Spot Self-Replication for Reaction-Diffusion Models in Two-Dimensional Domains

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Spot-Patterns: Qualitative I

Schnakenburg Model: 2-D domain Ω with $\partial_n u = \partial_n v = 0$ on $\partial\Omega$:

$$v_t = \varepsilon^2 \Delta v - v + uv^2 \,, \qquad \varepsilon^2 u_t = D\Delta u + a - \varepsilon^{-2} uv^2 \,.$$

Here $0 < \varepsilon \ll 1$, with $D > 0$, and $a > 0$, are parameters.

- **Spot pattern:** since $\varepsilon \ll 1$, v can concentrate at discrete points in Ω. **Semi-strong Regime:** $D = O(1)$ so that u is global.**Weak Interaction Regime:** $D = O(\varepsilon^2)$ so that u is localized.. We assume **semi-strong.**
- **Physical Experiments of Spot-Splitting:** The ferrocyanide-iodate-sulphite reaction (Swinney et al, Nature, 1994), the chloride-dioxide-malonic acid reaction (De Kepper et al, J. Phys. Chem, 1998), and certain semiconductor gas discharge systems (Purwins et al, Phys. Lett. A, 2001)
- **Numerical Results of Spot-Splitting in 2-D:** Many studies (Pearson, Nishiura, Muratov, Maini) for related models, i.e. the Gray-Scott (GS) model

$$v_t = \varepsilon^2 \Delta v - v + Auv^2 \,, \qquad \tau u_t = D\Delta u + (1-u) - uv^2 \,.$$

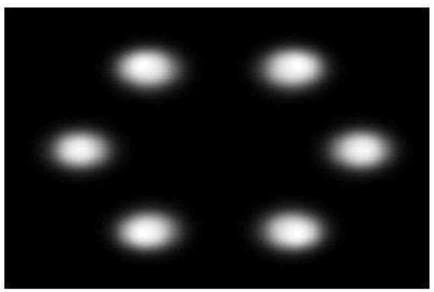
Spot Patterns; Qualitative II

Goal: Analyze spot dynamics and spot-splitting for Schnakenburg Model (Kolokolnikov, Ward, Wei, to appear, J. Nonlinear Science, 2008).

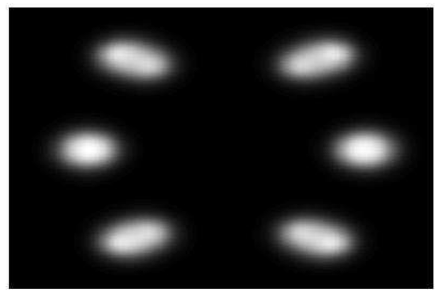
Outline of the Talk:

- **Slow Dynamics:** a DAE system for the evolution of K spots.
- **Spot-Splitting Criterion:** peanut-splitting and the splitting direction.
- **Numerical Confirmation of Asymptotic Theory**

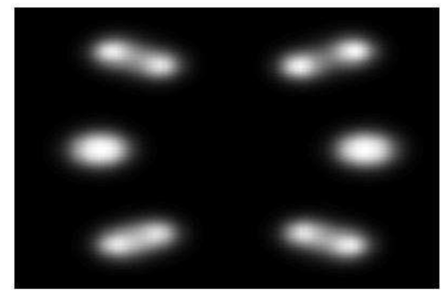
Example: $\Omega = [0,1]^2$, $\varepsilon = 0.02$, $a = 51$, $D = 0.1$. (movie 1). (movie 2)



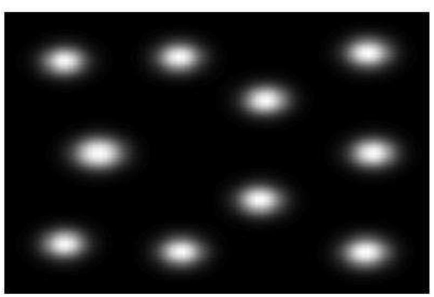
$t = 4.0$



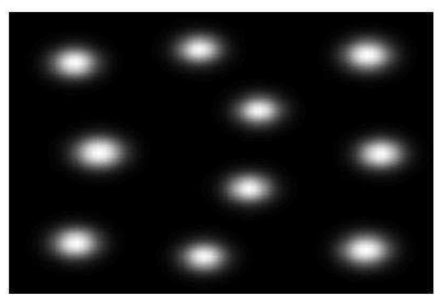
$t = 25.5$



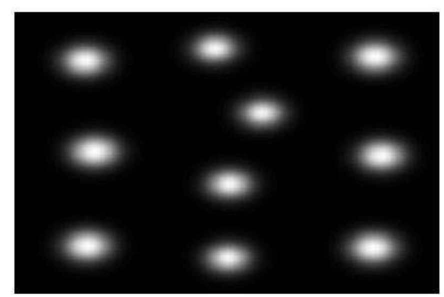
$t = 40.3.$



$t = 280.3$



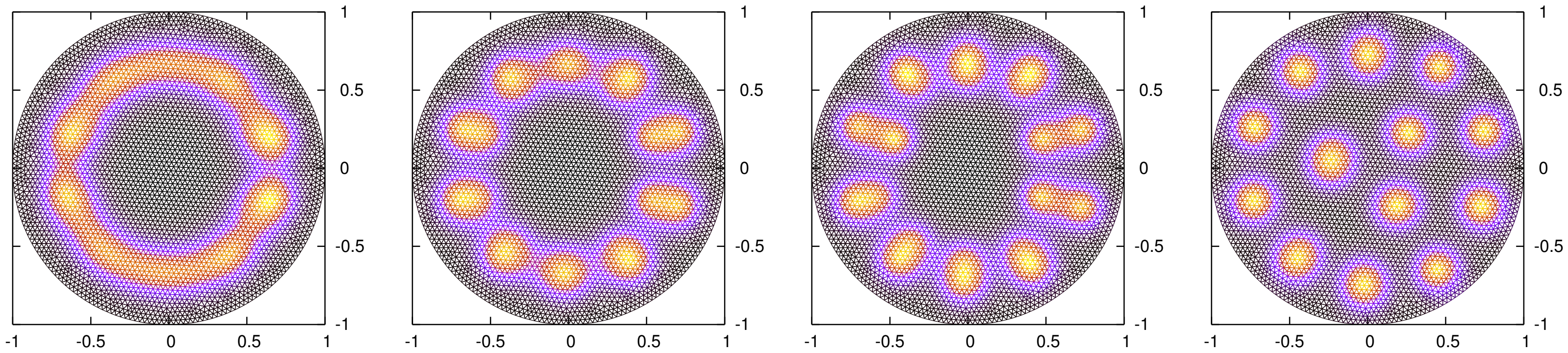
$t = 460.3$



$t = 940.3.$

Spot Patterns: Qualitative III

Spot patterns arise from generic initial conditions, or from the breakup of a stripe to varicose instabilities: Spot-replication appears here as a secondary instability GS Model: Semi-strong regime.



Some Previous Work:

- **1-D Theory:** Stability and dynamics of pulses for the GM and GS models in the semi-strong regime (Doelman, Kaper, Promisolow, Muratov, Osipov, Iron, MJW, Kolokolnikov, Wei), pulse-splitting for the GS model in the weak interaction regime $D = O(\varepsilon^2)$ (Nishiura, Ei, Ueyama).
- **2-D Theory:** NLEP stability theory for spot stability (Wei-Winter). One-Spot dynamics for GM (Chen, Kowalczyk, Kolokolnikov, MJW).
- **Growing Domains:** numerics showing spot-splitting for the Schnakenburg model on a slowly growing domain (Madvamuse, Maini, 2006)

The Quasi-Equilibrium Solution: I

Construction of a One-Spot Pattern by Singular Perturbation Techniques:

Inner Region: near the spot location $x_0 \in \Omega$ introduce $\mathcal{V}(y)$ and $\mathcal{U}(y)$ by

$$u = \frac{1}{\sqrt{D}} \mathcal{U} \,, \quad v = \sqrt{D} \mathcal{V} \,, \quad y = \varepsilon^{-1}(x - x_0) \,, \quad x_0 = x_0(\varepsilon^2 t) \,.$$

To leading order, $\mathcal{U} \sim U(\rho)$ and $\mathcal{V} \sim V(\rho)$ (radially symmetric) with $\rho = |y|$.

