The Stability of Periodic Patterns of Spots for RD Systems in $\mathbb{R}^2$

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Singularly Perturbed RD Models: Localization

Spatially localized solutions can occur for singularly perturbed RD models

$$v_t = \varepsilon^2 \triangle v + g(u,v)\,; \quad \tau u_t = D\triangle u + f(u,v)\,, \qquad x \in \mathbb{R}^2\,.$$

Assume semi-strong interactions for which $\varepsilon \ll 1$ and $D = \mathcal{O}(1)$.

Since $\varepsilon \ll 1$, v can be localized in space as a spot pattern, i.e. concentration at a discrete set of points.

Goal: Construct a steady-state periodic pattern of spots where v concentrates as $\varepsilon \to 0$ at lattice points of a Bravais lattice. Fix the fundamental Wigner-Seitz cell Ω to have unit area. By analyzing the spectrum of the linearization, identify the particular lattice arrangement that optimizes a certain stability threshold.

We consider three RD systems:

- **Gray-Scott Model:** (Pearson, Science 1993, Muratov scaling (1996))

$$g(u,v) = -v + Auv^2\,, \qquad f(u,v) = (1-u) - uv^2\,.$$

- **Schnakenburg Model:** $g(u,v) = -v + uv^2$ and $f(u,v) = a - uv^2$.
- **GM Model:** $g(u,v) = -v + v^2/u$ and $f(u,v) = -u + v^2$.

Brief History

- **Difficulties:** No variational structure; Patterns are "far-from- equilibrium". Turing and weakly nonlinear theories are not applicable.
- **A Stability Theory for Localized Pulse-Type Solutions (over past 15 years):**
  - **Tools for 1-D**: geometric singular perturbation, Lyapunov-Schmidt, NLEP analysis, matched asymptotics, renormalization group: (Doelman, Kaper, Gardner, Promislow, Van der Ploeg, ... ; Nishiura, Ueda, Ei, ...; Muratov, Osipov; Iron, Ward, Wei, Kolokolnikov,....)
  - **In 2-D:** NLEP stability studies of spots (Wei-Winter); Some specific case studies (X. Chen, W. Chen, Kowalzcyk, Kolokolnikov, Muratov, Osipov, Ward, Wei, Winter, etc..).
  - **Challenge in 2-D: Coulomb Interactions** with gauge $\nu = -1/\log \varepsilon$.

GM: In 1-D the stability of periodic pulses on $\mathbb{R}^1$:

- H. Van der Ploeg, A. Doelman, Indiana J., **54**(5), (2005).

GM: N pulses on finite 1-D domain with Neumann B.C.:

- D. Iron, M. J. Ward and J. Wei, Physica D, **150**(1-2), (2001).
- M. J. Ward, J. Wei, J. Nonl. Sci., **13**(2), (2003).

Leading Order NLEP Theory: I

Key: to leading order in $\nu \equiv -1/\log\varepsilon$, the stability theory is based on analysis of nonlocal eigenvalue problems (NLEP's) of the form:

$$\Delta\Phi - \Phi + 2w\Phi - \chi(\lambda) w^2 \frac{\int_{\mathbb{R}^2} w\Phi \, dy}{\int_{\mathbb{R}^2} w^2 \, dy} = \lambda\Phi \, .$$

Here $w(\rho) > 0$ is the unique radially symmetric ground state of

$$\Delta_\rho w - w + w^2 = 0 \, , \quad w(0) > 0 \, , \quad w(\infty) = 0 \, , \quad w'(0) = 0 \, .$$

Challenge: NLEP is nonlocal, non-self-adjoint, and $\chi = \chi(\lambda)$.

A Basic Result: If $\chi(0) < 1$, and $1/\chi(\lambda)$ analytic in $\Re(\lambda) \geq 0$, $\exists \; \lambda_0 > 0$ real.

GM Model: Construct an N-spot symmetric steady-state pattern (u_e, v_e) with spots of equal height for the GM model on finite 2-D domain Ω_f (with Neumann BC), and let $D = D_0/\nu \gg 1$ with $\nu = -1/\log\varepsilon \ll 1$:

$$v_t = \varepsilon^2 \triangle v - v + v^2/u \, ; \qquad \tau u_t = D \triangle u - u + v^2 \, , \qquad x \in \Omega_f \, .$$

Introduce perturbation of the form $v = v_e + \sum_{i=1}^{N} c_i e^{\lambda t} \hat{\Phi}\left(\frac{|\mathbf{x}-\mathbf{x}_i|}{\varepsilon}\right)$.

Leading Order NLEP Theory: II

A lengthy analysis shows that there are N choices for $\mathbf{c}_j \equiv (c_1, \ldots, c_N)^T$:

$$\mathbf{c}_1 = \mathbf{e} \equiv (1, \ldots, 1)^T \, ; \qquad \text{(synchronous mode)}$$

$$\mathbf{c}_j^T \mathbf{e} = 0 \, , \quad j = 2, \ldots, N \, ; \qquad \text{(competition modes)} \, .$$

For $\tau = \mathcal{O}(1)$ as $\varepsilon \to 0$, the two multipliers χ for the NLEP for the synchronous (s) and competition (c) modes are

$$\chi_c(\lambda) = \frac{2}{\mu + 1} \, , \qquad \mu \equiv \frac{2\pi N D_0}{|\Omega_f|} \, , \qquad D = \frac{D_0}{\nu} \, ,$$

$$\chi_s(\lambda) = \frac{2}{\mu + 1} \frac{(\mu + 1 + \tau\lambda)}{(1 + \tau\lambda)} \, .$$

Ref: [WWi,2001]) J. Wei, M. Winter, *Spikes for the two-dimensional Gierer-Meinhardt system: the weak coupling case*, J. Nonlinear Sci., **11**(6), (2001), pp. 415–458.

