The Slow Dynamics of Localized Spot Patterns for RD Systems on the Sphere

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Singularly Perturbed RD Models: Spots

Spatially localized solutions can occur for singularly perturbed RD models

$$u_t = \varepsilon^2 \triangle u + g(u,v)\,; \quad \tau v_t = D \triangle v + f(u,v)\,, \qquad x \in \mathbb{R}^2\,.$$

Assume semi-strong interactions: $\varepsilon \ll 1$ and $D = \mathcal{O}(1)$.

Key: Since $\varepsilon \ll 1$, u can be localized in space as a spot pattern, i.e. concentration at a discrete set of points.

Our Focus: Analysis of Localized Spot Patterns on the Sphere:

- **Brusselator Model:** $g(u,v) = E - (B+1)u + u^2 v$ and $f(u,v) = Bu - u^2 v$.
- **Schnakenburg Model:** $g(u,v) = -u + u^2 v$ and $f(u,v) = a - vu^2$.

Two Distinct Methodologies:

- **Classical Approach**: stability of spatially uniform states, Turing and weakly nonlinear analysis of small amplitude patterns, leading to normal form amplitude equations. (Key: equivariant bifurcation theory).
- **Localized Patterns:** “Far-from equilibrium patterns” (Nishiura) consisting of “particles” interacting through a “diffusion field”.

Outline

Part I: Phenomena: Localized Particle-Like Spot Solutions on the Sphere

1. **Terminology:** Competition Instabilities, Oscillatory Profile Instabilities, Spot Self-Replication Instabilities.
2. **Stability/Transitions Phenomena:** Localized Spot Patterns and Instabilities:
   - If $D = O(1/\nu) \gg 1$ with $\nu = -1/\log \varepsilon$ then competition and oscillatory instabilities are possible.
   - If $D = O(1)$, then for some models (GS, Brusselator, Schnakenburg) spot self-replication instabilities can occur if the “fuel” exceeds a threshold.

Part II: Specifics: Stability and Dynamics of Localized Spot Patterns on the Sphere

- Derive a differential algebraic system of ODE’s (DAE) for the dynamics of the centers of spots on the sphere (Brusselator, Schnakenburg)
- New class of “point-vortex models” on the sphere. (Eulerian fluids)
- Equilibria of the DAE system? Relationship to the elliptic Fekete point problem and diffusion problems in the presence of traps.

Asymptotic Limit of Brusselator

$$U_T = \varepsilon_0^2 \Delta_s U + E - (B+1)U + U^2 V \,, \qquad V_T = D \Delta_s V + BU - U^2 V \,.$$

Here Δ_s is the Laplace-Beltrami for the sphere.

Asymptotic Limit Where Spot Patterns Exist: Assume large diffusivity ratio and small “fuel” E:

$$\varepsilon_0 \ll 1 \,, \qquad D = \mathcal{O}(1) \,, \qquad E = \varepsilon_0 \mathrm{E}_0 = \mathcal{O}(\varepsilon) \,.$$

Introduce new variables: $U = Bu/\varepsilon_0$, $V = \varepsilon_0 v$, $T = t/(B+1)$. We get:

$$u_t = \varepsilon^2 \Delta_s u + \varepsilon^2 \mathrm{E} - u + f u^2 v \,, \qquad \tau v_t = \mathcal{D} \Delta_s v + \frac{1}{\varepsilon^2}(u - u^2 v) \,.$$

where

$$f \equiv \frac{B}{B+1} \,, \quad \tau \equiv \frac{1}{f^2} \,, \quad \mathrm{E} \equiv \frac{\mathrm{E}_0}{B} \,, \quad \mathcal{D} \equiv \frac{D(B+1)}{B^2} \,, \quad \varepsilon \equiv \frac{\varepsilon_0}{\sqrt{B+1}} \,.$$

Note: $0 < f < 1$. If $u = \mathcal{O}(1)$, then $U = \mathcal{O}(\varepsilon_0^{-1}) \gg 1$ near center of a spot.

Classical Theory: Turing-Type Analysis I

A further re-scaling of $u = \sqrt{\mathcal{D}}\tilde{u}$, $v = \tilde{v}/\sqrt{\mathcal{D}}$ yields

$$\tilde{u}_t = \varepsilon^2 \Delta_s \tilde{u} + \frac{\varepsilon^2 \mathrm{E}}{\sqrt{\mathcal{D}}} - \tilde{u} + f \tilde{u}^2 \tilde{v} \,, \qquad \frac{\tau}{\mathcal{D}} \tilde{v}_t = \Delta_s \tilde{v} + \frac{1}{\varepsilon^2} (\tilde{u} - \tilde{u}^2 \tilde{v}) \,.$$

Thus, we have the "fuel" $\mathrm{E}/\sqrt{\mathcal{D}}$. **Bifurcation parameters:** $\mathcal{D}$, E and f.

Classicial Approach: The spatially uniform state is

$$u_e = \varepsilon^2 \mathrm{E}/(1-f) \,, \quad v_e = (1-f)/(\varepsilon^2 \mathrm{E}) \,.$$

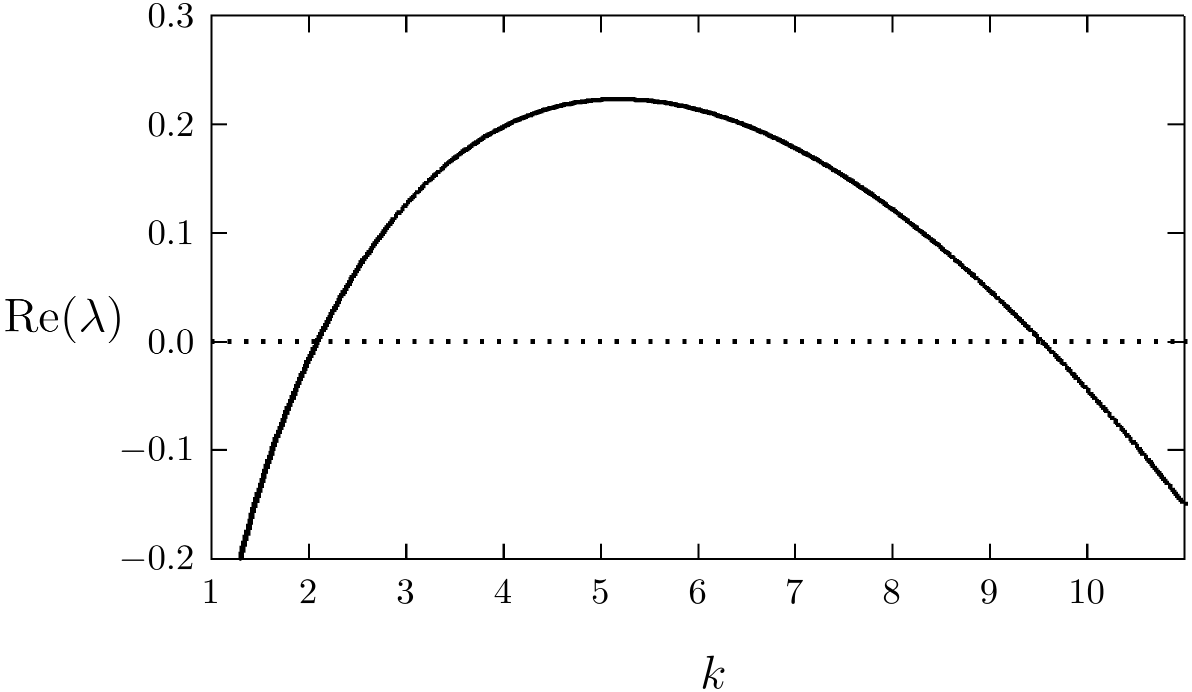
Turing-Type stability analysis: Introduce perturbation of the uniform state:

$$(u, v) = (u_e, v_e) + e^{\lambda t} Y_l^m(\theta, \phi)(\hat{u}, \hat{v}) \,, \qquad k^2 \equiv l(l+1) \,, \quad |m| \leq l \,.$$

Key: For $\varepsilon \to 0$ and $f > 1/2$ (correspnding to $B > 1$) the (wide) instability band where Re$(\lambda) > 0$ is

$$0 < k_{\text{low}} < k < k_{\text{up}} \sim \sqrt{2f - 1}/\varepsilon \,.$$

Classical Theory: Turing-Type Analysis II



Plot: Re(λ) versus k for $f = 0.8$, $\varepsilon = 0.075$, $\mathcal{D} = 0.2$, and $\mathrm{E} = 4$.

Key: For $\varepsilon \ll 1$, any $Y_l^m(\theta, \phi)$ with integers l and m satisfying $4.45 \leq l(l+1) \leq 91.6$ and $|m| \leq l$ is unstable. This gives $l = 2, \ldots, 9$.

Classical: Normal Form Theory

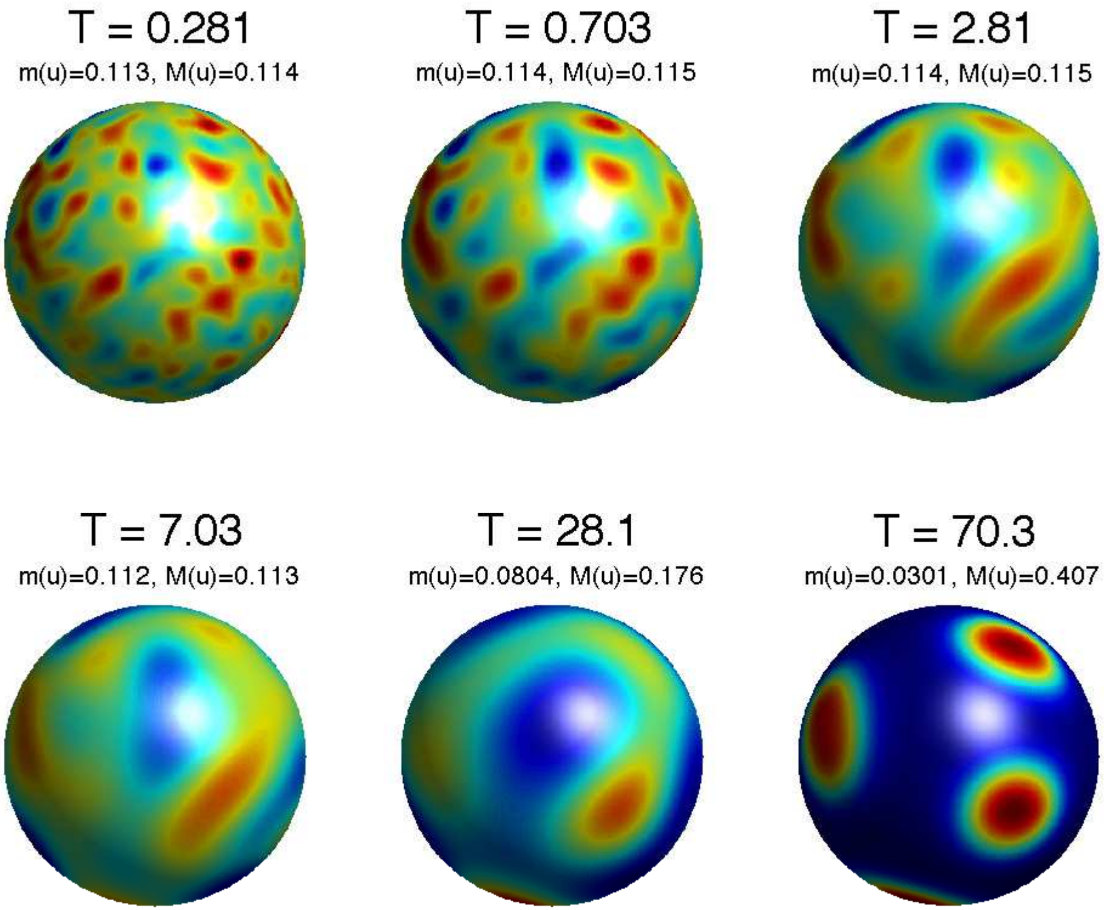
Fundamental Difficulty: The linear stability problem with $f > 1/2$ and $\varepsilon \ll 1$ is highly degenerate with many unstable modes of comparable growth rate. Pattern forms from weakly nonlinear interaction of many unstable modes with comparable growth rates.

Weakly Nonlinear Analysis: Non-Singular Limit

- Simultaneous zero-eigenvalue crossing due to degeneracy of spherical harmonics $Y_l^m(\theta, \phi)$.
- **Equivariant bifurcation theory** characterizes the form of coupled amplitude equations (Chossat, et al. ARMA 1990), which increase in size as l increases.
- However, quadratic terms predict instability, with possible re-stablization at cubic terms (through saddle-node).
- For $l = 1, \ldots, 6$ the coefficients in the normal form for the brusselator have been derived (Callaghan, Physica D 2003) in a tuning-limit where quadratic and cubic terms are of comparable order.

