Solutions to Problems in "Asymptotic Methods for PDE Problems in Fluid Mechanics and Related Systems with Strong Localized Perturbations in Two-Dimensional Domains"

M. J. WARD, M. C. KROPINSKI

Michael J. Ward; Department of Mathematics, University of British Columbia, Vancouver, British Columbia, V6T 1Z2, Canada, Mary Catherine Kropinski, Department of Mathematics, Simon Fraser University, Burnaby, British Columbia, V5A IS6, Canada

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1 Problem 1

Problem 1: *Consider a conventional infinite-order logarithmic expansion for the outer solution in the form*

$$W \sim \sum_{j=0}^{\infty} \left(\frac{-1}{\log(\varepsilon d)} \right)^j W_{0j}(\mathbf{x}) + \sigma(\varepsilon) W_1 + \cdots \,, \tag{1.1}$$

with $\sigma(\varepsilon) \ll \nu^k$ for any $k > 0$. By formulating a similar series for the inner solution, derive a recursive set of problems for the W_{0j} for $j \geq 0$ from the asymptotic matching of the inner and outer solutions. Show that this series can be summed and leads to the result in equation (2.17a) of the workshop notes.

Solution:

We consider the pipe flow problem of §2 of the notes, formulated as

$$\triangle w = -\beta \,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon \,, \tag{1.2 a}$$

$$w = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{1.2 b}$$

$$w = 0 \,, \quad \mathbf{x} \in \partial\Omega_\varepsilon \,. \tag{1.2 c}$$

In the outer region we expand the solution to (1.2) in an explicit infinite-order logarithmic expansion as

$$w(\mathbf{x};\varepsilon) = W_{0H}(\mathbf{x}) + \sum_{j=1}^{\infty} \nu^j W_{0j}(\mathbf{x}) + \cdots \,. \tag{1.3}$$

Here $\nu = \mathcal{O}(1/\log\varepsilon)$ is a gauge function to be chosen. The smooth function W_{0H} satisfies the unperturbed problem in the unperturbed domain, given by

$$\triangle W_{0H} = -\beta \,, \quad \mathbf{x} \in \Omega \,; \quad W_{0H} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,. \tag{1.4}$$

By substituting (1.3) into (1.2 a) and (1.2 b), and letting $\Omega_\varepsilon \to \mathbf{x}_0$ as $\varepsilon \to 0$, we get that W_{0j} for $j \geq 1$ satisfies

$$\triangle W_{0j} = 0 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\} \,, \tag{1.5 a}$$

$$W_{0j} = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{1.5 b}$$

$$W_{0j} \quad \text{is singular as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.5 c}$$

The matching of the outer and inner expansions will determine a singularity behavior for W_{0j} as $\mathbf{x} \to \mathbf{x}_0$ for each $j \geq 1$.

In the inner region near Ω_ε we introduce the inner variables

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0) \,, \qquad v(\mathbf{y};\varepsilon) = W(\mathbf{x}_0 + \varepsilon \mathbf{y};\varepsilon) \,. \tag{1.6}$$

We then pose the explicit infinite-order logarithmic inner expansion

$$v(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty} \gamma_j \nu^{j+1} v_c(\mathbf{y}) \,. \tag{1.7}$$

Here γ_j are ε-independent coefficients to be determined. Substituting (1.7) and (1.2 a) and (1.2 c), and allowing $v_c(\mathbf{y})$ to grow logarithmically at infinity, we obtain that $v_c(\mathbf{y})$ satisfies

$$\triangle_{\mathbf{y}} v_c = 0 \,, \quad \mathbf{y} \notin \Omega_1 \,; \qquad v_c = 0 \,, \quad \mathbf{y} \in \partial\Omega_1 \,, \tag{1.8 a}$$

$$v_c \sim \log|\mathbf{y}| \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{1.8 b}$$

The unique solution to (1.8) has the following far-field asymptotic behavior:

$$v_c(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1) \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{1.8 c}$$

Upon using the far-field behavior (1.8 c) in (1.7), and writing the resulting expression in terms of the outer variable $\mathbf{x} - \mathbf{x}_0 = \varepsilon \mathbf{y}$, we obtain that

$$v \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1} \log|\mathbf{x} - \mathbf{x}_0| + \gamma_j \right] \,. \tag{1.9}$$

The matching condition between the infinite-order outer expansion (1.3) as $\mathbf{x} \to \mathbf{x}_0$ and the far-field behavior (1.9) of the inner expansion is that

$$W_{0H}(\mathbf{x}_0) + \sum_{j=1}^{\infty} \nu^j W_{0j}(\mathbf{x}) \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1} \log|\mathbf{x} - \mathbf{x}_0| + \gamma_j \right] \,. \tag{1.10}$$

The leading-order match yields that

$$\gamma_0 = W_{0H}(\mathbf{x}_0) \,. \tag{1.11}$$

The higher-order matching condition, from (1.10), shows that the solution W_{0j} to (1.5) must have the singularity behavior

$$W_{0j} \sim \gamma_{j-1} \log|\mathbf{x} - \mathbf{x}_0| + \gamma_j \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.12}$$

The unknown coefficients γ_j for $j \geq 1$, starting with $\gamma_0 = W_{0H}(\mathbf{x}_0)$ are determined recursively from the infinite sequence of problems (1.5) and (1.12) for $j \geq 1$. The explicit solution to (1.5) with $W_{0j} \sim \gamma_{j-1} \log|\mathbf{x} - \mathbf{x}_0|$ as $\mathbf{x} \to \mathbf{x}_0$ is given explicitly in terms of the Dirichlet Green's function $G_d(\mathbf{x};\mathbf{x}_0)$ by

$$W_{0j}(\mathbf{x}) = -2\pi \gamma_{j-1} G_d(\mathbf{x};\mathbf{x}_0) \,, \tag{1.13}$$

where $G_d(\mathbf{x};\mathbf{x}_0)$ satisfies

$$\triangle G_d = -\delta(\mathbf{x} - \mathbf{x}_0) \,, \quad \mathbf{x} \in \Omega \,; \qquad G_d = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{1.14 a}$$

$$G_d(\mathbf{x};\mathbf{x}_0) = -\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_0| + R_d(\mathbf{x}_0;\mathbf{x}_0) + o(1) \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.14 b}$$

Next, we expand (1.13) as $\mathbf{x} \to \mathbf{x}_0$ and compare it with the required singularity structure (1.12). This yields

$$-2\pi\gamma_{j-1}\left[-\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_0| + R_{d00}\right] \sim \gamma_{j-1}\log|\mathbf{x}-\mathbf{x}_0| + \gamma_j\,, \tag{1.15}$$

where $R_{d00} \equiv R_d(\mathbf{x}_0;\mathbf{x}_0)$. By comparing the non-singular parts of (1.15), we obtain a recursion relation for the γ_j given by

$$\gamma_j = -2\pi R_{d00}\gamma_{j-1}\,, \qquad \gamma_0 = W_{0H}(\mathbf{x}_0)\,, \tag{1.16}$$

which has the explicit solution

$$\gamma_j = \left[-2\pi R_{d00}\right]^j W_{0H}(\mathbf{x}_0)\,, \quad j \geq 0\,. \tag{1.17}$$

Finally, to obtain the outer solution we substitute (1.13) and (1.17) into (1.3) to obtain

$$\begin{aligned} w &\sim W_{0H}(\mathbf{x}) + \sum_{j=1}^{\infty}\nu^j\left(-2\pi\gamma_{j-1}\right)G_d(\mathbf{x};\mathbf{x}_0)\,,\\ &\sim W_{0H}(\mathbf{x}) - 2\pi\nu G_d(\mathbf{x};\mathbf{x}_0)\sum_{j=0}^{\infty}\nu^j\gamma_j\\ &\sim W_{0H}(\mathbf{x}) - 2\pi\nu W_{0H}(\mathbf{x}_0)G_d(\mathbf{x};\mathbf{x}_0)\sum_{j=0}^{\infty}\left[-2\pi\nu R_{d00}\right]^j\\ &\sim W_{0H}(\mathbf{x}_0) - \frac{2\pi\nu W_{0H}(\mathbf{x}_0)}{1+2\pi\nu R_{d00}}G_d(\mathbf{x}_0;\mathbf{x}_0)\,. \end{aligned} \tag{1.18}$$

The last expression (1.18) agrees with equation (2.17a) of the notes. Similarly, upon substituting (1.17) into the infinite-order inner expansion (1.7), we obtain

$$v(\mathbf{y};\varepsilon) = \sum_{j=0}^{\infty}\gamma_j\nu^{j+1}v_c(\mathbf{y}) = \nu W_{0H}(\mathbf{x}_0)v_c(\mathbf{y})\sum_{j=0}^{\infty}\left[-2\pi R_{d00}\nu\right]^j = \frac{\nu W_{0H}(\mathbf{x}_0)}{1+2\pi\nu R_{d00}}v_c(\mathbf{y})\,, \tag{1.19}$$

which recovers equation (2.17b) of the notes.

2 Problem 2

Problem 2: *Consider the following problem in an arbitrary two-dimensional domain with N small inclusions:*

$$\triangle u - m(\mathbf{x})u = 0\,, \qquad \mathbf{x} \in \Omega \backslash \cup_{j=1}^{N}\Omega_{\varepsilon_j}\,, \tag{2.1 a}$$

$$u = f\,, \qquad \mathbf{x} \in \partial\Omega\,. \tag{2.1 b}$$

$$u = \alpha_j\,, \qquad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1,\ldots,N\,, \tag{2.1 c}$$

Here $m(\mathbf{x})$ is an arbitrary smooth function with $m(\mathbf{x}) \geq 0$ in Ω, f is an arbitrary function on $\partial\Omega$, and α_j are constants. Formulate a linear system, similar to equation (3.17) of the workshop notes, in terms of a certain Green's function, that effectively sums the infinite-order logarithmic series in the asymptotic expansion of the solution. Apply your general theory to the unit disk Ω for the case $N = 1$, $m \equiv 1$, $f \equiv 0$, and $\alpha_1 = 1$, and where there is an arbitrarily-shaped hole centered at the origin of the unit disk.