Leading Order NLEP Theory: III

Main Result (Competition Modes): [WWi,2001]:

- linearly stable $\forall \tau > 0$ iff $\mu < 1$.
- **Key:** to leading order in ν, $N-1$ competition modes simultaneously go unstable through a zero eigenvalue crossing as μ increases past $\mu = 1$.
- If $\mu = 1$, then $\Phi = w$ and $\lambda = 0$ is an eigenpair of the NLEP.

Main Result (Synchronous Mode):

- Since $\chi_s(0) = 2$, then $\forall \mu > 0$ and $\forall \tau > 0$ there are no zero eigenvalue crossings (**[WWi, 2001]**).
- If $\mu > 1$: there is a unique Hopf bifurcation value $\tau_H > 0$. (**[WWi, 2001]**).
- If $0 < \mu < 1$: then stability if $0 < \tau < \tau_2$ and for $\tau > \tau_3$, where $\tau_3 \geq \tau_2$ (**[WWi, 2001]**). Unresolved whether $\tau_3 > \tau_2$ or $\tau_3 = \tau_2$.
- More recently: stability $\forall \tau > 0$ with $\tau = \mathcal{O}(1)$ when $0 < \mu < 1$. For $0 < \mu < 1$, Hopf bifurcation stability threshold has anomalous scaling $\tau = \mathcal{O}(\varepsilon^{2(\mu-1)}/\nu)$ **[WardWei; 2014]**.

Central Problem: Unfolding $\lambda = 0$: I

Summary: On Ω_f, we have linear stability $\forall \tau > 0$ with $\tau = \mathcal{O}(1)$, when

$$D < D_c \sim \frac{D_{0c}}{\nu}\,, \qquad D_{0c} \equiv \frac{|\Omega_f|}{2\pi N}\,, \qquad \nu = -1/\log \varepsilon\,.$$

Specific Goal: Consider periodic pattern of spots in $\mathbb{R}^2$ concentrated at lattice points of Bravais lattice. Effectively, $|\Omega_f| \to N|\Omega|$, where $|\Omega|$ is the area of a Wigner Seitz cell. Consider the "blow up" neighborhood

$$D = \frac{|\Omega|}{2\pi\nu}(1 + \mu_1 \nu) + o(1)\,, \quad \mu_1 = \mathcal{O}(1) \text{ is a "de-tuning parameter"}\,,$$

- Calculate the band of continuous spectrum within an $\mathcal{O}(\nu)$ ball near the origin, i.e. within $|\lambda| = \mathcal{O}(\nu)$.
- For a given lattice, choose μ_1 sufficiently small, i.e. $\mu_1 < \mu_1^\star$, so that the entire band satisfies $\Re(\lambda) < 0$.
- Maximize μ_1 with respect to the lattice geometry to obtain the "optimal" lattice that allows for stability for the largest range of D.

Ref: [IRWW, 2013] Iron, Rumsey, Ward, Wei, *Logarithmic expansions and the Stability of Periodic Patterns of Localized Spots for RD Systems in* $\mathbb{R}^2$, submitted, J. Non. Science, (41 pages), (2013).

Central Problem: Unfolding $\lambda = 0$: II

Secondary Goal (Finite Domain Ω_f): Unfold the degenerate zero eigenvalue crossing for competition modes and calculate critical values μ_{1j} in

$$D_c \equiv \frac{D_{0c}}{\nu}\left(1 + \nu \mu_{1j}\right) + o(1)\,, \qquad j = 1, \ldots, N-1\,,$$

where individual modes cross through $\lambda = 0$. Choose $\min_j \mu_{1j}$ to get stability threshold. **Remark:** possible degeneracies at higher order as well. **Ref: [WaW, 2013]** Ward, Wei, to be submitted.

Broader: Similar problems with logarithmic interactions occur in other 2-D contexts including narrow escape problems in Biophysics, and eigenvalue problems in perforated domains, such as

$$\Delta u + \lambda u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_p\,, \quad \Omega_p \equiv \bigcup_{i=1}^{N} \Omega_{\varepsilon_i}\,,$$
$$\partial_n u = 0 \quad \mathbf{x} \in \partial\Omega\,; \qquad u = 0\,, \quad \mathbf{x} \in \partial\Omega_p\,,$$

where Ω_{ε_i} is a disk of radius ε centered at $\mathbf{x}_i \in \Omega$. Then, **Ref: [KTW, 2005],**

$$\lambda_0 \sim \frac{2\pi N \nu}{|\Omega|} - \frac{4\pi \nu^2}{|\Omega|}\mathbf{e}^T \mathcal{G} \mathbf{e} + \mathcal{O}(\nu^3)\,, \qquad \nu \equiv \frac{-1}{\log \varepsilon} \ll 1\,.$$

A Primer on Lattices: I

Consider the class of Bravais lattices Λ defined by

$$\Lambda \equiv \left\{ m\boldsymbol{l}_1 + n\boldsymbol{l}_2 \,\middle|\, m,\, n \in \mathbb{Z} \right\}.$$

WLOG, align $\boldsymbol{l}_1$ with positive x-axis.

Primitive cell: parallelogram generated by the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

Wigner Seitz cell centered at $\boldsymbol{l} \in \Omega$ is the set of all points closer to $\boldsymbol{l}$ than any other lattice point (Voronoi cell)

- The union of the WS cells tile $\mathbb{R}^2$.
- The Fundamental WS cell Ω (FWS) is centered at the origin. We set $|\Omega| = 1$. Note: $|\Omega| = |\boldsymbol{l}_1 \times \boldsymbol{l}_2|$.

