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STRONG LOCALIZED PERTURBATIONS OF EIGENVALUE PROBLEMS*

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Abstract. This paper considers the effect of three types of perturbations of large magnitude but small extent on a class of linear eigenvalue problems for elliptic partial differential equations in bounded or unbounded domains. The perturbations are the addition of a function of small support and large magnitude to the differential operator, the removal of a small subdomain from the domain of a problem with the imposition of a boundary condition on the boundary of the resulting hole, and a large alteration of the boundary condition on a small region of the boundary of the domain. For each of these perturbations, the eigenvalues and eigenfunctions for the perturbed problem are constructed by the method of matched asymptotic expansions for ϵ small, where ϵ is a measure of the extent of the perturbation. In some special cases, the asymptotic results are shown to agree well with exact results. The asymptotic theory is then applied to determine the exit time distribution for a particle undergoing Brownian motion inside a container having reflecting walls perforated by many small holes.

Key words. eigenvalues, strong localized perturbations, solvability conditions, asymptotic expansions

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Introduction. A perturbation of large magnitude but small extent will be called a strong localized perturbation. It may be contrasted with a weak perturbation, which is of small magnitude but may have large extent. We shall show how to calculate the effects of strong localized perturbations on the solutions of eigenvalue problems.

Three examples of strong localized perturbations that we will consider are the addition of a function of small support and large magnitude to a differential operator, the removal of a small subdomain from the domain of a problem with the imposition of a boundary condition on the boundary of the resulting hole, and a large alteration of the boundary condition on a small region of the boundary of a domain.

Problems analogous to those of the first kind arise in the measurement of the electrical properties of a sample of material by the change it produces in the resonant frequency of a cavity oscillator in which it is placed. Problems involving the effect of perturbing the boundary condition occur in calculating the change in resonant frequencies in room acoustics due to absorbing material on portions of the wall or ceiling. They also occur in analyzing the escape of particles from nearly closed containers.

Strong localized perturbations are singular perturbations in the sense that they produce large changes in the solutions of the problems in which they occur. However, these large changes are themselves localized. Consequently, the perturbed solutions can be constructed by the method of matched asymptotic expansions. An inner expansion can describe the large changes in the solution in a neighborhood of the strong perturbation. An outer expansion, valid in the region away from the strong perturbation, can account for the relatively small effects that the perturbation produces there.

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These two expansions can be matched to determine the undetermined coefficients in both of them and in the expansion of the eigenvalue.

In §§1 and 2 we consider the effect of a strong localized potential on the eigenvalues of the Schrodinger operator in a domain D of R^n where $n \geq 3$. In §3 we determine the effect of deleting a small subdomain from D when $n \geq 3$. The corresponding two-dimensional case is considered in §4. The results proved previously by Ozawa [3]–[6] in some special cases are reproduced in §§3 and 4. For some special geometries, we also show that the asymptotic results for the perturbed eigenvalues agree well with corresponding exact analytical results.

In §5 we treat the case of a strong localized perturbation of the boundary condition. The results are applied in §6 to determine the exit time distribution for a particle undergoing Brownian motion inside a container having reflecting walls perforated by many small holes.

The present analysis has been adapted in [9]–[11] to treat the effect of strong localized perturbations on nonlinear eigenvalue problems. Corrections to fold points of S-shaped response curves were obtained and applied in combustion theory. Some related work is found in [2].

1. Perturbation by a strong localized potential ($n \geq 3$). Let us consider the following unperturbed eigenvalue problem for the Schrodinger operator in a domain D of the n-dimensional space R^n:

$$\text{(1.1a)} \qquad \big[-\Delta + U(x)\big]\, u_0(x) = \lambda_0\, \rho(x)\, u_0(x)\,, \quad x \in D\,,$$

$$\text{(1.1b)} \qquad \big[\partial_n + b(x)\big] u_0(x) = 0\,, \quad x \in \partial D\,,$$

$$\text{(1.1c)} \qquad \int_D u_0^2(x)\, \rho(x)\, dx = 1\,.$$

The potential $U(x)$, the weight function $\rho(x) > 0$, and the boundary impedance $b(x)$ in (1.1) are given smooth functions. We assume that (1.1) has a simple isolated eigenvalue λ_0 with a corresponding eigenfunction $u_0(x)$. If $D = R^n$, condition (1.1b) is to be omitted.