Solution: In the outer region, defined away from Ω_{ε_j} for $j = 1,\ldots,N$, we expand

$$u(\mathbf{x};\varepsilon) \sim U_{0H}(\mathbf{x}) + U_0(\mathbf{x};\boldsymbol{\nu}) + \sigma(\varepsilon)U_1(\mathbf{x};\boldsymbol{\nu}) + \cdots\,. \tag{2.2}$$

Here $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ is a set of logarithmic gauge functions to be determined and $\sigma \ll \nu_j^k$ as $\varepsilon \to 0$ for $j = 1, \ldots, N$. In (2.2), $U_{0H}(\mathbf{x})$ is the smooth function satisfying the unperturbed problem in the unperturbed domain Ω

$$\triangle U_{0H} - m(\mathbf{x}) U_{0H} = 0\,, \quad \mathbf{x} \in \Omega\,; \qquad U_{0H} = f\,, \quad \mathbf{x} \in \partial\Omega\,. \tag{2.3}$$

Substituting (2.2) into (2.1 *a*) and (2.1 *b*), and letting $\Omega_{\varepsilon_j} \to \mathbf{x}_j$ as $\varepsilon \to 0$, we get that U_0 satisfies

$$\triangle U_0 - m(\mathbf{x}) U_0 = 0\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}\,, \tag{2.4 a}$$

$$U_0 = 0\,, \qquad \mathbf{x} \in \partial\Omega\,, \tag{2.4 b}$$

$$U_0 \quad \text{is singular as} \quad \mathbf{x} \to \mathbf{x}_j\,, \quad j = 1, \ldots, N\,. \tag{2.4 c}$$

The singularity behavior for U_0 as $\mathbf{x} \to \mathbf{x}_j$ will be found below by matching the outer solution to the far-field behavior of the inner solution to be constructed near each Ω_{ε_j}.

In the j^{th} inner region near Ω_{ε_j} we introduce the inner variables $\mathbf{y}$ and $v(\mathbf{y};\varepsilon)$ by

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)\,, \qquad v(\mathbf{y};\varepsilon) = u(\mathbf{x}_j + \varepsilon \mathbf{y};\varepsilon)\,. \tag{2.5}$$

We then expand $v(\mathbf{y};\varepsilon)$ as

$$v(\mathbf{y};\varepsilon) = \alpha_j + \nu_j \gamma_j v_{cj}(\mathbf{y}) + \mu_0(\varepsilon) V_{1j}(\mathbf{y}) + \cdots\,, \tag{2.6}$$

where $\gamma_j = \gamma_j(\boldsymbol{\nu})$ is a constant to be determined. Here $\mu_0 \ll \nu_j^k$ as $\varepsilon \to 0$ for any $k > 0$. In (2.6), the logarithmic gauge function ν_j is defined by

$$\nu_j = -1/\log(\varepsilon d_j)\,, \tag{2.7}$$

where d_j is specified below. By substituting (2.5) and (2.6) into (2.1 *a*) and (2.1 *c*), we conclude that $v_{cj}(\mathbf{y})$ is the unique solution to

$$\triangle_{\mathbf{y}} v_{cj} = 0\,, \quad \mathbf{y} \notin \Omega_j\,; \quad v_{cj} = 0\,, \quad \mathbf{y} \in \partial\Omega_j\,, \tag{2.8 a}$$

$$v_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d_j + o(1)\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{2.8 b}$$

Here $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$, and the logarithmic capacitance, d_j, is determined by the shape of Ω_j.

Writing (2.8 *b*) in outer variables and substituting the result into (2.6), we get that the far-field expansion of v away from each Ω_j is

$$v \sim \alpha_j + \gamma_j + \nu_j \gamma_j \log|\mathbf{x} - \mathbf{x}_j|\,, \qquad j = 1, \ldots, N\,. \tag{2.9}$$

Then, by expanding the outer solution (2.2) as $\mathbf{x} \to \mathbf{x}_j$, we obtain the following matching condition between the inner and outer solutions:

$$U_{0H}(\mathbf{x}_j) + U_0 \sim \alpha_j + \gamma_j + \nu_j \gamma_j \log|\mathbf{x} - \mathbf{x}_j|\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j\,, \quad j = 1, \ldots, N\,. \tag{2.10}$$

In this way, we obtain that U_0 satisfies (2.4) subject to the singularity structure

$$U_0 \sim \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu_j \gamma_j \log|\mathbf{x} - \mathbf{x}_j| + o(1)\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j\,, \quad j = 1, \ldots, N\,. \tag{2.11}$$

Observe that in (2.11) both the singular and regular parts of the singularity structure are specified. Therefore, (2.11) will effectively lead to a linear system of algebraic equations for γ_j for $j = 1, \ldots, N$.

The solution to (2.4 *a*) and (2.4 *b*), with $U_0 \sim \nu_j \gamma_j \log|\mathbf{x} - \mathbf{x}_j|$ as $\mathbf{x} \to \mathbf{x}_j$, can be written as

$$U_0(\mathbf{x};\boldsymbol{\nu}) = -2\pi \sum_{i=1}^{N} \nu_i \gamma_i G(\mathbf{x};\mathbf{x}_i)\,, \tag{2.12}$$

where $G(\mathbf{x};\mathbf{x}_j)$ is the Green's function satisfying

$$\triangle G - m(\mathbf{x})G = -\delta(\mathbf{x}-\mathbf{x}_j)\,, \quad \mathbf{x}\in\Omega\,; \quad G=0\,, \quad \mathbf{x}\in\partial\Omega\,, \tag{2.13 a}$$

$$G(\mathbf{x};\mathbf{x}_j) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}_j| + R(\mathbf{x}_j;\mathbf{x}_j) + o(1)\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_j\,. \tag{2.13 b}$$

Here $R_{jj}\equiv R(\mathbf{x}_j;\mathbf{x}_j)$ is the regular part of G.

Finally, we expand (2.12) as $\mathbf{x}\to\mathbf{x}_j$ and equate the resulting expression with the required singularity behavior (2.11) to get

$$\nu_j\gamma_j\log|\mathbf{x}-\mathbf{x}_j| - 2\pi\nu_j\gamma_j R_{jj} - 2\pi\sum_{\substack{i=1\\ i\neq j}}^{N}\nu_i\gamma_i G(\mathbf{x}_j;\mathbf{x}_i) = \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu_j\gamma_j\log|\mathbf{x}-\mathbf{x}_j|\,, \quad j=1,\ldots,N\,. \tag{2.14}$$

In this way, we get the following linear algebraic system for γ_j for $j=1,\ldots,N$:

$$-\gamma_j\left(1+2\pi\nu_j R_{jj}\right) - 2\pi\sum_{\substack{i=1\\ i\neq j}}^{N}\nu_i\gamma_i G_{ji} = \alpha_j - U_{0H}(\mathbf{x}_j)\,, \quad j=1,\ldots,N\,. \tag{2.15}$$

Here $G_{ji}\equiv G(\mathbf{x}_j;\mathbf{x}_i)$ and $\nu_j=-1/\log(\varepsilon d_j)$. We summarize the asymptotic construction as follows:

Principal Result: For $\varepsilon\ll 1$, the outer expansion for (2.1) is

$$u \sim U_{0H}(\mathbf{x}) - 2\pi\sum_{i=1}^{N}\nu_i\gamma_i G(\mathbf{x};\mathbf{x}_i)\,, \quad \text{for} \quad |\mathbf{x}-\mathbf{x}_j| = \mathcal{O}(1)\,. \tag{2.16 a}$$

The inner expansion near Ω_{ε_j} with $\mathbf{y}=\varepsilon^{-1}(\mathbf{x}-\mathbf{x}_j)$, is

$$u \sim \alpha_j + \nu_j\gamma_j v_{cj}(\mathbf{y})\,, \quad \text{for} \quad |\mathbf{x}-\mathbf{x}_j| = \mathcal{O}(\varepsilon)\,. \tag{2.16 b}$$

Here $\nu_j=-1/\log(\varepsilon d_j)$, d_j is defined in (2.8 b), $v_{cj}(\mathbf{y})$ satisfies (2.8), U_{0H} satisfies the unperturbed problem (2.3), while $G(\mathbf{x};\mathbf{x}_j)$ and $R(\mathbf{x}_j;\mathbf{x}_j)$ satisfy (2.13). Finally, the constants γ_j for $j=1,\ldots,N$ are obtained from the N dimensional linear algebraic system (2.15).

To illustrate the theory, let Ω be the unit disk containing one arbitrarily-shaped hole centered at the origin. Suppose that $m(\mathbf{x})=1$ and $f=0$. Then, $U_{0H}\equiv 0$, and the Green's function satisfying (2.13) is radially symmetric with a singularity at the center of the disk. The explicit Green's function is

$$G(\mathbf{x};\mathbf{0}) = \frac{1}{2\pi}\left[K_0(r) - \frac{K_0(1)}{I_0(1)}I_0(r)\right]\,, \quad 0<r<1\,, \tag{2.17}$$

where $r\equiv|\mathbf{x}|$. Here $I_0(r)$ and $K_0(r)$ are the modified Bessel functions of the first and second kind, respectively, of order zero. To identity the regular part of G at the origin, i.e. $R(\mathbf{0};\mathbf{0})$, we use the well-known asymptotics $K_0(r)\sim-\log r+\log 2-\gamma_e$ as $r\to 0$, where γ_e is Euler's constant. Then, from (2.17) and (2.13 b), we get that

$$R_{11}\equiv R(\mathbf{0};\mathbf{0}) = \frac{1}{2\pi}\left[\log 2 - \gamma_e - \frac{K_0(1)}{I_0(1)}\right]\,. \tag{2.18}$$

For $N=1$, $U_{0H}\equiv 0$, and $\alpha_1=1$, the system (2.15) then determines γ_1 in terms of R_{11} and $\nu=-1/\log(\varepsilon d_1)$ as

$$\gamma_1 = -\left[1+2\pi\nu_1 R_{11}\right]^{-1}\,. \tag{2.19}$$

Therefore, γ_1 is determined explicitly in terms of the logarithmic capacitance, d_1, of the arbitrarily-shaped hole centered at the origin.

3 Problem 3

Problem 3: *Consider the following problem in the disk $\Omega = \{\mathbf{x} \,|\, |\mathbf{x}| \leq 2\}$ that contains three small holes:*

$$\triangle u = 0\,, \qquad \mathbf{x} \in \Omega \backslash \cup_{j=1}^{3} \Omega_{\varepsilon_j}\,, \tag{3.1 a}$$

$$u = 4\cos(2\theta)\,, \qquad |\mathbf{x}| = 2\,. \tag{3.1 b}$$

$$u = \alpha_j\,, \qquad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1,2,3\,. \tag{3.1 c}$$

Suppose that each of the holes has an elliptical shape with semi-axes ε and 2ε. Apply the theory for summing infinite logarithmic expansions to first derive and then numerically solve a linear system for the source strengths. In your implementation assume that the holes are centered at $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$ and $\mathbf{x}_3 = (-1/4, 0)$. The boundary values on the holes are to be taken as $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 2$.