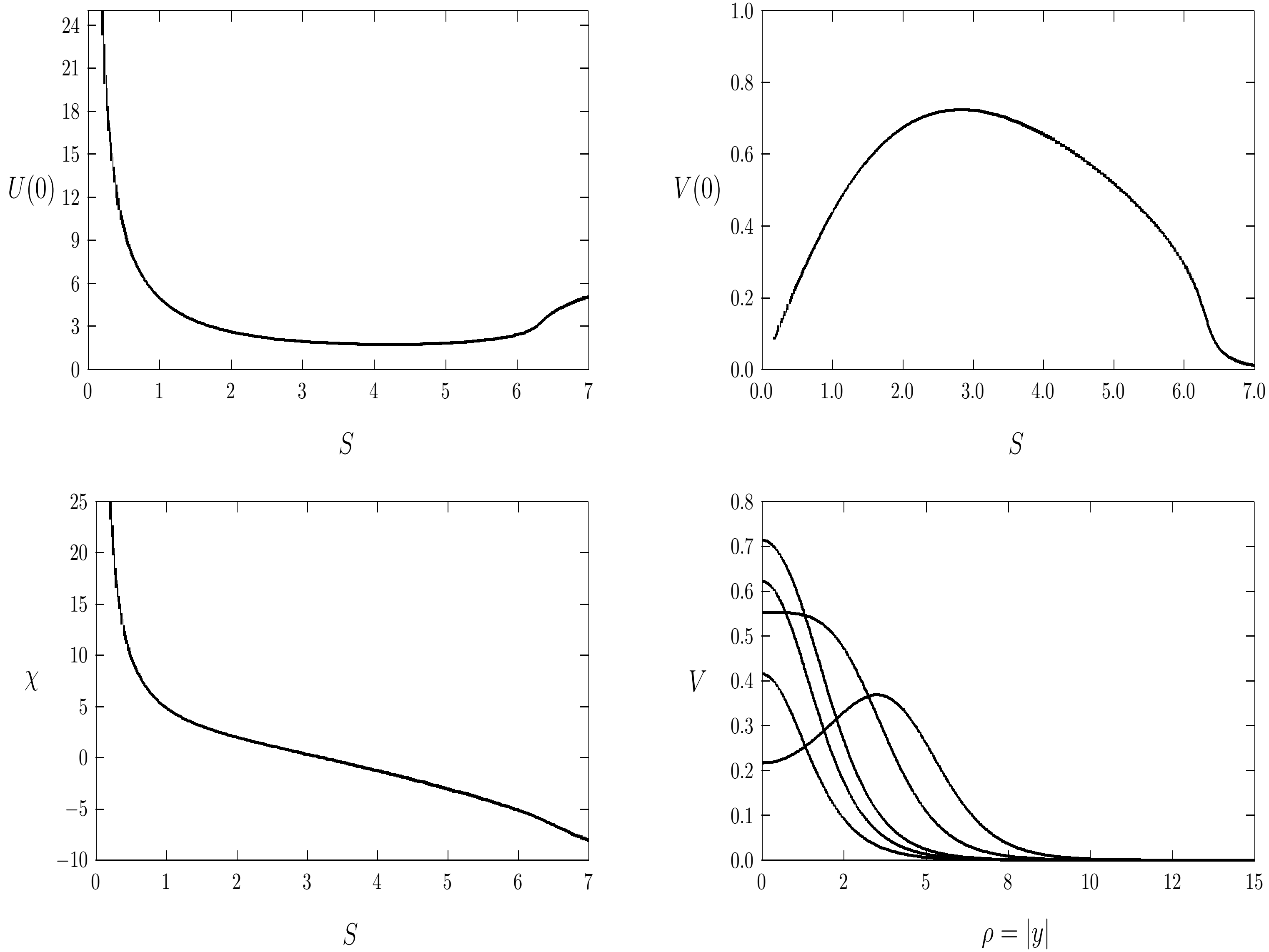
This yields the **coupled core problem** with $U'(0) = V'(0) = 0$, where:

$$V_{\rho\rho} + \frac{1}{\rho} V_\rho - V + UV^2 = 0 \,, \quad U_{\rho\rho} + \frac{1}{\rho} U_\rho - UV^2 = 0 \,, \qquad 0 < \rho < \infty \,,$$
$$V \to 0 \,, \qquad U \sim S \log \rho + \chi(S) + o(1) \,, \quad \text{as} \quad \rho \to \infty \,.$$

- Here $S > 0$ is called the “source strength” and is a parameter to be determined upon matching to an outer solution.
- The nonlinear function $\chi(S)$ must be computed numerically.

The Quasi-Equilibrium Solution: II

Plots of the Numerical Solution to the Core Problem:



The Quasi-Equilibrium Solution: III

Outer Region: $v \ll 1$ and $\varepsilon^{-2}uv^2 \to 2\pi\sqrt{D}S\delta(x-x_0)$. Hence,

$$\Delta u = -\frac{a}{D} + \frac{2\pi}{\sqrt{D}} S\,\delta(x-x_0)\,, \quad x \in \Omega\,; \partial_n u = 0\,, \quad x \in \partial\Omega\,,$$

$$u \sim \frac{1}{\sqrt{D}}\left[S \log|x-x_0| + \chi(S) + \frac{S}{\nu}\right] \quad \text{as} \quad x \to x_0\,, \quad \nu \equiv -1/\log\varepsilon\,.$$

Key Point: the regular part of this singularity structure is **specified** and was obtained from matching to the **inner core solution.**

- Divergence theorem yields S (and inner core solution U and V) as

$$S = \frac{a|\Omega|}{2\pi\sqrt{D}}\,.$$

- The outer solution is given uniquely by

$$u(x) = -\frac{2\pi}{\sqrt{D}}\left(SG(x;x_0) + u_c\right)\,,$$

where $\quad S + 2\pi\nu S R(x_0;x_0) + \nu\chi(S) = -2\pi\nu u_c\,, \qquad \nu \equiv -1/\log\varepsilon\,.$

The Quasi-Equilibrium Solution: IV

Neumann Green's Function: The Neumann Green's function $G(x;x_0)$ with regular part $R(x;x_0)$ satisfies

$$\Delta G = \frac{1}{|\Omega|} - \delta(x-x_0)\,, \quad x \in \Omega\,; \quad \partial_n G = 0 \quad \text{on} \quad \partial\Omega\,,$$

$$G(x;x_0) = -\frac{1}{2\pi}\log|x-x_0| + R(x;x_0)\,; \qquad \int_\Omega G\,dx = 0\,.$$

Remarks:

- The Neumann Green's function G, its regular part R, and their gradients, can be calculated for different Ω. (Simple formulae for a disk; more difficult for a rectangle, Ewald summation).
- Construction yields a quasi-equilibrium solution for any "frozen" x_0.
- No rigorous existence theory for solutions to the coupled core problem.
- The error is smaller than any power of $\nu = -1/\log\varepsilon$. Therefore, in effect, we have "summed" all the logarithmic terms.
- Related infinite log expansions: eigenvalue of the Laplacian in a domain with traps, slow viscous flow over a cylinder, etc.

The One-Spot Dynamics: I

<u>**Principal Result**</u>: *Provided that the one-spot solution is stable, the slow dynamics of a one-spot solution satisfies the gradient flow*

$$\frac{dx_0}{dt} \sim -2\pi\varepsilon^2 \gamma(S) S \;\; \nabla R(x_0; x_0) \,.$$

Here $\gamma(S)$ is determined from the **inner problem** by the following:

<u>**Lemma**</u>: *The constant* $\gamma(S)$ *is given by*

$$\gamma \equiv \gamma(S) = \frac{-2}{\int_0^\infty \rho V'(\rho) \hat{\Phi}^*(\rho)\, d\rho} \,.$$

Here $\hat{\Phi}^*(\rho)$ *is the first component of the radially symmetric adjoint solution* $\hat{P}^*(\rho) \equiv \left(\hat{\Phi}^*(\rho), \hat{\Psi}^*(\rho) \right)^t$ *satisfying*

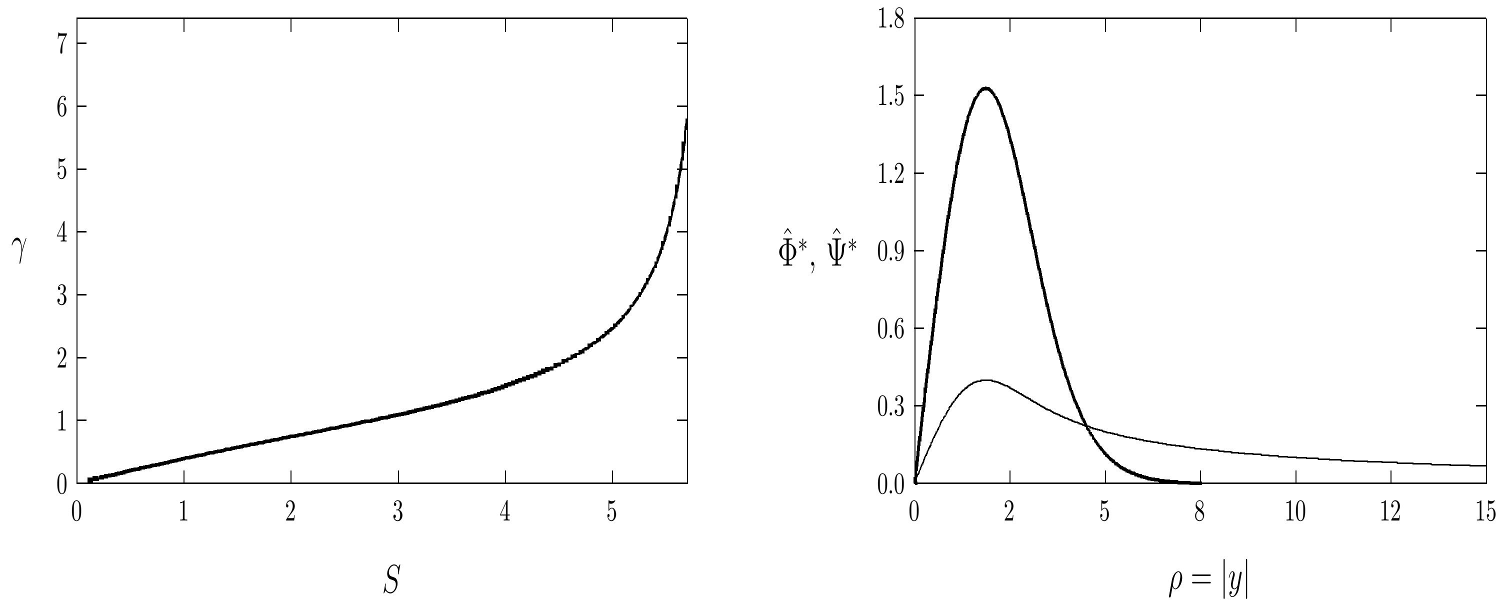
$$\partial_{\rho\rho} \hat{P}^* + \rho^{-1} \partial_\rho \hat{P}^* - \rho^{-2} \hat{P}^* + \mathcal{M}_0^t \hat{P}^* = \mathbf{0} \,, \quad 0 < \rho < \infty \,,$$

subject to $\hat{\Phi}^* \to 0$ *exponentially and* $\hat{\Psi}^* \sim 1/\rho$ *as* $\rho \to \infty$. *Here* $\mathcal{M}_0^t$ *is a* 2×2 *matrix associated with the core solution* U *and* V.

