Diffusion and Bifurcation Problems in Singularly Perturbed Domains

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Abstract

Diffusion problems under singular perturbations of the domain or the boundary conditions are analyzed. The first problem that we consider is the diffusion of a material from a domain that is nearly impermeable, having only several small patches on the boundary where the material can slowly leak out. The second problem that is studied is the diffusion of a material that originates from some localized regions in a two or three-dimensional domain. Steady-state solutions and the long-time behavior of solutions are analyzed in detail. Finally, the analysis is extended to determine the change in bifurcation values associated with nonlinear diffusion equations under singular perturbations of the domain. The results are then applied to a model in resource management.

1 Introduction

In this paper we study linear diffusion problems under different types of localized perturbations of a domain or the conditions on the boundary of a domain. The two specific types of perturbations that we will consider are: (A) the removal of small subdomains from the domain of a problem with the imposition of boundary conditions on the boundary of the resulting holes; (B) a large alteration of a no-flux boundary condition on small regions of the boundary of a domain. For each of these classes of problems our main goal is to calculate both the steady-state solution and the long-time behavior of solutions to the initial value problem. The two types of perturbations that we study are singular perturbations in the sense that they produce large, but localized, changes in the solution. Thus, the method of matched asymptotic expansions, as described more generally in [3] and [5], is used to calculate the perturbed solutions. The analysis will be done for both the case of two and three spatial dimensions and we highlight the qualitative differences between the two different dimensions.

The theory developed in this paper can be applied in several directions. First, problems of the type (A) can be used to model the passive diffusion of heat or of a polluting chemical species that is initially localized near some regions of a two or three-dimensional domain. The source of heat or pollution is assumed to be

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maintained for all time. Standard treatises on diffusive behavior such as [1] and [2] discuss such problems only for very specialized geometries for which the separation of variables technique can be used. Our goal is to asymptotically calculate, for an arbitrarily shaped multiply-connected domain with small holes, the steady-state solution and the asymptotic time-scale for which the time-dependent solution will approach its steady-state limit. On a bounded domain, the approach to the steady-state is determined in terms of the principal eigenvalue of a singularly perturbed eigenvalue problem. Asymptotic estimates for this principal eigenvalue, extending the work of [9], [10] and [6], are given.

A second application of problems of the type (A) is to calculate the change in the bifurcation values, as a result of a singular perturbation of the geometry of a domain, that determine the onset of stable spatially inhomogeneous solutions to nonlinear diffusion equations. These bifurcation values arise as the principal eigenvalues of certain singularly perturbed problems. The problem of this type that we consider is the determination of the size of the safe fishing zone for a fish population modeled by the competing effects of passive diffusion and logistic growth (cf. [8]). The theory can also be used to determine the critical patch size for a simple model of the budworm infestation (cf. [7]). Problems of the type (B) are relevant to calculating how long it will take a hazardous chemical species to slowly leak out of a container that has many small perforations on its boundary caused, perhaps, by a partial rusting of the boundary of the container.

The outline of the paper is as follows. In §2 we analyze the long-time behavior for the diffusion of a material through the wall of a nearly impermeable container. In §3 we study the diffusion of a material that originates from some localized regions in a two or three-dimensional domain. Steady-state solutions and the long-time behavior of solutions are analyzed in detail. Some examples of the theory are also given. Finally, in §4, we determine asymptotically the change in bifurcation values associated with nonlinear diffusion equations under singular perturbations of the geometry of a domain. The results are then applied to the model in resource management mentioned above.

2 Diffusion in an Almost Impermeable Container

For $\varepsilon \ll 1$, we consider the following diffusion problem for $u(\boldsymbol{x},t)$;

$$u_t = \nabla \cdot [p(\boldsymbol{x}) \nabla u] , \qquad \boldsymbol{x} \in D , \tag{2.1a}$$

$$\partial_n u = 0 , \qquad \boldsymbol{x} \in \partial D_0 , \tag{2.1b}$$

$$\varepsilon \partial_n u + b_j (u - u_b) = 0 , \qquad \boldsymbol{x} \in \partial D_{\varepsilon_j} , \qquad j = 1, \ldots, N , \tag{2.1c}$$

$$u(\boldsymbol{x},0) = f(\boldsymbol{x}) . \tag{2.1d}$$

Here $b_j > 0$ is constant, u_b is the constant ambient concentration of u outside the closed bounded domain D, $p(\boldsymbol{x}) > 0$ is the spatially inhomogeneous diffusivity, and ∂_n represents the outward normal derivative to the boundary. The patch $\partial D_{\varepsilon_j}$ represents that part of the boundary where the material can leak out of the domain, while ∂D_0 is that part of the boundary that is impermeable. We assume that for $\varepsilon \ll 1$ the length (area) of $\partial D_{\varepsilon_j}$ in two-dimensions (three-dimensions) is $O(\varepsilon)$ ($O(\varepsilon^2)$) and that the perturbing patches $\partial D_{\varepsilon_1}, \ldots, \partial D_{\varepsilon_N}$ are disjoint. Thus, the boundary ∂D of the domain is just the union $\partial D \equiv \partial D_0 \bigcup \sum_{j=1}^{N} \partial D_{\varepsilon_j}$. In the limit $\varepsilon \to 0$, we assume that $\partial D_{\varepsilon_j} \to \boldsymbol{x}_j \in \partial D$, and that there is a well-defined tangent line (plane) to ∂D at $\boldsymbol{x}_j$ in two-dimensions (three-dimensions) when $\varepsilon \ll 1$. This later assumption is satisfied when the boundary is smooth near $\partial D_{\varepsilon_j}$. Finally, we define the scaled patches that result from an $O(\varepsilon^{-1})$ magnification of the length scale by $\partial D_j \equiv \varepsilon^{-1} \partial D_{\varepsilon_j}$.

The solution to (2.1) is represented in terms of an eigenfunction expansion solution by

$$u(\boldsymbol{x}, t; \varepsilon) = u_b + \sum_{k=1}^{\infty} a_k \phi_k e^{-\lambda_k t}, \tag{2.2}$$

where $\phi_k = \phi_k(\boldsymbol{x}; \varepsilon)$ and $\lambda_k(\varepsilon)$ are the normalized eigenpairs of

$$\begin{aligned} \nabla \cdot [p(\boldsymbol{x}) \nabla \phi] \ &+ \ \lambda \phi = 0, \qquad \boldsymbol{x} \in D, & \text{(2.3a)} \\ \partial_n \phi \ &= \ 0, \qquad \boldsymbol{x} \in \partial D_0, & \text{(2.3b)} \\ \varepsilon \partial_n \phi + b_j \phi \ &= \ 0, \qquad \boldsymbol{x} \in \partial D_{\varepsilon_j}, \qquad j = 1, \ldots, N, & \text{(2.3c)} \\ \int_D \phi^2 \, d\boldsymbol{x} \ &= \ 1. & \text{(2.3d)} \end{aligned}$$

The coefficients $a_k(\varepsilon)$ in (2.2) satisfy

$$a_k = \int_D \left(f(\boldsymbol{x}) - u_b \right) \phi_k(x; \varepsilon) \, d\boldsymbol{x}. \tag{2.4}$$

For $t \gg 1$, the long-time behavior of the solution to (2.1) can be approximated in terms of the first eigenpair of (2.3) by

$$u(\boldsymbol{x}, t; \varepsilon) \sim u_b + a_1 \phi_1 \left(\boldsymbol{x}; \varepsilon \right) e^{-\lambda_1(\varepsilon) t}. \tag{2.5}$$

The singularly perturbed character of (2.1) is evident when we try to determine the steady-state solution to (2.1). When $b_j > 0$ for $j = 1, \ldots, N$ we have that $\lambda_1(\varepsilon) > 0$ and thus $u(\boldsymbol{x}, t; \varepsilon) \to u_b$ as $t \to \infty$. Alternatively, when $b_j = 0$ for $j = 1, \ldots, N$ then the boundary of D is impermeable and $\lambda_1 = 0$ and $\phi_1 = V^{-1/2}$, where V is the volume of D. Hence, in this case $u(\boldsymbol{x}, t; \varepsilon) \to u_b + a_1$ as $t \to \infty$, and from this we get

$$u(\boldsymbol{x}, t; \varepsilon) \to \frac{1}{V} \int_D f(\boldsymbol{x}) \, d\boldsymbol{x}. \tag{2.6}$$

In the analysis below we show how to use the method of matched asymptotic expansions to calculate the eigenpairs of (2.3), and in particular we get asymptotic formulae for ϕ_1 and λ_1. There are two specific cases that we will analyze:

$$\text{Case A:} \qquad b_j = \varepsilon b_j^{(0)} > 0 \quad \text{for} \quad j = 1, \ldots, N\,, \tag{2.7a}$$

$$\text{Case B:} \qquad b_j > 0 \quad \text{for} \quad j = 1, \ldots, N\,. \tag{2.7b}$$

A special limit of Case B is when $b_j = \infty$ for $j = 1, \ldots, N$, for which $u = u_b$ on $\partial D_{\varepsilon_j}$. The analysis to determine the spectrum of (2.3) will be done for both a two and a three-dimensional domain.