Solution: This is just a simple application of the theory in Problem 2 for the special case of a disk of radius 2 with $m(\mathbf{x}) \equiv 0$ and $f = 4\cos(2\theta) = 4(\cos^2\theta - \sin^2\theta) = x^2 - y^2$ on $(x^2+y^2)^{1/2} = 4$.

For this problem, the solution to the unperturbed problem (2.3) is simply

$$U_{0H}(x,y) = x^2 - y^2\,. \tag{3.2}$$

Next, the Green's function satisfying (2.13) of Problem 2 with $m(\mathbf{x}) \equiv 0$ and its regular part are calculated from the method of images as

$$G(\mathbf{x};\mathbf{x}_j) = -\frac{1}{2\pi}\log\left(\frac{2|\mathbf{x}-\mathbf{x}_j|}{|\mathbf{x}-\mathbf{x}_j'||\mathbf{x}_j|}\right)\,, \qquad R_{jj} \equiv R(\mathbf{x}_j;\mathbf{x}_j) = -\frac{1}{2\pi}\log\left[\frac{2}{|\mathbf{x}_j-\mathbf{x}_j'||\mathbf{x}_j|}\right]\,. \tag{3.3}$$

Here $\mathbf{x}_j'$ is the image point of $\mathbf{x}_j$ in the unit disk of radius two..

Next, we note that since each of the holes has an elliptic shape with semi-axes ε and 2ε, then from Table 1 of the notes their common logarithmic capacitance is $d = 3/2$. The holes are assumed to be centered at $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$ and $\mathbf{x}_3 = (-1/4, 0)$, and have the constant boundary values $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 2$.

Therefore, upon defining $\nu = -1/\log(3\varepsilon/2)$ we obtain from (2.15) of Problem 2 that γ_j for $j = 1, \ldots, 3$ is the solution of the linear system

$$-\gamma_1\left[1 + 2\pi\nu R_{11}\right] - 2\pi\nu\left[\gamma_2 G(\mathbf{x}_1;\mathbf{x}_2) + \gamma_3 G(\mathbf{x}_1;\mathbf{x}_3)\right] = 1\,, \tag{3.4 a}$$

$$-\gamma_2\left[1 + 2\pi\nu R_{22}\right] - 2\pi\nu\left[\gamma_1 G(\mathbf{x}_2;\mathbf{x}_1) + \gamma_3 G(\mathbf{x}_2;\mathbf{x}_3)\right] = -1/4\,, \tag{3.4 b}$$

$$-\gamma_3\left[1 + 2\pi\nu R_{33}\right] - 2\pi\nu\left[\gamma_1 G(\mathbf{x}_3;\mathbf{x}_1) + \gamma_2 G(\mathbf{x}_3;\mathbf{x}_2)\right] = 31/16\,. \tag{3.4 c}$$

Here R_{jj} and $G(\mathbf{x}_j;\mathbf{x}_i)$ are to be evaluated from (3.3).

We solve this linear system numerically for γ_j as a function of ε. The curves $\gamma_j(\varepsilon)$ as a function of ε are plotted in Fig. 1. We observe that the leading-order approximation to (3.4), valid for $\nu \ll 1$, is simply $\gamma_1 = -1$, $\gamma_2 = 1/4$ and $\gamma_3 = -31/16$. From Fig. 1 we observe that this approximation, which neglects interaction effects between the holes, is rather inaccurate unless ε is very small.

4 Problem 4

Problem 4: *Beginning with the steady, incompressible, Navier-Stokes equations in velocity-pressure form, derive the problem in equation (4.1) of the notes for the streamfunction for steady viscous flow over an infinite cylinder.*

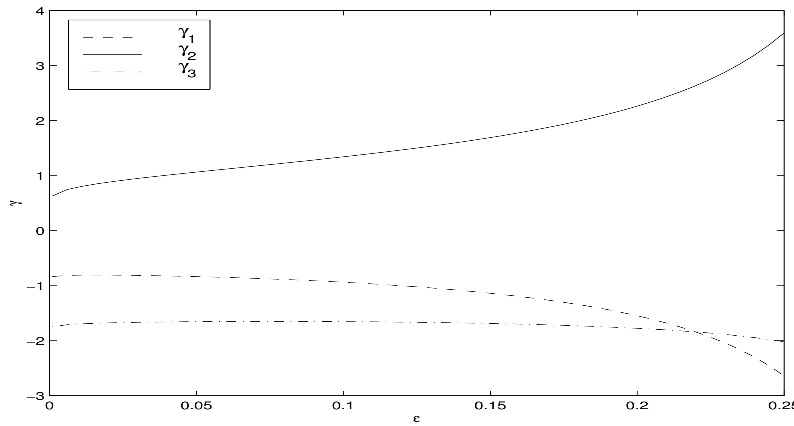

FIGURE 1. Plot of $\gamma_j = \gamma_j(\epsilon)$ for $j = 1, 2, 3$ obtained from the numerical solution to (3.4).

Solution: We begin with the steady-state incompressible dimensionless Navier-Stokes equations in two space dimensions for the velocity $\mathbf{u}(\mathbf{x})$ and the pressure $p(\mathbf{x})$, with $\mathbf{x} = (x_1, x_2, 0)$, given by

$$\nabla \cdot \mathbf{u} = 0\,, \qquad \varepsilon\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla p + \triangle \mathbf{u}\,. \tag{4.1}$$

Lengths are made dimensionless with respect to the radius a of the cross-section of the infinitely-long cylinder, the dynamic viscosity ν, and the free-stream speed u_∞ in the x_1 direction at infinity. The parameter $\varepsilon = u_\infty a/\nu$ is the Reynolds number, and it is assumed to be small. As $|\mathbf{x}| \to \infty$, we have $\mathbf{u} \to \hat{\imath}$, indicating a uniform flow in the x_1 direction. The no slip condition $\mathbf{u} = \mathbf{0}$ is to hold on the body.

To eliminate the pressure, we take the curl of the momentum equation and then use $\nabla \times \nabla p = 0$ to get

$$(\mathbf{u}\cdot\nabla)\,(\nabla\times\mathbf{u}) - [(\nabla\times\mathbf{u})\cdot\mathbf{u}] = \frac{1}{\varepsilon}\triangle\,(\nabla\times\mathbf{u})\,. \tag{4.2}$$

Next, since the flow is two dimensional, we introduce the scalar Lagrangian stream function $\psi(\rho,\theta)$ defined in terms of the velocity components $\mathbf{u} = u_\rho \hat{r} + u_\theta \hat{\theta}$ in the radial and tangential directions by

$$u_\rho = \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\,, \qquad u_\theta = \frac{\partial\psi}{\partial\rho}\,. \tag{4.3}$$

With this choice, the divergence-free condition $\nabla\cdot\mathbf{u} = 0$ holds automatically. Then, by calculating the cross-products in cylindrical coordinates, (4.3) readily transforms to

$$\triangle^2\psi = -\frac{\varepsilon}{\rho}\left[\partial_\rho\psi\partial_\theta\,(\triangle\psi) - \partial_\theta\psi\partial_\rho\,(\triangle\psi)\right]\,. \tag{4.4}$$

which agrees with equation (3.1) of the notes.

Next, we take $\psi = 0$ to correspond to the boundary of the cross-section of the cylinder. Next, if $\mathbf{u} = 0$ on the body, then this corresponds from (4.3) to $\partial_n\psi = 0$, where ∂_n is the outward normal derivative on the body. Finally, if we put $\psi \sim \rho\sin\theta$ as $\rho\to\infty$, then from (4.3), we get $u_r \to \cos\theta$ and $u_\theta \to \sin\theta$ as $r \to \infty$. In terms of cartesian coordinates this corresponds to uniform flow in the x direction at infinity.

5 Problem 5

Problem 5: *Consider the Biharmonic equation in the two-dimensional concentric annulus, formulated as*

$$\triangle^2 u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon\,, \tag{5.1 a}$$

$$u = f\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.1 b}$$

$$u = u_r = 0\,, \quad r = \varepsilon\,. \tag{5.1 c}$$

Here Ω is the unit disk centered at the origin, containing a small hole of radius ε centered at $\mathbf{x} = 0$, i.e. $\Omega_\varepsilon = \{\mathbf{x} \mid |\mathbf{x}| \leq \varepsilon\}$. Consider the following two choices for f: **Case I:** *$f = 1$.* **Case II:** *$f = \sin\theta$. For each of these two cases calculate the exact solution, and from it determine an approximation to the solution in the outer region $|\mathbf{x}| \gg \mathcal{O}(\varepsilon)$. Can you re-derive these results from singular perturbation theory in the limit $\varepsilon \to 0$? (Hint: the leading-order outer problem for Case I is different from what you might expect).*

Solution:

Case I: We consider the perturbed problem

$$\triangle^2 u = 0\,, \quad \varepsilon < r < 1\,, \tag{5.2 a}$$

$$u = 1\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.2 b}$$

$$u = u_r = 0\,, \quad \text{on} \quad r = \varepsilon\,. \tag{5.2 c}$$

We first find the exact solution of (5.2) and then expand it for $\varepsilon \to 0$. Since the radially symmetric solutions to (5.2 a) are linear combinations of $\{r^2, r^2\log r, \log r, 1\}$, we can write the solution to (5.2 a), which satisfies (5.2 b), as

$$u = A\left(r^2 - 1\right) + Br^2\log r - (2A + B)\log r + 1\,, \tag{5.3}$$

for any constants A and B. Then, imposing that $u = u_r = 0$ on $r = \varepsilon$, we get two equations for A and B:

$$2A\left(1 - \varepsilon^2\right)A + B\left(1 - \varepsilon^2 - 2\varepsilon^2\log\varepsilon\right) = 0\,, \tag{5.4 a}$$

$$A\left(1 + 2\log\varepsilon - \varepsilon^2\right) + B\left(1 - \varepsilon^2\right)\log\varepsilon = 1\,. \tag{5.4 b}$$

Equation (5.4 a) gives

$$A = -\frac{B}{2}\left(1 - \frac{2\varepsilon^2\log\varepsilon}{1 - \varepsilon^2}\right)\,. \tag{5.5}$$