The One-Spot Dynamics: II

- The ODE for x_0 is derived from a higher-order matching condition together with an appropriate solvability condition.
- The function $\gamma(S)$ must be computed numerically. We obtain $\gamma(S) > 0$.
- **Implication:** a stable equilibrium occurs at a minimum point of $R(x_0; x_0)$.

Left: $\gamma(S)$ vs. S. Right: adjoint eigenfunction when $S = 3.5$



The Stability of a One-Spot Solution: I

We seek fast $\mathcal{O}(1)$ time-scale instabilities relative to slow time-scale of x_0. Let $u = u_e + \mathrm{e}^{\lambda t}\eta$ and $v = v_e + \mathrm{e}^{\lambda t}\phi$. In the **inner region** we introduce the local angular mode $m = 0, 2, 3, \ldots$ by

$$\eta = \frac{1}{D} e^{im\theta} N(\rho)\,, \quad \phi = e^{im\theta}\Phi(\rho)\,, \quad \rho = |y|\,, \qquad y = \varepsilon^{-1}(x - x_0)\,.$$

Then, on $0 < \rho < \infty$, we get the two-component eigenvalue problem

$$\mathcal{L}_m\Phi - \Phi + 2UV\Phi + V^2 N = \lambda\Phi\,, \qquad \mathcal{L}_m N - 2UV\Phi - V^2 N = 0\,,$$

$$\mathcal{L}_m\Phi \equiv \partial_{\rho\rho}\Phi + \rho^{-1}\partial_\rho\Phi - m^2\rho^{-2}\Phi\,.$$

Let $\lambda_0(S, m)$ denote the eigenvalue with the largest real part, with Σ_m be the value of S such that Re$\lambda_0(\Sigma_m, m) = 0$.

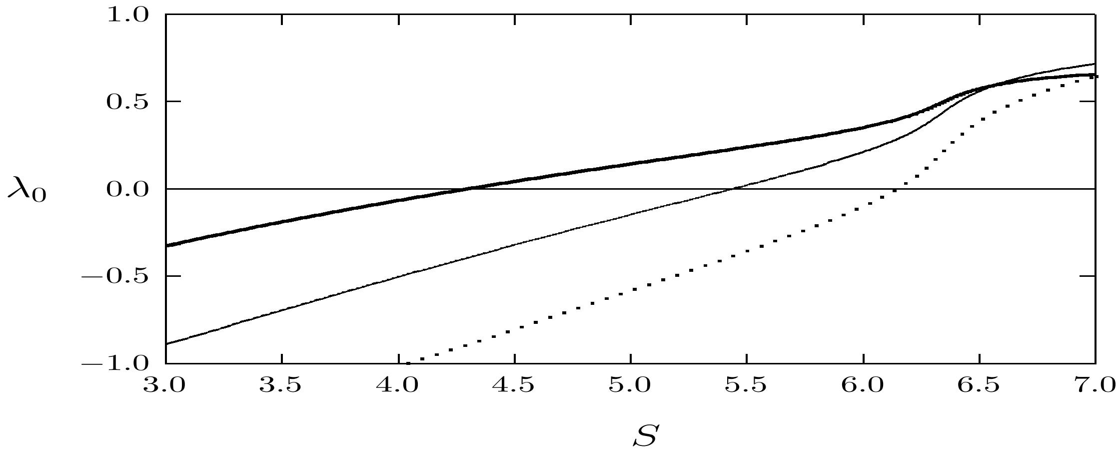
- U and V are computed from the core problem and depend on S.
- **Key Point:** This is a two-component eigenvalue problem, in contrast to the scalar problem of NLEP theory. Hence, with no maximum principle there is no ordering principle for eigenvalues wrt number of nodal lines of eigenfunctions.

The Stability of a One-Spot Solution: II

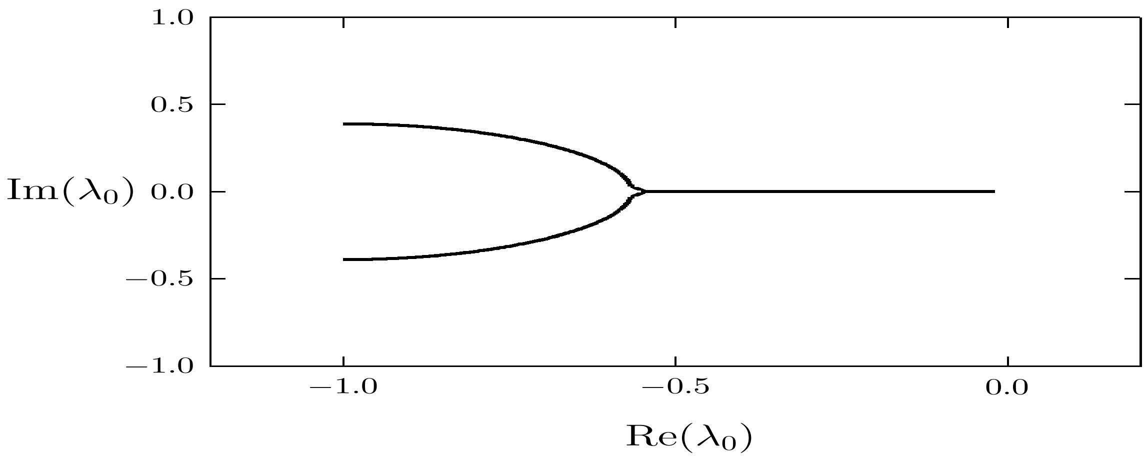
The Modes $m \geq 2$: We must impose $N \sim \rho^{-2}$ as $\rho \to \infty$. We compute

$$\Sigma_2 = 4.303\,, \quad \Sigma_3 = 5.439\,, \quad \Sigma_4 = 6.143\,.$$

Key point: the peanut-splitting instability is the dominant instability.



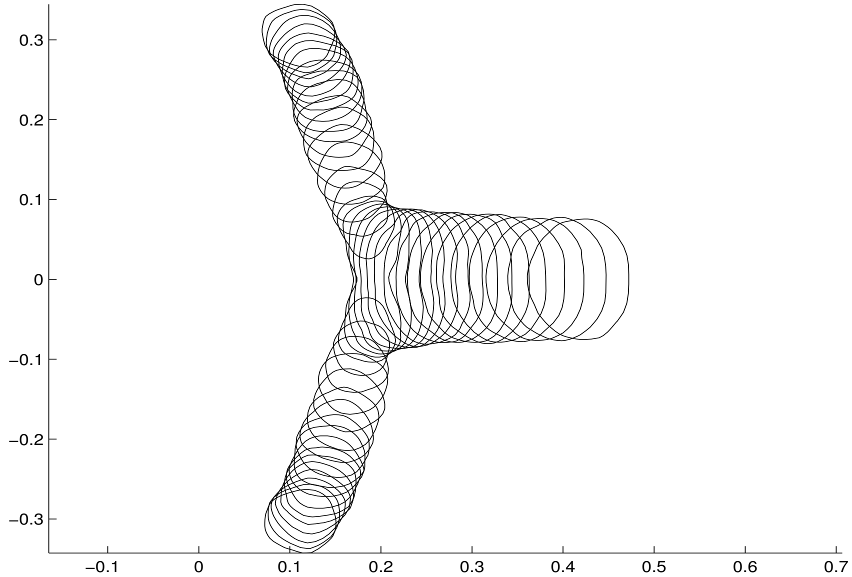
The Mode $m = 0$: After matching to the outer solution, we require that N is bounded as $\rho \to \infty$. Key: stability wrt this mode at least up to $S = 7.8$.

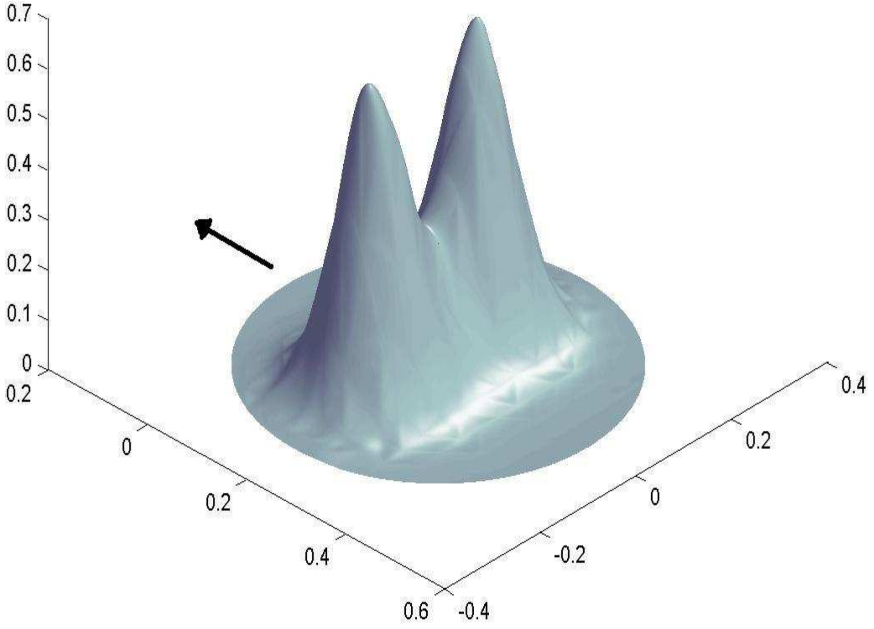


The Direction of Splitting

- For $S \approx \Sigma_2$, the linearization of the core problem has an approximate four-dimensional null-space (two translation and splitting modes).
- By deriving a certain solvability condition, we show that splitting occurs in a direction perpendicular to the motion when $\varepsilon \ll 1$.