Now we introduce a strongly perturbed form of this problem by adding to (1.1a) the perturbing potential $\epsilon^{-2}V(y)$, where $y = (x - x_0)/\epsilon$. Here x_0 is an interior point of D, and ϵ is a small parameter proportional to the range of the perturbing potential. We require that $V(y)$ tend to zero sufficiently rapidly as $|y|$ tends to infinity. Some examples of the perturbing potentials that we can treat are those that decay exponentially as $y \to \infty$, those that have sufficiently fast algebraic decay as $y \to \infty$ in a sense to be made precise below, or those that have compact support. Then as $\epsilon \to 0$, $\epsilon^{-2}\, V[(x - x_0)/\epsilon]$ tends to zero for all $x \neq x_0$. In order that the perturbation produce an appreciable effect on the eigenvalue and eigenfunction for small values of ϵ, the strength of the perturbation potential is made proportional to ϵ^{-2} for $x - x_0 = O(\epsilon)$. Thus the strongly perturbed problem is to find $u(x, \epsilon)$ and $\lambda(\epsilon)$ satisfying

$$\text{(1.2a)} \quad \big[-\Delta + U(x) + \epsilon^{-2}V[(x - x_0)/\epsilon]\big] u(x,\epsilon) = \lambda(\epsilon)\, \rho(x)\, u(x,\epsilon)\,, \quad x \in D\,,$$

$$\text{(1.2b)} \qquad \big[\partial_n + b(x)\big] u(x,\epsilon) = 0\,, \quad x \in \partial D\,,$$

$$\text{(1.2c)} \qquad \int_D u^2(x,\epsilon)\, \rho(x)\, dx = 1\,.$$

To reveal the nature of the spectrum associated with (1.2), we introduce the local variables $y = \epsilon^{-1}(x - x_0)$ and $v(y, \epsilon) = u(x_0 + \epsilon y)$, defined near the support of V.

Then we obtain the following inner eigenvalue problem:

$$-\Delta_y v + V(y)\, v = \mu\, \rho(x_0) v\,, \quad y \in R^n\,; \qquad v \to 0 \quad \text{as } y \to \infty\,. \tag{1.3}$$

This problem has a finite number of discrete eigenvalues μ_k as well as a continuous spectrum. The full problem (1.2) has a discrete eigenvalue near the eigenvalue $\mu_k \epsilon^{-2}$ associated with the inner problem. The full problem (1.2) also has a discrete eigenvalue near any isolated eigenvalue of the outer problem (1.1). The analysis below is limited to this latter case in which $\lambda(\epsilon)$ tends to a simple isolated eigenvalue λ_0 of (1.1). A technical assumption that we shall need throughout the analysis below is that (1.3) does not have a zero eigenvalue.

We seek a solution of (1.2) for ϵ small, for which $\lambda(\epsilon) \to \lambda_0$ as $\epsilon \to 0$. We expect that the corresponding perturbed eigenfunction $u(x, \epsilon)$ will differ appreciably from $u_0(x)$ for x near x_0, but that it will differ little from u_0 for x far from x_0. Therefore, we shall represent $u(x, \epsilon)$ by two different asymptotic expansions for $\epsilon \ll 1$, an "inner" expansion for x near x_0 and an "outer" expansion for x far from x_0.

The outer expansion of u must begin with u_0, and the expansion of λ must begin with λ_0; so we write

$$u(x, \epsilon) = u_0(x) + \nu_1(\epsilon)\, u_1(x) + \nu_2(\epsilon)\, u_2(x) + \nu_3(\epsilon)\, u_3(x) + \cdots\,, \qquad |x - x_0| \gg O(\epsilon), \tag{1.4}$$

$$\lambda(\epsilon) = \lambda_0 + \nu_1(\epsilon)\, \lambda_1 + \nu_2(\epsilon)\, \lambda_2 + \nu_3(\epsilon)\, \lambda_3 + \cdots\,. \tag{1.5}$$