For a regular hexagonal lattice with $|\Omega| = 1$, we have

$$\boldsymbol{l}_1 = \left(\left(\frac{4}{3} \right)^{1/4}, 0 \right) \qquad \text{and} \qquad \boldsymbol{l}_2 = \left(\frac{4}{3} \right)^{1/4} \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right).$$

A Primer on Lattices: II

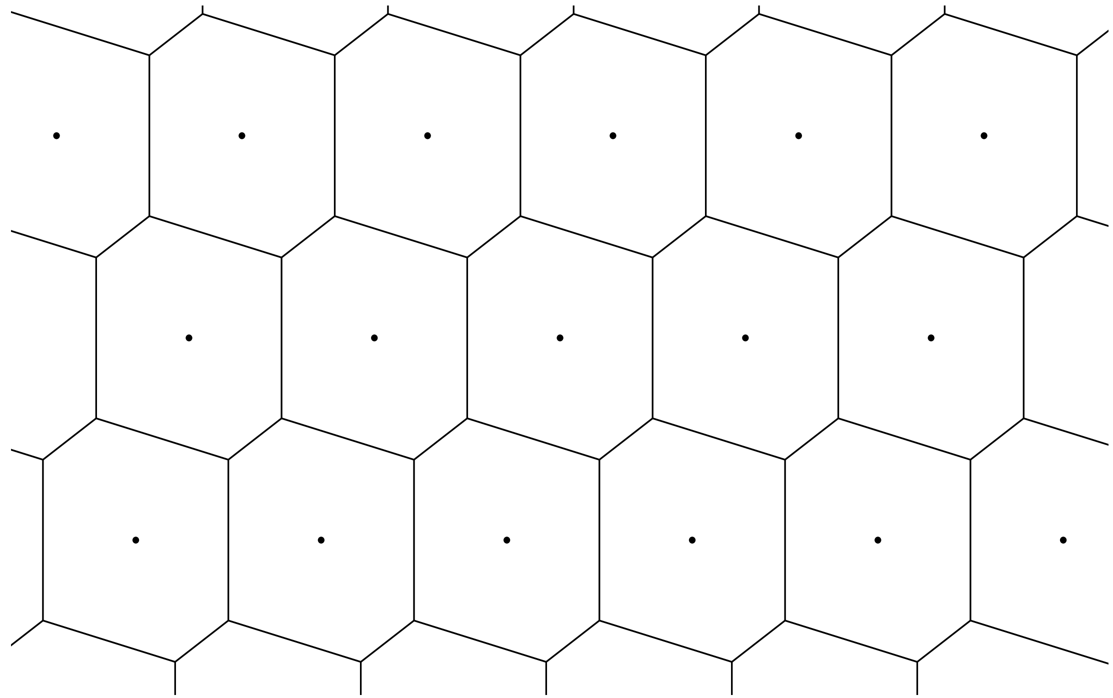


Figure 1: WS cells for an oblique lattice with $\boldsymbol{l}_1 = (1, 0)$, $\boldsymbol{l}_2 = (\cot\theta, 1)$, and $\theta = 74^\circ$, so that $|\Omega| = 1$. These cells tile the plane.

- Generically, the FWS cell has three pairs of parallel sides of equal length (except for rectangular cells).
- Triangular lattices are excluded since we cannot tile $\mathbb{R}^2$ with translates of a FWS.

A Primer on Lattices: III

Reciprocal lattice: $\Lambda^\star$ is defined in terms of two independent vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$, satisfying

$$\boldsymbol{d}_i \cdot \boldsymbol{l}_j = \delta_{ij}\,, \qquad \Lambda^\star \equiv \left\{ m\boldsymbol{d}_1 + n\boldsymbol{d}_2 \;\middle|\; m,\, n \in \mathbb{Z} \right\}.$$

First Brillouin zone Ω_B**:** is the Fundamental WS cell in reciprocal space.

Poisson Summation Formula (PSF) between direct and reciprocal lattices:

$$\sum_{\boldsymbol{l}\in\Lambda} f(\mathbf{x}+\boldsymbol{l})\, e^{i\boldsymbol{k}\cdot\boldsymbol{l}} = \frac{1}{|\Omega|} \sum_{\boldsymbol{d}\in\Lambda^*} \hat{f}(2\pi\boldsymbol{d}-\boldsymbol{k})\, e^{i\mathbf{x}\cdot(2\pi\boldsymbol{d}-\boldsymbol{k})}\,, \qquad \boldsymbol{k}/(2\pi) \in \Omega_B\,,$$

where $\hat{f}$ is the Fourier transform of f, defined by

$$\hat{f}(\boldsymbol{p}) = \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-i\mathbf{x}\cdot\boldsymbol{p}}\, \mathrm{d}\mathbf{x}\,, \qquad f(\mathbf{x}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{p})\, e^{i\boldsymbol{p}\cdot\mathbf{x}}\, \mathrm{d}\boldsymbol{p}\,.$$

Ref: G. Beylkin, C. Kurcz, L. Monzón, *Fast algorithms for Helmholtz Green's functions*, Proc. R. Soc. A, **464**, (2008), pp. 3301-3326.

Key: PSF is critical for readily calculating a required Bloch Green's function.