Brusselator Patterns: Numerics

Full Numerics: $f = 0.8$, $\varepsilon = 0.075$, $\mathcal{D} = 0.2$, **and** $\mathrm{E} = 4$. **The initial data is a** 2% **random perturbation from the spatially uniform state.**



Numerics: New PDE methodologies: "Closest Point Algorithms to Compute PDE's on Surfaces"; S. Ruuth (SFU) , C. McDonald (Oxford, UBC).

Brusselator Patterns: Challenges

- Very complicated transient dynamics due to the weakly nonlinear interaction of many unstable spherical harmonics.
- A distinct localized pattern with 8 localized spots at $t \approx 70.3$.
- Due to mode degeneracy, linear and weakly nonlinear analysis is of only limited use in predicting pattern development.

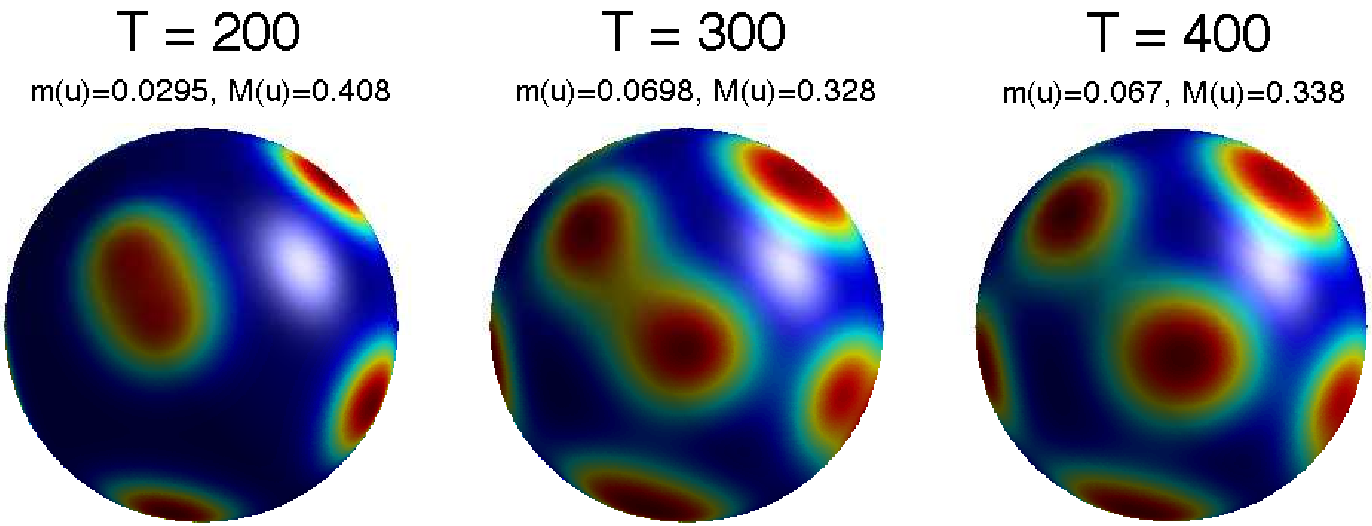
Challenge: Develop a mathematical theory to analyze the existence, stability, and dynamics, of localized "far-from equilibrium" spot patterns.

- **Question 1:** Do such localized patterns undergo secondary instabilities on longer time-scales? (competition, self-replication, etc.)
- **Question 2:** Are spot dynamics similar to that of Eulerian point vortices?
- **Question 3:** Are steady-state spot patterns related to elliptic Fekete point distributions of point charges?

References:

- **[RRW] Rozada, Ruuth, Ward; The Stability of Localized Spot Patterns for the Brusselator on the Sphere**, SIADS **13**(1), (2014), pp. 564–627.
- **[TW] Trinh, Ward; Dynamics of Localized Spot Patterns for RD Systems on the Sphere**, submitted Nonlinearity, (November 2014), (37 pages).