Here the gauge functions $\nu_i(\epsilon)$, which satisfy $\nu_i(\epsilon) \ll 1$ and $\nu_i(\epsilon) \gg \nu_{i+1}(\epsilon)$ as $\epsilon \to 0$, are to be found. Now we substitute (1.4) and (1.5) into (1.2) and equate terms of each order in ϵ. The terms of order ϵ^0 yield (1.1) for $x \neq x_0$ while the terms of order $\nu_1(\epsilon)$ yield

$$\Delta u_1 - U(x)\, u_1 + \lambda_0\, \rho(x)\, u_1 = -\lambda_1\, \rho(x)\, u_0\,, \qquad |x - x_0| \gg O(\epsilon)\,, \tag{1.6a}$$

$$\big[\partial_n + b(x)\big]\, u_1(x) = 0\,, \qquad x \in \partial D\,, \tag{1.6b}$$

$$\int_D u_0(x)\, u_1(x)\, \rho(x)\, dx = 0\,. \tag{1.6c}$$

Since the behavior of $u_1(x)$ as x tends to x_0 is not yet known, the function u_1 is not yet completely defined. However, u_0 was defined as a solution of (1.1), so it is regular at x_0.

To write the inner expansion of u, we introduce the stretched variable $y = (x - x_0)/\epsilon$ and set $v(y, \epsilon) = u(x_0 + \epsilon y, \epsilon)$. Then (1.2a) becomes

$$-\Delta_y v + \big[V(y) + \epsilon^2\, U(x_0 + \epsilon y)\big]\, v = \epsilon^2\, \lambda\, \rho(x_0 + \epsilon y)\, v\,. \tag{1.7}$$

Now for $n \geq 3$ we write the inner expansion of u as

$$u(x_0 + \epsilon y, \epsilon) = v(y, \epsilon) = v_0(y) + \epsilon v_1(y) + \epsilon^2 v_2(y) + \cdots\,. \tag{1.8}$$

Next we substitute (1.5) and (1.8) into (1.7) and equate coefficients of ϵ^0, ϵ, and ϵ^2 to get

$$\Delta_y v_0 - V(y)\, v_0 = 0\,, \tag{1.9}$$

$$\Delta_y v_1 - V(y)\, v_1 = 0\,, \tag{1.10}$$

$$\Delta_y v_2 - V(y)\, v_2 = -\lambda_0\, \rho(x_0)\, v_0 + U(x_0)\, v_0\,. \tag{1.11}$$

The inner and outer expansions must be asymptotically equal in some overlap domain within which y is large and $x - x_0$ is small. Thus in this domain the matching condition is

$$u_0(x) + \nu_1(\epsilon)u_1(x) + \nu_2(\epsilon)u_2(x) + \cdots \sim v_0(y) + \epsilon v_1(y) + \epsilon^2 v_2(y) + \cdots . \tag{1.12}$$

From the terms of order ϵ^0 in (1.12), we obtain the first matching condition, $v_0(y) \to u_0(x_0)$ as $y \to \infty$. Under the assumption that $\mu = 0$ is not an eigenvalue of (1.3), there exists a unique solution $\hat{v}_0(y)$ of (1.9) that satisfies $\hat{v}_0(y) \to 1$ as $y \to \infty$. Therefore, the leading-order inner solution is $v_0(y) = u_0(x_0)\hat{v}_0(y)$. To proceed further with the matching, we need more terms in the behavior of $\hat{v}_0(y)$ for $y \to \infty$. We assume that $|y|^k V(y) \to 0$ as $|y| \to \infty$ for $k = n+1$. Then (1.9) reduces to Laplaces's equation for $|y| \to \infty$ and

$$\hat{v}_0(y) = 1 - \frac{C}{|y|^{n-2}} + \frac{C_i\, y_i}{|y|^n} + \cdots . \tag{1.13}$$

The coefficients C and C_i in (1.13) are uniquely determined by the solution $\hat{v}_0(y)$, so they are determined by the perturbing potential $V(y)$. In three dimensions ($n = 3$), C has the dimensions of length, and it is called the "scattering length" of V.

We now use $v_0(y) = u_0(x_0)\hat{v}_0(y)$ in (1.12) and set $y = (x - x_0)/\epsilon$. Then the second term in the far field form of $u_0(x_0)\hat{v}_0(y)$ becomes $-\epsilon^{n-2}u_0(x_0)\, C/|x - x_0|^{n-2}$, which must be matched by the second term $\nu_1(\epsilon)\, u_1(x)$ on the left-hand side of (1.12). Therefore, the first gauge function must be $\nu_1(\epsilon) = \epsilon^{n-2}$, and then

$$u_1(x) \sim -u_0(x_0)\, C/|x - x_0|^{n-2} \quad \text{as } x \to x_0 . \tag{1.14}$$

This condition determines the previously unknown behavior of $u_1(x)$ as x tends to x_0.