We now outline the portion of the analysis that is independent of the dimension. Let ψ_k and μ_k denote the k^{th} eigenpair of the unperturbed problem corresponding to (2.3). Thus ψ_k satisfies (2.3b) on all of ∂D and (2.3c) is vacuous. We shall assume for simplicity that D is such that these eigenpairs have multiplicity one. We then look for an eigenpair of (2.3) for which $\phi_k \to \psi_k$ as $\varepsilon \to 0$ away from $O(\varepsilon)$ regions near the perturbing patches $\partial D_{\varepsilon_j}$. The perturbed eigenvalue satisfies $\lambda_k(\varepsilon) \to \mu_k$ as $\varepsilon \to 0$, and hence we expand

$$\lambda_k(\varepsilon) \sim \mu_k + \nu(\varepsilon)\lambda_k^{(1)} + \cdots\,. \tag{2.8}$$

Here $\nu(\varepsilon) \ll 1$ is to be determined. The eigenfunction ϕ_k is represented by two separate expansions; an *outer* expansion where ϕ_k is close to ψ_k and an *inner* expansion near each $\partial D_{\varepsilon_j}$ where ϕ_k is significantly different from ψ_k. In the outer region we expand

$$\phi_k(\boldsymbol{x};\varepsilon) \sim \psi_k(\boldsymbol{x}) + \nu(\varepsilon)\psi_k^{(1)}(\boldsymbol{x}) + \cdots\,, \tag{2.9}$$

for some $\psi_k^{(1)}$. Substituting (2.8) and (2.9) into (2.3), and noting that $\partial D_{\varepsilon_j} \to \boldsymbol{x}_j$ as $\varepsilon \to 0$, we find that $\psi_k^{(1)}$ satisfies

$$L\psi_k^{(1)} \equiv \nabla \cdot \left[p(\boldsymbol{x})\nabla\psi_k^{(1)}\right] + \mu_k\psi_k^{(1)} = -\lambda_k^{(1)}\psi_k\,, \qquad \boldsymbol{x} \in D\,, \tag{2.10a}$$

$$\partial_n\psi_k^{(1)} = 0\,, \qquad \boldsymbol{x} \in \partial D\,, \tag{2.10b}$$

$$\psi_k^{(1)} \quad \text{is singular as } \boldsymbol{x} \to \boldsymbol{x}_j\,, \; j = 1, \ldots, N\,, \tag{2.10c}$$

$$\int_D \psi_k\psi_k^{(1)}\, d\boldsymbol{x} = 0\,. \tag{2.10d}$$

The singularity behavior in (2.10c) is determined only after asymptotically matching the outer expansion (2.9) to appropriate inner expansions to be constructed below near each patch $\partial D_{\varepsilon_j}$. This behavior, as well as the form of the gauge function $\nu(\varepsilon)$, depends critically on the range of b_j in (2.7) and on the dimension of D.

For each of these cases we will derive below that the singularity for $\psi_k^{(1)}$ as $\boldsymbol{x} \to \boldsymbol{x}_j$ is a certain multiple of the free-space Green's function for the Laplacian in the appropriate dimension. Hence we can re-write (2.10) in all of D as

$$L\psi_k^{(1)} = -\lambda_k^{(1)}\psi_k + \sum_{j=1}^{N} \gamma_j \delta(\boldsymbol{x} - \boldsymbol{x}_j) , \qquad \boldsymbol{x} \in D , \tag{2.11a}$$

$$\partial_n \psi_k^{(1)} = 0 , \qquad \boldsymbol{x} \in \partial D , \tag{2.11b}$$

for some γ_j. Here the singular point $\boldsymbol{x}_j$ lies on the boundary of the domain. Since ψ_k is a solution to the homogeneous problem, Green's identity can be used to derive the solvability condition

$$\lambda_k^{(1)} = \sum_{j=1}^{N} \frac{\gamma_j}{2} \psi_k(\boldsymbol{x}_j) . \tag{2.12}$$

Hence, from (2.8) the expansion of the eigenvalue is

$$\lambda_k(\varepsilon) \sim \mu_k + \sum_{j=1}^{N} \frac{\nu(\varepsilon)\gamma_j}{2} \psi_k(\boldsymbol{x}_j) + \cdots . \tag{2.13}$$

We now show how to determine the strengths γ_j of the singularities as well as the gauge function $\nu(\varepsilon)$ for Cases A and B in both two and three dimensions.

2.1 A Three-Dimensional Domain: Case A

For fixed j we construct an inner expansion near $\partial D_{\varepsilon_j}$. We first introduce a locally orthogonal coordinate system (s_1, s_2, n) centered at $\boldsymbol{x}_j$ on $\partial D_{\varepsilon_j}$. Here $-n$ is the distance from $\boldsymbol{x} \in D$ to ∂D and s_1, s_2 are the orthogonal coordinates through the two principal directions at $\boldsymbol{x}_j$. The coordinate change is summarized by

$$\boldsymbol{x} = (x_1, x_2, x_3) \to (s_1, s_2, n) ; \qquad \phi_k(x_1, x_2, x_3) \to v_k(s_1, s_2, n) . \tag{2.14}$$

Since the patch $\partial D_{\varepsilon_j}$ has area $O(\varepsilon^2)$, we introduce the scaled coordinates (ξ_1, ξ_2, η) by

$$\xi_1 = s_1/\varepsilon , \quad \xi_2 = s_2/\varepsilon , \quad \eta = n/\varepsilon , \tag{2.15}$$

and we define $\boldsymbol{y} = (\xi_1, \xi_2, \eta)$. Since $p(\boldsymbol{x})$ can be expanded in a Taylor series near $\boldsymbol{x}_j$, (2.3a)-(2.3c) transforms near $\partial D_{\varepsilon_j}$ to

$$p(\boldsymbol{x}_j)\triangle' v_k + \varepsilon F\left(\boldsymbol{x} - \boldsymbol{x}_j, \partial_{\xi_1} v_k, \partial_{\xi_2} v_k, \partial_n v_k\right) + O(\varepsilon^2) = 0 , \qquad \eta < 0 , \tag{2.16a}$$

$$\partial_n v_k = 0 , \qquad (\xi_1, \xi_2) \notin \partial D_j \quad \text{on} \quad \eta = 0 , \tag{2.16b}$$

$$\partial_n v_k + \varepsilon b_j^{(0)} v_k = 0 , \qquad (\xi_1, \xi_2) \in \partial D_j \quad \text{on} \quad \eta = 0 , \tag{2.16c}$$

for some smooth function $F(u,v,w,z)$. Here $\triangle'$ indicates the Laplacian in the $\boldsymbol{y}$ variable. This suggests that the appropriate inner expansion for $v_k(\boldsymbol{y};\varepsilon)$ is

$$v_k(\boldsymbol{y};\varepsilon) \sim v_k^{(0)}(\boldsymbol{y}) + \varepsilon v_k^{(1)}(\boldsymbol{y}) + \cdots \,. \tag{2.17}$$

Here $v_k^{(0)}$ is a constant and $v_k^{(1)}$ satisfies the half-space problem

$$\triangle' v_k^{(1)} = 0\,, \qquad \eta < 0\,, \tag{2.18a}$$
$$\partial_n v_k^{(1)} = 0\,, \qquad (\xi_1,\xi_2) \notin \partial D_j \quad \text{on} \quad \eta = 0\,, \tag{2.18b}$$
$$\partial_n v_k^{(1)} = -b_j^{(0)} v_k^{(0)}\,, \qquad (\xi_1,\xi_2) \in \partial D_j \quad \text{on} \quad \eta = 0\,. \tag{2.18c}$$

By applying the Divergence Theorem to (2.18) we can derive the far-field behavior

$$v_k^{(1)} = \frac{-b_j^{(0)} A_j v_k^{(0)}}{2\pi|\boldsymbol{y}|} + O\left(\frac{1}{|\boldsymbol{y}|^2}\right)\,, \qquad |\boldsymbol{y}| \gg 1\,. \tag{2.19}$$

Here A_j is the area of ∂D_j. The far-field form of the inner expansion (2.17), written in terms of the outer variable, is

$$v_k \sim v_k^{(0)} \left(1 - \frac{\varepsilon^2 b_j^{(0)} A_j}{2\pi|\boldsymbol{x}-\boldsymbol{x}_j|} + \dots\right)\,. \tag{2.20}$$

Matching (2.20) to the outer solution (2.9) we require that

$$v_k^{(0)} = \psi_k(\boldsymbol{x}_j)\,; \qquad \nu(\varepsilon) = \varepsilon^2\,; \qquad \psi_k^{(1)} \sim -\frac{b_j^{(0)} A_j \psi_k(\boldsymbol{x}_j)}{2\pi|\boldsymbol{x}-\boldsymbol{x}_j|} \quad \text{as} \quad \boldsymbol{x} \to \boldsymbol{x}_j\,. \tag{2.21}$$

Finally, since the free-space Green's function for the Laplacian in three-dimensions is $-1/4\pi r$, we require that γ_j in (2.11a) satisfy

$$\gamma_j = 2\psi_k(\boldsymbol{x}_j) b_j^{(0)} A_j p(\boldsymbol{x}_j)\,. \tag{2.22}$$

Therefore, from (2.13) we obtain the following result:

Proposition 2.1: *Let A_j be the area of the scaled patch ∂D_j, where $\partial D_j = \varepsilon^{-1}\partial D_{\varepsilon_j}$. Then, for $\varepsilon \ll 1$, the eigenvalue λ_k of (2.3) in three-dimensions and for Case A has the expansion*

$$\lambda_k(\varepsilon) \sim \mu_k + \varepsilon^2 \sum_{j=1}^{N} p(\boldsymbol{x}_j) A_j b_j^{(0)} \left[\psi_k(\boldsymbol{x}_j)\right]^2 + \cdots \,. \tag{2.23}$$

By evaluating a_1 in (2.4), and by noting that $\mu_1 = 0$ and $\psi_1 = V^{-1/2}$, we get the following long-time result:

Corollary 2.1: *Under the conditions of Prop. 2.1 above, the principal eigenvalue λ_1 of (2.3) satisfies*

$$\lambda_1(\varepsilon) \sim \frac{\varepsilon^2}{V} \sum_{j=1}^{N} p(\boldsymbol{x}_j) A_j b_j^{(0)} \,, \tag{2.24a}$$

where V is the volume of D. The long-time behavior of u away from the perturbing patches is

$$u(\boldsymbol{x}, t; \varepsilon) \sim u_b + \frac{1}{V} \int_D (f(\boldsymbol{x}) - u_b)\, d\boldsymbol{x} \left(1 + \varepsilon^2 \psi_k^{(1)} V^{1/2}\right) e^{-\lambda_1 t} \,. \tag{2.24b}$$

Hence, for Case A it takes a time $t \gg O(\varepsilon^{-2})$ for (2.1) to approach the steady-state limit u_b.