Floquet-Bloch Theory

- Localize as $\varepsilon \to 0$ a steady-state spot for the GM system at $0 \in \Omega$. Extend periodically to $\mathbb{R}^2$.
- Linearize GM system around this steady-state solution. For $\varepsilon \to 0$, the eigenfunction Ψ corresponding to the long-range component u satisfies an elliptic PDE with coefficients that are spatially periodic on Λ.
- Thus, by the Floquet-Bloch theorem, we impose for ψ that

$$\psi(\mathbf{x} + \boldsymbol{l}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{l}}\, \psi(\mathbf{x})\,, \qquad \boldsymbol{l} \in \Lambda\,.$$

- **Formulate Boundary Operator on $\partial\Omega$:** Let L_i and L_{-i} be two parallel Bragg lines on opposite sides of $\partial\Omega$ for $i = 1, \ldots, L/2$, with $L = \{4, 6\}$. Let $\mathbf{x}_{i1} \in L_i$ and $\mathbf{x}_{i2} \in L_{-i}$ be any two opposing points on these Bragg lines. We define the boundary operator $\mathcal{P}_k \Psi$ by

$$\mathcal{P}_k \Psi \equiv \left\{ \Psi \,\middle|\, \begin{pmatrix} \Psi(\mathbf{x}_{i1}) \\ \partial_n \Psi(\mathbf{x}_{i1}) \end{pmatrix} = e^{-i\mathbf{k}\cdot\boldsymbol{l}_i} \begin{pmatrix} \Psi(\mathbf{x}_{i2}) \\ \partial_n \Psi(\mathbf{x}_{i2}) \end{pmatrix} , \right.$$
$$\left. \forall\, \mathbf{x}_{i1} \in L_i\,,\ \forall\, \mathbf{x}_{i2} \in L_{-i}\,,\ \boldsymbol{l}_i \in \Lambda,\ \ i = 1, \ldots, L/2 \right\}.$$

- The boundary operator $\mathcal{P}_0 \Psi$ simply corresponds to periodic BC on $\partial\Omega$.

Key Results For Certain Green's Functions

There are two key Green's functions on Ω that play a central role:

Key 1: The source-neutral or periodic G-function is

$$\Delta G_{0p} = \frac{1}{|\Omega|} - \delta(\mathbf{x}) \,, \quad x \in \Omega \,; \qquad \mathcal{P}_0 G_{0p} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,,$$

$$G_{0p} \sim -\frac{1}{2\pi} \log|\mathbf{x}| + R_{0p} + o(1) \,, \quad \text{as} \quad \mathbf{x} \to 0 \,; \quad \int_\Omega G_{0p} \, d\mathbf{x} = 0 \,.$$

Theorem (Chen-Oshita, 2007): Fix $|\Omega| = 1$. The regular part R_{0p} is minimized for a regular hexagon. $\exists$ an explicit formula for R_{0p}.

Key 2: The Bloch Green's function for $\mathbf{k}/(2\pi) \in \Omega_B$ satisfies

$$\Delta G_{b0} = -\delta(\mathbf{x}) \,, \qquad \mathbf{x} \in \Omega \,; \qquad \mathcal{P}_k G_{b0} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,,$$

$$G_{b0} \sim -\frac{1}{2\pi} \log|\mathbf{x}| + R_{b0} + o(1) \,, \quad \text{as} \quad \mathbf{x} \to 0 \,.$$

Lemma [IRWW, 2013]: $R_{b0}(\mathbf{k})$ is real-valued, with $R_{b0}(\mathbf{k}) = \mathcal{O}(\left[\mathbf{k}^T Q \mathbf{k}\right]^{-1}) = \mathcal{O}(|\mathbf{k}|^{-2})$ as $|\mathbf{k}| \to 0$ for an orthogonal matrix Q.

Brief Sketch of Analysis: I

Steady-State Construction: Inner Region: Near $\mathbf{x} = 0 \in \Omega$, let $u = D\,U(\rho)$, $v = DV(\rho)$, $\rho \equiv \varepsilon^{-1}|x|$. This gives the radially symmetric core problem

$$\Delta_\rho V - V + V^2/U = 0\,, \qquad \Delta_\rho U = -V^2\,, \quad \rho > 0$$
$$V \to 0\,, \quad U \sim -S\log\rho + \chi(S) + o(1)\,, \quad \text{as} \quad \rho \to \infty\,.$$

It is readily shown that $\chi(S) = \mathcal{O}(S^{1/2})$ as $S \to 0$.

Lemma [IRWW, 2013]: *For $S = S_0\nu^2 + S_1\nu^3 + \cdots$, where $\nu \equiv -1/\log\varepsilon \ll 1$, the asymptotics of the core solution for $S \to 0$ is*

$$V \sim \nu\left[\chi_0 w + \nu\left(\chi_1 w + S_0 V_{1p}\right) + \cdots\right]\,, \qquad \chi \sim \nu\left(\chi_0 + \nu\chi_1 + \cdots\right)\,,$$

where $w(\rho)$ is the ground state. Here V_{1p} is the unique solution to

$$L_0 V_{1p} \equiv \Delta_\rho V_{1p} - V_{1p} + 2wV_{1p} = w^2 U_{1p}\,; \quad V_{1p} \to 0\,, \ \text{as} \ \rho \to \infty\,,$$
$$\Delta_\rho U_{1p} = -w^2/b\,; \quad U_{1p} \to -\log\rho + o(1)\,, \ \text{as} \ \rho \to \infty\,,$$

where $b \equiv \int_0^\infty \rho w^2\, d\rho$. Finally, χ_0 and χ_1 are related to S_0 and S_1 by

$$\chi_0 = \sqrt{\frac{S_0}{b}}\,, \qquad \chi_1 = \frac{S_1}{2\chi_0 b} - \frac{S_0}{b}\int_0^\infty wV_{1p}\rho\, d\rho\,.$$