Problems (1.6) and (1.14) are an inhomogeneous form of (1.1), so they will have a solution only if the inhomogeneous terms satisfy a solvability condition. To derive it, we multiply (1.6a) by u_0 and integrate the resulting equation over the region outside a small sphere D_σ of radius σ centered at x_0. Upon using Green's theorem and the boundary conditions (1.1b) and (1.6b), or the assumed decay of u_0 and u_1 at infinity if $D = R^n$, we obtain

$$-\lambda_1 \int_{D \backslash D_\sigma} u_0^2(x)\, \rho(x)\, dx = \int_{\partial D_\sigma} (u_0 \partial_n u_1 - u_1 \partial_n u_0)\, dx . \tag{1.15}$$

Here ∂_n corresponds to the outward normal derivative to $D\backslash D_\sigma$. Now, by using the behavior (1.14) of u_1 near x_0, we can evaluate the limit of the right-hand side of (1.15) as σ tends to zero. The integral on the left-hand side tends to 1 because u_0 is normalized, and we obtain

$$\lambda_1 = (n-2)\, \omega_n\, C\, \big[u_0(x_0)\big]^2 . \tag{1.16}$$

Here $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in n dimensions, and $\Gamma(z)$ is the gamma function.

With λ_1 given by (1.16), problems (1.6) and (1.14) can be solved for u_1. The solution is made unique by the orthogonality condition (1.6c). Substituting $u_1(x)$ and $\nu_1(\epsilon) = \epsilon^{n-2}$ into (1.4) gives the first correction term in the outer expansion of $u(x, \epsilon)$. The leading term in the inner expansion is $v_0(y) = u_0(x_0)\hat{v}_0(y)$. The first correction to the eigenvalue is λ_1 given by (1.16). We summarize our results in the following statement.

PROPOSITION 1. *For dimension $n \geq 3$, assume that $|y|^k V(y) \to 0$ as $|y| \to \infty$ for $k = n+1$ and that $\mu = 0$ is not an eigenvalue of* (1.3). *Assume that $\lambda(\epsilon)$ is an eigenvalue of* (1.2) *that tends to a simple isolated eigenvalue λ_0 of* (1.1) *as $\epsilon \to 0$. Then, for $\epsilon \ll 1$,*

$$\lambda(\epsilon) = \lambda_0 + \epsilon^{n-2}(n-2)\frac{2\pi^{n/2}}{\Gamma(n/2)} C\left[u_0(x_0)\right]^2 + O(\epsilon^{n-1}), \qquad n \geq 3. \tag{1.17}$$

Here $u_0(x)$ is the normalized eigenfunction of (1.1) *corresponding to λ_0. The normalized eigenfunction $u(x,\epsilon)$ corresponding to $\lambda(\epsilon)$ is given by the two expansions*

$$u(x,\epsilon) = u_0(x) + \epsilon^{n-2} u_1(x) + O(\epsilon^{n-1}), \quad |x-x_0| \gg \epsilon, \tag{1.18}$$

$$u(x,\epsilon) = u_0(x_0)\,\hat{v}_0\left(\frac{x-x_0}{\epsilon}\right) + O(\epsilon), \quad |x-x_0| = O(\epsilon). \tag{1.19}$$

The function u_1 satisfies (1.6) *and* (1.14), *while $\hat{v}_0$ satisfies* (1.9) *and $\hat{v}_0(\infty) = 1$. The constant C in* (1.17) *is determined by the solution $\hat{v}_0$ via* (1.13). *The composite expansion of $u(x,\epsilon)$, valid everywhere in D, obtained by adding* (1.18) *and* (1.19) *and subtracting their common part, is given by*

$$u(x,\epsilon) = u_0(x) + \epsilon^{n-2}u_1(x) + u_0(x_0)\left[\hat{v}_0\left(\frac{x-x_0}{\epsilon}\right) - \left(1 - \frac{\epsilon^{n-2}C}{|x-x_0|^{n-2}}\right)\right] + \cdots. \tag{1.20}$$

2. Higher-order terms ($n \geq 3$). Now we will extend the calculations of §1 to obtain further terms in the expansions. These terms are essential when $u_0(x_0) = 0$, for then the correction term λ_1, given in (1.16), vanishes.