As a remark, using the result from the analysis above that $\phi_k - \psi_k = O(\varepsilon)$ *uniformly* in D, the result (2.23) can be derived more readily by regular perturbation theory. To do so, we re-write (2.3) in Case A as

$$\nabla \cdot [p(\boldsymbol{x}) \nabla \phi_k] \;+\; \mu_k \phi_k = (\mu_k - \lambda_k) \phi_k \,, \qquad \boldsymbol{x} \in D \,, \tag{2.25a}$$

$$\partial_n \phi_k \;=\; -\sum_{j=1}^{N} b_j^{(0)} I_j \phi_k \,, \qquad \boldsymbol{x} \in \partial D \,. \tag{2.25b}$$

Here I_j is the indicator function defined by

$$I_j = \begin{cases} 1 \,, & \text{when } x \in \partial D_{\varepsilon_j} \,, \\ 0 \,, & \text{when } x \notin \partial D_{\varepsilon_j} \,, \end{cases} \tag{2.25c}$$

Now applying Green's identity to the eigenfunctions ϕ_k of (2.25) and ψ_k of the unperturbed problem, we derive

$$(\mu_k - \lambda_k)\,(\phi_k, \psi_k) = -\sum_{j=1}^{N} b_j^{(0)} \int_{\partial D_{\varepsilon_j}} p(\boldsymbol{x}) \phi_k \psi_k \, d\boldsymbol{x} \,. \tag{2.26a}$$

Here $(u, v) \equiv \int_D uv d\boldsymbol{x}$. Now since the estimate $\phi_k - \psi_k = O(\varepsilon)$ holds even in the vicinity of the perturbing patch $\partial D_{\varepsilon_j}$, we can replace ϕ_k by ψ_k in (2.26a) to derive

$$(\mu_k - \lambda_k)\,(\psi_k, \psi_k) \sim -\varepsilon^2 \sum_{j=1}^{N} p(\boldsymbol{x}_j) A_j b_j^{(0)} \left[\psi_k(\boldsymbol{x}_j)\right]^2 \,. \tag{2.26b}$$

Using the fact that ψ_k is normalized, it follows that (2.26b) reduces to (2.23).

2.2 A Three-Dimensional Domain: Case B

In this case we proceed as in (2.14)-(2.15) to derive in place of (2.16)

$$p(\boldsymbol{x}_j) \triangle' v_k \;+\; \varepsilon F(\boldsymbol{x} - \boldsymbol{x}_j, \partial_{\xi_1} v_k, \partial_{\xi_2} v_k, \partial_n v_k) + O(\varepsilon^2) = 0 \,, \qquad \eta < 0 \,, \tag{2.27a}$$

$$\partial_n v_k \;=\; 0 \,, \qquad (\xi_1, \xi_2) \notin \partial D_j \quad \text{on} \quad \eta = 0 \,, \tag{2.27b}$$

$$\partial_n v_k + b_j v_k \;=\; 0 \,, \qquad (\xi_1, \xi_2) \in \partial D_j \quad \text{on} \quad \eta = 0 \,. \tag{2.27c}$$

The inner solution is then expanded as in (2.17) where $v_k^{(0)}$ now satisfies

$$\begin{aligned} \triangle' v_k^{(0)} &= 0\,, \qquad \eta < 0\,, && (2.28a)\\ \partial_n v_k^{(0)} &= 0\,, \qquad (\xi_1,\xi_2) \notin \partial D_j \quad \text{on} \quad \eta = 0\,, && (2.28b)\\ \partial_n v_k^{(0)} + b_j v_k^{(0)} &= 0\,, \qquad (\xi_1,\xi_2) \in \partial D_j \quad \text{on} \quad \eta = 0\,. && (2.28c) \end{aligned}$$

The far-field expansion of the solution has the form

$$v_k^{(0)} = v_\infty \left[1 - \frac{C(b_j)}{|\boldsymbol{y}|} + O\left(\frac{1}{|\boldsymbol{y}|^2}\right)\right]\,, \qquad |\boldsymbol{y}| \gg 1\,. \tag{2.28d}$$

Here for each b_j, $C = C(b_j)$ is a constant determined uniquely by the solution to (2.28) satisfying $v_k^{(0)} \to v_\infty$ as $|\boldsymbol{y}| \to \infty$. When $b_j = \infty$, $C(\infty)$ is called the capacitance of the patch ∂D_j. In general $C(b_j)$ must be found numerically. However, in the special case where $b_j = \infty$ and ∂D_j is a circular patch of radius one, the exact solution to (2.28) can be found by integral transform techniques (cf. [12]) as

$$v_k^{(0)} = v_\infty \left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega} e^{\omega\eta} J_0(\omega\rho)\, d\omega \right]\,, \qquad \rho \equiv (\xi_1^2 + \xi_2^2)^{1/2}\,. \tag{2.29}$$

Here $J_0(z)$ is the Bessel function of order zero. By expanding the integral asymptotically as $\eta \to -\infty$ we obtain $C(\infty) = 2/\pi$, which is the well-known capacitance of a circular disk of radius one.

Matching (2.28d) to the outer expansion (2.9) we require that

$$v_\infty = \psi_k(\boldsymbol{x}_j)\,; \qquad \nu(\varepsilon) = \varepsilon\,; \qquad \psi_k^{(1)} \sim \frac{-C(b_j)\psi_k(\boldsymbol{x}_j)}{|\boldsymbol{x} - \boldsymbol{x}_j|} \quad \text{as} \quad \boldsymbol{x} \to \boldsymbol{x}_j\,. \tag{2.30}$$

Hence, γ_j in (2.11a) satisfies

$$\gamma_j = 4\pi C(b_j)\psi_k(\boldsymbol{x}_j) p(\boldsymbol{x}_j)\,. \tag{2.31}$$

Therefore, from (2.13) we obtain the following result:

Proposition 2.2: *Let $C = C(b_j)$ be defined by the solution to (2.28) for each b_j. Then, for $\varepsilon \ll 1$, the eigenvalue λ_k of (2.3) in three-dimensions and for Case B has the expansion*

$$\lambda_k(\varepsilon) \sim \mu_k + 2\pi\varepsilon \sum_{j=1}^{N} p(\boldsymbol{x}_j) C(b_j) \left[\psi_k(\boldsymbol{x}_j)\right]^2 + \cdots\,. \tag{2.32}$$

In analogy with Corollary 2.1 we obtain the following long-time result:

Corollary 2.2: *Under the conditions of Prop. 2.2 above, the principal eigenvalue λ_1 of (2.3) satisfies*

$$\lambda_1(\varepsilon) \sim \frac{2\pi\varepsilon}{V} \sum_{j=1}^{N} p(\boldsymbol{x}_j) C(b_j)\,, \tag{2.33a}$$

where V is the volume of D. The long-time behavior of u away from the perturbing patches is

$$u(\boldsymbol{x},t;\varepsilon) \sim u_b + \frac{1}{V}\int_D (f(\boldsymbol{x}) - u_b)\, d\boldsymbol{x} \left(1 + \varepsilon \psi_k^{(1)} V^{1/2}\right) e^{-\lambda_1 t}\,. \tag{2.33b}$$

Hence, for Case B it takes a time $t \gg O(\varepsilon^{-1})$ for (2.1) to approach the steady-state limit. Thus the material will leak out quicker than for Case A by a factor of ε.

To illustrate these results we consider a specific example. We let D be the hemisphere $x_1^2 + x_2^2 + x_3^2 \le 1$ with $x_3 \le 0$. The volume is $V = 2\pi/3$. Assume that the boundary of the hemisphere is perfectly insulating except on a small portion of the plane $x_3 = 0$ with $x_1^2 + x_2^2 \le 1$. On this portion of the boundary we assume that there are N non-overlapping circular patches $\partial D_{\varepsilon_j}$ centered at $\boldsymbol{x}_j = (x_{1j}, x_{2j}, 0)$ with radii εr_j for $j = 1, \ldots, N$ where the material can pass through the boundary of the domain. Then, for Case A, we get from Corollary 2.1 that

$$\lambda_1 \sim \frac{3\varepsilon^2}{2}\sum_{j=1}^N r_j^2 b_j^{(0)} p(\boldsymbol{x}_j)\,. \tag{2.34a}$$

For Case B with $b_j = \infty$, we get $C(\infty) = 2r_j/\pi$ for the j^{th} circular patch. Thus, from (2.33a) we obtain that

$$\lambda_1 \sim \frac{6\varepsilon}{\pi}\sum_{j=1}^N r_j p(\boldsymbol{x}_j)\,. \tag{2.34b}$$

2.3 A Two-Dimensional Domain: Cases A and B

We first consider **Case A**. In this case we can proceed as in §2.1 to construct inner expansions near each $\partial D_{\varepsilon_j}$ and then match to the outer solution in order to determine the coefficients γ_j in (2.11a). This analysis, which we shall omit, again shows that $\phi_k - \psi_k = o(1)$ as $\varepsilon \to 0$ uniformly in D. Hence, we can derive an asymptotic formula for λ_k using regular perturbation theory. We re-write (2.3) as in (2.25) and then use Green's identity to obtain (2.26a). Then, using the fact that $\phi_k - \psi_k = o(1)$ as $\varepsilon \to 0$ uniformly in D we get

$$(\mu_k - \lambda_k)(\psi_k, \psi_k) \sim -\sum_{j=1}^N b_j^{(0)} \int_{\partial D_{\varepsilon_j}} p(\boldsymbol{x})\, [\psi_k(\boldsymbol{x})]^2\, d\boldsymbol{x}\,. \tag{2.35}$$

Now let $q(\boldsymbol{x})$ be a smooth function. Then $\int_{\partial D_{\varepsilon_j}} q(\boldsymbol{x})\, d\boldsymbol{x} \sim \varepsilon q(\boldsymbol{x}_j) L_j$, where εL_j is the length of $\partial D_{\varepsilon_j}$. Using this result in (2.35) we obtain the following results analogous to Prop. 2.1 and Corollary 2.1:

Proposition 2.3: *Let εL_j be the length of the segment $\partial D_{\varepsilon_j}$. Then, for $\varepsilon \ll 1$, the eigenvalue λ_k of (2.3) in two-dimensions and for Case A has the expansion*

$$\lambda_k(\varepsilon) \sim \mu_k + \varepsilon \sum_{j=1}^N p(\boldsymbol{x}_j) L_j b_j^{(0)} \left[\psi_k(\boldsymbol{x}_j)\right]^2 + \cdots\,. \tag{2.36a}$$

The long-time behavior of the solution to (2.1) away from the perturbing segments is

$$u(\boldsymbol{x},t;\varepsilon) \sim u_b + \frac{1}{A}\int_D (f(\boldsymbol{x}) - u_b)\, d\boldsymbol{x}\; e^{-\lambda_1 t}\,, \tag{2.36b}$$

where λ_1 *satisfies*

$$\lambda_1(\varepsilon) \sim \frac{\varepsilon}{A}\sum_{j=1}^{N} p(\boldsymbol{x}_j) L_j b_j^{(0)}\,. \tag{2.36c}$$

Here A *is the area of* D.