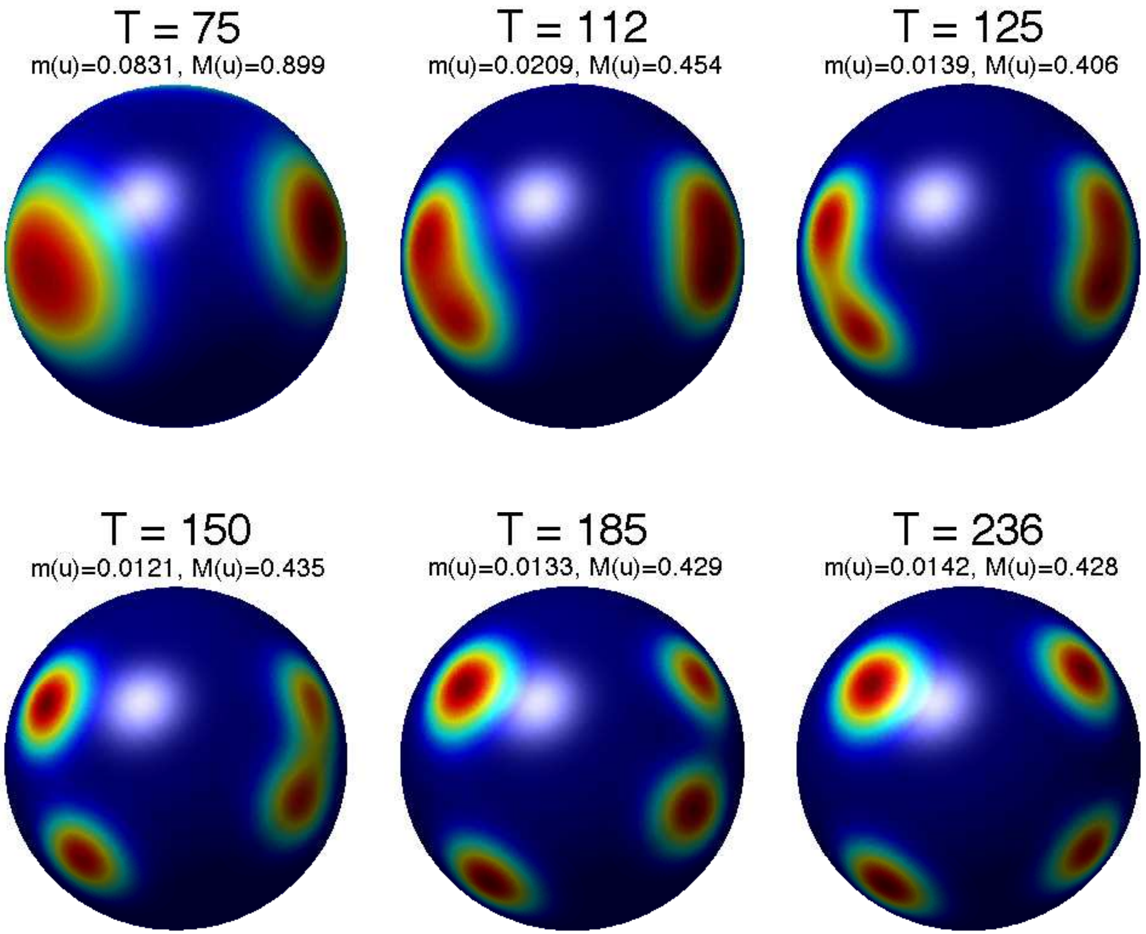
Spot Self-Replication



Plot: Solution at later times for $f = 0.8$, $\varepsilon = 0.075$, and $\mathcal{D} = 0.2$. For $t \leq 70$ we take $\mathrm{E} = 4$. and then slowly increase E as $\mathrm{E} = \min(4 + 0.05(t-70), 6)$.
Key: Spot self-replication occurs over a rather long time-scale due to dynamically-triggered bifurcation.

Spot self-replication (Another Example): Take $f = 0.7$, $\varepsilon = 0.06$, $\mathcal{D} = 0.7$, and $\mathrm{E} = 2.5$ for $0 < t < 50$. Increase the fuel E as $\mathrm{E} = 2.5 + \sigma(t-50)$ with $\sigma = 0.05$ for $t \geq 50$. Roughly simultaneous spot splitting occurs. (Movie)

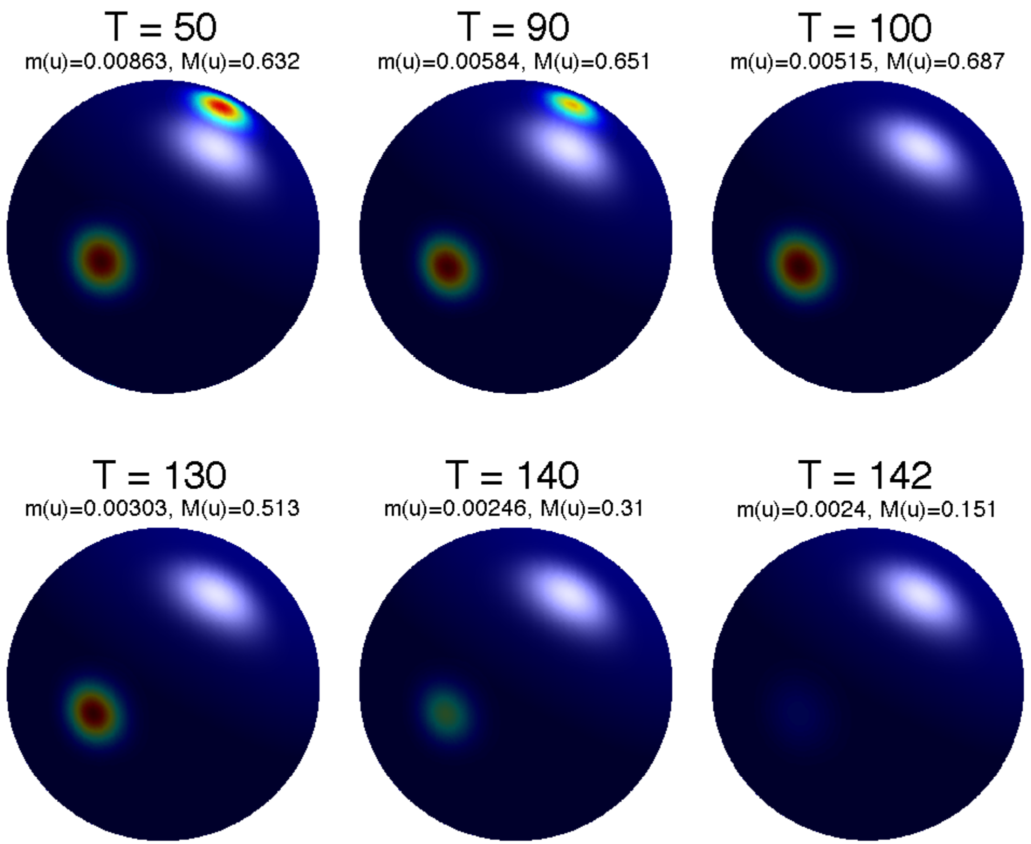
Self-Replication: Growing Domain



Plot: Solution at later times for $f = 0.8$, $\varepsilon = 0.12$, and $\mathcal{D} = 1.0$. For $t \leq 70$ we take $\mathrm{E} = 4$. For $t \geq 75$ we take $\varepsilon = 0.12/L$ and $\mathcal{D} = 1/L^2$, where $L = \min(1.0 + 0.02(t - 75), 2.0)$, for sphere of radius L.

Key: Four spots undergo (roughly) simultaneous spot self-replication.

Competitition Instabilities



Plot: Solution for $f = 0.7$, $\varepsilon = 0.06$, $\mathcal{D} = 0.7$, and $\mathrm{E} = 2.5$. For $0 < t < 50$, we take $\mathrm{E} = 2.5$ for $0 \leq t \leq 50$, and then slowly decrease E as $\mathrm{E} = 2.5 - \sigma(t - 50)$ with $\sigma = 0.02$ for $t \geq 50$. The four-spot pattern that exists at the end of the transient dynamics undergoes a competition instability destroying two spots.

Three Main $\mathcal{O}(1)$-Time Scale Instabilities

Competition Instability: instability from a positive real eigenvalue with sign fluctuating eigenfunction. Triggers monotonic collapse of spots.

Oscillatory Instability: An instability due to a Hopf bifurcation with in-phase eigenfunction that triggers a synchronous oscillatory collapse of spots (subcritical?)

Self-Replication Instability: An instability of the shape of the spot profile to locally non-radially symmetric perturbations. This linear instability triggers a nonlinear spot-splitting event.

All Can Be Dynamically triggered: i.e. each $\mathcal{O}(1)$ time-scale instability can be triggered only at some later time through the collective slow dynamics of the spot locations. Bifurcations induced by intrinsic spot motion, not by externally varying control parameter.

Mathematical Challenge: Classify instability types and determine instability thresholds in a phase diagram in parameter space for certain equilibrium and quasi-equilibrium spot patterns. Characterize the slow dynamics of quasi-equilibria before/after fast instabilities.

Brief History: Localized Patterns

Difficulties: PDE's with No variational structure; Turing and weakly nonlinear theories are not applicable.