Upon substituting this into (5.4 b), we obtain that B satisfies

$$-\frac{B}{2}\left(1 - \frac{2\varepsilon^2\log\varepsilon}{1 - \varepsilon^2}\right)\left(1 + \frac{2\log\varepsilon}{1 - \varepsilon^2}\right) + B\log\varepsilon = \frac{1}{1 - \varepsilon^2} \tag{5.6 a}$$

$$-\frac{B}{2} - \frac{B\log\varepsilon}{1 - \varepsilon^2} + \frac{B\varepsilon^2\log\varepsilon}{1 - \varepsilon^2} + \frac{2\varepsilon^2 B(\log\varepsilon)^2}{(1 - \varepsilon^2)^2} + B\log\varepsilon = \frac{1}{1 - \varepsilon^2} \tag{5.6 b}$$

$$-\frac{B}{2} - B(1 + \varepsilon^2)\log\varepsilon + B\varepsilon^2(1 + \varepsilon^2)\log\varepsilon + 2\varepsilon^2(\log\varepsilon)^2 B + B\log\varepsilon \sim 1 + \varepsilon^2 \tag{5.6 c}$$

$$-\frac{B}{2} + 2\varepsilon^2(\log\varepsilon)^2 B \sim 1 + \mathcal{O}(\varepsilon^2)\,. \tag{5.6 d}$$

The last line of (5.6) determines B, while (5.5) determines A. In this way, we get

$$B \sim -2 - 8\varepsilon^2\left(\log\varepsilon\right)^2\,, \qquad A \sim 1 + 4\varepsilon^2\left(\log\varepsilon\right)^2\,. \tag{5.7}$$

Upon substituting (5.7) into (5.3), we obtain the following two-term expansion in the outer region $r \gg \mathcal{O}(\varepsilon)$:

$$u \sim u_0(r) + \varepsilon^2 (\log \varepsilon)^2 u_1(r) + \cdots , \tag{5.8}$$

where $u_0(r)$ and $u_1(r)$ are defined by

$$u_0(r) = r^2 - 2r^2 \log r , \qquad u_1 = 4\left(r^2 - 1\right) - 8r^2 \log r . \tag{5.9}$$

It is interesting to note that the leading-order outer solution $u_0(r)$ is not a C^2 smooth function, but that it does satisfy the point constraint $u_0(0) = 0$. Hence, in the limit of small hole radius the ε-dependent solution does not tend to the unperturbed solution in the absence of the hole. This unperturbed solution would have $B = 0$ and $A = 0$ in (5.3), and consequently $u = 1$ in the outer region.

Next, we show how to recover (5.8) from a matched asymptotic expansion analysis. In the outer region we expand the solution as

$$u \sim w_0 + \sigma w_1 + \cdots , \tag{5.10}$$

where $\sigma \ll 1$ is an unknown gauge function, and where w_0 satisfies the following problem with a point constraint:

$$\triangle^2 w_0 = 0 , \quad 0 < r < 1 ; \quad w_0(1) = 1 , \quad w_{0r}(1) = 0 , \quad w_0(0) = 0 . \tag{5.11}$$

The solution is readily calculated as

$$w_0 = r^2 - 2r^2 \log r . \tag{5.12}$$

The problem for w_1 is

$$\triangle^2 w_1 = 0 , \quad 0 < r < 1 ; \quad w_1(1) = w_{1r}(1) = 0 . \tag{5.13}$$

The solution to (5.13) is given in terms of unknown coefficients α_1 and β_1 as

$$w_1 = \alpha_1 \left(r^2 - 1\right) + \beta_1 r^2 \log r - (2\alpha_1 + \beta_1) \log r . \tag{5.14}$$

The behavior of w_1 as $r \to 0$, as found below by matching to the inner solution, will determine α_1 and β_1.

In the inner region we set $r = \varepsilon\rho$ and obtain from (5.12) that the terms of order $\mathcal{O}(\varepsilon^2 \log \varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ will be generated in the inner region. Therefore, this suggests that in the inner region we expand the solution as

$$v(\rho) = u(\varepsilon\rho) = \left(\varepsilon^2 \log \varepsilon\right) v_0(\rho) + \varepsilon^2 v_1(\rho) + \cdots . \tag{5.15}$$

The functions v_0 and v_1 must satisfy $v_j(1) = v_{j\rho}(1) = 0$. Therefore, we obtain for $j = 0, 1$ that

$$v_j = A_j \left(\rho^2 - 1\right) + B_j \rho^2 \log \rho - (2A_j + B_j) \log \rho . \tag{5.16}$$

We substitute (5.16) into (5.15), and write the resulting expression in terms of the outer variable $r = \varepsilon\rho$. A short calculation gives that the far-field behavior of (5.15) is

$$v \sim -(\log \varepsilon)^2 B_0 r^2 + (\log \varepsilon)\left[(A_0 - B_1) r^2 + B_0 r^2 \log r\right] + A_1 r^2 + B_1 r^2 \log r + 2A_0 \varepsilon^2 (\log \varepsilon)^2 + \mathcal{O}(\varepsilon^2 \log \varepsilon) . \tag{5.17}$$

In contrast, the two-term outer solution from (5.10), (5.12), and (5.14), is

$$u \sim r^2 - 2r^2 \log r + \sigma \left[\alpha_1 \left(r^2 - 1\right) + \beta_1 r^2 \log r - (2\alpha_1 + \beta_1) \log r\right] + \cdots . \tag{5.18}$$

Upon comparing (5.18) with (5.17), we conclude that

$$B_0 = 0 , \quad B_1 = A_0 , \quad A_1 = 1 , \quad B_1 = -2 , \quad \sigma = \varepsilon^2 (\log \varepsilon)^2 . \tag{5.19}$$

This leaves the unmatched constant term $-4\varepsilon^2(\log\varepsilon)^2$ on the right-hand side of (5.17). Consequently, it follows that the outer correction w_1 is bounded as $r\to 0$ and has the point value $w_1(0)=-4$. Consequently, $2\alpha_1+\beta_1=0$ and $\alpha_1=4$ in (5.18). This gives $\beta_1=-8$, and specifies the second-order term as

$$w_1 = 4\left(r^2-1\right) - 8r^2\log r\,. \tag{5.20}$$

This expression reproduces that obtained in (5.9) from the perturbation of the exact solution.

In Problem 9 below we elaborate on why it is impossible to match to an outer solution u_0 that does not satisfy $u_0(0)=0$. In addition, we further remark that point constraints are possible with the Biharmomic operator, since the free-space Green's function has singularity $\mathcal{O}\left(|\mathbf{x}-\mathbf{x}_0|^2\log|\mathbf{x}-\mathbf{x}_0|\right)$ as $\mathbf{x}\to\mathbf{x}_0$. However, with a point constraint we will not have C^2 smoothness.

Case II: Next, we consider the perturbed problem

$$\triangle^2 u = 0\,, \quad \varepsilon < r < 1\,, \tag{5.21 a}$$

$$u=\sin\theta\,, \quad u_r = 0\,, \quad \text{on} \quad r=1\,, \tag{5.21 b}$$

$$u = u_r = 0\,, \quad \text{on} \quad r=\varepsilon\,. \tag{5.21 c}$$

We first find the exact solution of (5.21) and then expand it for $\varepsilon\to 0$. Since the solutions to (5.21 *a*) proportional to $\sin\theta$ are linear combinations of $\{r^3, r\log r, r, r^{-1}\}\sin\theta$, then we can write the solution to (5.21 *a*), which satisfies (5.21 *b*), as

$$u = \left(Ar^3 + Br\log r + \left(-2A+\frac{1}{2}-\frac{B}{2}\right)r + \left(\frac{1}{2}+A+\frac{B}{2}\right)\frac{1}{r}\right)\sin\theta\,, \tag{5.22}$$

for any constants A and B. Then, imposing that $u=u_r=0$ on $r=\varepsilon$, we get two equations for A and B:

$$A\varepsilon^3 + B\varepsilon\log\varepsilon + \left(-2A+\frac{1}{2}-\frac{B}{2}\right)\varepsilon + \left(\frac{1}{2}+A+\frac{B}{2}\right)\varepsilon^{-1} = 0\,, \tag{5.23 a}$$

$$3A\varepsilon^2 + B + B\log\varepsilon + \left(-2A+\frac{1}{2}-\frac{B}{2}\right) - \left(\frac{1}{2}+A+\frac{B}{2}\right)\varepsilon^{-2} = 0\,. \tag{5.23 b}$$

By comparing the $\mathcal{O}(\varepsilon^{-1})$ and $\mathcal{O}(\varepsilon^{-2})$ terms in (5.23), it follows that

$$\frac{1}{2}+A+\frac{B}{2} = \kappa\varepsilon^2\,, \tag{5.24}$$

where κ is an $\mathcal{O}(1)$ constant to be found. Substituting (5.24) into (5.23), and neglecting the higher order $A\varepsilon^3$ and $3A\varepsilon^2$ terms in (5.23), we obtain the approximate system

$$B\log\varepsilon + \left(-2A+\frac{1}{2}-\frac{B}{2}\right) \approx -\kappa\,, \qquad B + B\log\varepsilon + \left(-2A+\frac{1}{2}-\frac{B}{2}\right) \approx \kappa\,. \tag{5.25}$$

By adding the two equations above to eliminate κ, we obtain that

$$B + 2B\log\varepsilon + (-4A+1-B) = 0\,. \tag{5.26}$$

From (5.26), together with $A\sim -(1+B)/2$ from (5.24), we obtain that

$$B\sim \frac{3\nu}{2-\nu}\,, \qquad A = 1-\frac{3}{2-\nu}\,, \quad \text{where} \quad \nu\equiv\frac{-1}{\log\left[\varepsilon e^{1/2}\right]}\,. \tag{5.27}$$

Finally, substituting (5.27) into (5.22), we obtain that the outer solution has the asymptotics

$$u\sim\left((1-\tilde{A})r^3 + \nu\tilde{A}r\log r + \tilde{A}r\right)\sin\theta\,, \quad r\gg\mathcal{O}(\varepsilon)\,. \tag{5.28 a}$$

where $\tilde{A}$ is defined by

$$\tilde{A} \equiv \frac{3}{2-\nu}\,, \qquad \nu \equiv \frac{-1}{\log\left[\varepsilon e^{1/2}\right]}\,. \tag{5.28 b}$$

We remark that (5.28) is an infinite-order logarithmic series approximation to the exact solution. However, it does not contain transcendentally small terms of algebraic order in ε as $\varepsilon \to 0$.

Next, we show how to derive (5.28) by employing the hybrid formulation used in the low Reynolds number flow problem of §4.