In **Case B** we must construct an inner expansion near $\partial D_{\varepsilon_j}$. We introduce an orthogonal coordinate system (s,n) centered at $\boldsymbol{x}_j$ on $\partial D_{\varepsilon_j}$, where $-n$ is the distance from $x \in D$ to ∂D and s is arclength along ∂D. Since $\partial D_{\varepsilon_j}$ has length $O(\varepsilon)$, we introduce the local scaled variables ξ and η by $\xi = s/\varepsilon$ and $\eta = n/\varepsilon$, and we define $\boldsymbol{y} = (\xi,\eta)$. Thus, the coordinate change is

$$\boldsymbol{x} = (x_1,x_2) \to (\xi,\eta)\,; \qquad \phi_k(x_1,x_2) \to v_k(\xi,\eta)\,. \tag{2.37}$$

The inner solution $v_k(\boldsymbol{y};\varepsilon)$ is expanded as

$$v_k(\boldsymbol{y};\varepsilon) \sim g_0(\varepsilon) v_k^{(0)}(\boldsymbol{y}) + g_1(\varepsilon) v_k^{(1)}(\boldsymbol{y}) + \cdots\,, \tag{2.38}$$

where the undetermined gauge functions $g_0(\varepsilon)$ and $g_1(\varepsilon)$ satisfy $g_1 \ll g_0 \ll 1$ as $\varepsilon \to 0$. Substituting the inner expansion (2.38) and the local coordinate system into (2.3), we find that $v_k^{(0)}$ satisfies

$$\triangle' v_k^{(0)} = 0\,, \qquad \eta < 0\,, \tag{2.39a}$$

$$\partial_n v_k^{(0)} = 0\,, \qquad \xi \notin \partial D_j \quad \text{on} \quad \eta = 0\,, \tag{2.39b}$$

$$\partial_n v_k^{(0)} + b_j v_k^{(0)} = 0\,, \qquad \xi \in \partial D_j \quad \text{on} \quad \eta = 0\,. \tag{2.39c}$$

There is a solution to (2.39) with the far-field asymptotic behavior

$$v_k^{(0)} = v_\infty^{(0)}\left(\| \boldsymbol{y}| - \log\left[d(b_j)\right] + O\left(\frac{1}{|\boldsymbol{y}|}\right)\right)\,, \qquad |\boldsymbol{y}| \gg 1\,. \tag{2.39d}$$

Here for each b_j, $d = d(b_j)$ is a constant determined uniquely by the solution to (2.39). When $b_j = \infty$, $d(\infty)$ is called the logarithmic capacitance of ∂D_j. In general $d(b_j)$ must be found numerically. However, in the special case where $b_j = \infty$ and ∂D_j is a segment of length 2, we can obtain $d(\infty) = 1/2$ from a certain exact solution.

The far-field form of the inner expansion, written in terms of the outer variable, is

$$v_k^{(0)} \sim g_0(\varepsilon) v_\infty^{(0)}\left(-\log\left[\varepsilon d(b_j)\right] + \log|\boldsymbol{x} - \boldsymbol{x}_j| + O(\varepsilon)\right)\,. \tag{2.40}$$

To match to the outer expansion (2.9) we require that

$$g_0(\varepsilon) = -\frac{1}{\log\left[\varepsilon d(b_j)\right]} = \nu(\varepsilon)\,; \qquad v_\infty^{(0)} = \psi_k(\boldsymbol{x}_j)\,; \qquad \psi_k^{(1)} \sim \psi_k(\boldsymbol{x}_j)\log|\boldsymbol{x}-\boldsymbol{x}_j| \quad \text{as} \quad \boldsymbol{x}\to\boldsymbol{x}_j\,. \tag{2.41}$$

Now since the free-space Green's function for the Laplacian in two-dimensions is $(2\pi)^{-1}\log r$, we get that the strength γ_j of the singularity at $\boldsymbol{x}_j$ in (2.11a) is

$$\gamma_j = 2\pi\psi_k(\boldsymbol{x}_j)\,. \tag{2.42}$$

Finally, substituting (2.41) and (2.42) into (2.13) we get the following key results:

Proposition 2.4: *Let $d = d(b_j)$ be defined by the solution to the half-plane problem (2.39). Then, for $\varepsilon \ll 1$, the eigenvalue λ_k of (2.3) in two-dimensions and for Case B has the expansion*

$$\lambda_k(\varepsilon) \sim \mu_k + \sum_{j=1}^{N}\left(-\frac{1}{\log\left[\varepsilon d(b_j)\right]}\right)\pi p(\boldsymbol{x}_j)\left[\psi_k(\boldsymbol{x}_j)\right]^2 + \cdots\,. \tag{2.43a}$$

The long-time behavior of the solution to (2.1) away from the perturbing segments is given by (2.36b) where λ_1 satisfies

$$\lambda_1(\varepsilon) \sim \sum_{j=1}^{N}\left(-\frac{1}{\log\left[\varepsilon d(b_j)\right]}\right)\frac{\pi p(\boldsymbol{x}_j)}{A} + \cdots\,, \tag{2.43b}$$

and A is the volume of D.

This result shows that λ_1 can be rather large in Case B when ε is a small fixed value. Hence, in this case, the material will leak out of the domain rather quickly.

We remark that the expansion (2.43a) represents the leading term in an infinite logarithmic expansion for $\lambda_k - \mu_k$ in powers of $-1/\log\varepsilon$. A hybrid asymptotic-numerical technique to sum such infinite logarithmic series, which then provides an error estimate that is algebraic in ε, was developed for related problems in [13].

3 Diffusion in a Singularly Perturbed Domain

For $\varepsilon \ll 1$, we consider the following singularly perturbed diffusion problem for $u(\boldsymbol{x},t)$;

$$u_t = \nabla\cdot\left[p(\boldsymbol{x})\nabla u\right] - m(\boldsymbol{x})u\,, \qquad \boldsymbol{x}\in D\backslash\bigcup_{j=1}^{N} D_j^\varepsilon\,, \tag{3.1a}$$

$$u = u_b\,, \qquad \boldsymbol{x}\in\partial D\,, \tag{3.1b}$$

$$u = \alpha_j\,, \qquad \boldsymbol{x}\in\partial D_j^\varepsilon\,, \qquad j=1,\ldots,N\,, \tag{3.1c}$$

$$u(\boldsymbol{x},0) = f(\boldsymbol{x})\,. \tag{3.1d}$$

For simplicity we assume that u_b and α_j for $j = 1, \ldots, N$ are constants. Also, D is a closed bounded domain in $\mathcal{R}^N$ with $N = 2$ or $N = 3$, $p(\boldsymbol{x}) > 0$ is the spatially inhomogeneous diffusivity and $m(\boldsymbol{x}) > 0$ represents a bulk consumption of material. The subdomain D_j^ε of radius $O(\varepsilon)$, which models a localized sink or source of material, is strictly contained within D and the subdomains are assumed to be non-overlapping. We denote the scaled subdomain that results from an $O(\varepsilon^{-1})$ magnification of the length scale of D_j^ε by $D_j \equiv \varepsilon^{-1} D_j^\varepsilon$. Finally, we assume that $D_j^\varepsilon \to \boldsymbol{x}_j \in D$ uniformly as $\varepsilon \to 0$. A schematic plot of the geometry of the domain for the two-dimensional case is shown in Fig. 1.

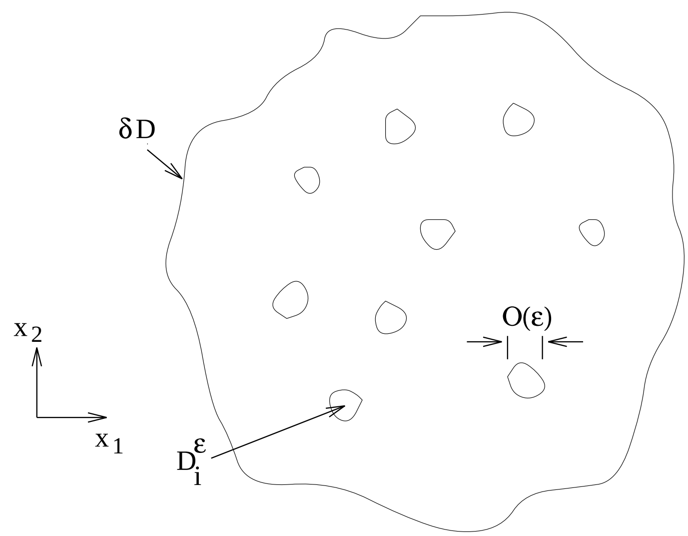

Figure 1: A schematic plot of the geometry of the singularly perturbed domain for (3.1). The domain D is assumed to have N small holes.