By using $y = (x-x_0)/\epsilon$ in (1.13), we find that the third term is of order ϵ^{n-1}. Thus, when (1.13) is used in the matching condition (1.12), it follows that $\nu_2(\epsilon) = \epsilon^{n-1}$. Then, with this gauge function, and assuming a sufficiently rapid decay of $V(y)$ as $y \to \infty$, we substitute (1.4) and (1.5) into (1.2) to obtain the following equation for u_2:

$$\Delta u_2 - U(x)\,u_2 + \lambda_0\,\rho(x)\,u_2 = -\lambda_2\rho(x)\,u_0 - \lambda_1\,\rho(x)\,u_1\delta_{n3}, \qquad |x-x_0| \gg O(\epsilon), \tag{2.1a}$$

$$\left[\partial_n + b(x)\right]u_2 = 0, \quad x \in \partial D, \tag{2.1b}$$

$$\int_D \left[2u_0u_2 + u_1^2\delta_{n3}\right]\rho(x)\,dx = 0. \tag{2.1c}$$

We now must determine the singular behavior of u_2 as x tends to x_0.

To do so, we first calculate the next term in u_1 for x near x_0 by using (1.6) and (1.14). We find that it is given by

$$u_1(x) \sim -\frac{u_0(x_0)\,C}{|x-x_0|^{n-2}} - \alpha\,u_0(x_0)\,C \begin{cases} |x-x_0|^{4-n} & \text{for } n \geq 5, \\ \log|x-x_0| & \text{for } n = 4, \\ 1 & \text{for } n = 3. \end{cases} \tag{2.2}$$

Here α is a constant, independent of C but depending on n, which is determined uniquely by the solution u_1. Writing (2.2) in inner variables, expanding $u_0(x_0+\epsilon y) = u_0(x_0) + \epsilon\left[\partial_{x_i}u_0(x_0)\right]y_i + \cdots$, and then using the matching condition (1.12), we obtain

$$v_1(y) \sim \left[\partial_{x_i}u_0(x_0)\right]y_i - \alpha\,u_0(x_0)\,C\,\delta_{n3} \quad \text{as } |y| \to \infty. \tag{2.3}$$

This condition determines a unique solution of (1.10) for $v_1(y)$, provided that we require that further terms in the expansion of v_1 vanish as $|y| \to \infty$. Then, as in the case of v_0, we can express the solution $v_1(y)$ in the form

$$v_1(y) = [\partial_{x_i} u_0(x_0)]\, \hat{v}_{1i}(y) - \alpha\, u_0(x_0)\, C\, \delta_{n3}\, \hat{v}_0(y)\,. \tag{2.4}$$

Here $\hat{v}_0(y)$ was defined below (1.12), and $\hat{v}_{1i}(y)$ is the unique solution of (1.10) satisfying $\hat{v}_{1i}(y) \sim y_i + o(1)$ as $y \to \infty$. In particular, assuming that $|y|^k V(y) \to 0$ for $k = n+2$ then the far field form of $\hat{v}_{1i}(y)$ is

$$\hat{v}_{1i}(y) = \left[y_i + \frac{B_i}{|y|^{n-2}} + \frac{B_{ij} y_j}{|y|^n} + \cdots \right] \quad \text{as } |y| \to \infty\,. \tag{2.5}$$

The constants B_i and B_{ij} are determined by the solution $\hat{v}_{1i}(y)$ and thus by the potential $V(y)$.

We now use (2.4), (2.5), and (1.13) in the matching condition (1.12) with $y = \epsilon^{-1}(x - x_0)$. Then the terms of order ϵ^{n-1} yield the following condition on u_2 as x tends to x_0:

$$u_2(x) \sim \frac{u_0(x_0) C_i\, (x_i - x_{0i})}{|x - x_0|^n} + \frac{[\partial_{x_i} u_0(x_0) B_i + \alpha\, u_0(x_0)\, C^2\, \delta_{n3}]}{|x - x_0|^{n-2}}\,. \tag{2.6}$$

By proceeding as before and using (1.15), with u_1, λ_1 replaced by u_2, λ_2, we derive the solvability condition for (2.1) and (2.6), and from it we get

$$\lambda_2 = -u_0(x_0)\, \omega_n\, C_i\, \partial_{x_i} u_0(x_0) + (2-n)\, \omega_n\, u_0(x_0)\, [\partial_{x_i} u_0(x_0) B_i + \alpha\, u_0(x_0)\, C^2\, \delta_{n3}]\ . \tag{2.7}$$

Then problems (2.1), (2.6) can be solved uniquely for u_2. Now both (1.17) for $\lambda(\epsilon)$ and (1.19) for the inner expansion of u can be extended to another term. We summarize our results as follows.