Brief Sketch of Analysis: II

Match to an outer solution for u. For $D = D_0/\nu$, S satisfies

$$\left[1 + \mu + 2\pi\nu R_{0p} + \mathcal{O}(\nu^2)\right] S = \nu\chi(S)\,, \qquad \mu \equiv \frac{2\pi D_0}{|\Omega|}\,. \qquad (\star)$$

To leading-order, $\lambda = 0$ when $\mu = 1$. Thus, we expand

$$\lambda = \nu\lambda_1 + \cdots\,, \qquad \text{for} \qquad \mu = 1 + \nu\mu_1 + \cdots\,.$$

From asymptotics of core problem and $(\star)$, we relate χ_1 to μ_1 by

$$\chi_1 = -\frac{\mu_1}{4b} - \frac{\pi R_{0p}}{2b} - \frac{1}{2b^2}\int_0^\infty w V_{1p}\rho\, d\rho\,.$$

Stability problem: expand $\Phi = w + \nu\Phi_1 + \cdots$ for $\mu = 1 + \nu\mu_1 + \cdots$. We get

$$\mathcal{L}\Phi_1 \equiv L_0\Phi_1 - w^2\frac{\int_0^\infty w\Phi_1\rho\, d\rho}{\int_0^\infty w^2\rho\, d\rho} = \mathcal{F} + \lambda_1 w\,; \qquad \Phi_1 \to 0\,, \quad \text{as} \quad \rho \to \infty\,,$$

$$\mathcal{F} \equiv 2\pi R_{b0}w^2 + 2\chi_1 b w^2 + \frac{1}{2b}w^2\int_0^\infty w V_{1p}\rho\, d\rho + w^2 U_{1p}\,.$$

Solvability Condition: Since $\mathcal{L}^\star\Psi^\star = 0$ with $\Psi^\star \equiv w + \rho w'/2$, we must have $\lambda_1\int_0^\infty w\Psi^\star\rho\, d\rho + \int_0^\infty \mathcal{F}\Psi^*\rho\, d\rho = 0$. This, ultimately, relates λ_1 to μ_1.

GM Model: Main Result for Periodic Patterns

$$v_t = \varepsilon^2 \Delta v - v + v^2/u\,, \qquad \tau u_t = D\Delta u - u + v^2\,; \qquad \text{(GM Model)}\,.$$

Principal Result [IRWW, 2013]: *For* $D \sim \frac{|\Omega|}{2\pi\nu}\left(1+\nu\mu_1\right)$, *the portion of the continuous spectrum satisfying* $|\lambda| \leq \mathcal{O}(\nu)$ *is*

$$\lambda = \nu\lambda_1 + \cdots\,, \qquad \lambda_1 = \mu_1 - 4\pi R_{b0} + 2\pi R_{0p} - \frac{1}{b}\int_0^\infty \rho w V_{1p}\, d\rho\,.$$

Thus, a periodic arrangement of spots on Λ *is linearly stable when*

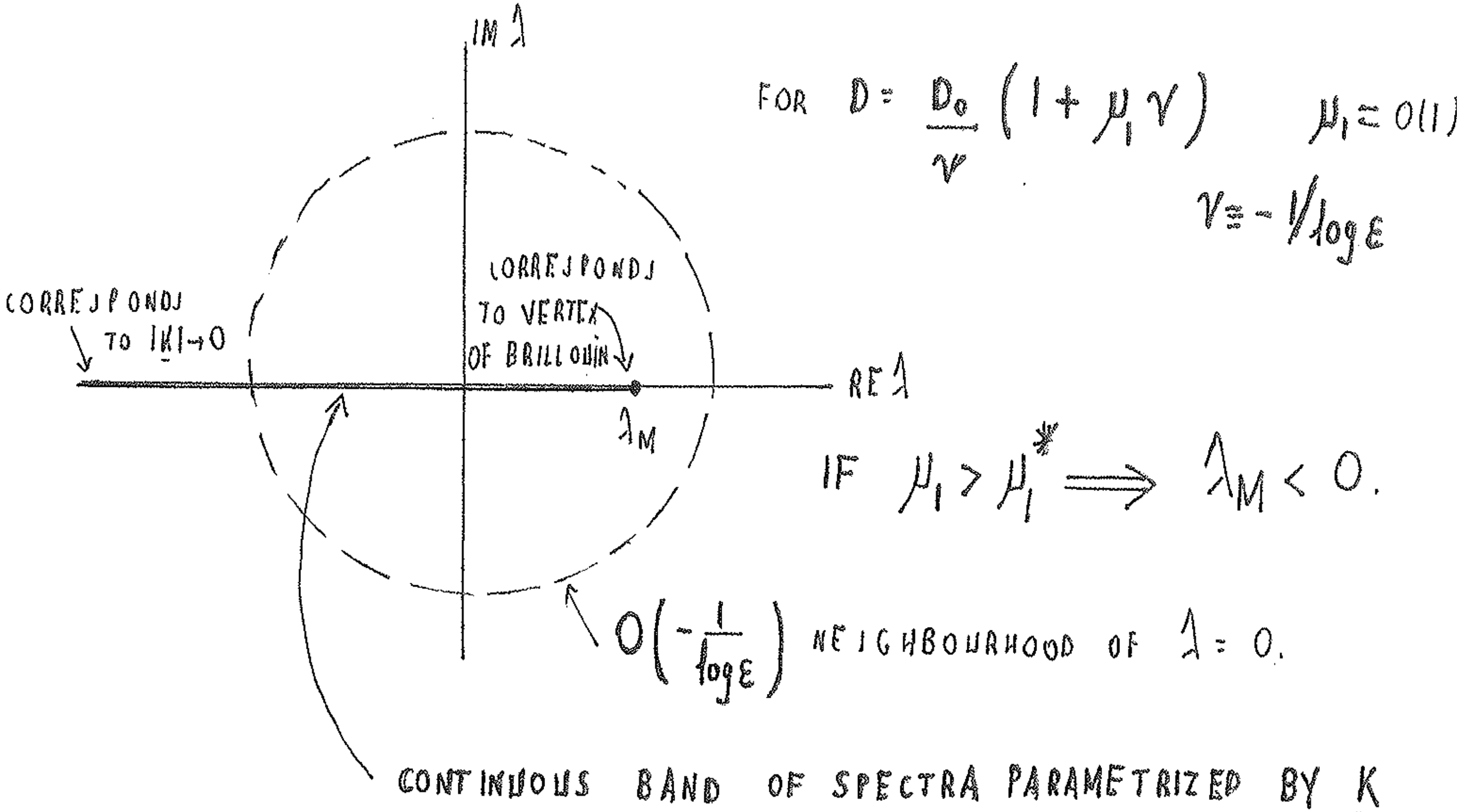
$$\mu_1 < \mu_1^\star \equiv 4\pi R_{b0}^\star - 2\pi R_{0p} + \frac{1}{b}\int_0^\infty w V_{1p}\rho\, d\rho\,, \quad R_{b0}^\star \equiv \min_{\mathbf{k}} R_{b0}(\mathbf{k})\,.$$