Spot-Splitting in the Unit Disk: $x_0(0) = (0.5, 0.0)$, $\varepsilon = 0.03$, $D = 1$, and $a = 8.8$. Left: Trace of the contour $v = 0.5$ from $t = 15$ to $t = 175$ with increments $\Delta t = 5$. Right: spatial profile of v at $t = 105$ during the splitting.





The DAE System for a K-Spot Pattern: I

Collective Coordinates: S_j, x_j, for $j = 1, \ldots, K$.

Principal Result: (DAE System): *For "frozen" spot locations x_j, the source strengths S_j and u_c satisfy the nonlinear algebraic system*

$$S_j + 2\pi\nu \left(S_j R_{j,j} + \sum_{\substack{i=1 \\ i\neq j}}^{K} S_i G_{j,i} \right) + \nu \chi(S_j) = -2\pi\nu u_c \,, \quad j = 1, \ldots, K \,,$$

$$\sum_{j=1}^{K} S_j = \frac{a|\Omega|}{2\pi\sqrt{D}} \,, \qquad \nu \equiv \frac{-1}{\log \varepsilon} \,.$$

The slow dynamics of the spots with speed $O(\varepsilon^2)$ satisfies

$$x_j' \sim -2\pi\varepsilon^2 \gamma(S_j) \left(S_j \nabla R(x_j; x_j) + \sum_{\substack{i=1 \\ i\neq j}}^{K} S_i \nabla G(x_j; x_i) \right) \,, \quad j = 1, \ldots, K \,.$$

Here $G_{j,i} \equiv G(x_j; x_i)$ and $R_{j,j} \equiv R(x_j; x_j)$ (Neumann G-function).

The DAE System II: Qualitative Comments

- **Vortices in GL Theory:** some similarities for the law of motion.
- **Spot-Splitting Criterion:** For $D = O(1)$ and $K \geq 1$ the q. e. solution is stable wrt the local angular modes $m \geq 2$ iff $S_j < \Sigma_2 \approx 4.303$ for all $j = 1, \ldots, K$. The J^{th} spot is unstable to the $m = 2$ peanut-splitting mode when $S_J > \Sigma_2$, which triggers a nonlinear spot self-replication process. **Note: asymptotically no inter-spot coupling when $m \geq 2$.**
- **Stability to Locally Radially Symmetric Fluctuations:** For $D = O(1)$, and to leading order in ν, a K-spot q. e. solution with $K > 1$ is stable wrt $m = 0$. A one-spot solution is always stable wrt $m = 0$.
- **NLEP theory when $\mathbf{D = 0(\nu^{-1}) \gg 1}$:** Yields a scalar inner eigenvalue problem, so that the $m = 2$ mode is always stable. For $K \geq 2$, the $m = 0$ mode is stable only when

$$D \leq \frac{a^2|\Omega|^2\nu^{-1}}{4\pi^2 K^2 b_0}\,; \quad b_0 \equiv \int_0^\infty \rho\, [w(\rho)]^2 \, d\rho\,.$$

- **Universality:** Similar DAE systems, but with different $\gamma(S)$ and $\chi(S)$ (through different core problems), and possibly with different Green's functions (such as the reduced wave Green's function), can be derived for other RD models such as the GS or GM models.

Comparison: Asymptotics with Full Numerics

Asymptotic Theory:

- **Inner:** Compute $\gamma(S)$ and $\chi(S)$ from core problem at discrete points in S. Then, interpolate with a spline.
- **Domain:** Calculate G, its regular part R, and gradients of G, R.
- Solve DAE system numerically using Newton's method for nonlinear algebraic part, and a Runge-Kutta ODE solver for the dynamics.
- For special geometries, the algebraic part of the DAE system can be solved analytically (ring patterns in a disk).

Full Numerics:

- Adaptive grid finite-difference code VLUGR2 (P. Zegeling, J.Blom, J. Verwer) to compute solutions in a square. Use finite-element code of W. Sun (U. Calgary) for a disk. "Prepared" initial data:

$$v = \sqrt{D} \sum_{j=1}^{K} v_j \text{sech}^2 \left(\frac{|x - x_j|}{2\varepsilon} \right) , \quad u = -\frac{2\pi}{\sqrt{D}} \left(\sum_{j=1}^{K} S_j G(x; x_j) + u_c \right) .$$

- Find the location of maxima of v on the computational grid

Neumann Green's Function Formulae: I

Unit Disk: Let $\Omega := \{x \,|\, |x| \le 1\}$ and represent $x \in \Omega$ as a complex number. Then

$$G(x;\xi) = \frac{1}{2\pi}\left(-\log|x-\xi| - \log\left|x|\xi| - \frac{\xi}{|\xi|}\right| + \frac{1}{2}(|x|^2 + |\xi|^2) - \frac{3}{4}\right)\,,$$
$$R(\xi;\xi) = \frac{1}{2\pi}\left(-\log\left|\xi|\xi| - \frac{\xi}{|\xi|}\right| + |\xi|^2 - \frac{3}{4}\right)\,.$$

Furthermore, for the dynamics in the DAE system, we use

$$\nabla G(x;\xi) = -\frac{1}{2\pi}\left[\frac{(x-\xi)}{|x-\xi|^2} + \frac{|\xi|^2}{\bar{x}|\xi|^2 - \bar{\xi}} - x\right]\,,$$
$$\nabla R(\xi;\xi) = \frac{1}{2\pi}\left(\frac{2-|\xi|^2}{1-|\xi|^2}\right)\xi\,,$$

Neumann Green's Function Formulae: II

Rectangle: Let $\Omega \equiv [0, L] \times [0, d]$ and $|\Omega| = Ld$. Then, with $x = (x, y)$,

$$G(x; x') = \frac{2}{|\Omega|} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2} + \frac{2}{|\Omega|} \sum_{m=1}^{\infty} \frac{\cos\left(\frac{m\pi y}{d}\right)\cos\left(\frac{m\pi y'}{d}\right)}{\left(\frac{m\pi}{d}\right)^2}$$

$$+ \frac{4}{|\Omega|} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x'}{L}\right)\cos\left(\frac{m\pi y}{d}\right)\cos\left(\frac{m\pi y'}{d}\right)}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{d}\right)^2} ,$$

- This series does not converge at the source point $x = x'$, and converges extremely slowly at other points in Ω.
- Require an Ewald summation type method to first extract the singular part. Such techniques are ubiquitous in the study of homogenization problems on infinite periodic lattices.

A lengthy calculation yields an infinite-image type expansion

$$G(x; x') = -\frac{1}{2\pi} \log |x - x'| + R(x; x') ,$$

Neumann Green's Function Formulae: III

$$R(x;x') = -\frac{1}{2\pi}\sum_{n=0}^{\infty}\log\left(|1-q^n z_{+,+}||1-q^n z_{+,-}||1-q^n z_{-,+}|\right)$$

$$-\frac{1}{2\pi}\sum_{n=0}^{\infty}\log\left(|1-q^n \zeta_{+,+}||1-q^n \zeta_{+,-}||1-q^n \zeta_{-,+}||1-q^n \zeta_{-,-}|\right)$$

$$-\frac{1}{2\pi}\log\frac{|1-z_{-,-}|}{|r_{-,-}|}+\frac{1}{d}H(x,x')-\frac{1}{2\pi}\sum_{n=1}^{\infty}\log|1-q^n z_{-,-}|\,.$$

Here the points, and the one-dimensional Green's function H are

$$z_{\pm,\pm} \equiv \mathrm{e}^{\mu r_{\pm,\pm}/2}\,,\quad \zeta_{\pm,\pm} \equiv \mathrm{e}^{\mu \rho_{\pm,\pm}/2}\,,\quad \mu \equiv \frac{2\pi}{d}\quad q \equiv \mathrm{e}^{-\mu L} < 1\,,$$

$$r_{+,\pm} \equiv -|x+x'| + i(y \pm y')\,,\qquad r_{-,\pm} \equiv -|x-x'| + i(y \pm y')\,,$$

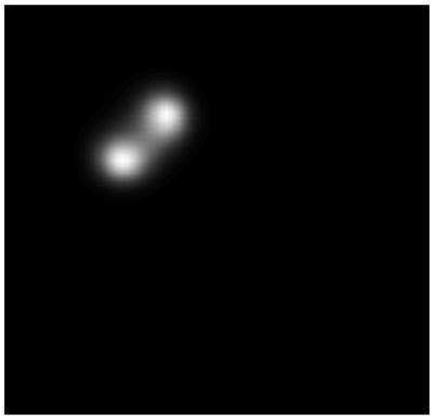
$$\rho_{+,\pm} \equiv |x+x'| - 2L + i(y \pm y')\,,\qquad \rho_{-,\pm} \equiv |x-x'| - 2L + i(y \pm y')\,,$$

$$H(x,x') \equiv \frac{L}{12}\left[h\left(\frac{x-x'}{L}\right)+h\left(\frac{x+x'}{L}\right)\right]\,,\quad h(\theta) \equiv 2 - 6|\theta| + 3\theta^2\,,$$

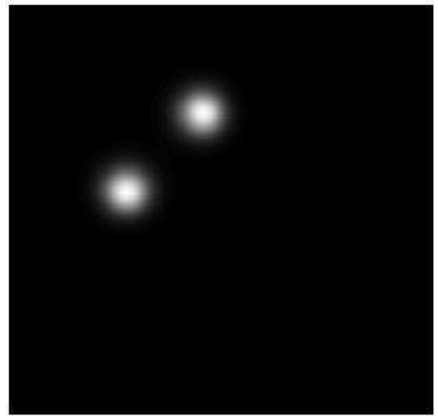
For the unit square, $q = e^{-2\pi}$ and $q^n \to 0$. Thus, rapid convergence.