Our goal is to determine the long-time behavior of solutions to (3.1) in both two and three dimensions. In §3.1 we determine the asymptotic steady-state solution to (3.1) for $\varepsilon \ll 1$. In §3.2 we determine the long-time behavior of time-dependent solutions as they tend to the steady-state solution. Some examples of the theory are given in §3.3. In §4 we consider a class of nonlinear problems whose linearizations around bifurcation points yield problems of the form (3.1).

3.1 The Steady-State Solution

Using singular perturbation techniques, we first construct the steady-state solution to (3.1) in both a two-dimensional and a three-dimensional bounded domain D. The asymptotic solution to (3.1) for $\varepsilon \ll 1$ is carried out in two different regions: an *outer region* defined at an $O(1)$ distance from the perturbing subdomains and N distinct *inner regions* defined in $O(\varepsilon)$ neighborhoods of each subdomain. In these inner regions the solution must change quickly in order to satisfy the boundary condition on each ∂D_j^ε.

In the *outer region* we expand the steady-state solution $U = U(\boldsymbol{x};\varepsilon)$ to (3.1) in a regular perturbation series

$$U(\boldsymbol{x};\varepsilon) \sim U_0(\boldsymbol{x}) + \nu(\varepsilon)U_1(\boldsymbol{x}) + \cdots \,, \tag{3.2}$$

where $\nu(\varepsilon) \ll 1$ is a gauge function to be determined. Here U_0 is the solution to the unperturbed problem (with no subdomains) and satisfies

$$\nabla \cdot [p(\boldsymbol{x})\nabla U_0] - m(\boldsymbol{x})U_0 = 0\,, \qquad \boldsymbol{x} \in D\,, \tag{3.3a}$$

$$U_0 = u_b\,, \qquad \boldsymbol{x} \in \partial D\,. \tag{3.3b}$$

Since D_j^ε shrinks to a point $\boldsymbol{x}_j$ as $\varepsilon \to 0$, we get that U_1 satisfies

$$\nabla \cdot [p(\boldsymbol{x})\nabla U_1] - m(\boldsymbol{x})U_1 = 0\,, \qquad \boldsymbol{x} \in D\backslash\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N\}\,, \tag{3.4a}$$

$$U_1 = 0\,, \qquad \boldsymbol{x} \in \partial D\,, \tag{3.4b}$$

$$U_1 \quad \text{is singular as } \boldsymbol{x} \to \boldsymbol{x}_j\,, \;\; j = 1,\ldots,N\,. \tag{3.4c}$$

The singularity behavior is determined by the condition that the outer expansion asymptotically match with the inner solution to be constructed near each subdomain D_j^ε. Typically, we will obtain from this analysis that U_1 tends, as $\boldsymbol{x} \to \boldsymbol{x}_j$, to a certain multiple of the free-space Green's function for the Laplacian in the appropriate dimension. Hence, in all of D, we can re-write (3.4) as

$$\nabla \cdot [p(\boldsymbol{x})\nabla U_1] - m(\boldsymbol{x})U_1 = \sum_{j=1}^{N} \gamma_j \delta(\boldsymbol{x} - \boldsymbol{x}_j)\,, \qquad \boldsymbol{x} \in D\,, \tag{3.5a}$$

$$U_1 = 0\,, \qquad \boldsymbol{x} \in \partial D\,, \tag{3.5b}$$

$$\tag{3.5c}$$

for some constant γ_j to be determined. Here $\delta(\boldsymbol{x})$ is the usual delta function.

Now let $G(\boldsymbol{x};\boldsymbol{\xi})$ be the Green's function satisfying

$$\nabla \cdot [p(\boldsymbol{x})\nabla G] - m(\boldsymbol{x})G = \delta(\boldsymbol{x} - \boldsymbol{\xi})\,, \qquad \boldsymbol{x} \in D\,, \tag{3.6a}$$

$$G = 0\,, \qquad \boldsymbol{x} \in \partial D\,, \tag{3.6b}$$

for some fixed $\boldsymbol{\xi} \in D$. Then, the solution to (3.5) is simply a superposition of Green's functions with singularity at $\boldsymbol{x}_j$. In this way, we obtain that the outer expansion of the steady-state solution is

$$U(\boldsymbol{x};\varepsilon) \sim U_0(\boldsymbol{x}) + \sum_{j=1}^{N} \nu(\varepsilon)\gamma_j G(\boldsymbol{x};\boldsymbol{x}_j) + \cdots . \tag{3.7}$$

We now separately carry out the details of the analysis in the inner region, and the determination of the multipliers γ_j, for the two-dimensional and the three-dimensional cases.

Consider first the **three-dimensional** case. In the inner region near D_j^ε we introduce the stretched variables $\boldsymbol{y}$ and V by

$$\boldsymbol{y} = \varepsilon^{-1}(\boldsymbol{x} - \boldsymbol{x}_j)\,, \qquad V(\boldsymbol{y};\varepsilon) = U(\boldsymbol{x}_j + \varepsilon\boldsymbol{y};\varepsilon)\,. \tag{3.8}$$

In terms of these new variables, (3.1a) and (3.1c) become

$$\nabla^{'} \cdot \left[p(\boldsymbol{x}_j + \varepsilon\boldsymbol{y})\nabla^{'} V\right] - \varepsilon^2 m(\boldsymbol{x}_j + \varepsilon\boldsymbol{y})V = 0\,, \qquad \boldsymbol{y} \notin D_j\,, \tag{3.9a}$$

$$V = \alpha_j\,, \qquad \boldsymbol{y} \in \partial D_j\,. \tag{3.9b}$$

Here $\nabla^{'}$ indicates differentiation in the $\boldsymbol{y}$ variable, and D_j is the $O(\varepsilon^{-1})$ magnification of the original subdomain D_j^ε.

We then expand the solution $V(\boldsymbol{y};\varepsilon)$ as

$$V(\boldsymbol{y};\varepsilon) \sim V_0(\boldsymbol{y}) + \varepsilon V_1(\boldsymbol{y}) + \cdots . \tag{3.10}$$

Substituting (3.10) into (3.9), and equating powers of ε, we find that V_0 satisfies

$$\triangle^{'} V_0 = 0\,, \qquad \boldsymbol{y} \notin D_j\,, \tag{3.11a}$$

$$V_0 = \alpha_j\,, \qquad \boldsymbol{y} \in \partial D_j\,. \tag{3.11b}$$

Here $\triangle^{'}$ indicates the Laplacian in the $\boldsymbol{y}$ variable. In order to match to the outer expansion (3.2) we require that V_0 satisfy the far-field condition

$$V_0 \sim U_0(\boldsymbol{x}_j)\,, \qquad \text{as} \quad |\boldsymbol{y}| \to \infty\,. \tag{3.11c}$$

The solution to (3.11) can then be written as

$$V_0(\boldsymbol{y}) = U_0(\boldsymbol{x}_j) + [\alpha_j - U_0(\boldsymbol{x}_j)]\, \hat{V}_0(\boldsymbol{y})\,, \tag{3.12}$$

where $\hat{V}_0$ is the unique solution to

$$\triangle^{'} \hat{V}_0 = 0\,, \qquad \boldsymbol{y} \notin D_j\,, \tag{3.13a}$$

$$\hat{V}_0 = 1\,, \qquad \boldsymbol{y} \in \partial D_j\,, \tag{3.13b}$$

$$\hat{V}_0 \to 0\,, \qquad \text{as} \quad |\boldsymbol{y}| \to \infty\,. \tag{3.13c}$$

The far-field behavior as $|\boldsymbol{y}| \to \infty$ of the solution is given in terms of some constant $C_j > 0$ by

$$\hat{V}_0 \sim \frac{C_j}{|\boldsymbol{y}|} + O\left(\frac{1}{|\boldsymbol{y}|^2}\right) . \tag{3.13d}$$

The problem for $\hat{V}_0$ is a classical problem in electromagnetic theory (see [4]). In that context, $\hat{V}_0$ is called the capacitative potential and the constant C_j, which depends on the shape of the subdomain D_j, is called the capacitance of D_j. When D_j is a sphere of radius r_j, we have $C_j = r_j$; for a circular disk of radius r_j, $C_j = 2r_j/\pi$; for an oblate spheroid with semi-axes $a_1 = a_2 > a_3$, $C_j = (a_1^2 - a_3^2)^{1/2}/\cos^{-1}(a_3/a_1)$; for a prolate spheroid with semi-axes $a_1 > a_2 = a_3$, $C_j = (a_1^2 - a_2^2)^{1/2}/\cosh^{-1}(a_1/a_2)$. For a general ellipsoid,

$$C_j = 2\left(\int_0^\infty d\eta/R(\eta)\right)^{-1}, \quad \text{where} \quad R(\eta) \equiv (a_1^2+\eta)^{1/2}(a_2^2+\eta)^{1/2}(a_3^2+\eta)^{1/2} . \tag{3.14}$$

Next, substituting (3.13d) into (3.12) we find that the far-field behavior of V_0, when written in terms of the outer variable $\boldsymbol{x}$, is

$$V_0 \sim U_0(\boldsymbol{x}_j) + \frac{\varepsilon C_j}{|\boldsymbol{x} - \boldsymbol{x}_j|}\left[\alpha_j - U_0(\boldsymbol{x}_j)\right] . \tag{3.15}$$

To match to the outer solution we require that $\nu(\varepsilon) = \varepsilon$ and that U_1 have the singular behavior

$$U_1 \sim \left[\alpha_j - U_0(\boldsymbol{x}_j)\right]\frac{C_j}{|\boldsymbol{x} - \boldsymbol{x}_j|}, \qquad \text{as} \quad \boldsymbol{x} \to \boldsymbol{x}_j , \quad j = 1, \ldots, N . \tag{3.16}$$

Since the free-space Green's function for the Laplacian in three dimensions is $-1/4\pi r$, we require that γ_j in (3.5a) satisfy

$$\gamma_j = -4\pi C_j\left[\alpha_j - U_0(\boldsymbol{x}_j)\right]p(\boldsymbol{x}_j) . \tag{3.17}$$

Therefore, from (3.7) we obtain the following steady-state result:

Proposition 3.1: *For $\varepsilon \ll 1$, and in a three-dimensional bounded domain, the steady-state solution $U(\boldsymbol{x};\varepsilon)$ to (3.1) in the outer region satisfies*

$$U(\boldsymbol{x};\varepsilon) \sim U_0(\boldsymbol{x}) - 4\pi\varepsilon\sum_{j=1}^{N} C_j\left[\alpha_j - U_0(\boldsymbol{x}_j)\right]p(\boldsymbol{x}_j)G(\boldsymbol{x};\boldsymbol{x}_j) + \cdots , \qquad \text{for} \quad |\boldsymbol{x} - \boldsymbol{x}_j| \gg O(\varepsilon) , \tag{3.18}$$

and $j = 1, \ldots, N$. Here U_0 satisfies (3.3), G satisfies (3.6), and C_j is a constant defined by the solution to the inner problem (3.13).