1-D Theory: Stability and Dynamics of Localized Pulse-Type Solutions (over past 15 years): geometric singular perturbation, Lyapunov-Schmidt, NLEP analysis, matched asymptotics, renormalization group: (Doelman, Kaper, Gardner, Promislow, Rademacher, Van der Ploeg, ... ; Nishiura, Ueda, Ei, ...; Muratov, Osipov; Iron, Ward, Wei, Kolokolnikov,....)

2-D Theory: "Particle+Field Descriptions": Coulomb Interactions, with gauge $\nu = -1/\log \varepsilon$

- **NLEP Stability Theory for Spots:** (Wei-Winter, 2000-2004)
- **One-Spot Dynamics for D Large:** X. Chen, M. Kowalzcyk (2002); Kolokolnikov, Ward (2003).
- **Slow Dynamics of a Collection of Localized Spots for $D = \mathcal{O}(1)$:** on the plane: Kolokolnikov, Ward, Wei, (Schnakenburg Model, J. Nonlinear Science, 2009); Chen, Ward (GS Model, SIADS 2011).
- **Weakly Interacting Spots** on the plane: Ei, Mimura, Nishiura

Part II: Quasi-Equilibrium Patterns I

$$u_t = \varepsilon^2 \Delta_s u + \varepsilon^2 \mathrm{E} - u + f u^2 v \,, \qquad \tau v_t = \Delta_s v + \frac{1}{\varepsilon^2}(u - u^2 v)\,.$$

MMAE approach: Construct quasi-equilibrium localized spot patterns using MMAE tailored to problems with logarithmic expansions (Ward, Henshaw, Keller (1993).

Core Problem: Consider a collection of N spots centered at $\mathbf{x}_j \,,\, j = 1, \ldots, N$. On the local-in-$\varepsilon$-tangent-plane at $\mathbf{x}_j$, we have the core problem:

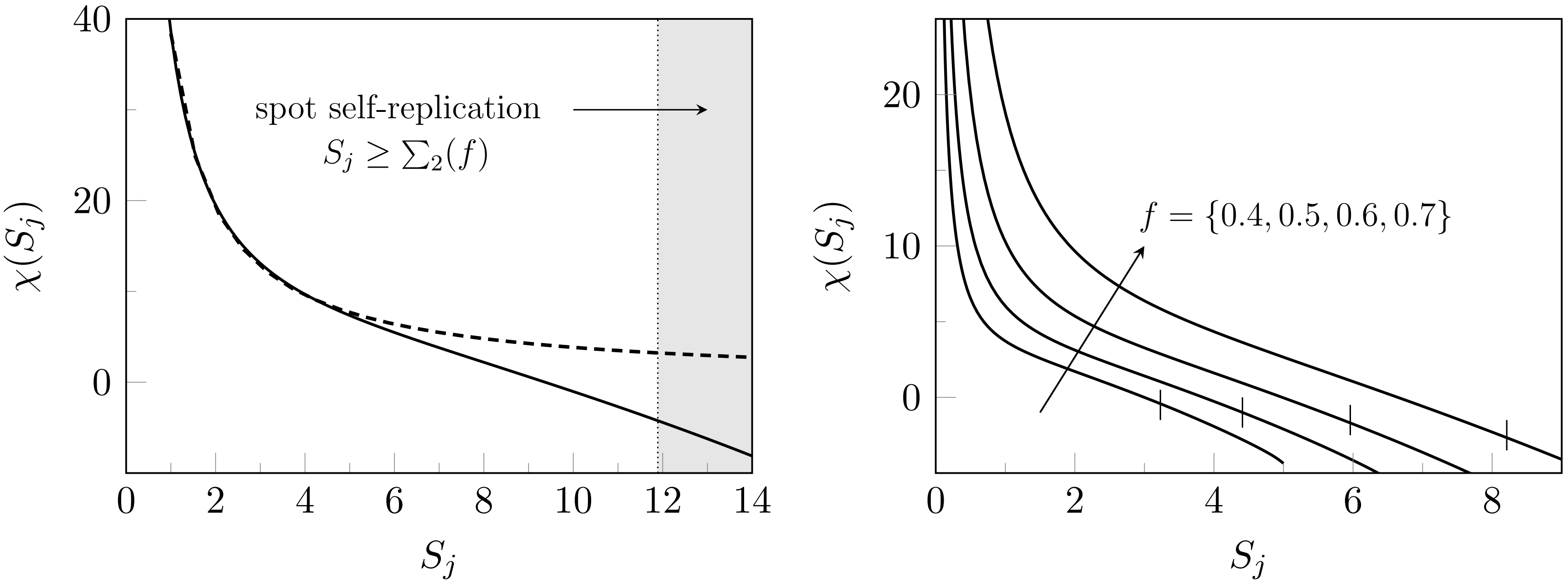
$$\Delta_\rho U_{j0} - U_{j0} + f U_{j0}^2 V_{j0} = 0\,, \qquad \Delta_\rho V_{j0} + U_{j0} - U_{j0}^2 V_{j0} = 0\,,$$

$$U_{j0} \to 0\,, \quad V_{j0} \sim S_j \log \rho + \chi + o(1) \quad \text{as } \rho \to \infty.$$

Here $\chi = \chi(S_j; f)$ must be computed numerically.

Peanut-Splitting Instability: Numerical computations of a 1-D eigenvalue problem show that for $S_j > \Sigma_2(f)$, the j-th spot is linearly unstable on an $\mathcal{O}(1)$ time-scale to peanut-splitting. This is the trigger of a nonlinear spot self-replication event.

Part II: Quasi-Equilibrium Patterns II



Caption: Left: χ versus S_j for $f = 0.3$ (heavy solid), compared with asymptotics. Right: χ versus S_j for $f = 0.4$, $f = 0.5$, $f = 0.6$, and $f = 0.7$, showing spot self-replication threshold $S_j = \Sigma_2(f)$ as thin vertical line.