Numerical Validation for 1-Spot: II

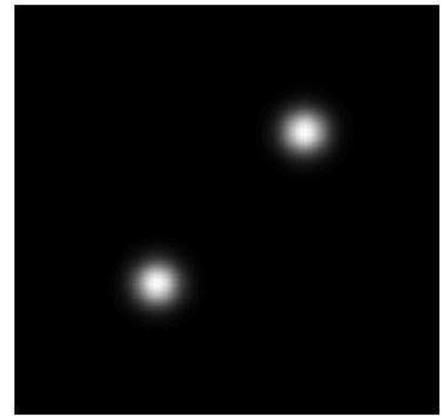
Splitting of One Spot: Let $\Omega = [0,1]^2$ and fix $\varepsilon = 0.02$, $x_0 = (0.2, 0.8)$, $a = 10$, and $D = 0.1$. Then, $S \approx 5.03 > \Sigma_2$. We predict a spot-splitting event beginning at $t = 0$. The growth rate is $\lambda_0(S, 2) \approx 0.15$. (movie)



$t = 23.6$



$t = 40.2$



$t = 322.7.$

- For $\varepsilon = .02$, full numerics gives a threshold in $4.15 < S < 4.28$.
- In a slowly growing square $\Omega = [0, L]^2$, we predict spot-splitting when

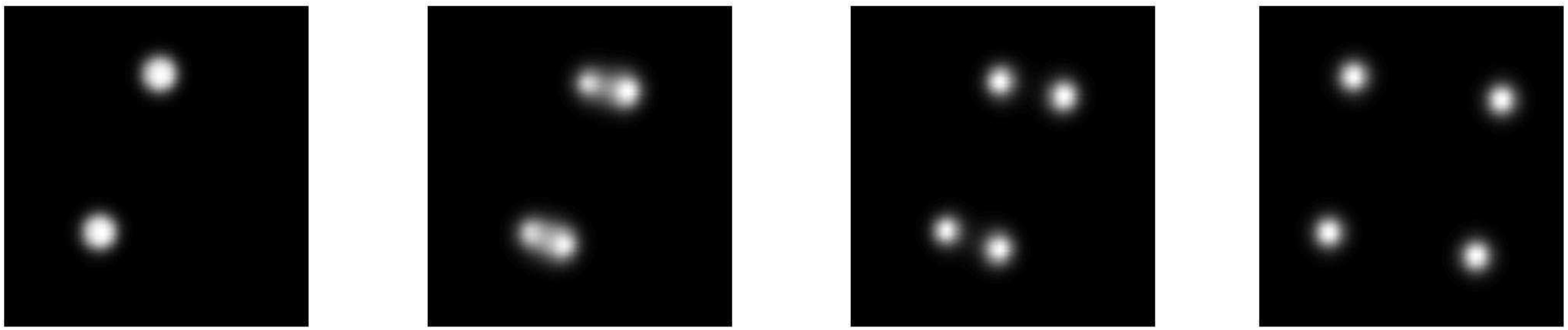
$$L > L_1 = \left(\frac{2\pi\sqrt{D}\Sigma_2}{a} \right)^{1/2} .$$

Numerical Validation, 2-Spot Solutions: I

Let $\Omega = [0,1]^2$. Fix $\varepsilon = 0.02$, $x_1(0) = (0.3, 0.3)$, $a = 18$, and $D = 0.1$.

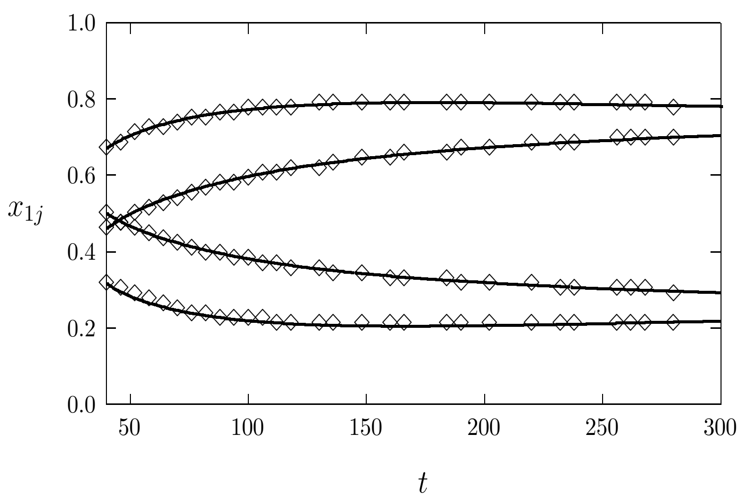
We only only vary $x_2(0)$, the initial location of the second spot.

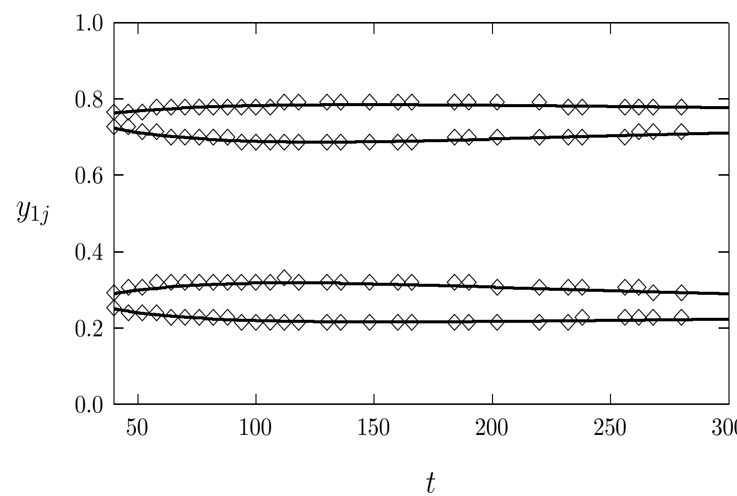
(I): $x_2(0) = (0.5, 0.8)$; $S_1 = 4.61$, $S_2 = 4.46$; Both spots split; (movie)



$t = 2.0$ $t = 33.5$ $t = 46.3$ $t = 280.3.$

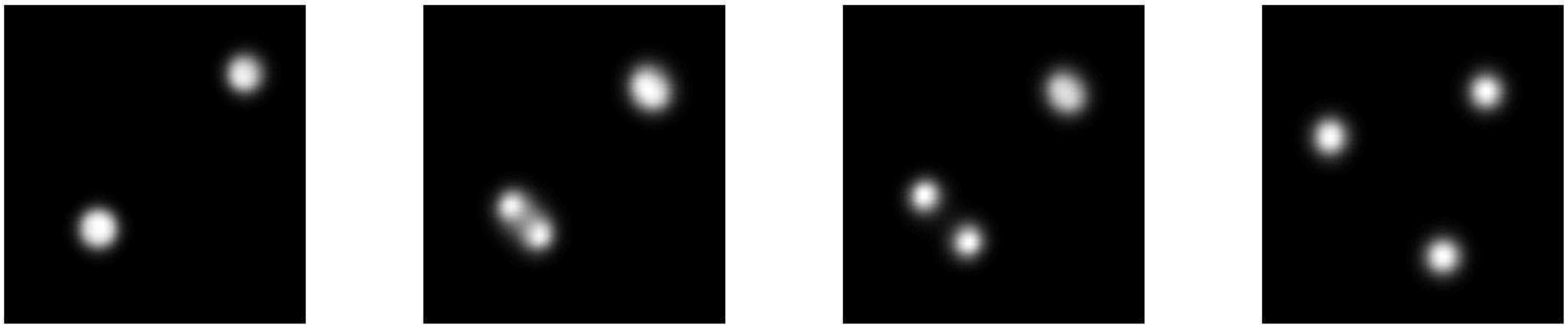
The DAE system tracks spot trajectories closely after the splitting





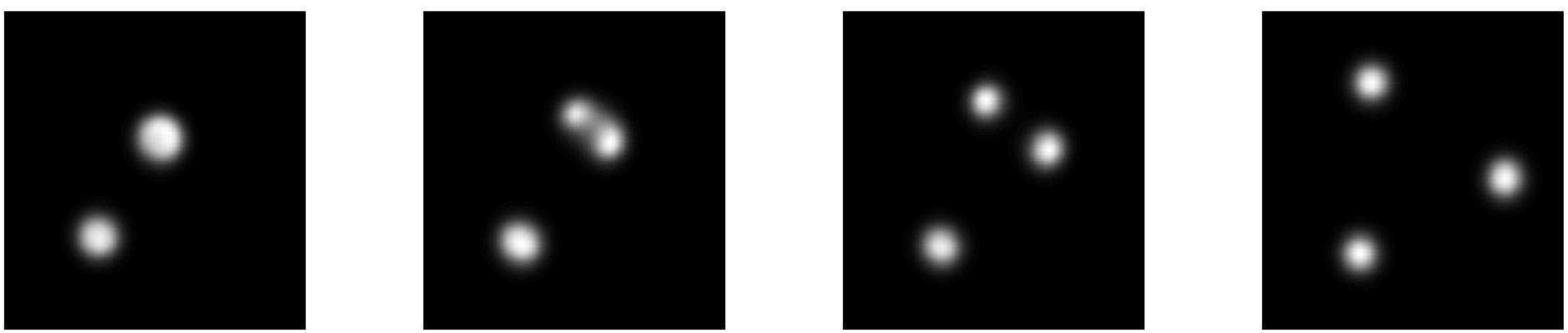
Numerical Validation, 2-Spot Solutions: II

(II): $x_2(0) = (0.8, 0.8)$; $S_1 = 5.27$, $S_2 = 3.79$; x_1 splits; (movie)



$t = 2.5$ $t = 19.9$ $t = 29.4$ $t = 220.3$.