Next, we consider the **two-dimensional** case. In the inner region near D_j^ε we again introduce the stretched variables $\boldsymbol{y}$ and V as in (3.8) to transform (3.1a) and (3.1c) to (3.9). We then expand the inner solution $V(\boldsymbol{y};\varepsilon)$ as

$$V(\boldsymbol{y};\varepsilon) \sim V_0(\boldsymbol{y}) + \mu(\varepsilon)V_1(\boldsymbol{y}) + \cdots . \tag{3.19}$$

Substituting (3.19) into (3.9), and assuming that $O(\varepsilon) \ll \mu(\varepsilon) \ll 1$, we find that V_0 and V_1 satisfy

$$\begin{aligned} \triangle' V_k &= 0\,, \qquad \boldsymbol{y} \notin D_j\,, && (3.20\text{a}) \\ V_k &= \alpha_j \delta_{k,0}\,, \qquad \boldsymbol{y} \in \partial D_j\,, && (3.20\text{b}) \end{aligned}$$

for $k = 0$ and $k = 1$. Here $\delta_{k,0}$ is the Kronecker symbol with $\delta_{0,0} = 1$ and $\delta_{k,0} = 0$ for $k \neq 0$. In order to match to the outer expansion (3.2) we require that the inner expansion $V(\boldsymbol{y};\varepsilon)$ satisfy $V(\boldsymbol{y};\varepsilon) \to U_0(\boldsymbol{x}_j)$ as $|\boldsymbol{y}| \to \infty$. In contrast to the three-dimensional case, it is impossible to enforce that V_0 satisfy (3.20) together with the far-field condition that $V_0 \to U_0(\boldsymbol{x}_j)$ as $|\boldsymbol{y}| \to \infty$. To overcome this difficulty with the leading order asymptotic matching condition it is necessary to take $V_0 \equiv \alpha_j$ everywhere outside D_j and allow V_1 to grow logarithmically as $|\boldsymbol{y}| \to \infty$. Matching is then possible when $\mu(\varepsilon) = O(-1/\log \varepsilon)$. More precisely, we define $\hat{V}_1(\boldsymbol{y})$ by $V_1 = v_\infty \hat{V}_1$, where v_∞ is a constant to be determined and $\hat{V}_1$ is the unique solution to

$$\begin{aligned} \triangle' \hat{V}_1 &= 0\,, \qquad \boldsymbol{y} \notin D_j\,, && (3.21\text{a}) \\ \hat{V}_1 &= 0\,, \qquad \boldsymbol{y} \in \partial D_j\,. && (3.21\text{b}) \\ \hat{V}_1 &\sim \log|\boldsymbol{y}|\,, \qquad \text{as} \quad |\boldsymbol{y}| \to \infty\,. && (3.21\text{c}) \end{aligned}$$

In terms of this solution, we define the unique constant d_j, referred to as the logarithmic capacitance of D_j, by

$$\hat{V}_1 \sim \log|\boldsymbol{y}| - \log d_j\,, \qquad \text{as} \quad |\boldsymbol{y}| \to \infty\,. \tag{3.21d}$$

For some specific shapes of subdomains D_j, the constant d_j can be found analytically (see [11] for a tabulation of d_j for different shapes). In particular, when D_j is a circle of radius r_j, we have $d_j = r_j$; for an ellipse with semi-axes a_1 and a_2 we have $d_j = (a_1 + a_2)/2$. For an arbitrarily shaped domain D_j, the constant d_j must be determined numerically.

Next, substituting (3.21d) and $V_0 = \alpha_j$ into (3.19) we find that the far-field behavior of V, when written in terms of the outer variable $\boldsymbol{x}$, is

$$V \sim \alpha_j + \mu(\varepsilon) v_\infty \left(\log|\boldsymbol{x} - \boldsymbol{x}_j| - \log[\varepsilon d_j] \right)\,. \tag{3.22}$$

To match to the outer solution we require that the gauge functions satisfy

$$\mu(\varepsilon) = -\frac{1}{\log[\varepsilon d_j]} = \nu(\varepsilon)\,, \tag{3.23a}$$

and that

$$v_\infty = U_0(\boldsymbol{x}_j) - \alpha_j\,; \qquad U_1 \sim [U_0(\boldsymbol{x}_j) - \alpha_j] \log|\boldsymbol{x} - \boldsymbol{x}_j|\,, \qquad \text{as} \quad \boldsymbol{x} \to \boldsymbol{x}_j\,. \tag{3.23b}$$

Now since the free-space Green's function for the Laplacian in two dimensions is $(2\pi)^{-1}\log r$, we get that γ_j in (3.5a) satisfies

$$\gamma_j = 2\pi \left[U_0(\boldsymbol{x}_j) - \alpha_j \right] p(\boldsymbol{x}_j) \,. \tag{3.24}$$

Therefore, from (3.7) we obtain the following steady-state result:

Proposition 3.2: *For $\varepsilon \ll 1$, and in a two-dimensional bounded domain, the steady-state solution $U(\boldsymbol{x};\varepsilon)$ to (3.1) in the outer region satisfies*

$$U(\boldsymbol{x};\varepsilon) \sim U_0(\boldsymbol{x}) + 2\pi \sum_{j=1}^{N} \left(\frac{-1}{\log\left[\varepsilon d_j\right]} \right) \left[U_0(\boldsymbol{x}_j) - \alpha_j \right] p(\boldsymbol{x}_j) G(\boldsymbol{x};\boldsymbol{x}_j) + \cdots \,, \qquad \text{for} \quad |\boldsymbol{x}-\boldsymbol{x}_j| \gg O(\varepsilon) \,, \tag{3.25}$$

and $j = 1, \ldots, N$. Here U_0 satisfies (3.3), G satisfies (3.6), and d_j is the constant defined by the solution to the inner problem (3.21).

The expansion (3.25) represents the leading term in an infinite logarithmic expansion for U in powers of $-1/\log\varepsilon$. Such a series can be summed by using ideas in [13], but we shall not carry out the details here. The advantages of using the asymptotic formulation over a direct numerical approach on the original problem (3.1) are that: (i) The problem for U_1 is not stiff as $\varepsilon \to 0$; (ii) Upon changing the shape of D_j one needs only to recompute the constants C_j and d_j for the three and two-dimensional cases, respectively. In contrast, when numerically computing for the full steady-state solution corresponding to (3.1) one must first re-mesh the domain to account for the change of shape of D_j. Then one must undertake a full series of computations for smaller and smaller ε.

3.2 The Long-Time Behavior of Solutions

To study the long-time behavior of solutions to (3.1) we represent the solution to (3.1) in terms of an eigenfunction expansion as

$$u(\boldsymbol{x},t;\varepsilon) = U(\boldsymbol{x};\varepsilon) + \sum_{k=1}^{\infty} a_k \phi_k e^{-\lambda_k t} \,. \tag{3.26}$$

Here $U(\boldsymbol{x};\varepsilon)$ denotes the steady-state solution to (3.1), which was constructed asymptotically in §3.1, and $\phi_k = \phi_k(x;\varepsilon)$, $\lambda_k(\varepsilon)$ are the normalized eigenpairs of

$$\nabla \cdot \left[p(\boldsymbol{x}) \nabla \phi \right] - m(\boldsymbol{x})\phi = -\lambda\phi \,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} \partial D_j^{\varepsilon} \,, \tag{3.27a}$$

$$\phi = 0 \,, \qquad \boldsymbol{x} \in \partial D \,, \tag{3.27b}$$

$$\phi = 0 \,, \qquad \boldsymbol{x} \in \partial D_j^{\varepsilon} \,, \qquad j = 1, \ldots, N \,, \tag{3.27c}$$

$$\int_D \phi^2 \, d\boldsymbol{x} = 1 \,. \tag{3.27d}$$

The coefficients $a_k(\varepsilon)$ in (3.26) satisfy

$$a_k = \int_D (f(\boldsymbol{x}) - U)\, \phi_k(x;\varepsilon)\, d\boldsymbol{x} \,. \tag{3.28}$$

For $t \gg 1$, the long-time behavior of the solution to (3.1) can be approximated in terms of the first eigenpair of (3.27) by

$$u(\boldsymbol{x}, t; \varepsilon) \sim U(\boldsymbol{x};\varepsilon) + a_1 \phi_1\,(\boldsymbol{x};\varepsilon)\, e^{-\lambda_1(\varepsilon)t} \,. \tag{3.29}$$

Hence, we must determine the first eigenpair of (3.27) in both the two and three-dimensional cases.