In order to sum the infinite logarithmic series we formulate a hybrid method by following equations (4.13)–(4.15) of the workshop notes. In the inner region, with inner variable $\rho \equiv \varepsilon^{-1} r$, we look for an inner solution in the form (see equations (4.14) and (4.21) of the notes)

$$v(\rho,\theta) = u(\varepsilon\rho,\theta) \sim \varepsilon\nu\tilde{A}(\nu)\left(\rho\log\rho - \frac{\rho}{2} + \frac{1}{2\rho}\right)\sin\theta\,. \tag{5.29}$$

Here $\nu \equiv -1/\log\left[\varepsilon e^{1/2}\right]$ and $\tilde{A} \equiv \tilde{A}(\nu)$ is a function of ν to be found. The extra factor of ε in (5.29) is needed since the solution in the outer region is not algebraically large as $\varepsilon \to 0$. Now letting $\rho \to \infty$, and writing (5.29) in terms of the outer variable $r = \varepsilon\rho$, we obtain that the far-field form of (5.29) is

$$v \sim \left(\tilde{A}\nu r\log r + \tilde{A} r\right)\sin\theta\,. \tag{5.30}$$

Therefore, the approximate outer hybrid solution w_H to (5.21) that sums all the logarithmic terms must satisfy

$$\triangle^2 w_H = 0\,, \quad 0 < r < 1\,, \tag{5.31 a}$$

$$w_H = \sin\theta\,, \quad w_{Hr} = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.31 b}$$

$$w_H \sim \left(\tilde{A}\nu r\log r + \tilde{A} r\right)\sin\theta\,, \quad \text{as} \quad r \to 0\,. \tag{5.31 c}$$

The solution to (5.31 a) and (5.31 b) has the explicit given in

$$w_H = \left(\alpha r^3 + \beta r\log r + \left(-2\alpha + \frac{1}{2} - \frac{\beta}{2}\right) r + \left(\frac{1}{2} + \alpha + \frac{\beta}{2}\right)\frac{1}{r}\right)\sin\theta\,. \tag{5.32}$$

The condition (5.31 c) then yields the three equations

$$\beta = \tilde{A}\nu\,, \qquad -2\alpha + \frac{1}{2} - \frac{\beta}{2} = \tilde{A}\,, \qquad \frac{1}{2} + \alpha + \frac{\beta}{2} = 0\,, \tag{5.33}$$

for α, β, and $\tilde{A}$. We solve this system to obtain

$$\beta = \tilde{A}\nu\,, \qquad \tilde{A} = \frac{3}{2-\nu}\,, \qquad \alpha = 1 - \tilde{A}\,. \tag{5.34}$$

Upon substituting (5.34) into (5.32), we obtain that the resulting expression agrees exactly with the result (5.28) obtain from the asymptotics of the exact solution.

This simple example of Case II has shown explicitly, without numerical methods, that the hybrid asymptotic numerical method for summing infinite logarithmic expansions agrees with the results that can be obtained from the exact solution.

6 Problem 6

Problem 6: *Consider the following convection-diffusion equation for $T(\mathbf{X})$, with $\mathbf{X} = (X_1, X_2)$ posed outside two circular disks Ω_j for $j = 1, 2$ of a common radius a, and with a center-to-center separation $2L$ between the two disks:*

$$\kappa \triangle T = \mathbf{U} \cdot \nabla T \,, \qquad \mathbf{X} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} \Omega_j \,, \tag{6.1 a}$$

$$T = T_j \,, \qquad \mathbf{X} \in \partial \Omega_j \,, \quad j = 1, 2 \,, \tag{6.1 b}$$

$$T \sim T_\infty \,, \qquad |\mathbf{X}| \to \infty \,. \tag{6.1 c}$$

Here $\kappa > 0$ is constant, T_j for $j = 1, 2$ and T_∞ are constants, and $\mathbf{U} = \mathbf{U}(\mathbf{X})$ is a given bounded flow field with $\mathbf{U}(\mathbf{X}) \to (\mathbf{U}_\infty, \mathbf{0})$ as $|\mathbf{X}| \to \infty$, where U_∞ is constant.

- *Non-dimensionalize (6.1) in terms of U_∞ and the length-scale $\gamma = \kappa/U_\infty$ to derive a convection-diffusion equation outside of two circular disks of radii $\varepsilon \equiv U_\infty a/\kappa$, with inter-disk separation $2L\varepsilon/a$. Here ε is the Peclet number.*
- *In the low Peclet number limit $\varepsilon \to 0$ show how a hybrid asymptotic-numerical solution can be implemented to sum the infinite logarithmic expansions for two different distinguished limits:* **Case 1:** *$L/a = \mathcal{O}(1)$.* **Case 2:** *$L/a = \mathcal{O}(\varepsilon^{-1})$. For Case 1, we require an explicit formula for the logarithmic capacitance, d, of two disks of a common radius, a, and with a center-to-center separation of $2l$. By using bipolar coordinates, the result for d is (see the workshop notes for the precise reference)*

$$\log d = \log (2\beta) - \frac{\xi_c}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\xi_c}}{m \cosh(m\xi_c)} \,, \tag{6.2}$$

where β and ξ_c are determined in terms of a and l by

$$\beta = \sqrt{l^2 - a^2} \,; \qquad \xi_c = \log \left[\frac{l}{a} + \sqrt{\left(\frac{l}{a}\right)^2 - 1} \right] \,. \tag{6.3}$$

- *For a uniform flow with $\mathbf{U} = (U_\infty, 0)$ for $\mathbf{X} \in \mathbb{R}^2$, determine the required Green's function and its regular part.*

Solution:

We introduce the dimensionless variables $\mathbf{x}$, $\mathbf{u}(\mathbf{x})$, and $w(\mathbf{x})$ by

$$\mathbf{x} = \mathbf{X}/\gamma \,, \quad T = T_\infty w \,, \quad u(\mathbf{x}) = \mathbf{U}(\gamma \mathbf{x})/U_\infty \,, \quad \gamma \equiv \kappa/U_\infty \,. \tag{6.4}$$

We define the dimensionless centers of the two circular disks by $\mathbf{x}_j$ for $j = 1, 2$, and their constant boundary temperatures α_j for $j = 1, 2$, by

$$\mathbf{x}_j = \mathbf{X}_j/\gamma \,, \qquad \alpha_j = w_j/T_\infty \,, \qquad j = 1, 2 \,. \tag{6.5}$$

Then, (6.1) transforms in dimensionless form to

$$\triangle w = \mathbf{u} \cdot \nabla w \,, \qquad \mathbf{x} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_{\varepsilon j} \,, \tag{6.6 a}$$

$$w = \alpha_j \,, \qquad \mathbf{x} \in \partial D_{\varepsilon j} \,, \quad j = 1, 2 \,, \tag{6.6 b}$$

$$w \sim 1 \,, \qquad |\mathbf{x}| \to \infty \,. \tag{6.6 c}$$

Here $D_{\varepsilon j} = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_j| \leq \varepsilon\}$ is the circular disk of radius ε centered at $\mathbf{x}_j$. The center-to-center separation is

$$|\mathbf{x}_2 - \mathbf{x}_1| = 2l\varepsilon\,, \qquad l \equiv L/a\,. \tag{6.7}$$

The dimensionless flow has limiting behavior $\mathbf{u} \sim (1,0)$ as $|\mathbf{x}| \to \infty$.

<u>Case 1:</u> We assume that $l = \mathcal{O}(1)$ as $\varepsilon \to 0$, so that $|\mathbf{x}_2 - \mathbf{x}_1| = \mathcal{O}(\varepsilon)$. This is the case where the bodies are close together. It leads below to a new inner problem, not considered previously in the notes.

We assume without loss of generality that $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}$. We then introduce the inner variables $\mathbf{y}$ and $v(\mathbf{y})$ by

$$\mathbf{y} = \varepsilon^{-1}\mathbf{x}\,, \qquad v(\mathbf{y}) = w(\varepsilon \mathbf{y})\,. \tag{6.8}$$

Then, we obtain that (6.6 a) and (6.6 b) transform to

$$\triangle_{\mathbf{y}} v = \varepsilon \mathbf{u}_0 \cdot \nabla_{\mathbf{y}} v\,, \qquad \mathbf{y} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_j\,, \tag{6.9 a}$$

$$v = \alpha_j\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1,2\,, \tag{6.9 b}$$

Here $D_j = \{\mathbf{y} \mid |\mathbf{y} - \mathbf{y}_j| \leq 1\}$ is the circular disk centered at $\mathbf{y}_j = \mathbf{x}_j/\varepsilon$ of radius one, and $\mathbf{u}_0 \equiv \mathbf{u}(\mathbf{0})$. The inter-disk separation is

$$|\mathbf{y}_2 - \mathbf{y}_1| = 2l\,. \tag{6.10}$$

We then look for a solution to (6.9) in the form

$$v = v_0 + \nu A v_c\,, \tag{6.11}$$

where $\nu = \mathcal{O}(-1/\log \varepsilon)$ and $A = A(\nu)$ is to be found. Here v_0 is the solution to

$$\triangle_{\mathbf{y}} v_0 = 0\,, \qquad \mathbf{y} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_j\,, \tag{6.12 a}$$

$$v_0 = \alpha_j\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1,2\,, \tag{6.12 b}$$

$$v_0 \text{ bounded as } |\mathbf{y}| \to \infty\,. \tag{6.12 c}$$

Moreover, $v_c(\mathbf{y})$ is the solution to

$$\triangle_{\mathbf{y}} v_c = 0\,, \qquad \mathbf{y} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_j\,, \tag{6.13 a}$$

$$v_c = 0\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1,2\,, \tag{6.13 b}$$

$$v_c \sim \log|\mathbf{y}|\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{6.13 c}$$

Since D_j for $j = 1,2$ are non-overlapping circular disks, the problem (6.12) can be solved explicitly using conformal mapping and the introduction of symmetric points. In this way, we can derive that

$$v_0 \sim v_{0\infty} + o(1)\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{6.14}$$

The simple calculation of $v_{0\infty}$ is omitted. When $\alpha_1 = \alpha_2 = \alpha_c$, then clearly $v_{0\infty} = \alpha_1$. Next, we can solve (6.13) exactly by introducing bipolar coordinates as suggested in the hint, which was motivated by Appendix B of Reference [10] of the workshop notes. In this way, we calculate that

$$v_c(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1)\,, \quad |\mathbf{y}| \to \infty\,, \tag{6.15}$$

where d is given by setting $a = 1$ in (6.2) and (6.3). Therefore, in this analysis we have neglected the transcendentally small $\mathcal{O}(\varepsilon)$ term in (6.9), representing a weak drift in the inner region.