(III): $x_2(0) = (0.5, 0.6)$; $S_1 = 3.67$, $S_2 = 5.39$; x_2 splits; (movie)



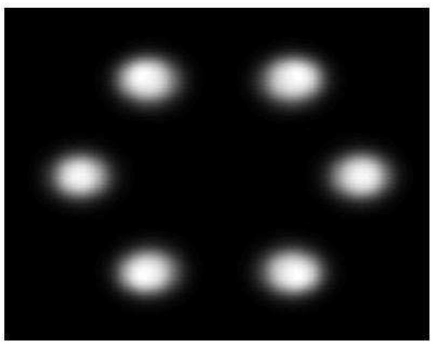
$t = 4.0$ $t = 16.5$ $t = 29.4$ $t = 322.7$.

Numerical Validation, Another Example

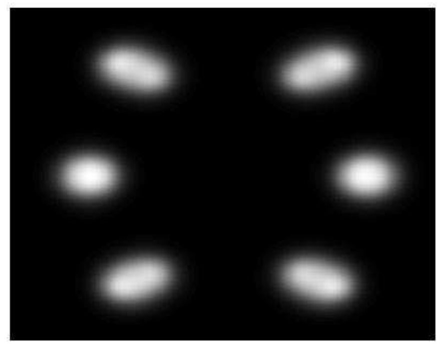
(IV): Let $\Omega = [0,1]^2$, $\varepsilon = 0.02$, $a = 51$, $D = 0.1$ and let

$$x_j = x_c + 0.33e^{i\pi(j-1)/3}\,, \quad j = 1, \ldots, 6\,;$$

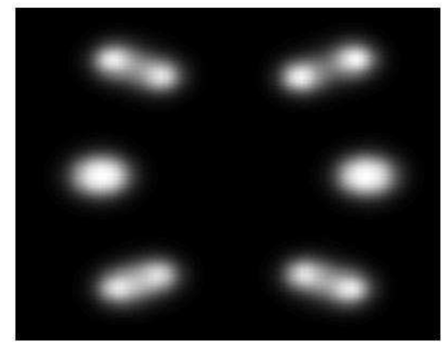
The DAE system gives $S_1 = S_4 \approx 4.01$, and $S_2 = S_3 = S_5 = S_6 \approx 4.44$. We predict that four spots will split (movie). The DAE system closely tracks the spot locations after the splitting. (another movie)



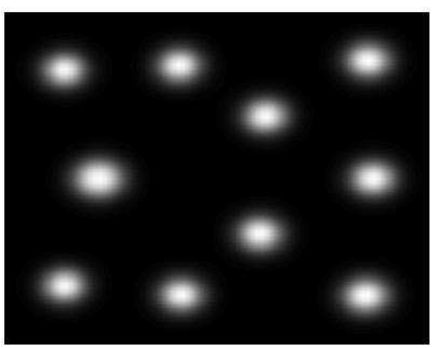
$t = 4.0$



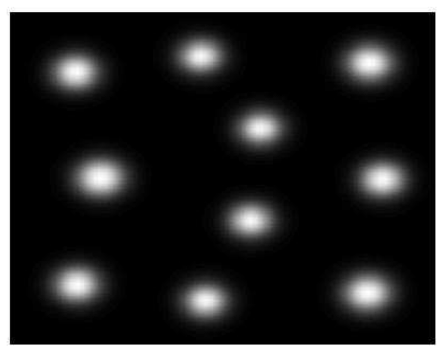
$t = 25.5$



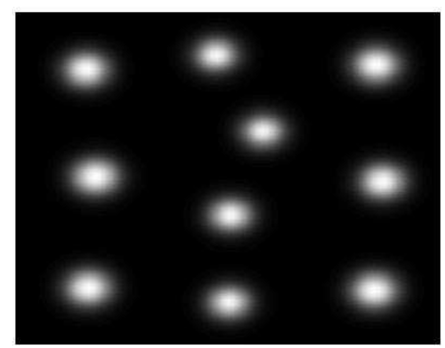
$t = 40.3.$



$t = 280.3$



$t = 460.3$



$t = 940.3.$

Ring Patterns in the Unit Disk: I

Let $\mathcal{G}$ be the (symmetric) Green's function matrix with entries $\mathcal{G}_{ii} = R$ and $\mathcal{G}_{ij} = G_{ij}$. Then:

Proposition: *Consider the nonlinear algebraic system in the DAE system for the spot strengths S_j, for $j = 1, \ldots, K$. Suppose that the spot locations x_j for $j = 1, \ldots, K$ are arranged so that $\mathcal{G}$ is a circulant matrix. Then, with $e = (1, \ldots, 1)^t$,*

$$\mathcal{G} e = \frac{p}{K} e \,, \qquad p = p(x_1, \ldots, x_K) \equiv \sum_{i=1}^{K} \sum_{j=1}^{K} \mathcal{G}_{ij} \,,$$

and the spots have the common strength

$$S_j \equiv S_c \equiv \frac{a|\Omega|}{2\pi K \sqrt{D}} \,, \quad j = 1, \ldots, K \,.$$

Remark: For a ring pattern of spots in the unit disk, $\mathcal{G}$ is circulant. Hence, we predict the possibility of simultaneous spot-splitting events. In addition, we can derive a simple ODE for the ring radius in terms of p.

Ring Patterns in the Unit Disk: II

Analysis of the DAE system is possible for a ring pattern in the unit disk:

$$x_j = re^{2\pi i j/K}\,, \quad j = 1, \ldots, K\,, \qquad \text{(Pattern I)}\ .$$

Then, $\mathcal{G}$ is circulant with eigenpair $e = (1, \ldots, 1)^t$ and $p_K(r)/K$, where

$$p_K(r) \equiv \frac{1}{2\pi}\left[-K\log(Kr^{K-1}) - K\log\left(1 - r^{2K}\right) + r^2K^2 - \frac{3K^2}{4}\right]\,.$$

There is a common source strength $S_c \equiv a|\Omega|/(2\pi K\sqrt{D})$. For $S_c < \Sigma_2 \approx 4.3$, the spot locations x_j satisfy the ODE's

$$x_j' \sim -\pi\varepsilon^2\gamma(S_c)S_c\frac{1}{K}p_K'(r)e^{2\pi i j/K}\,, \quad j = 1, \ldots, K\,.$$

This yields an ODE for the ring radius

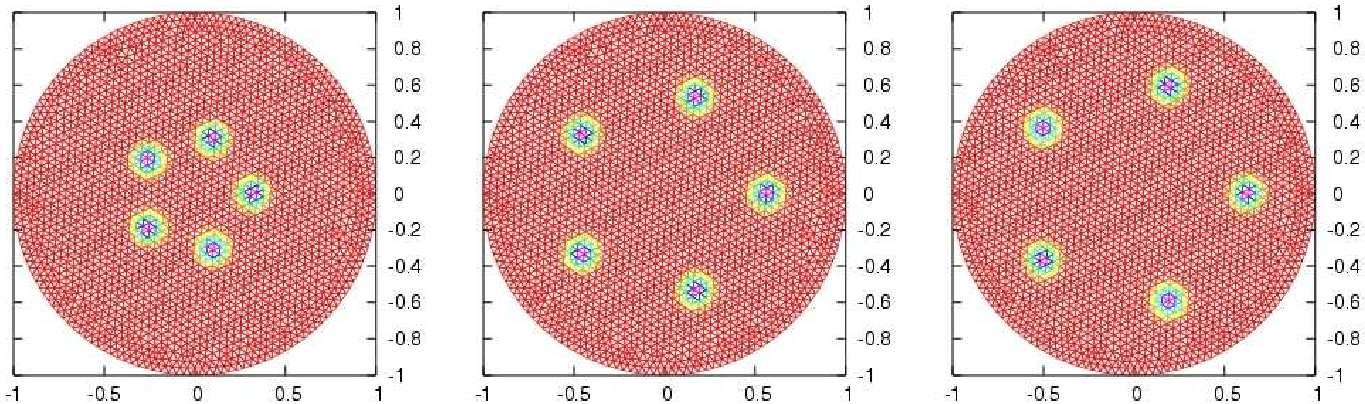
$$r' = -\varepsilon^2\gamma(S_c)S_c\left[-\frac{(K-1)}{2r} + \frac{Kr^{2K-1}}{1 - r^{2K}} + rK\right]\,.$$

