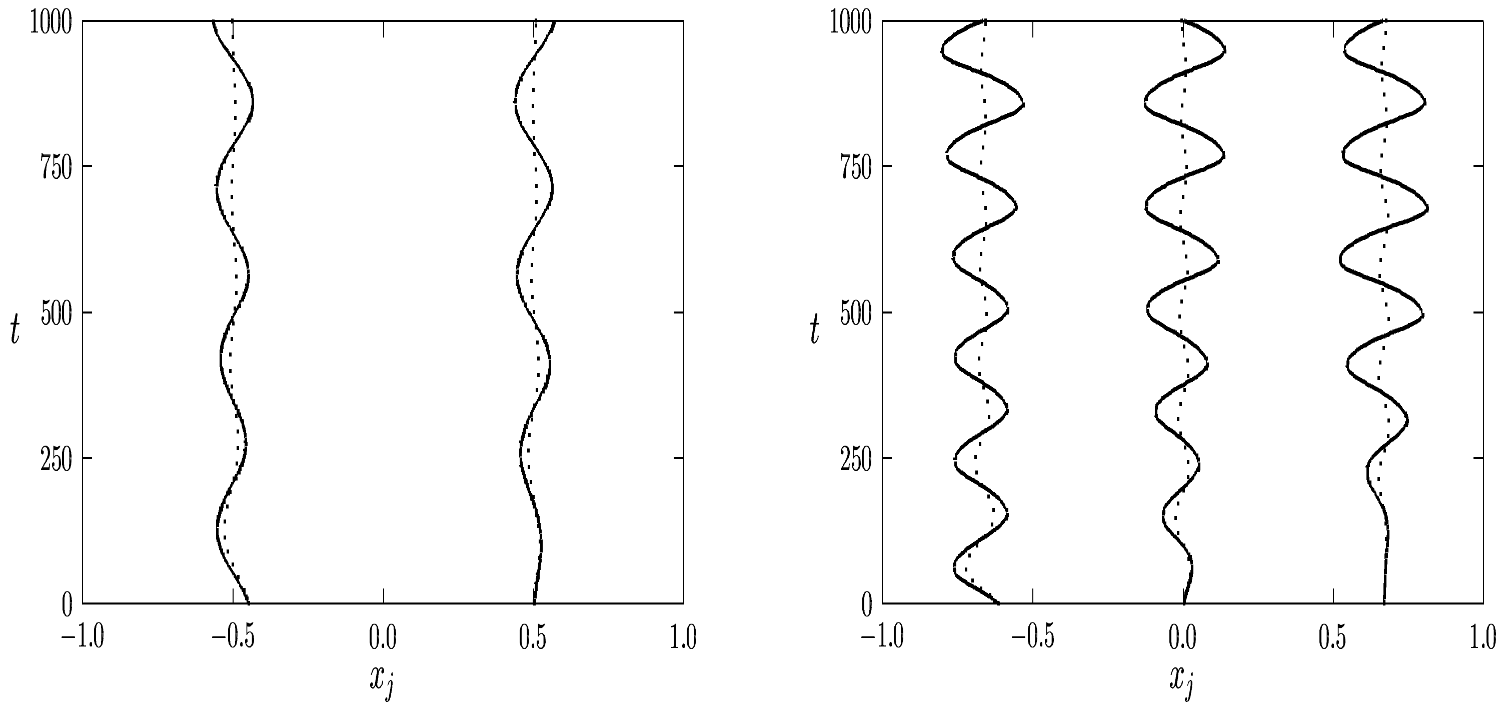
$$x_j(t) \sim x_j(0) + \delta c_j \cos\left(\frac{\varepsilon \alpha A \omega_k t}{\sqrt{D}} - \phi\right) \,,$$

where $\delta \ll 1$, and ϕ is arbitrary.

Here $c_j = (-1)^j$ and $x_j(0) = -1 + (2j-1)/k$ for $j = 1, \ldots, k$. Also $\omega_k = \xi_{Ik}/\tau_{dk}$, where $F_k(i\xi_{Ik}) = 0$.

In other words the minimum for τ_{dj} is obtained when $j = k$.