The optimal lattice arrangement maximizes $\mathcal{K}_{gm} \equiv 4\pi R_{b0}^\star - 2\pi R_{0p}$. *The stability threshold on the optimum lattice is*

$$D_{\text{optim}} \sim \frac{|\Omega|}{2\pi\nu}\left[1 + \nu\left(\max_{\Lambda}\mathcal{K}_{\text{gm}} + \frac{1}{b}\int_0^\infty w V_{1p}\rho\, d\rho\right)\right]\,,$$

Numerical computations yield $b \approx 4.93$ *and* $\int_0^\infty w V_{1p}\rho\, d\rho \approx -0.945$.

Plot of the Spectrum Near Criticality



Remarks:

- Since $R_{b0} = \mathcal{O}(|\mathbf{k}|^{-2})$ as $|\mathbf{k}| \to 0$, long-wavelength perturbations are at a safe distance $\mathcal{O}(\nu/|\mathbf{k}|^2)$ along the negative real axis.
- Recall $R_{b0}(\mathbf{k})$ is real-valued. Thus, the band is along real axis. Need only locate right-most edge of band.

GM Model: Main Result for Finite Domain Ω_f

Principal Result [WaW, 2013]: *Let $N \geq 2$, and suppose $\mathbf{e} = (1. \ldots, 1)^T$ is an eigenvector of Neumann Green's matrix $\mathcal{G}$. Then, for $D \sim \frac{|\Omega_f|}{2\pi N \nu}(1 + \nu \mu_1)$, there are $N - 1$ eigenvalues (counting multiplicity) in an $|\lambda| \leq \mathcal{O}(\nu)$ ball:*

$$\lambda \sim \nu \lambda_{1i}\,, \quad \lambda_{1i} = \mu_1 - 4\pi\kappa_i + 2\pi\kappa_N - \frac{1}{b}\int_0^\infty \rho w V_{1p}\, d\rho\,, \quad i = 1, \ldots, N-1\,.$$

Here $\mathcal{G}\mathbf{e} = \kappa_N \mathbf{e}$ and $\mathcal{G}\mathbf{q}_i = \kappa_i \mathbf{q}_i$, with $\mathbf{q}_i^T \mathbf{e} = 0$ for $i = 1, \ldots, N-1$. Also, $\mathcal{G}_{ii} = R_i$, and $\mathcal{G}_{ij} = G_{ij}$ for $i \neq j$, where the Neumann $G(\mathbf{x};\xi)$ satisfies

$$\Delta G = \frac{1}{|\Omega_f|} - \delta(\mathbf{x} - \xi)\,, \quad x \in \Omega_f\,; \qquad \partial_n G = 0\,, \quad \mathbf{x} \in \partial\Omega_f\,,$$

$$G \sim -\frac{1}{2\pi}\log|\mathbf{x} - \xi| + R(\xi) + o(1)\,, \quad \text{as} \quad \mathbf{x} \to \xi\,; \quad \int_{\Omega_f} G\, d\mathbf{x} = 0\,.$$

Remarks:

- **Correspondence to periodic problem:** $R_{0p} \to \kappa_N$ and $R_{b0}(\mathbf{k}) \to \kappa_i$, for $i = 1, \ldots, N-1$.
- **Interesting Issue:** Establish the correspondence when $N \to \infty$.

Schnakenburg Model (Main Result)

$$v_t = \varepsilon^2 \Delta v - v + uv^2 \,, \qquad \tau u_t = D\Delta u + a - uv^2 \,; \qquad \text{(Schnakenburg)}\,.$$

Principal Result [IRWW, 2013]: *For* $D \sim \frac{a^2|\Omega|^2}{4\pi^2 b\nu}(1+\mu_1\nu)$ *the portion of the continuous spectrum satisfying* $|\lambda| \leq \mathcal{O}(\nu)$ *is*

$$\lambda = \nu\lambda_1 + \cdots \,, \qquad \lambda_1 = \mu_1 - 2\pi R_{b0} - \frac{1}{b^2}\int_0^\infty \rho V_{1p}\, d\rho \,,$$

where $b = \int_0^\infty w^2 \rho \, d\rho$*, and* $R_{b0} = R_{b0}(\mathbf{k})$ *with* $\mathbf{k}/2\pi \in \Omega_B$*. Thus, a periodic arrangement of spots on* Λ *is linearly stable when*

$$\mu_1 < \mu_1^\star \equiv 2\pi R_{b0}^\star + \frac{1}{b^2}\int_0^\infty V_{1p}\rho\, d\rho \,, \qquad R_{b0}^\star \equiv \min_{\mathbf{k}} R_{b0}(\mathbf{k})\,.$$