This ODE has a unique stable equilibrium in $0 < r_e < 1$, satisfying

$$(K-1)/(2K) - r^2 = r^{2K}/(1 - r^{2K})\,.$$

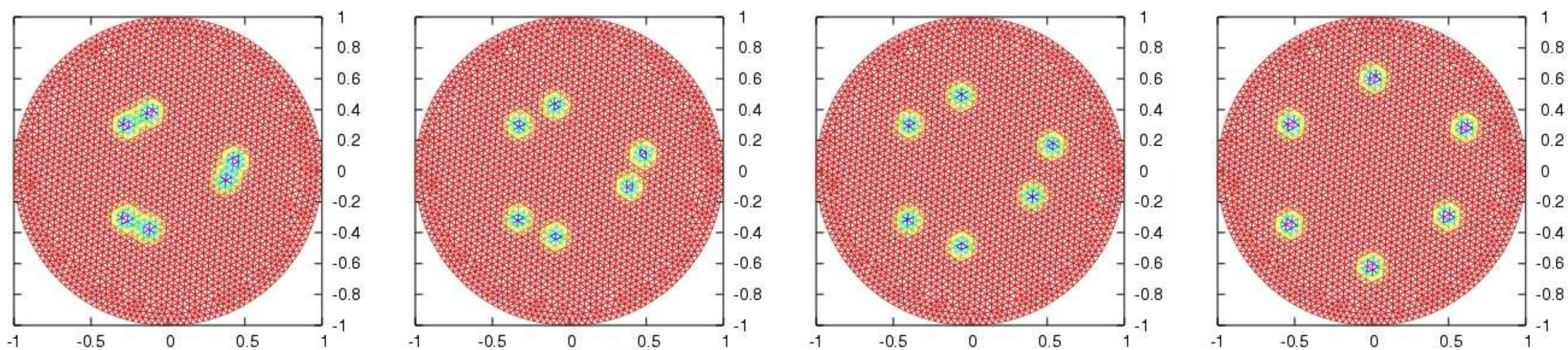
Ring Patterns in the Unit Disk: III

Experiment (Expanding Ring): $\varepsilon = 0.02$, $K = 5$, $a = 35$, and $D = 1$. Then, $S_c = 3.5 < \Sigma_2$, and the ring expands to $r_e \approx 0.625$.



$t = 8$ $\quad$ $t = 90$ $\quad$ $t = 297.$

Experiment (Spot-Splitting on a Ring): $\varepsilon = 0.02$, $K = 3$, $a = 30$, and $D = 1$. Then, $S_c = 5.0 > \Sigma_2$. Final state has 6 spots with $r_e \approx 0.642$. (movie)

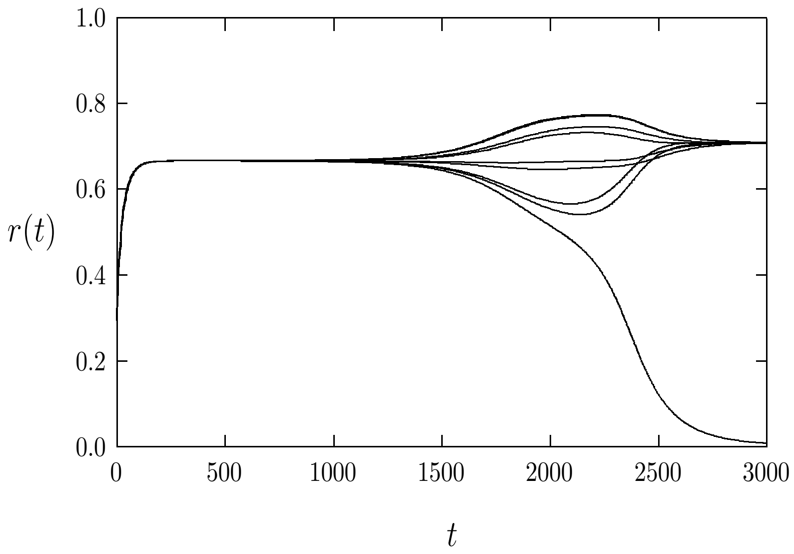


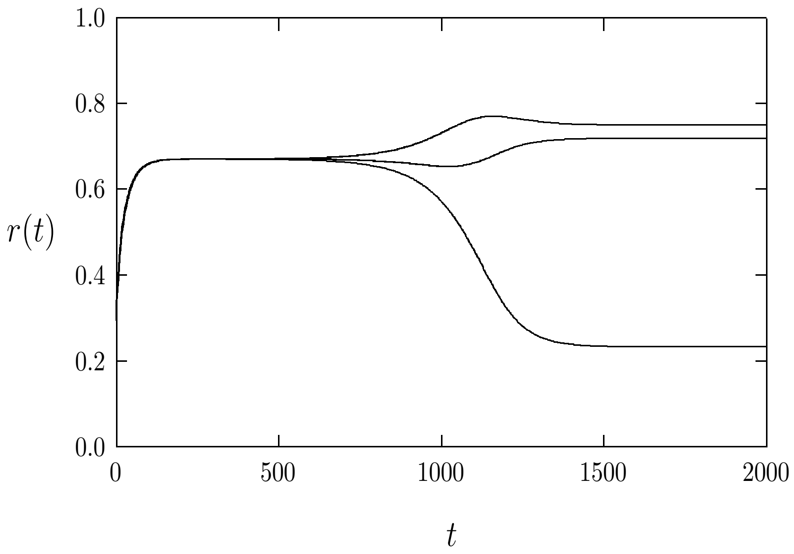
$t = 30$ $\quad$ $t = 45$ $\quad$ $t = 75$ $\quad$ $t = 135.$

Ring Patterns in the Unit Disk: IV

Although the ODE for the ring radius has a stable equilibrium, the DAE system has an instability if too many spots are on one ring.

Experiment (Small Eigenvalue Instability): Left: $\varepsilon = 0.02$, $a = 60$, $K = 9$, and $D = 1$. Initially nine spots remain on a slowly expanding ring. However, the equilibrium has eight spots on a ring with a center-spot. Right: $\varepsilon = 0.02$, $a = 66$, $K = 10$, and $D = 1$. The equilibrium has three rings with four spots each on two large rings and two spots on a smaller ring.





Similar instability to that found by S. Gueron, I. Shafir, “On a Discrete Variational Principle Involving Interacting Particles”, SIMA, 1999.

Ring Patterns in the Unit Disk: V

Now consider a ring with center-spot pattern in the unit disk

$$x_j = re^{2\pi i j/(K-1)}\,, \quad j = 1, \ldots, K-1\,, \qquad x_K = 0\,, \quad \text{(Pattern II)}\;.$$

Source Strengths: $S_j = S_c$ for $j = 1, \ldots, K-1$, and S_K for the center-spot.

Principal Result: *Let $K-1$ spots, with $K \geq 3$, be initially equi-distributed on a ring and put a center-spot at $x_K = 0$. Then, if $S_c, S_K < \Sigma_2$, the center-spot remains at $x_K = 0$, while the ring radius for the other $K-1$ spots satisfies*

$$r' = -\varepsilon^2 \gamma(S_c) S_c \left[-\frac{(K-2)}{2r} + \frac{(K-1)r^{2K-3}}{1-r^{2K-2}} + r(K-1) + \frac{S_K}{S_c}\left(r - \frac{1}{r}\right)\right]\;.$$

Here $S_c(r)$ and $S_K(r)$ are found from the nonlinear algebraic equation:

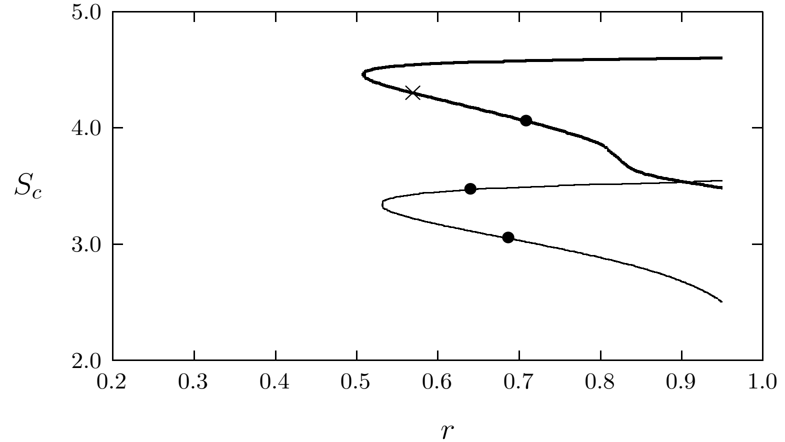
$$c_1 S_c + c_2 = \chi(\mu - S_c(K-1)) - \chi(S_c)\,, \quad S_K = \mu - (K-1)S_c\,, \quad \mu \equiv \frac{a|\Omega|}{2\pi\sqrt{D}}\,,$$

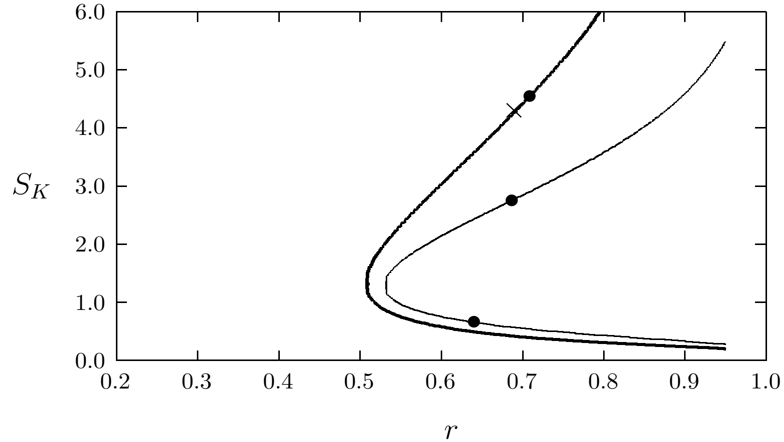
where c_1 and c_2 are constants depending on K, μ, and r.