Upon substituting (6.14) and (6.15) into (6.11), and writing $\mathbf{y} = \varepsilon^{-1}\mathbf{x}$, we obtain in terms of outer variables that the far-field behavior of v is

$$v \sim v_{0\infty} + A + \nu A \log|\mathbf{x}|\,, \quad \nu \equiv \frac{-1}{\log(\varepsilon d)}\,. \tag{6.16}$$

The behavior (6.16) is the singularity behavior for the infinite-logarithmic series approximation $V_0(\mathbf{x};\mu)$ to the outer solution as $\mathbf{x} \to \mathbf{0}$. This approximation satisfies

$$\triangle V_0 = \mathbf{u}\cdot\nabla V_0\,, \qquad \mathbf{x} \in \mathbb{R}^2\backslash\{\mathbf{0}\}\,; \qquad V_0 \sim 1\,, \quad |\mathbf{x}| \to \infty\,, \tag{6.17}$$

with singularity behavior (6.16) as $\mathbf{x} \to \mathbf{0}$.

To solve this problem we introduce the Green's function $G(\mathbf{x};\boldsymbol{\xi})$ satisfying

$$\triangle G = \mathbf{u}\cdot\nabla G - \delta(\mathbf{x}-\boldsymbol{\xi})\,, \quad \mathbf{x} \in \mathbb{R}^2\,, \tag{6.18 a}$$

$$G(\mathbf{x};\boldsymbol{\xi}) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}| + R(\boldsymbol{\xi};\boldsymbol{\xi}) + o(1)\,, \quad \mathbf{x} \to \boldsymbol{\xi}\,, \tag{6.18 b}$$

with $G(\mathbf{x};\boldsymbol{\xi}) \to 0$ as $|\mathbf{x}| \to \infty$. Here $R(\boldsymbol{\xi};\boldsymbol{\xi})$ is the regular part of this Green's function at $\mathbf{x} = \boldsymbol{\xi}$.

The solution to (6.17) with singular behavior $V_0 \sim \nu A \log|\mathbf{x}|$ as $\mathbf{x} \to \mathbf{0}$ is

$$V_0 = 1 - 2\pi\nu A G(\mathbf{x};\mathbf{0})\,. \tag{6.19}$$

By expanding (6.19) as $\mathbf{x} \to 0$, and equating the regular part of the resulting expression with that in (6.16), we get $1 - 2\pi\nu A R_{00} = A + v_{0\infty}$. This determines $A = A(\nu)$ by

$$A = \frac{1 - v_{0\infty}}{1 + 2\pi\nu R_{00}}\,, \qquad \nu \equiv \frac{-1}{\log(\varepsilon d)}\,, \tag{6.20}$$

where $R_{00} \equiv R(\mathbf{0};\mathbf{0})$. The outer and inner solutions are then given in terms of A. Finally, one can calculate the Nusselt number, representing the average heat flux across the bodies, by using the divergence theorem together with the form (6.16) of the far-field behavior in the inner region.

Case 2: We assume that $l = \mathcal{O}(\varepsilon^{-1})$ as $\varepsilon \to 0$, and define $l = l_0/\varepsilon$ with $l_0 = \mathcal{O}(1)$, so that $|\mathbf{x}_2 - \mathbf{x}_1| = 2l_0$. This is the case where the small disks of radius ε are separated by $\mathcal{O}(1)$ distances in (6.6). In the analysis there are two distinct inner regions; one near $\mathbf{x}_1$ and the other at an $\mathcal{O}(1)$ distance away centered at $\mathbf{x}_2$. Since each separated disk is a circle of radius ε, it has a logarithmic capacitance $d = 1$. Therefore, the infinite-logarithmic series approximation $V_0(\mathbf{x};\mu)$ to the outer solution satisfies

$$\triangle V_0 = \mathbf{u}\cdot\nabla V_0\,, \qquad \mathbf{x} \in \mathbb{R}^2\backslash\{\mathbf{0}\}\,; \qquad V_0 \sim 1\,, \quad |\mathbf{x}| \to \infty\,, \tag{6.21 a}$$

$$V_0 \sim \alpha_j + A_j + \nu A_j \log|\mathbf{x}-\mathbf{x}_j|\,, \quad \nu \equiv \frac{-1}{\log\varepsilon}\,. \tag{6.21 b}$$

The solution to (6.21) is given explicitly by

$$V_0 = 1 - 2\pi\nu\sum_{i=1}^{2} A_i G(\mathbf{x};\mathbf{x}_i)\,. \tag{6.22}$$

We then let $\mathbf{x} \to \mathbf{x}_j$ for $j = 1, 2$ in (6.22) and equate the nonsingular part of the resulting expression with the regular part of the singularity structure in (6.21 b). This yields that A_1 and A_2 satisfy the linear algebraic system

$$A_1\,(1 + 2\pi\nu R_{11}) + 2\pi\nu A_2 G_{12} = 1 - \alpha_1\,; \qquad A_2\,(1 + 2\pi\nu R_{22}) + 2\pi\nu A_1 G_{21} = 1 - \alpha_2\,. \tag{6.23}$$

Here $G_{ij} = G(\mathbf{x}_j;\mathbf{x}_i)$ and $R_{jj} = R(\mathbf{x}_j;\mathbf{x}_j)$ are the Green's function and its regular part as defined by (6.18).

Finally, we remark that for the case of a uniform flow where $\mathbf{u} = (1, 0)$, then the explicit solution to (6.18) is

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi} \exp\left[\frac{x_1 - \xi_1}{2}\right] K_0\left(|\mathbf{x} - \boldsymbol{\xi}|\right) , \tag{6.24 a}$$

where $\mathbf{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. By letting $\mathbf{x} \to \boldsymbol{\xi}$, and using $K_0(r) \sim -\log r + \log 2 - \gamma_e$, as $r \to 0^+$, where γ_e is Euler's constant, we readily calculate that

$$R(\boldsymbol{\xi}, \boldsymbol{\xi}) = \frac{1}{2\pi}\left(\log 2 - \gamma_e\right) . \tag{6.24 b}$$

These results for G and its regular part can be used in the results of either (6.20) or (6.23) for Case I or Case II, respectively.

7 Problem 7

Problem 7: *Let Ω be the unit disk containing one arbitrarily-shaped hole Ω_ε centered at the origin. Our goal is to calculate the principal eigenvalue of*

$$\Delta u + \lambda u = 0 , \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon ; \quad \int_{\Omega\backslash\Omega_\varepsilon} u^2 \, d\mathbf{x} = 1 , \tag{7.1 a}$$

$$u = 0 , \quad \mathbf{x} \in \partial\Omega ; \qquad u = 0 , \quad \mathbf{x} \in \partial\Omega_\varepsilon . \tag{7.1 b}$$

This eigenvalue problem is slightly different from the problem (5.1) of the notes in that here we pose the Dirichlet condition $u = 0$ on $\partial\Omega$. For (7.1), derive an explicit transcendental equation for the infinite-order logarithmic series approximation to the principal eigenvalue.

Solution:

The analysis in §5 of the notes can be repeated, and we readily obtain equation (5.12) of the notes. Thus, the infinite-order logarithmic series approximation $\lambda^\star$ to the principal eigenvalue λ satisfies the transcendental equation

$$R_h(\mathbf{x}_0; \mathbf{x}_0, \lambda^*) = -\frac{1}{2\pi\nu} , \qquad \nu = -\frac{1}{\log(\varepsilon d)} , \tag{7.2}$$

where $R_h(\mathbf{x}_0; \mathbf{x}_0, \lambda^*)$ is the regular part of the Helmholtz Green's function, satisfying

$$\Delta G_h + \lambda^* G_h = -\delta(\mathbf{x} - \mathbf{x}_0) , \quad \mathbf{x} \in \Omega ; \quad G_h = 0 , \quad \mathbf{x} \in \partial\Omega , \tag{7.3 a}$$

$$G_h(\mathbf{x}; \mathbf{x}_0, \lambda^*) \sim -\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_0| + R_h(\mathbf{x}_0; \mathbf{x}_0, \lambda^*) + o(1) , \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 . \tag{7.3 b}$$

Notice that $G_h = 0$ on $\partial\Omega$. Since the hole is centered at the origin then $\mathbf{x}_0 = \mathbf{0}$.

When Ω is the unit disk with a hole centered at the origin, then (7.3) becomes a radially symmetric problem whose solution can be found explicitly. A simple calculation gives

$$G = -\frac{1}{4}\left[Y_0\left(\sqrt{\lambda^\star} r\right) - \frac{Y_0\left(\sqrt{\lambda^\star}\right)}{J_0\left(\sqrt{\lambda^\star}\right)} J_0\left(\sqrt{\lambda^\star} r\right)\right] , \quad 0 < r < 1 , \tag{7.4}$$

where $r = |\mathbf{x}|$. Here $J_0(z)$ and $Y_0(z)$ are the Bessel functions of the first and second kind, of order zero. By using the well-known asymptotic behavior $Y_0(z) \sim 2\pi^{-1}\left[\log z - \log 2 + \gamma_e + o(1)\right]$ and $J_0(z) \sim 1 + o(1)$ as $z \to 0^+$, we obtain from (7.4) that the local behavior for G as $\mathbf{x} \to 0$ is given by

$$G(\mathbf{x};\mathbf{0}) \sim -\frac{1}{2\pi}\log|\mathbf{x}| + R_h + o(1)\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{0}\,, \tag{7.5 a}$$

$$R_h \equiv -\frac{1}{2\pi}\left(-\log 2 + \gamma_e + \log\left(\sqrt{\lambda^\star}\right)\right) + \frac{1}{4}\left(\frac{Y_0\left(\sqrt{\lambda^\star}\right)}{J_0\left(\sqrt{\lambda^\star}\right)}\right)\,, \tag{7.5 b}$$

where γ_e is Euler's constant. Finally, upon substituting (7.5 *b*) for R_h into (7.2), we conclude that $\lambda^\star(\varepsilon d)$ is the first root of the transcendental equation

$$\log 2 - \gamma_e - \log\left(\sqrt{\lambda^\star}\right) + \frac{\pi}{2}\left(\frac{Y_0\left(\sqrt{\lambda^\star}\right)}{J_0\left(\sqrt{\lambda^\star}\right)}\right) = -\frac{1}{\nu} = \log(\varepsilon d)\,. \tag{7.6}$$

Here d is the logarithmic capacitance of the arbitrarily-shaped hole centered at the origin of the unit disk.