GS High-Feed Regime: Drift Instability V



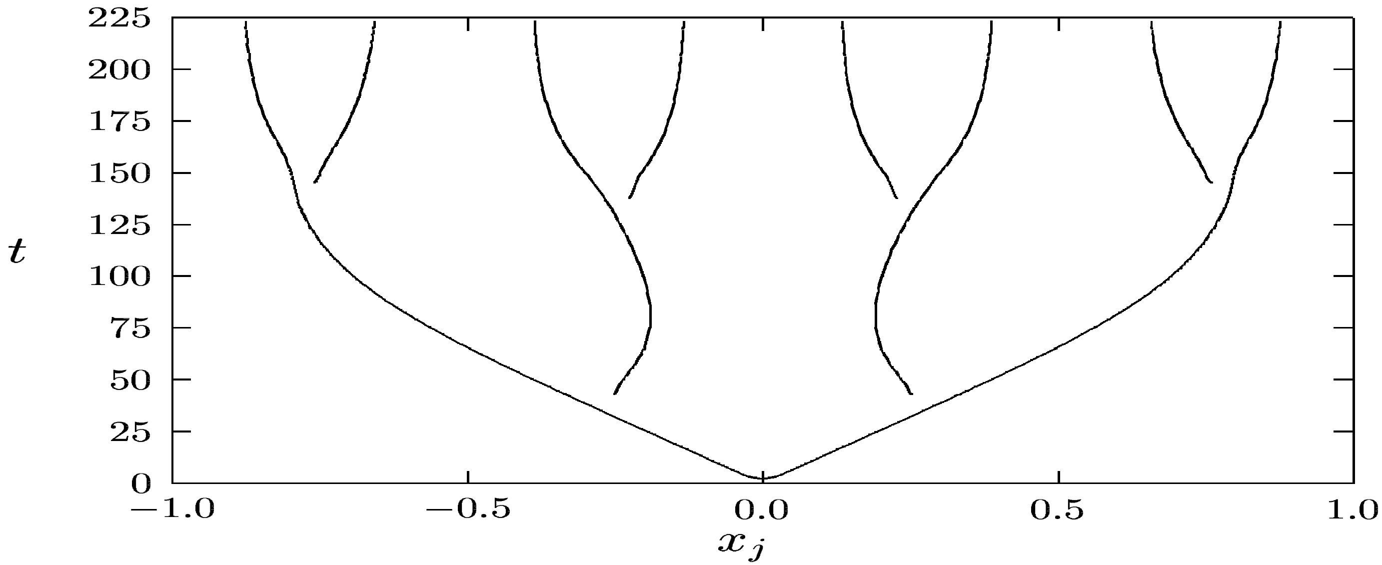
Left: Breather instability for two spikes when $A = 1.4$, $D = 0.1$, $\varepsilon = 0.01$, with $\tau = 35$ (dashed curve) and $\tau = 55$ (solid curve). Here $\tau_{TW} = 39$.
Right: Breather instability for three spikes when $A = 1.6$, $D = 0.1$, $\varepsilon = 0.01$, with $\tau = 75$. Here $\tau_{TW} = 49$.

GS High-Feed Regime: Drift Instability VI

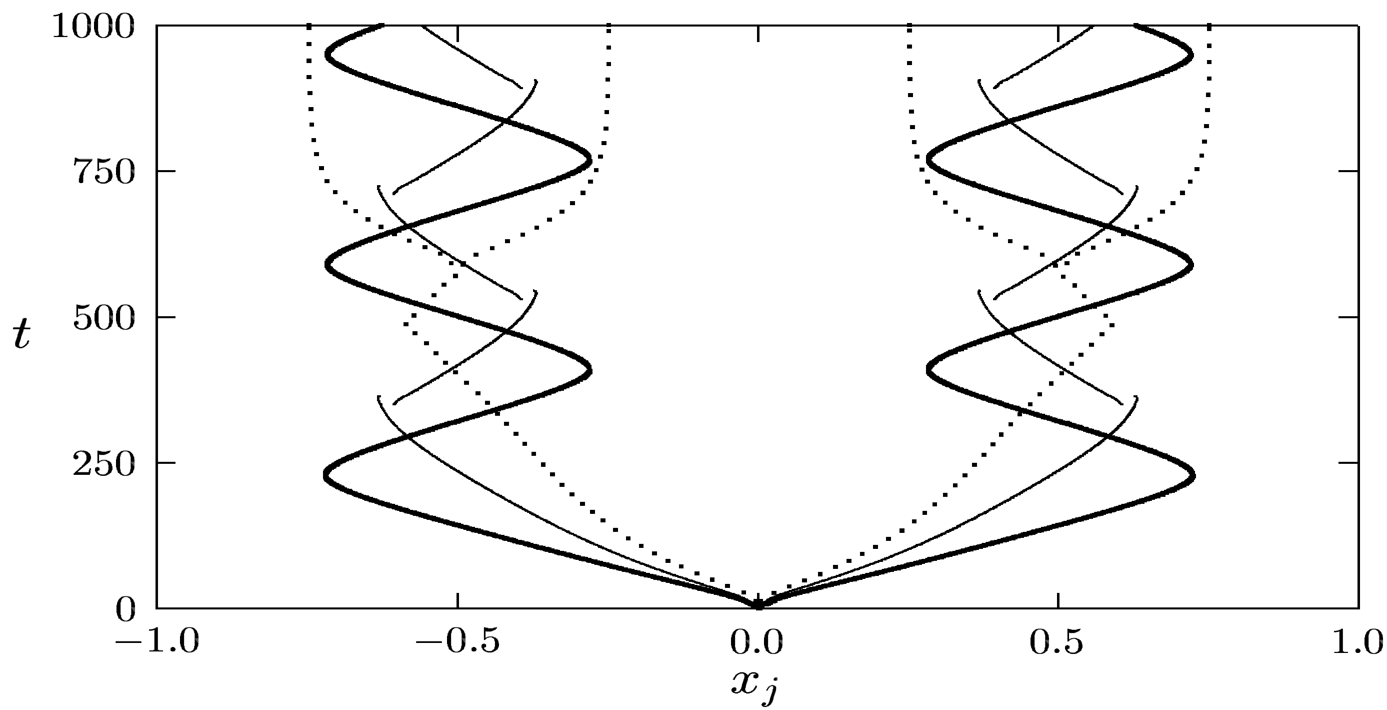
Conjecture : Let $\varepsilon \ll 1$, $\tau \ll O(\varepsilon^{-1})$, and $\varepsilon A/\sqrt{D} \ll 1$. Consider one-spike initial data centered at the origin. Then, the final equilibrium state is stable wrt both the large and small eigenvalues, and it has 2^m spikes where, for some integer $m \geq 0$, A satisfies