Ring Patterns in the Unit Disk: VI

Numerical solution of the nonlinear algebraic system reveals bistability

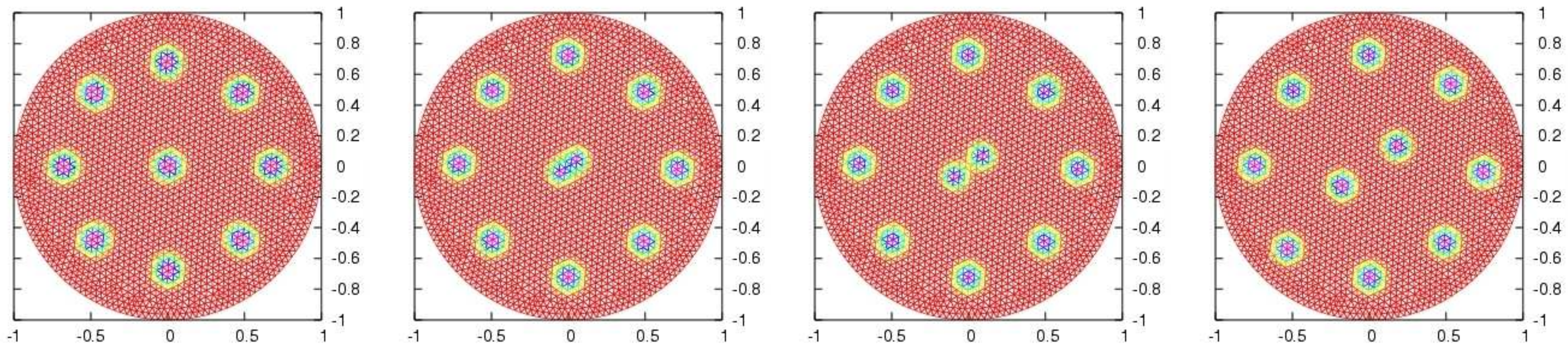
$\varepsilon = 0.02$, $D = 1$. **Solid:** $K = 6$, $a = 36$. **Heavy Solid:** $K = 9$, $a = 74$.





Dynamic Spot-Splitting Instability: A ring pattern II that is stable at $t = 0$ can become unstable at some $t > 0$ when S_K exceeds $\Sigma_2 \approx 4.3$.

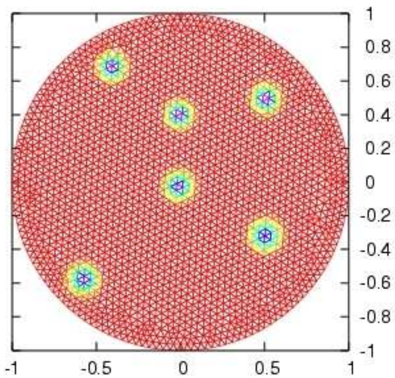
Experiment: $\varepsilon = 0.02$, $K = 9$, $a = 74$, and $D = 1$. The center-spot eventually splits since $S_K > \Sigma_2$ at some $t > 0$. (movie).



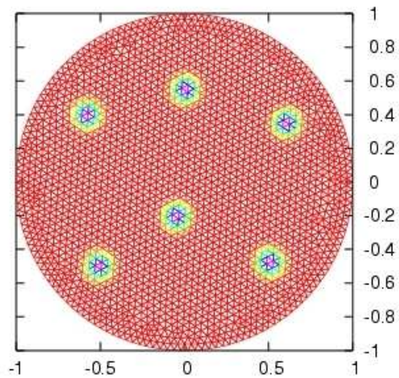
$t = 54$ $t = 372$ $t = 387$ $t = 800.$

Ring Patterns in the Unit Disk: VII

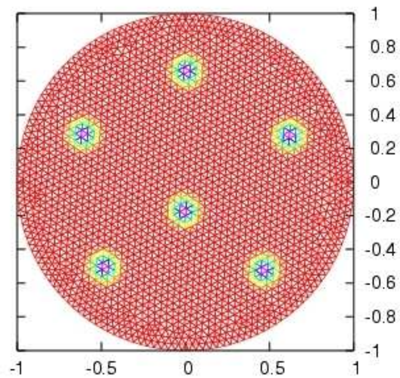
Experiment (Convergence to Pattern II): Let $\varepsilon = 0.02$, $K = 6$, $a = 36$, and $D = 1$, with given $x_j(0)$. The solution to the DAE system converges to a ring with center-spot pattern as t increases, representing a point on the lower branch of the S_c versus r diagram. (movie).



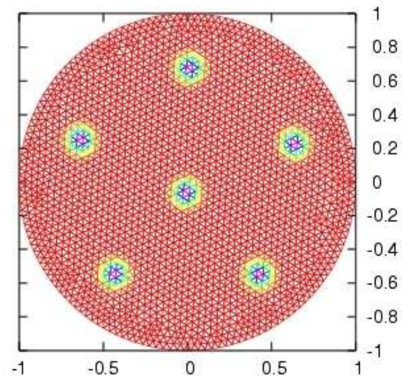
$t = 23$



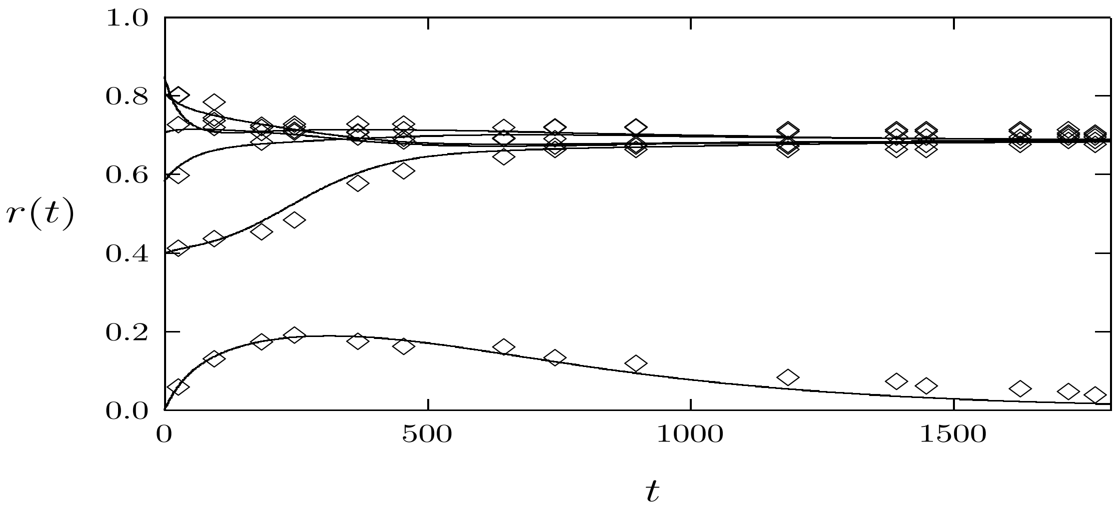
$t = 418$



$t = 922$



$t = 1880$.



A Related Problem: GS Model

GS model with $0 < \varepsilon \ll 1$ and parameters $D > 0$, $\tau > 0$, and $A > 0$:

$$v_t = \varepsilon^2 \Delta v - v + A u v^2 \,, \qquad \tau u_t = D \Delta u + (1-u) - u v^2 \,.$$

Key: inner core problem is the same as for the Schnakenburg model: Wan Chen, (UBC) (ongoing thesis) has derived and is studying:

Principal Result: (DAE System): *Let $\mathcal{A} = \varepsilon A / (\nu \sqrt{D})$ and $\nu = -1/\log \varepsilon$. The DAE system for the source strengths S_j and spot locations x_j is*

$$\mathcal{A} = S_j + 2\pi\nu \left(S_j R_{j,j} + \sum_{\substack{i=1 \\ i \neq j}}^{K} S_i G_{j,i} \right) + \nu \chi(S_j) \,, \quad j = 1, \ldots, K$$

$$x_j' \sim -2\pi\varepsilon^2 \gamma(S_j) \left(S_j \nabla R(x_j; x_j) + \sum_{\substack{i=1 \\ i \neq j}}^{K} S_i \nabla G(x_j; x_i) \right) \,, \quad j = 1, \ldots, K \,.$$

Here $G_{j,i} \equiv G(x_j; x_i)$ and $R_{j,j} \equiv R(x_j; x_j)$, where G is the Reduced Wave Green's function with regular part R, i.e. $\Delta G - \frac{1}{D} G = -\delta(x - x_j)$.

Open Issues and Further Directions

Open Issues:

- **Rigour:** existence and stability theory for core problem, derivation of DAE system?
- **DAE System:** Bifurcation study of solutions to the DAE system. Multiple equilibria and bistability? Dynamic Instabilities?
- **Beyond NLEP theory:** stability theory for the $m = 0$ mode to include all powers of ν?
- **Universality:** similar analyses for other RD models such as GS and GM models leading to similar DAE systems and spot-splitting criteria?
- **Construction of Attractors:** can one construct an attractor or “loop” that is composed of splitting events, leading to spot creation, followed by an over-crowding instability (competition instability) from an NLEP-type theory, leading to spot death? This should be possible for the Gray-Scott model