It is interesting to note that the result (7.6) can also be obtained by first finding the exact eigenvalue relation for the concentric annulus $\varepsilon < |\mathbf{x}| < 1$ with $u = 0$ on $|\mathbf{x}| = \varepsilon$ and on $|\mathbf{x}| = 1$, and then letting $\varepsilon \to 0$ in this resulting expression. The eigenfunction is proportional to

$$u = \left[J_0\left(\sqrt{\lambda}r\right) - \frac{J_0\left(\sqrt{\lambda}\right)}{Y_0\left(\sqrt{\lambda}\right)}Y_0\left(\sqrt{\lambda}r\right)\right]\,, \quad 0 < r < 1\,, \tag{7.7}$$

and upon setting $u = 0$ at $r = \varepsilon$, we get the eigenvalue relation

$$Y_0\left(\sqrt{\lambda}\varepsilon\right) = J_0\left(\sqrt{\lambda}\varepsilon\right)\frac{Y_0\left(\sqrt{\lambda}\right)}{J_0\left(\sqrt{\lambda}\right)}\,. \tag{7.8}$$

Next, in (7.8) we use the small argument expansions of $Y_0(z)$ and $J_0(z)$ as $z \to 0^+$, and then, finally, replace ε by εd in the resulting expression by recalling Kaplun's equivalence principle. In this way, we readily recover the transcendental equation (7.6) for the approximation $\lambda^\star$ to λ.

8 Problem 8

Problem 8: *For the eigenvalue problem (5.1) of the notes, consider the special case of K holes that have a common logarithmic capacitance $d = d_1 = \ldots, d_K$. By introducing two-term expansions directly in equation (5.1) of the notes for the eigenvalue and the outer and inner approximations to the eigenfunction, re-derive the two-term approximation in equation (5.27) of the Corollary.*

Solution: We write the eigenvalue problem as

$$\triangle u + \lambda u = 0\,, \quad \mathbf{x} \in \Omega\backslash\Omega_p\,; \qquad \Omega_p \equiv \cup_{j=1}^{K}\Omega_{\varepsilon_j}\,, \tag{8.1 a}$$

$$\partial_n u = 0\,, \quad \mathbf{x}\in\partial\Omega\,; \qquad \int_{\Omega\backslash\Omega_p} u^2\,d\mathbf{x} = 1 \tag{8.1 b}$$

$$u = 0\,, \quad \mathbf{x}\in\partial\Omega_{\varepsilon_j}\,, \quad j = 1,\ldots,N\,. \tag{8.1 c}$$

We assume that each hole Ω_{ε_j} is centered at $\mathbf{x}_j \in \Omega$ and has the same logarithmic capacitance d.

We look for a two-term expansion for the principal eigenvalue $\lambda_0(\varepsilon)$ as

$$\lambda_0(\varepsilon) = \lambda_1\nu + \lambda_2\nu^2 + \cdots\,, \qquad \nu = -1/\log(\varepsilon d)\,. \tag{8.2}$$

In the outer region, away from $\mathcal{O}(\varepsilon)$ neighborhoods of the holes, we expand the outer solution for u as

$$u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots \,. \tag{8.3}$$

The leading-order term is

$$u_0 = |\Omega|^{-1/2} \,, \tag{8.4}$$

where $|\Omega|$ is the area of Ω. Upon substituting (8.2) and (8.3) into (8.1 a) and (8.1 b), and collecting powers of ν, we obtain that u_1 satisfies

$$\triangle u_1 = -\lambda_1 u_0 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_K\} \,; \qquad \int_\Omega u_1 \, d\mathbf{x} = 0 \,, \tag{8.5 a}$$

$$\partial_n u_1 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad u_1 \text{ singular as } \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,, \tag{8.5 b}$$

while u_2 satisfies

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_K\} \,; \qquad \int_\Omega \left(u_1^2 + 2u_0 u_2\right) \, d\mathbf{x} = 0 \,, \tag{8.6 a}$$

$$\partial_n u_2 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad u_2 \text{ singular as } \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,. \tag{8.6 b}$$

Now in the j^{th} inner region we introduce the new variables by

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j) \,, \qquad v(\mathbf{y}) = u(\mathbf{x}_j + \varepsilon \mathbf{y}) \,. \tag{8.7}$$

We then expand the inner solution as

$$v(\mathbf{y}) = \nu A_{0j} v_{cj}(\mathbf{y}) + \nu^2 A_{1j} v_{cj}(\mathbf{y}) + \cdots \,. \tag{8.8}$$

Upon substituting (8.7) and (8.8) into (8.1 a) and (8.1 c), we obtain that v_{cj} satisfies

$$\triangle_{\mathbf{y}} v_{cj} = 0 \,, \quad \mathbf{y} \notin \Omega_j \,; \quad v_{cj} = 0 \,, \quad \mathbf{y} \in \partial\Omega_j \,, \tag{8.9 a}$$

$$v_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1) \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{8.9 b}$$

Here $\triangle_{\mathbf{y}}$ is the Laplacian in the $\mathbf{y}$ variable, and $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$. We consider the special case where d is independent of j.

Upon using the far-field form (8.9 b) in (8.8), and writing the resulting expression in outer variables, we get

$$v = A_{0j} + \nu \left[A_{0j} \log|\mathbf{x} - \mathbf{x}_j| + A_{1j}\right] + \nu^2 \left[A_{1j} \log|\mathbf{x} - \mathbf{x}_j| + A_{2j}\right] + \cdots \,. \tag{8.10}$$

The far-field behavior (8.10) must agree with the local behavior of the outer expansion (8.3). Therefore, we obtain that

$$A_{0j} = u_0 = |\Omega|^{-1/2} \,, \quad j = 1, \ldots K \,, \tag{8.11 a}$$

$$u_1 \sim u_0 \log|\mathbf{x} - \mathbf{x}_j| + A_{1j} \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,, \tag{8.11 b}$$

$$u_2 \sim A_{1j} \log|\mathbf{x} - \mathbf{x}_j| + A_{2j} \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,. \tag{8.11 c}$$

Equations (8.11 b) and (8.11 c) give the required singularity structure for u_1 and u_2 in (8.5) and (8.6), respectively.

The problem for u_1 with singular behavior (8.11 b) can be written in terms of the delta function as

$$\triangle u_1 = -\lambda_1 u_0 + 2\pi A_0 \sum_{j=1}^{K} \delta(\mathbf{x}-\mathbf{x}_j)\,, \quad \mathbf{x}\in\Omega\,; \qquad \int_\Omega u_1\, d\mathbf{x} = 0\,, \tag{8.12 a}$$

$$\partial_n u_1 = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{8.12 b}$$

Upon using the divergence theorem we obtain that $-\lambda_1 u_0 \int_\Omega 1 d\mathbf{x} + 2\pi A_0 K = 0$, so that with $u_0 = A_0$ from (8.11 a), we get

$$\lambda_1 = \frac{2\pi K}{|\Omega|}\,. \tag{8.13}$$

The solution to (8.12) can be written in terms of the Neumann Green's function as

$$u_1 = -2\pi u_0 \sum_{i=1}^{K} G_N(\mathbf{x};\mathbf{x}_i)\,, \tag{8.14}$$

where the Neumann Green's function $G_N(\mathbf{x};\boldsymbol{\xi})$ satisfies

$$\Delta G_N = \frac{1}{|\Omega|} - \delta(\mathbf{x}-\boldsymbol{\xi})\,, \quad \mathbf{x}\in\Omega\,; \quad \partial_n G_N = 0\,, \quad \mathbf{x}\in\partial\Omega\,, \tag{8.15 a}$$

$$G_N(\mathbf{x};\boldsymbol{\xi}) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}| + R_N(\boldsymbol{\xi};\boldsymbol{\xi}) + o(1)\,, \quad \text{as} \quad \mathbf{x}\to\boldsymbol{\xi}\,; \quad \int_\Omega G_N(\mathbf{x};\boldsymbol{\xi})\, d\mathbf{x} = 0\,. \tag{8.15 b}$$

The constant $R_N(\boldsymbol{\xi};\boldsymbol{\xi})$ is the regular part of G_N at the singularity. Since G_N has a zero spatial average, it follows from (8.14) that $\int_\Omega u_1\, d\mathbf{x} = 0$, as required in (8.12 a).

Next, we expand u_1 as $\mathbf{x}\to\mathbf{x}_j$. We use the local behavior for G_N, given in (8.15 b), to obtain from (8.14) that

$$u_1 \sim u_0 \log|\mathbf{x}-\mathbf{x}_j| - 2\pi u_0 \left[R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nij} \right]\,, \quad \mathbf{x}\to\mathbf{x}_j\,, \tag{8.16}$$

where $G_{Nji} = G_N(\mathbf{x}_j;\mathbf{x}_i)$ and $R_{Njj} = R_N(\mathbf{x}_j;\mathbf{x}_j)$. Comparing (8.16) and the required singularity behavior (8.11 b), we obtain that

$$A_{1j} = -2\pi u_0 \left[R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nij} \right]\,, \qquad j = 1,\ldots,N\,. \tag{8.17}$$

Next, we write the problem (8.6) in Ω as

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 + 2\pi \sum_{j=1}^{K} A_{1j}\delta(\mathbf{x}-\mathbf{x}_j)\,, \quad \mathbf{x}\in\Omega\,; \quad \partial_n u_2 = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{8.18}$$

Since $\int_\Omega u_1\, d\mathbf{x} = 0$ and $u_0 = |\Omega|^{-1/2}$, the divergence theorem applied to (8.18) determines λ_2 as $\lambda_2 u_0 |\Omega| = 2\pi \sum_{j=1} A_{1j}$. Finally, we use (8.17) for A_{1j}, we get

$$\lambda_2 = -\frac{4\pi^2}{|\Omega|} p(\mathbf{x}_1,\ldots,\mathbf{x}_K)\,, \qquad p(\mathbf{x}_1,\ldots,\mathbf{x}_K) \equiv \sum_{j=1}^{N} \left(R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nji} \right)\,. \tag{8.19}$$

Combining (8.2) with (8.13) and (8.19) we get the two-term expansion given in equations (5.27) and (5.28) of the Corollary in §5 of the workshop notes given by

$$\lambda_0(\varepsilon) \sim \frac{2\pi\nu K}{|\Omega|} - \frac{4\nu\pi^2}{|\Omega|} p(\mathbf{x}_1,\ldots,\mathbf{x}_K) + \cdots\,, \qquad \nu = -1/\log(\varepsilon d)\,. \tag{8.20}$$