Quasi-Equilibrium Patterns III

Outer Approximation: The leading-order inhibitor field satisfies

$$\Delta_S v = -\mathrm{E} + 2\pi \sum_{i=1}^{N} S_i \delta(\mathbf{x} - \mathbf{x}_i)\,, \qquad \sum_{i=1}^{N} S_i = 2\mathrm{E}\,,$$

$$v \sim S_j \log|\mathbf{x} - \mathbf{x}_j| - S_j \log\varepsilon + \chi(S_j)\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j\,.$$

In terms of the well-known source-neutral Green's function G,

$$v = -2\pi \sum_{i=1}^{N} S_i G(\mathbf{x};\mathbf{x}_i) + \overline{v} = \sum_{i=1}^{N} S_i L_i(\mathbf{x}) - (2\log 2 - 1)\mathrm{E} + \overline{v}\,,$$

Matching: gives a nonlinear algebraic system (NAS) for the S_j:

$$S_j + \nu\chi(S_j; f) - \nu \sum_{\substack{i=1 \\ i\neq j}}^{N} S_i L_{ij} = \overline{v}_c\,, \quad j = 1\,,\ldots,N\,; \qquad \sum_{i=1}^{N} S_i = 2\mathrm{E}\,,$$

where ν, L_{ij}, and $\overline{v}_c$ are defined by

$$\nu \equiv -1/\log\varepsilon\,, \qquad L_{ij} = \log|\mathbf{x}_i - \mathbf{x}_j|\,, \qquad \overline{v} \equiv \frac{\overline{v}_c}{\nu} + (2\log 2 - 1)\mathrm{E}\,.$$

NLEP Theory: Competition Instabilities

Consider $\mathrm{E} = \mathcal{O}(\sqrt{\nu})$, and spot patterns for which $S_j = \mathcal{O}(\sqrt{\nu}) \ \forall j$.

To leading-order in ν, the stability of these patterns is characterized in terms of the discrete eigenvalues of a class of NLEP of the form

$$L_0 \Phi - \beta(\lambda) w^2 \frac{\int_0^\infty w \Phi \rho \, d\rho}{\int_0^\infty w^2 \rho \, d\rho} = \lambda \Phi \,, \qquad \Phi \to 0 \quad \text{as} \quad \rho \to \infty \,;$$

where $\Delta_\rho w - w + w^2 = 0$ and $L_0 \Phi \equiv \Delta \Phi - \Phi + 2w\Phi$, and

$$\beta(\lambda) = \frac{a_1 + b_1 \lambda + c_1 \lambda^2}{a_2 + b_2 \lambda + c_2 \lambda^2} \,, \qquad \text{bi-quadratic}$$

- This NLEP is independent of the spot locations.
- Wei-Winter NLEP theory for GS and GM needed bilinear $\beta(\lambda)$.

Main Result: For $\mathrm{E} < \frac{N\sqrt{\nu d_0}}{2}$ where $d_0 \equiv b(1-f)/f^2$ and $b \approx 4.934$, there are no positive real eigenvalues and hence no competition instabilities.

Scaling Law: The number N of spots with a common S_j that are stable is

$$\frac{2\mathrm{E}}{\Sigma_2(f)} < N < \frac{2\mathrm{E}}{\sqrt{\nu}} \frac{f}{\sqrt{b(1-f)}} \,, \qquad b \approx 4.934.$$

Brusselator: Slow Spot Dynamics I

A higher-order calculation yields the following reduced DAE system:

Principal Result: *For $\varepsilon \to 0$, and provided that the q.e. pattern is stable on an $\mathcal{O}(1)$ time-scale, the dynamics of a collection of N spots satisfies the DAE system for $S_1, \ldots, S_N$ and $\mathbf{x}_1, \ldots, \mathbf{x}_N$ on long time-scale $\varepsilon^2 t$:*

$$\frac{d\mathbf{x_j}}{dt} = \frac{2\varepsilon^2}{\mathcal{A}_j} \left(\mathrm{I} - \mathcal{Q}_j \right) \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{S_i \mathbf{x}_i}{|\mathbf{x}_i - \mathbf{x}_j|^2} \,, \qquad \mathcal{Q}_j \equiv \mathbf{x}_j \mathbf{x}_j^T \,, \qquad j = 1, \ldots, N \,,$$

coupled to the nonlinear algebraic constraint (NAS)

$$\mathcal{N}(\mathbf{S}) \equiv \left[\mathrm{I} - \nu(\mathrm{I} - \mathcal{E}_0)\mathcal{G} \right] \mathbf{S} + \nu(\mathrm{I} - \mathcal{E}_0)\chi(\mathbf{S}; f) - \frac{2\mathrm{E}}{N}\mathbf{e} = \mathbf{0} \,.$$

Here $\mathcal{A}_j = \mathcal{A}(S_j; f) < 0$ is defined via an integral. Also I is the identity matrix, $(\mathbf{S})_i = S_i$, $(\mathcal{E}_0)_{ij} = \frac{1}{N}$, $(\chi(\mathbf{S}; f))_i = \chi(S_i)$, $(\mathbf{e})_i = 1$, $\nu = -1/\log\varepsilon$, and the Green's matrix $\mathcal{G}$ is

$$(\mathcal{G})_{ij} = \log|\mathbf{x}_i - \mathbf{x}_j| \,, \quad i \neq j \,; \qquad (\mathcal{G})_{ii} = 0 \,.$$

Slow Spot Dynamics: Basics I

Derivation: Higher-order matching between core solution and inhibitor field. Inhibitor field represented in terms of source-neutral Green's function on the sphere. Key Difficulty: Must go beyond simply projecting the local core solution centered at $\mathbf{x}_j$ onto the tangent-plane to the sphere. Then, apply solvability condition.

Property I: $\mathrm{I} - \mathcal{Q}_j$ projects onto the sphere, i.e. if $\mathbf{x}_j(0)$ satisfies $|\mathbf{x}_j(0)| = 1$, then $|\mathbf{x}_j(\sigma)| = 1\,,\ \forall \sigma > 0$.

Property 2: Equilibria of the dynamics are invariant under multiplication by an orthogonal matrix $\mathcal{R}$.

Property 3: If the pattern $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ of spots are such that $\mathcal{G}\mathbf{e} = \kappa_1 \mathbf{e}$ then, the pattern has a common source strength, i.e.

$$S_1 =, \ldots, S_N = S_c \equiv \frac{2\mathrm{E}}{N}\,.$$

This property holds for spots equi-distantly placed on a ring, for all platonic solids, any two-spot pattern, twisted cuboids, the icosahedron.

Slow Spot Dynamics: Basics II

Property 4: If we set $S_j = S_c$ for all j in the dynamics, then stable equilibria of the dynamics are local minima of the discrete logarithmic energy

$$\mathcal{H}_L(x_1, \ldots, x_N) \equiv - \sum_{i=1}^{N} \sum_{j>i}^{N} \log |\mathbf{x}_i - \mathbf{x}_j| \,, \qquad |x_j| = 1 \,, \quad j = 1, \ldots, N \,.$$

Elliptic fekete point sets are global minimizers of $\mathcal{H}_L$ **Proof:** Use Lagrange multipliers and project to the sphere.

Property 5: To leading order in ν, the NAS has solutions $S_j = S_c + \mathcal{O}(\nu)$ for all j. Thus, to leading-order in ν, elliptic Fekete point sets are stable equilibria of DAE dynamics.