Since the analysis to determine the first eigenpair of (3.27) is very similar to that described in §2, we shall omit the details of the analysis and instead give only the main results. In the results below we have defined μ_1 and ψ_1 to be the principal eigenpair of the unperturbed problem corresponding to (3.27). Thus, ψ_1 and μ_1 satisfy (3.27a) in all of D together with the condition (3.27b) on ∂D, and the normalization condition (3.27d). In the three-dimensional case, the asymptotic estimates for λ_1 and for the long-time behavior are given in the following results:

Proposition 3.3: *Let C_j be defined as in (3.13). Then, for $\varepsilon \ll 1$, the principal eigenvalue λ_1 of (3.27) in three dimensions satisfies*

$$\lambda_1(\varepsilon) \sim \mu_1 + 4\pi\varepsilon \sum_{j=1}^{N} p(\boldsymbol{x}_j) C_j \left[\psi_1(\boldsymbol{x}_j)\right]^2 + \cdots \,. \tag{3.30a}$$

The long-time behavior of the solution is

$$u(\boldsymbol{x}, t; \varepsilon) \sim U(x;\varepsilon) + \int_D (f(\boldsymbol{x}) - U)\;(\psi_1 + 0(\varepsilon))\, d\boldsymbol{x}\, e^{-\lambda_1 t} \,. \tag{3.30b}$$

A similar result holds in the two-dimensional case:

Proposition 3.4: *Let d_j be defined as in (3.21). Then, for $\varepsilon \ll 1$, the principal eigenvalue λ_1 of (3.27) in two dimensions satisfies*

$$\lambda_1(\varepsilon) \sim \mu_1 + 2\pi \sum_{j=1}^{N} \left(-\frac{1}{\log\left[\varepsilon d_j\right]} \right) p(\boldsymbol{x}_j) \left[\psi_1(\boldsymbol{x}_j)\right]^2 + \cdots \,. \tag{3.31a}$$

The long-time behavior of the solution is

$$u(\boldsymbol{x}, t; \varepsilon) \sim U(x;\varepsilon) + \int_D (f(\boldsymbol{x}) - U_b) \left[\psi_1 + 0\left(\frac{-1}{\log \varepsilon} \right) \right] d\boldsymbol{x}\, e^{-\lambda_1 t} \,. \tag{3.31b}$$

Once again we remark that the expansion (3.31a) represents the leading term in an infinite logarithmic expansion for $\lambda_1 - \mu_1$ in powers of $-1/\log\varepsilon$.

3.3 Some Examples of the Theory

In order to be able to determine solutions analytically we shall assume in all of the examples below that $p(\boldsymbol{x})$ and $m(\boldsymbol{x})$ are constants with $p(\boldsymbol{x}) \equiv 1$. We first consider some examples in **three dimensions**.

Example 1 (Ellipsoid in a Sphere): Assume that $m > 0$ and let D be a sphere of radius one containing an ellipsoid with semi-axes εa_1, εa_2 and εa_3. We assume that the ellipsoid is centered at the origin and that $u = u_b$ on the boundary $r = 1$ of the sphere and $u = \alpha$ on the boundary of the ellipsoid. In this case, the solution U_0 to the unperturbed problem (3.3) is

$$U_0(r) = \frac{u_b}{r}\left(\frac{\sinh(\sqrt{m}r)}{\sinh(\sqrt{m})}\right), \tag{3.32}$$

where $r = |\boldsymbol{x}|$. In addition, we can calculate that the Green's function satisfying (3.6), where the singular point is at the origin, is

$$G(\boldsymbol{x};0) = \frac{1}{4\pi r}\left[\coth\left(\sqrt{m}\right)\sinh\left(\sqrt{m}r\right) - \cosh(\sqrt{m}r)\right], \qquad r = |\boldsymbol{x}|. \tag{3.33}$$

Therefore, from (3.18) the outer expansion of the steady-state solution for $\varepsilon \ll 1$ is

$$U(r;\varepsilon) \sim U_0(r) - 4\pi\varepsilon C\left[\alpha - U_0(0)\right]G(\boldsymbol{x};0), \tag{3.34}$$

where U_0 and G are defined in (3.32) and (3.33), respectively. The capacitance C for an ellipsoid is given in (3.15).

To determine the approach to the steady-state solution, we first calculate the principal eigenpair of the unperturbed problem. We readily obtain that

$$\mu_1 = m + \pi^2, \qquad \psi_1 = N_1 j_0(\pi r), \qquad j_0(z) \equiv \frac{\sin z}{z}, \tag{3.35}$$

where the normalization constant N_1 is $N_1 = \sqrt{\pi/2}$. Substituting (3.35) into (3.30a) we get that

$$\lambda_1 \sim \mu_1 + 2\varepsilon C\mu_1 + \cdots. \tag{3.36}$$

Further analytical examples of the three-dimensional theory can be given. In particular, if $m = 0$ we can easily determine G analytically for a sphere and for a parallelpiped. Hence, we can obtain explicit formulae for the eigenvalue λ_1 and the steady-state solution U in terms of the capacitances of the various small objects.

Example 2 (A Collection of Ellipsoids Below the Ground): For this example we let D be the unbounded domain $x_3 < 0$ and we calculate the steady-state solution to (3.1) when $p(\boldsymbol{x}) = 1$ and m is a constant. This problem models the spread of a hazardous material that can diffuse through the boundaries of a collection of small underground storage tanks. The mathematical problem that we must solve is

$$\mathcal{L}U \equiv \triangle U - mU = 0\,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} D_j^{\varepsilon}\,, \tag{3.37a}$$

$$U = 0\,, \qquad x_3 = 0\,, \tag{3.37b}$$

$$u = \alpha_j\,, \qquad \boldsymbol{x} \in \partial D_j^{\varepsilon}\,, \qquad j = 1, \ldots, N\,. \tag{3.37c}$$

The unperturbed steady-state solution is $U_0 = 0$. Hence, (3.18) can be written as

$$U(\boldsymbol{x};\varepsilon) \sim -4\pi\varepsilon \sum_{j=1}^{N} C_j \alpha_j G(\boldsymbol{x};\boldsymbol{x}_j)\,, \tag{3.38}$$

where $G(\boldsymbol{x};\boldsymbol{x}_j)$ satisfies $\mathcal{L}G = \delta(\boldsymbol{x}-\boldsymbol{x}_j)$ in all of D with $G = 0$ on the plane $x_3 = 0$. The free-space Green's function is for $\mathcal{L}$ is simply $G_f = -\left(4\pi\rho\right)^{-1} e^{-\sqrt{m}\rho}$, where $\rho = |\boldsymbol{x}|$. Hence, using the method of images, we obtain that G in (3.38) is given by

$$G(\boldsymbol{x};\boldsymbol{x}_j) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_j|} e^{-\sqrt{m}|\boldsymbol{x}-\boldsymbol{x}_j|} + \frac{1}{4\pi|\boldsymbol{x}-\bar{\boldsymbol{x}}_j|} e^{-\sqrt{m}|\boldsymbol{x}-\bar{\boldsymbol{x}}_j|}\,. \tag{3.39}$$

Here $\bar{\boldsymbol{x}}_j$ is the reflection of the point $\boldsymbol{x}_j$ in the coordinate axis $x_3 = 0$. The constants C_j in (3.38) are determined by the solution to (3.13). Using the asymptotic solution (3.38) we can easily calculate the flux through the ground given by the surface integral of $\partial U/\partial x_3$ over the plane $x_3 = 0$.

Next, we consider some examples in two dimensions.

Example 3 (Ellipse in a Circle): Assume that $m > 0$ and let D be a circle of radius one containing an ellipse with semi-axes εa_1 and εa_2. We assume that the ellipse is centered at the origin and that $u = u_b$ and $u = \alpha$ on the boundary of the circle and the ellipse, respectively. For this example, the solution U_0 to the unperturbed problem (3.3) is

$$U_0(r) = u_b \left(\frac{I_0\left(\sqrt{m}r\right)}{I_0\left(\sqrt{m}\right)} \right)\,, \tag{3.40}$$

where $r = |\boldsymbol{x}|$. The Green's function, which satisfies (3.6), where the singular point at the origin is,

$$G(\boldsymbol{x};0) = -\frac{1}{2\pi}\left[K_0\left(\sqrt{m}r\right) - K_0\left(\sqrt{m}\right) \frac{I_0\left(\sqrt{m}r\right)}{I_0\left(\sqrt{m}\right)} \right]\,. \tag{3.41}$$

In (3.40) and (3.41), K_0 and I_0 are the modified Bessel functions of order zero. Therefore, from (3.25) the outer expansion of the steady-state solution is

$$U(r;\varepsilon) \sim U_0(r) + 2\pi \left(\frac{-1}{\log[\varepsilon d]} \right) \left[U_0(0) - \alpha \right] G\,, \tag{3.42}$$

where U_0 and G are defined in (3.40) and (3.41), respectively. The logarithmic capacitance d for the ellipse is $d = (a_1 + a_2)/2$.

To determine the approach to the steady-state solution, we need to calculate the principal eigenpair of the unperturbed problem. In this case, we find

$$\mu_1 = m + z_1^2 \,, \qquad \psi_1 = N_1 J_0\left(z_1 r\right) \,. \tag{3.43}$$

Here z_1 is the first positive zero of $J_0(z)$, and the normalization constant N_1 is $N_1 = \pi^{-1/2}/J_0'(z_1)$. Substituting (3.43) into (3.31a) we get for $\varepsilon \ll 1$ that

$$\lambda_1 \sim \mu_1 + \frac{2}{\left[J_0'(z_1)\right]^2}\left(\frac{-1}{\log[\varepsilon d]}\right) \,, \tag{3.44}$$

where $d = (a_1 + a_2)/2$.

Further analytical examples are possible. In particular, if $m = 0$ we can easily determine G analytically for a circle and for a rectangle. This allows us to assign explicit values for the various constants in (3.25) and (3.31a).