The optimal lattice arrangement maximizes the objective function $\mathcal{K}_s \equiv R_{b0}^*$*. The stability threshold on the optimum lattice is*

$$D_{\text{optim}} \sim \frac{a^2|\Omega|^2}{4\pi^2 b\nu}\left[1 + \nu\left(2\pi \max_{\Lambda} \mathcal{K}_s + \frac{1}{b^2}\int_0^\infty V_{1p}\rho\, d\rho\right)\right]\,.$$

Numerical computations yield $b \approx 4.93$ *and* $\int_0^\infty V_{1p}\rho\, d\rho \approx 0.481$.

The Gray-Scott Model

$$v_t = \varepsilon^2 \Delta v - v + A u v^2 \, ; \qquad \tau u_t = D \Delta u + (1-u) - u v^2 \, , \qquad \text{(GS Model)} \, .$$

Fix $D = D_0/\nu$. For $A > A_c$, we have stability wrt competition. Hence, want to minimize A_c for optimal lattice.

Principal Result [IRWW, 2013]: *The optimal lattice arrangement for a steady-state periodic pattern of spots for the GS model in the regime $D = D_0/\nu \gg 1$ and $A = \mathcal{O}(\varepsilon)$ is the one for which $\mathcal{K}_{gs}$ is maximized:*

$$\mathcal{K}_{\text{gs}} \equiv \pi \mu R_{b0}^{\star} - 2\pi R_{0p} \, , \qquad R_{b0}^{\star} \equiv \min_{\mathbf{k}} R_{b0}(\mathbf{k}) \, , \qquad \mu \equiv \frac{2\pi D_0}{|\Omega|} \, .$$

For $\nu = -1/\log \varepsilon \ll 1$, a two-term asymptotic expansion for the competition instability threshold of A on the optimal lattice is

$$A_{\text{optim}} = \varepsilon \sqrt{\frac{2\pi b}{|\Omega| \mu}} \mathcal{A}_{\text{optim}} \, , \qquad b = \int_0^\infty w^2 \rho \, d\rho \, ,$$

$$\mathcal{A}_{\text{optim}} \sim (2+\mu) + \nu \left(-\max_{\Lambda} \mathcal{K}_{\text{gs}} + \frac{1}{b^2} \left(1 - \frac{\mu}{2} \right) \int_0^\infty V_{1p} \rho \, d\rho \right) + \cdots \, ,$$

Here $\max_{\Lambda} \mathcal{K}_{gs}$ is taken over all lattices Λ for which FWS has $|\Omega| = 1$.

Analytical Formula for R_{b0} of Bloch G-Function

Challenge: Infinite series representation of G_{b0} in physical space has poor convergence properties. Challenging to efficiently compute R_{b0}.

Ref: G. Beylkin, C. Kurcz, L. Monzón, *Fast algorithms for Helmholtz Green's functions*, Proc. R. Soc. A, **464**, (2008), pp. 3301-3326.

Introduce cut-off $\eta > 0$ representing "portion" of terms obtained from direct and reciprocal lattice. By using PSF between Λ and Λ^*, we get

$$R_{b0} = \sum_{\boldsymbol{d}\in\Lambda^*} \exp\left(-\frac{|2\pi\boldsymbol{d}-\boldsymbol{k}|^2}{4\eta^2}\right) \frac{1}{|2\pi\boldsymbol{d}-\boldsymbol{k}|^2} + \sum_{\substack{\boldsymbol{l}\in\Lambda \\ \boldsymbol{l}\neq 0}} e^{i\boldsymbol{k}\cdot\boldsymbol{l}}\, F_{\text{sing}}(\boldsymbol{l}) - \frac{\gamma}{4\pi} - \frac{\log\eta}{2\pi},$$

where γ is Euler's constant, $F_{\text{sing}}(\boldsymbol{l}) = E_1(|\boldsymbol{l}|^2\eta^2)/(4\pi)$, and $E_1(z)$ is exponential integral. Need only consider $\mathbf{k}/(2\pi) \in \Omega_B$.

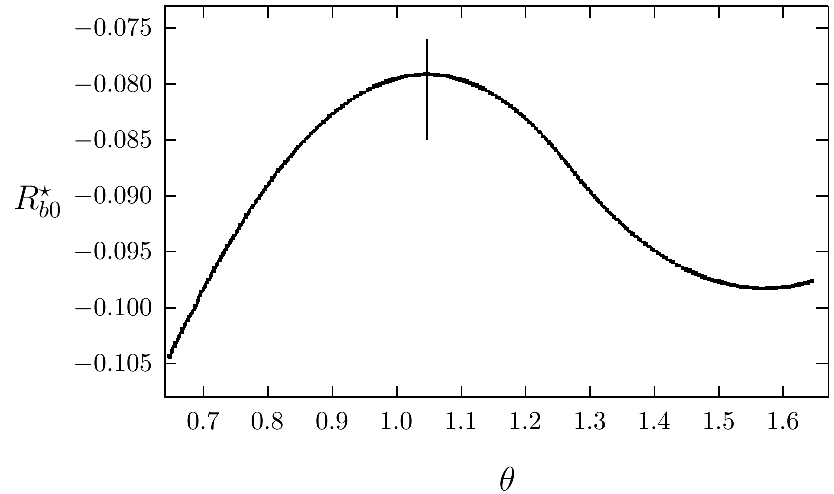
Numerical Computations of R_{b0}

Sweep I: $|\boldsymbol{l}_1| = |\boldsymbol{l}_2|$, with $|\Omega| = |\boldsymbol{l}_1||\boldsymbol{l}_2|\sin\theta = 1$. Let $\boldsymbol{l}_1 = (1/\sqrt{\sin(\theta)}, 0)$ and $\boldsymbol{l}_2 = (\cos(\theta)/\sqrt{\sin(\theta)}, \sqrt{\sin(\theta)})$ and sweep $0 < \theta < \pi/2$.