Recall: For $i = 1, \ldots, N$, the well-known point vortex dynamics are

$$\mathbf{x}_j' = \frac{1}{2\pi} \sum_{\substack{i=1 \\ i \neq j}}^{N} \Gamma_i \frac{\mathbf{x}_i \times \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^2} \,, \qquad j = 1, \ldots, N \,; \qquad \sum_{i=1}^{N} \Gamma_i = 0 \,.$$

Slow Spot Dynamics: Small N Results I

Main Goal: For fixed N, we start with 50 random initial locations $\mathbf{x}_i(0)$, for $i = 1, \ldots, N$, representing a collection of spots on the sphere. Our goal is to characterize equilibrium states of DAE with large basin of attractions.

Small N Equilibria: $N = 2, .., 8$ **(All are Elliptic Fekete Point Sets):**

- **N=2:** two antipodal spots ($S_1 = S_2$)
  Lemma (2-Spot Dynamics): Let $\mathbf{x}_2^T \mathbf{x}_1 = \cos\gamma_{1,2}$. Then, $S_1 = S_2 = \mathrm{E}$, and the solution of the DAE is

$$\cos\left(\gamma_{1,2}/2\right) = \cos\left(\gamma_{1,2}(0)/2\right) e^{-\mathrm{E}\varepsilon^2 t/|\mathcal{A}(\mathrm{E})|} \,,$$

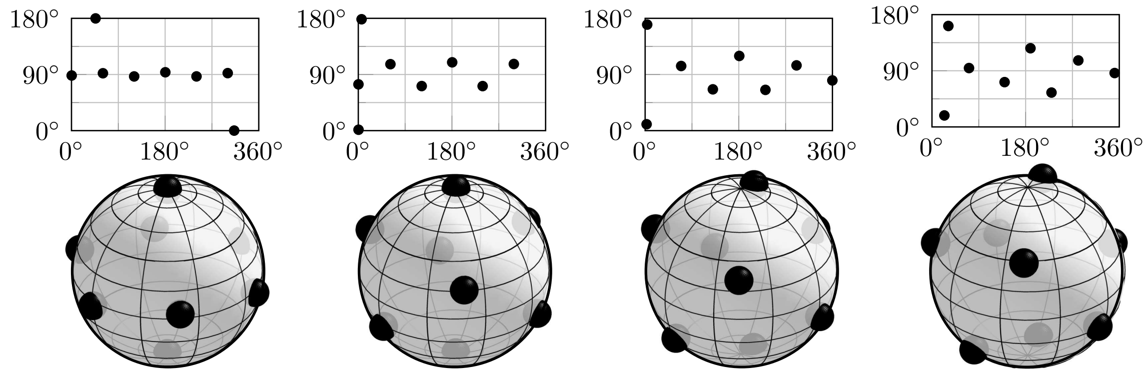
  so that $\gamma_{1,2} \to \pi$ as $\sigma \to \infty$ for any $\gamma_{1,2}(0)$.
- **N=3:** Three equally-spaced spots on a equator ($S_1 = S_2 = S_3$).
- **N=4:** Spots at vertices of a tetrahedron ($S_j = S_c$ for $j = 1, \ldots, 4$).
- **N=5,6,7:** Two antipodal spots, with $N - 2$ spots equally-spaced on the mid-plane. Two source strengths S_p and S_c.
- **N=8:** “Twisted Cuboid”. ($S_j = S_c \; \forall j$)

Slow Spot Dynamics: Small N Results II

Define: Ring pattern: consists of N particles on an equator. $(N-2)+2$ pattern: consists of $N-2$ particles on an equator with two polar spots.

Remarks (Numerics):

- Ring patterns are unstable for $N>3$ (orbitally stable if $N \leq 3$)
- $(N-2)+2$ patterns are orbitally stable for $N=4,5,6,7$, but unstable if $N \geq 8$.

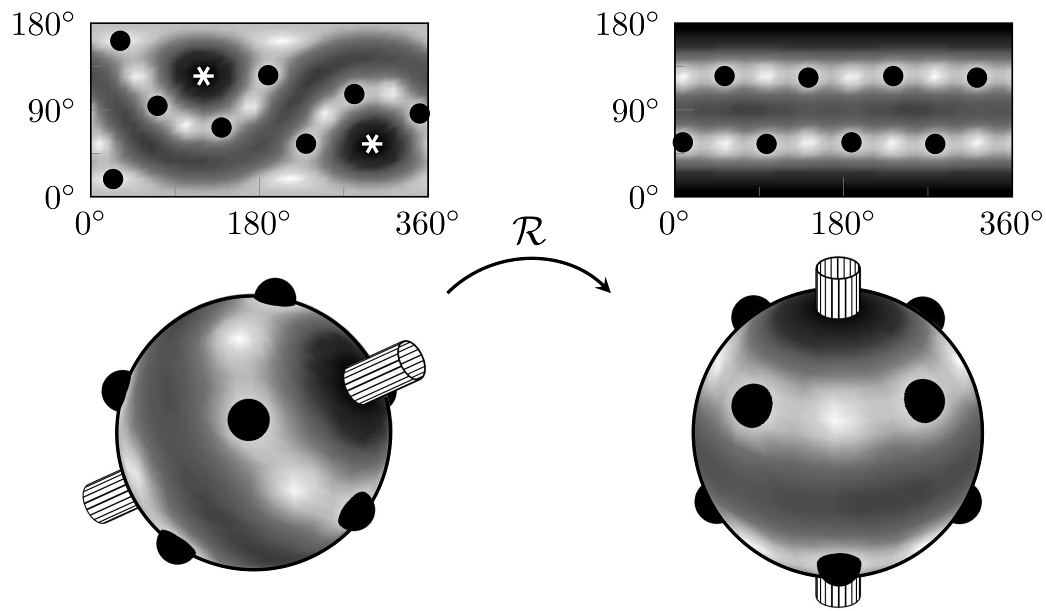


(a) $\sigma = 2$ (b) $\sigma = 3.5$ (c) $\sigma = 7$ (d) $\sigma = 25$

Caption: For $f=0.5$, $\mathrm{E}=16$, and $\varepsilon = 0.02$. Eight spots in an $(N-2)+2$ pattern undergo a 1% random perturbation at time $t=0$. The initial $(N-2)+2$ pattern is unstable. Plot at different $\sigma = \varepsilon^2 t$.

Remark: Patterns become more difficult to visualize at larger N.