Example 4 (A Collection of Cylinders Below the Ground): Finally, we consider the two-dimensional analogue of Example 2 where the domain D is the half-plane $x_2 < 0$. This problem models the spread of a hazardous material that can diffuse through the boundaries of a collection of underground pipes of constant cross-section. Typically we would like to determine the flux of this material through the ground $x_2 = 0$. The problem to solve is

$$\begin{aligned}
\mathcal{L}U \equiv \Delta U - mU &= 0 \,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} D_j^{\varepsilon} \,, && (3.45\text{a}) \\
U &= 0 \,, \qquad x_2 = 0 \,, && (3.45\text{b}) \\
U &= \alpha_j \,, \qquad \boldsymbol{x} \in \partial D_j^{\varepsilon} \,, \qquad j = 1, \ldots, N \,. && (3.45\text{c})
\end{aligned}$$

The unperturbed steady-state solution is $U_0 = 0$. Hence, (3.25) can be written as

$$U(\boldsymbol{x};\varepsilon) \sim -2\pi \sum_{j=1}^{N} \left(\frac{-1}{\log[\varepsilon d_j]}\right) \alpha_j G(\boldsymbol{x};\boldsymbol{x}_j) \,. \tag{3.46}$$

Here $G(\boldsymbol{x};\boldsymbol{x}_j)$ satisfies $\mathcal{L}G = \delta(\boldsymbol{x} - \boldsymbol{x}_j)$ in all of D with $G = 0$ on $x_2 = 0$. The free-space Green's function for $\mathcal{L}$ is simply $G_f = -\left(2\pi\right)^{-1} K_0(\sqrt{m}\rho)$, where $\rho = |\boldsymbol{x}|$. Hence, using the method of images, we obtain that G in (3.46) is given by

$$G(\boldsymbol{x};\boldsymbol{x}_j) = -\frac{1}{2\pi}\left[K_0\left(\sqrt{m}|\boldsymbol{x} - \boldsymbol{x}_j|\right) - K_0\left(\sqrt{m}|\boldsymbol{x} - \bar{\boldsymbol{x}}_j|\right)\right] \,. \tag{3.47}$$

Here $\bar{\boldsymbol{x}}_j$ is the reflection of the point $\boldsymbol{x}_j$ in the line $x_2 = 0$. The constant d_j in (3.46) is determined by the solution to (3.21).

4 Some Bifurcation Problems

In this section we apply the results of §3 to determine a bifurcation point of the nonlinear parameter dependent problem

$$\begin{aligned} u_t &= \nabla \cdot [p(\boldsymbol{x}) \nabla u] + \sigma f(u) \,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} D_j^{\varepsilon} \,, && (4.1a) \\ \partial_n u + bu &= 0 \,, \qquad \boldsymbol{x} \in \partial D \,, && (4.1b) \\ u &= 0 \,, \qquad \boldsymbol{x} \in \partial D_j^{\varepsilon} \,, \qquad j = 1, \ldots, N \,, && (4.1c) \\ u(\boldsymbol{x}, 0) &= f(\boldsymbol{x}) \,. && (4.1d) \end{aligned}$$

Here $\sigma > 0$ is a positive parameter, ∂_n indicates the outward normal derivative, and b may depend on $\boldsymbol{x} \in \partial D$. The other assumptions are the same as those described following (3.1) above. We assume that $f(u)$ is such that $f(0) = 0$ and $f'(0) > 0$. Notice that $u = 0$ is a steady-state solution for any value of σ. Our goal is to determine the location of the first bifurcation point that is associated with the bifurcation off of this trivial solution.

We first linearize (4.1) around $u = 0$. We set $u = v$, where $v \ll 1$ and we then write $v = e^{-\omega t} \phi$. The corresponding eigenvalue problem for ϕ and λ is

$$\begin{aligned} \nabla \cdot [p(\boldsymbol{x}) \nabla \phi] + \lambda \phi &= 0 \,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} \partial D_j^{\varepsilon} \,, && (4.2a) \\ \partial_n \phi + b \phi &= 0 \,, \qquad \boldsymbol{x} \in \partial D \,, && (4.2b) \\ \phi &= 0 \,, \qquad \boldsymbol{x} \in \partial D_j^{\varepsilon} \,, \qquad j = 1, \ldots, N \,, && (4.2c) \\ \int_D \phi^2 \, d\boldsymbol{x} &= 1 \,. && (4.2d) \end{aligned}$$

Here we have defined λ by

$$\lambda = \sigma f^{'}(0) + \omega \,. \tag{4.2e}$$

The first bifurcation point occurs at the minimum value of σ for which $\omega = 0$. Labeling this point by σ_c, we get

$$\sigma_c = \lambda_1 / f^{'}(0) \,, \tag{4.3}$$

where λ_1 is the principal eigenvalue of (4.2). As σ increases past σ_c, the trivial solution $u = 0$ becomes unstable.

Thus, to determine the bifurcation point we must calculate the principal eigenvalue for (4.2). This was done in §3 (for slightly different boundary conditions). The results for λ_1 given in Prop. 3.3 and Prop. 3.4 for the three and two-dimensional cases, respectively, still hold provided that in (3.30a) and (3.31a) we use the principal eigenpair μ_1 and ψ_1 of the unperturbed problem associated with (4.2).

We now give an example of the theory.

Example (The Safe Fishing Zone Problem): Let P denote the population of a fish species in coastal waters that can reproduce by logistic growth and then diffuse. We assume that there is no flux of fish through the coastline and that at some distance $X = L$ perpendicular to the coastline all of the fish are removed by the presence of foreign fish boats. We assume that there is some limited fishing taking place between $0 < X < L$, where local fishing boats are removing all of the fish. In dimensional variables, we formulate this problem for $P(X, Y, \tau)$ in two spatial dimensions as

$$
\begin{aligned}
P_\tau &= \nu \triangle P + rP\,(1 - P/K)\,, \quad (X,Y) \in D \backslash \bigcup_{j=1}^{N} R_j\,, && (4.4a)\\
P_X(0,Y,\tau) &= P_Y(X, \pm H, \tau) = 0\,; \qquad P(L,Y,\tau) = 0\,. && (4.4b)\\
P &= 0\,, \qquad (X,Y) \in \partial R_j\,, \qquad j = 1,\ldots,N\,, && (4.4c)\\
P(X,Y,0) &= P_0(X,Y)\,. && (4.4d)
\end{aligned}
$$

Here r is the growth rate of fish for small P, K is the carrying capacity, D is the region $[0, L] \times [-H, H]$, and R_j is one of the regions where the local, or near-coast, fishing is taking place. Our assumption is that the region where local fishing occurs is much smaller than L, i. e. $L \gg \text{Radius}(R_j)$, for $j = 1,\ldots,N$. Also, we assume that $H > L$.

We non-dimensionalize (4.4) by introducing the scaled variables u, x_1, x_2 and t by

$$
t = \tau\nu/L^2\,, \qquad x_1 = X/L\,, \qquad x_2 = Y/L\,, \qquad u = P/K\,. \tag{4.5}
$$

Then (4.4) becomes

$$
\begin{aligned}
u_t &= \triangle u + \sigma f(u)\,, \qquad \boldsymbol{x} \in D \backslash \bigcup_{j=1}^{N} D_j^{\varepsilon}\,, && (4.6a)\\
u_{x_1}(0,x_2,t) &= u_{x_2}(x_1, \pm\beta, t) = 0\,; \qquad u(1,x_2,t) = 0\,. && (4.6b)\\
u &= 0\,, \qquad \boldsymbol{x} \in D_j^{\varepsilon}\,, \qquad j = 1,\ldots,N\,, && (4.6c)\\
u(X,Y,0) &= u_0(X,Y)\,. && (4.6d)
\end{aligned}
$$

Here $\boldsymbol{x} = (x_1, x_2)$, $\beta = H/L$, $D = [0,1] \times [-\beta, \beta]$, and D_j^{ε} is one of the regions where local fishing occurs. The size of D_j^{ε} is $O(\varepsilon)$, where $\varepsilon \ll 1$ is the ratio of the 'radius' of R_j to L. Finally, σ and $f(u)$ are defined by

$$\sigma = rL^2/\nu \,, \qquad f(u) = u(1-u)\,. \tag{4.6e}$$

If we then linearize (4.6) about the trivial solution $u = 0$ corresponding to a depleted fish stock, and then separate variables, we get (4.2) with $p(\boldsymbol{x}) \equiv 1$ and where $b = 0$ on all sides of the rectangle except along $x_1 = 1$ where $b = \infty$. The principal eigenvalue for the unperturbed problem, described in [8], has an eigenfunction that is independent of x_2 and we obtain

$$\mu_1 = \pi^2/4 \,, \qquad \psi_1 = \sqrt{2}\cos\left(\pi x_2/2\right) \,. \tag{4.7}$$

Therefore, since $f'(0) = 1$, we get for the unperturbed problem that $\sigma_c = \mu_1$. Using the definition (4.6e) for σ, this implies that if $L < \nu^{1/2}\pi/(2r^{1/2})$ that the depleted fish stock solution $u = 0$ is stable. Substituting (4.7) into (3.31a) we obtain that the depleted fish stock solution for the perturbed problem is stable when $L < L_c(\varepsilon)$, where

$$L_c(\varepsilon) = \sqrt{\frac{\nu}{r}}\left(\frac{\pi^2}{4} + S(\varepsilon)\right)^{1/2} \,, \qquad S(\varepsilon) \equiv 4\pi \sum_{j=1}^{N} \left(\frac{-1}{\log[\varepsilon d_j]}\right) \cos^2\left(\frac{\pi x_{1j}}{2}\right) \,. \tag{4.8}$$

Here x_{1j} is the x_1 coordinate of the center of the local fishing zones, and d_j depends on the shape of these zones as defined by (3.21). As a result of the fact that $-\log\varepsilon$ increases very slowly as ε decreases, we conclude that the presence of the local fishing zones forces L to have to increase by a significant amount, even when ε is rather small, in order to de-stabilize the zero-fish equilibrium solution.

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