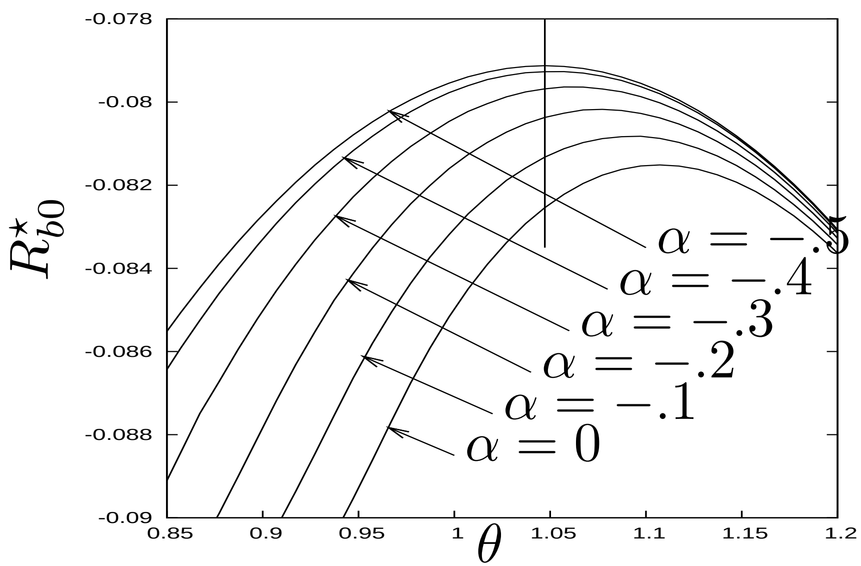
Sweep II: Let $\boldsymbol{l}_1 = (a, 0)$ and $\boldsymbol{l}_2 = (b, c)$, and introduce parameter α. Define $a = (\sin\theta)^{\alpha}$, $c = (\sin\theta)^{-\alpha}$ and $b = \cos\theta\ (\sin\theta)^{-\alpha-1}$. Then, $|\Omega| = 1$. Note: regular hexagon occurs only when $\alpha = -0.5$.

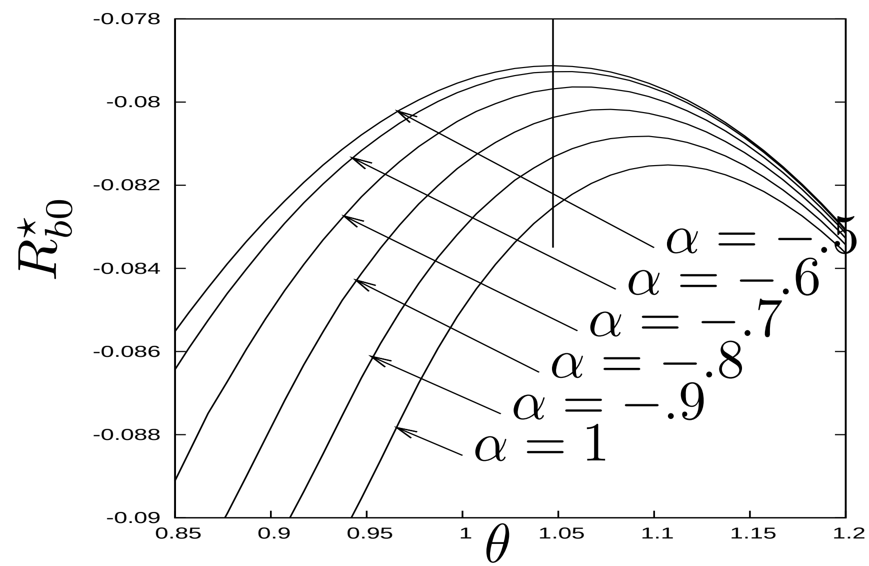
Numerical Computation of Optimal R^*_{b0}

Conjecture (based on numerics): The regular Hexagon maximizes R^*_{b0}.



Sweep I: R^*_{b0} versus θ (above). **Sweep II:** $R^\star_{b0}$ versus θ for various α (below).





Final Comment and Open Issues

Remark: In comparison to the di-block copolymer droplet problem of Chen, Oshita (2007), the optimal lattice is identified not through an energy minimization criterion, but instead from a detailed analysis that determines the spectrum of the linearization near the origin in the spectral plane near a bifurcation point (of D). Similar comment for GL vortices on Abrikisov lattices.

A Few Open Problems:

- Prove that $R_{b0}^{\star}$ is maximized for a regular hexagon.
- Our analysis is based partially on formal asymptotics. Provide a more rigorous proof.
- Calculate stability thresholds for small eigenvalues with $\lambda = \mathcal{O}(\varepsilon^2)$. These are the translation modes. In contrast to our NLEP analysis for competition instabilities, is the right-most edge of the continuous spectrum arising from the long-wavelength limit $|\mathbf{k}| \to 0$?. For periodic patterns in the Turing weakly nonlinear regime, long-wavelength instabilities set stability thresholds (Ref: Callahan, Knobloch, Phys. Rev. E. 2001).