$$A_{p2^{m-1}} < A < A_{p2^m} \qquad A_{pk} \equiv 1.347 \coth\left(\frac{1}{k\sqrt{D}}\right) .$$

Example: Let $A = 2.4$, $D = 0.1$, $\varepsilon = 0.01$ and $\tau = 2$. Then, $2.05 = A_{p4} < A < A_{p8} = 3.58$. We predict an eight-spike final state.



GS High-Feed: Large Drift Instability I



Parameters are: $A = 1.5$, $D = 0.04$, $\varepsilon = 0.0077$, and $\tau = 6.8$ (dashed), $\tau = 20$ (solid), and $\tau = 30$ (heavy solid). Since $A_{p2} = 1.365$ and $A_{p4} = 1.588$, and $A_{p2} < A < A_{p4}$ we predict a four-spike final equilibrium state. For $\tau = 6.76$ (dashed) this is observed. For $\tau = 30$ (heavy solid) get two-spike oscillatory state. Borderline case is $\tau = 20$. This suggests that large τ can allow for the existence of a time-periodic two-spike solution even when the two-spike equilibrium does not exist.

GS High-Feed: Large Drift Instability II

An asymptotic analysis leads to the following formulation for the dynamics of a spike solution in the high-feed rate regime with $\tau = \varepsilon^{-1}\tau_0$ and $\tau_0 = O(1)$ on the slow time-scale $\sigma = \varepsilon t$:

Solve the core problem on $-\infty < y < \infty$ with boundary conditions:

$$-x_0' V_y = V_{yy} - V + UV^2 \,, \qquad U_{yy} - UV^2 = 0 \,,$$

$$V'(0) = 0 \,, \quad V \to 0 \;\; |y| \to \infty \,,$$

$$U_y \to \pm A\sqrt{D} u_x(x_0^{\pm}, \sigma) \;\; y \to \pm\infty \,,$$

and couple it to the time-dependent heat equation with moving source

$$\tau_0 u_\sigma = D u_{xx} + (1 - u) - \gamma \delta \left(x - x_0(\sigma) \right) \,, \quad |x| < 1 \,,$$

$$u_x(\pm 1, \sigma) = 0 \,, \quad u(x_0(\sigma), \sigma) = 0 \,.$$

Here $\gamma = \gamma(\sigma)$ is determined by the constraint $u(x_0, \sigma) = 0$.

Preliminary results indicate that large oscillatory instabilities can occur instead of pulse-splitting.

References

- D. Iron, M. J. Ward, J. Wei, *The Stability of Spike Solutions to the One-Dimensional Gierer-Meinhardt Model*, Physica D. Vol. 150, (2001) pp. 25-62.
- D. Iron, M. J. Ward, *The Dynamics of Multi-Spike Solutions for the One-Dimensional Gierer-Meinhardt Model*, SIAM J. Appl. Math., Vol. 62, No. 6, (2002), pp. 1924-1951.
- W. Sun, M. J. Ward, R. Russell, *The Slow Dynamics of Two-Spike Solutions for the Gray-Scott and Gierer-Meinhardt Systems: Competition and Oscillatory Instabilities*, SIADS, Dec. (2005).
- T. Kolokolnikov, M. J. Ward, *Reduced Wave Green's Functions and their Effect on the Dynamics of a Spike for the Gierer-Meinhardt Model*, EJAM, Vol. 14, No. 5, (2003), pp. 513-545.
- W. Chen, M. J. Ward, *Oscillatory Drift and Profile Instabilities for the Gray-Scott Model on a Finite Interval*, in preparation.