9 Problem 9

Problem 9: *Consider the following Biharmonic eigenvalue problem in a two-dimensional bounded domain* Ω *containing a small circular hole* Ω_ε *of radius* ε *centered at* $\mathbf{x}_0 \in \Omega$*,*

$$\triangle^2 u - \lambda u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon\,; \quad u = \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,; \quad u = \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega_\varepsilon\,. \tag{9.1}$$

Let $\lambda_{0\varepsilon}$ *denote the first positive eigenvalue of this problem. Let* λ_0 *be the first eigenvalue of the unperturbed problem with no hole, with corresponding eigenfunction* $u_0(\mathbf{x})$*. Assume that* $u_0(\mathbf{x}_0) \neq 0$*. By using a matched asymptotic expansion argument, show that* $\lambda_{0\varepsilon}$ *does not approach* λ_0 *as* $\varepsilon \to 0$*, in contrast to that for Laplacian eigenvalue problems in perforated domains. Instead, show that* $\lambda_{0\varepsilon} \to \lambda_0^\star$ *as* $\varepsilon \to 0$*, where* $\lambda_0^\star$ *is the first eigenvalue of the following problem with a point constraint:*

$$\triangle^2 u^\star - \lambda^\star u^\star = 0\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\}\,; \quad u^\star = \partial_n u^\star = 0\,, \quad \mathbf{x} \in \partial\Omega\,; \quad u^\star(\mathbf{x}_0) = 0\,. \tag{9.2}$$

Finally, calculate the asymptotic behavior of the difference $\lambda_{0\varepsilon} - \lambda_0^\star$ *as* $\varepsilon \to 0$ *using a matched asymptotic analysis.*

Solution: Let $\lambda_{0\varepsilon}$ and $u_{0\varepsilon}(\mathbf{x})$ be the principal eigenvalue of the Biharmonic eigenvalue problem with a hole, given by (9.1) with normalization condition $\int_{\Omega\backslash\Omega_\varepsilon} u_{0\varepsilon}^2 \, d\mathbf{x} = 1$. Next, let λ_0 and $u_0(\mathbf{x})$ be the first eigenpair of the unperturbed problem with no hole

$$\triangle^2 u - \lambda u = 0\,, \quad \mathbf{x} \in \Omega\,; \quad u = \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{9.3}$$

with normalization condition $\int_\Omega u^2 \, d\mathbf{x} = 1$.

We now show that $\lambda_{0\varepsilon}$ does not tend to λ_0 as $\varepsilon \to 0$. To show this, suppose to the contrary that for some $\sigma \ll 1$ we have

$$\lambda_{0\varepsilon} = \lambda_0 + \sigma\lambda_1 + \cdots\,. \tag{9.4}$$

In the outer region we expand the outer eigenfunction as

$$u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x}) + \sigma u_1(\mathbf{x}) + \cdots\,. \tag{9.5}$$

Now at $\mathbf{x} = \mathbf{x}_0$, we assume that $u_0(\mathbf{x}_0) \neq 0$.

In the inner region we introduce the new variables $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0)$ and $v_\varepsilon(\mathbf{y}) = u_\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{y})$. Then, for some gauge function μ, we put

$$v_\varepsilon(\mathbf{y}) = \mu v_0(\rho)\,, \qquad \rho \equiv |\mathbf{y}|\,. \tag{9.6}$$

Upon substituting (9.6) into (9.1), we obtain that v_0 satisfies

$$\triangle^2 v_0 = 0\,, \quad \rho = |\mathbf{y}| \geq 1\,; \qquad v_0(1) = v_{0\rho}(1) = 0\,. \tag{9.7}$$

The general solution of this problem has the form

$$v_0 = a\rho^2 + b\rho^2 \log\rho + c\log\rho + d\,, \quad \rho \geq 1\,. \tag{9.8}$$

The matching condition is that the outer solution as $\mathbf{x} \to \mathbf{x}_0$ must agree with the inner expansion as $\rho = |\mathbf{y}| \to \infty$. Therefore,

$$u_0(\mathbf{x}_0) + \cdots + \sigma u_1 \sim \mu v_0(\rho) + \cdots\,. \tag{9.9}$$

The only possibility for matching is that $a = b = 0$, and that $c = u_0(\mathbf{x}_0)$ with $\mu = -1/\log\varepsilon$. However, this choice leaves only one free parameter d to satisfy the two boundary conditions $v_0(1) = v_{0\rho}(1) = 0$, which is impossible.

Therefore, we conclude that if we assume that the perturbed eigenfunction is close to the unperturbed eigenfunction with no hole in the outer region, then asymptotic matching is impossible. This suggests that this assumption must be modified, and that the limiting problem as $\varepsilon \to 0$ is not the problem with no hole.

Instead, we let $\lambda_0^\star$ and $u_0^\star(\mathbf{x})$ be the principal eigenpair of the Biharmonic eigenvalue problem with a point constraint, given by (9.3). In other words, we claim that the limiting problem $\varepsilon \to 0$ corresponds to the eigenvalue problem (9.2) with point constraint. Point constraints are compatible with Biharmonic problems, but not with Laplace's equation.

We then look for an eigenvalue of (9.1) close to $\lambda_0^\star$. For some gauge function $\sigma \ll 1$, we expand

$$\lambda_{0\varepsilon} = \lambda_0^\star + \sigma\lambda_1 + \cdots \,. \tag{9.10}$$

In the outer region, we expand the eigenfunction as

$$u_{0\varepsilon}(\mathbf{x}) = u_0^\star(\mathbf{x}) + \sigma u_1(\mathbf{x}) + \cdots \,. \tag{9.11}$$

Substituting (9.10) and (9.11) into (9.1), we obtain that λ_1 and $u_1(\mathbf{x})$ satisfy

$$\triangle^2 u_1 - \lambda_0^\star u_1 = \lambda_1 u_0^\star \,, \quad \mathbf{x} \in \Omega\backslash\{\mathbf{x}_0\} \,, \tag{9.12 a}$$

$$u_1 = \partial_n u_1 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad \int_\Omega u_0^\star u_1 \, d\mathbf{x} = 0 \,, \tag{9.12 b}$$

$$u_1 \quad \text{singular as } \mathbf{x} \to \mathbf{x}_0 \,. \tag{9.12 c}$$

Next, we must derive a singularity condition for u_1 as $\mathbf{x} \to \mathbf{x}_0$.

In the inner region, we introduce the new variables

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0) \,, \qquad v(\mathbf{y}) = u(\mathbf{x}_0 + \varepsilon\mathbf{y}) \,. \tag{9.13}$$

In terms of the gauge function $\mu \ll 1$, we then expand

$$v_\varepsilon(\mathbf{y}) = \mu v_0(\mathbf{y}) \,, \quad \rho = |\mathbf{y}| \,. \tag{9.14}$$

Since $u_0(\mathbf{x}_0) = 0$, the matching condition is that the outer expansion of the eigenfunction as $\mathbf{x} \to \mathbf{x}_0$ must agree with the far-field form of the inner expansion as $\mathbf{y} \to \infty$,

$$\nabla u_0^\star \cdot (\mathbf{x} - \mathbf{x}_0) + \cdots + \sigma u_1 \sim \mu v_0(\mathbf{y}) + \cdots \,. \tag{9.15}$$

Here we have defined

$$\nabla u_0^\star \equiv \nabla u_0^\star(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0} \,. \tag{9.16}$$

The problem for v_0 is

$$\triangle^2 v_0 = 0 \,, \quad \rho = |\mathbf{y}| \geq 1 \,; \qquad v_0(1) = v_{0\rho}(1) = 0 \,. \tag{9.17}$$

For any vector $\mathbf{a}$, there is a solution to (9.17) of the form

$$v_0 = \mathbf{A} \cdot \mathbf{e}_\theta v_c(\rho) \,, \tag{9.18 a}$$

where $\mathbf{e}_\theta \equiv (\cos\theta, \sin\theta)$ and $v_c(\rho)$ is given by

$$v_c = \rho \log \rho - \rho \log \left[e^{1/2}\right] + \frac{1}{2\rho} \,. \tag{9.18 b}$$

Notice that this is the Stokes solution given in equation (4.21) of the workshop notes.

We then write the far-field expansion of the inner solution in terms of the outer variables as

$$\mu v_0(\mathbf{y}) \sim \varepsilon^{-1}\mu \mathbf{A}\cdot\mathbf{e}_\theta|\mathbf{x}-\mathbf{x}_0|\left[\log|\mathbf{x}-\mathbf{x}_0| - \log\left(\varepsilon e^{1/2}\right)\right]. \tag{9.19}$$

This far-field expression suggests that we define μ and ν by

$$\nu = -\frac{1}{\log\left[\varepsilon e^{1/2}\right]}, \qquad \mu = \varepsilon\nu \tag{9.20}$$

Then, the matching condition (9.15) becomes

$$\nabla u_0^\star\cdot(\mathbf{x}-\mathbf{x}_0) + \cdots + \sigma u_1 \sim \mathbf{A}\cdot\mathbf{e}_\theta|\mathbf{x}-\mathbf{x}_0| + \mathbf{A}\cdot\mathbf{e}_\theta\nu|\mathbf{x}-\mathbf{x}_0|\log|\mathbf{x}-\mathbf{x}_0| + \cdots. \tag{9.21}$$

Therefore, we conclude that

$$\mathbf{A} = \nabla u_0^\star, \qquad \sigma = \nu \tag{9.22}$$

The matching condition (9.21) shows that the solution u_1 to (9.12) must have the singularity behavior

$$u_1 \sim \nabla u_0^\star\cdot(\mathbf{x}-\mathbf{x}_0)\log|\mathbf{x}-\mathbf{x}_0|, \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_0. \tag{9.23}$$

Finally, we apply the divergence theorem to (9.12) over Ω_0, where $\Omega_0 \equiv \Omega\backslash\Omega_\gamma$, and Ω_γ is a small disk of radius $\gamma \ll 1$, centered at $\mathbf{x}_0$. In this way, we get

$$\lambda_1 = -4\pi|\nabla u_0^\star|^2, \qquad \sigma = \nu. \tag{9.24}$$

In summary, the principal eigenvalue of (9.1) has the two-term asymptotic expansion

$$\lambda_{0\varepsilon} \sim \lambda_0^\star - 4\pi\nu|\nabla u_0^\star|^2 + \cdots, \qquad \nu = -\frac{1}{\log\left[\varepsilon e^{1/2}\right]}. \tag{9.25}$$

Here $u_0^\star$ and $\lambda_0^\star$ are the principal eigenpair of the problem (9.2) with point constraint.