Twisted Cuboid Equilibrium: $N = 8$



Caption: Left: The shading on the sphere and top (ϕ, θ) plane indicate with two asterisks a better location to place the polar axis of the sphere (marked by a cylinder). **Right:** After an orthogonal transformation, $\mathcal{R}$.

N=8 Equilibria Pattern is a 45° twisted cuboid consisting of two parallel planes, each with 4 equally-spaced spots. The spots on the two planes are 45° phase-shifted. The ratio of the distance between neighboring spots on a ring to the perpendicular distance between the planes is 0.967. This yields the plane latitudes $\theta \approx 55.6^\circ$ and $\theta \approx 124.4^\circ$. ($S_j = S_c$ for $j = 1, \ldots, 8$). This is the elliptic Fekete point pattern for 8 particles.

Slow Spot Dynamics: Elliptic Fekete Points

Numerical Observation: Elliptic Fekete points lead to a $\mathcal{G}$ matrix for which $\mathbf{e}$ is nearly an eigenvector (see Table). **Q:** Can this numerical finding be "justified" theoretically, i.e. is there an asymptotic equi-partition of the discrete logarithmic energy for large N?

Number of points N	$\|\|\xi - \mathbf{e}\|\|_2$
8	$6.0301e-16$
10	0.0216
20	0.0050
50	0.0029
100	0.0029
120	0.0015
130	0.0027

Caption: The distance $\|\| \ldots \|\|_2$ between the principal eigenvector ξ of $\mathcal{G}$ and $\mathbf{e}$, both normalized, for different elliptic Fekete point configurations. For $N = 8$, where the elliptic Fekete point set is a rotated cuboid, the first row yields to machine precision that $\mathbf{e}$ is an eigenvector of $\mathcal{G}$. For comparison, a random distribution of 100 points on the sphere has a norm difference of 0.2512.

Schnakenburg: Slow Spot Dynamics

$$u_t = \varepsilon^2 \Delta_s u - u + u^2 v \,, \qquad \tau v_t = \Delta_s v + a - \frac{1}{\varepsilon^2} u^2 v \,.$$

Principal Result: *For $\varepsilon \to 0$, and provided that the q.e. pattern is stable on an $\mathcal{O}(1)$ time-scale, the dynamics of a collection of N spots satisfies the DAE system for $S_1, \ldots, S_N$ and $\mathbf{x}_1, \ldots, \mathbf{x}_N$ on long time-scale $\varepsilon^2 t$:*

$$\frac{d\mathbf{x_j}}{dt} = \frac{2\varepsilon^2}{\mathcal{A}_j} \left(\mathrm{I} - \mathcal{Q}_j\right) \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{S_i \mathbf{x}_i}{|\mathbf{x}_i - \mathbf{x}_j|^2} \,, \qquad \mathcal{Q}_j \equiv \mathbf{x}_j \mathbf{x}_j^T \,, \qquad j = 1, \ldots, N \,,$$

coupled to the nonlinear algebraic constraint (NAS)

$$\mathcal{N}(\mathbf{S}) \equiv \left[\mathrm{I} - \nu(\mathrm{I} - \mathcal{E}_0)\mathcal{G}\right]\mathbf{S} + \nu(\mathrm{I} - \mathcal{E}_0)\chi(\mathbf{S}) - \frac{2a}{N}\mathbf{e} = \mathbf{0} \,, \,.$$

The Green's matrix $\mathcal{G}$ is again

$$(\mathcal{G})_{ij} = \log|\mathbf{x}_i - \mathbf{x}_j| \,, \quad i \neq j \,; \qquad (\mathcal{G})_{ii} = 0 \,.$$

Universality Feature: Only difference to the Brusselator is in the NAS and the model specific functions $\mathcal{A}_j$ and $\chi(S_j)$.

Broader: Narrow Capture Problems

Similar problems with logarithmic interactions occur in other 2-D such as for narrow capture problems involving a Brownian particle on the surface Ω of the unit sphere in the presence of localized traps:

$$\Delta_s u + \lambda u = 0 \,, \quad \mathbf{x} \in \Omega \backslash \Omega_p \,, \quad \Omega_p \equiv \cup_{i=1}^{N} \Omega_{\varepsilon_i} \,,$$

where Ω_{ε_i} is a "disk" of radius ε centered at $\mathbf{x}_i \in \Omega$. Then,

$$\lambda_0(\varepsilon) \sim \frac{\mu N}{2} + \mu^2 \left[-\frac{N^2}{4} \left(2 \log 2 - 1 \right) - \mathcal{H}_L(\mathbf{x}_1, \ldots, \mathbf{x}_N) \right] + \mathcal{O}(\mu^3) \,,$$

where with $\mu \equiv -1/\log \epsilon$ *and* $\mathcal{H}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ *is the discrete logarithmic energy*

$$\mathcal{H}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \equiv - \sum_{i=1}^{N} \sum_{j>i}^{N} \log |x_i - x_j| \,.$$

Key Point: $\lambda_0(\varepsilon)$ is maximized at the elliptic Fekete points.

Reference: [CSW] Coombs, Straube, Ward, SIAM 2009.

Open Questions and Extensions

A Few Open Problems:

- Use symmetry groups, and point matching algorithms from CS to classify all equilibria of DAE system for larger N. What is basin of attraction of equilibria? Stability of Equilibria of DAE system?
- Explore relation between elliptic Fekete points and equilibria of DAE system for large N. When is $\mathbf{e}$ an eigenvector of the Green's matrix $\mathcal{G}$?
- Our analysis is based (largely) on formal asymptotics. Provide a rigorous proof of the DAE system.

Extensions of the Model and Methodology:

- Formulate and analyze coupled bulk-surface RD systems. These have novel stability properties.
- Analyze localized spot patterns for Brusselator on manifolds. The key here is to determine $\Delta_s G = |\Omega|^{-1} - \delta(x - x_0)$. Hybrid asymptotic numerical description of pattern formation. (Movie)
- Analysis currently relies on inhibitor field satisfying $\Delta_s v - \kappa v = \sum_j S_j \delta(x - x_j)$. Relax this requirement.

Final Comments

Thanks: Stefanella Boatto (Rio Federal) and Paul Newton (USC) (Point vortices); Paul Matthews (Patterns on the Sphere and the $N = 8$ Problem), Juncheng Wei (NLEP theory)

Related Topics at Snowbird:

Narrow Escape and Collective Dynamics Problems: MS105 (Wed) (Kolokolnikov, Redner), MS94 (Wed) (Lindsay, Tzou)

Localized Patterns in a Plant Root Hair Model: CP29 (Tuesday) V. Brena-Medina, A. Champneys, et al.