COROLLARY 1. *For dimension $n \geq 3$, assume that $|y|^k V(y) \to 0$ as $|y| \to \infty$ for $k = n+2$ and that $\mu = 0$ is not an eigenvalue of* (1.3). *Assume that $\lambda(\epsilon)$ is an eigenvalue of* (1.2) *that tends to a simple isolated eigenvalue λ_0 of* (1.1) *as $\epsilon \to 0$. Then, for $\epsilon \ll 1$,*

$$\begin{aligned}\lambda(\epsilon) =& \lambda_0 + \epsilon^{n-2}(n-2)\, \omega_n\, C\, \big[u_0(x_0)\big]^2 \\ &+ \epsilon^{n-1} u_0(x_0) \omega_n \Big[\partial_{x_i} u_0(x_0) \Big((2-n) B_i - C_i\Big) \\ &\qquad + (2-n)\, \alpha\, u_0(x_0) C^2\, \delta_{n3}\Big] + \cdots\,,\end{aligned} \tag{2.8}$$

$$u(x,\epsilon) = u_0(x) + \epsilon^{n-2}\, u_1(x) + \epsilon^{n-1}\, u_2(x) + \cdots\,, \qquad |x - x_0| \gg \epsilon\,, \tag{2.9}$$

$$\begin{aligned}u(x,\epsilon) =& u_0(x_0)\, \hat{v}_0\left(\frac{x - x_0}{\epsilon}\right) \\ &+ \epsilon \left[\partial_{x_i} u_0(x_0) \hat{v}_{1i}\left(\frac{x - x_0}{\epsilon}\right) - \alpha\, u_0(x_0)\, C \delta_{n3} \hat{v}_0\left(\frac{x - x_0}{\epsilon}\right)\right] + \cdots\,, \\ & \qquad |x - x_0| = O(\epsilon)\,.\end{aligned} \tag{2.10}$$

Further terms can be obtained by continuing in the same way.

We will not calculate any further in general. However, in the special case when $u_0(x_0) = 0$, both corrections λ_1 and λ_2 vanish, and then we must proceed further to obtain a nonzero correction to λ_0. Since $\lambda_1 = 0$ and $u_0(x_0) = 0$, we find from (1.6) and (1.14) that $u_1 \equiv 0$. However, we observe from (2.1) and (2.6) that u_2 does not vanish identically. To determine the first nonvanishing correction to the eigenvalue, we choose $\nu_3(\epsilon) = \epsilon^n$ so that the next term in expansions (1.4) and (1.5) are $\epsilon^n u_3(x)$ and $\epsilon^n \lambda_3$, respectively. Using these expansions in (1.2), and assuming a sufficiently rapid decay of $V(y)$ as $y \to \infty$, we find that u_3 satisfies

$$\Delta u_3 - U(x)\, u_3 + \lambda_0\, \rho(x)\, u_3 = -\lambda_3\, \rho(x)\, u_0\,, \qquad |x - x_0| \gg O(\epsilon)\,, \tag{2.11a}$$

$$\big[\partial_n + b(x)\big] u_3 = 0\,, \quad x \in \partial D\,, \tag{2.11b}$$

$$\int_D u_0\, u_3\, \rho(x)\, dx = 0\,. \tag{2.11c}$$

To determine the singular behavior of u_3, we first must calculate the next term in u_2 for x near x_0 when $u_0(x_0) = 0$. From (2.1) and (2.6), we find that

$$u_2(x) \sim \frac{\partial_{x_i} u_0(x_0)\, B_i}{|x - x_0|^{n-2}} + \partial_{x_i} u_0(x_0)\, B_i\, \gamma \begin{cases} |x - x_0|^{4-n} & \text{for} \quad n \geq 5, \\ \log|x - x_0| & \text{for} \quad n = 4, \\ 1 & \text{for} \quad n = 3. \end{cases} \tag{2.12}$$

Here γ is a constant, independent of B_i, determined uniquely by the solution u_2. We now write (2.12) in inner variables, expand $u_0(x_0 + \epsilon y)$ through terms of order ϵ^2, and use the matching condition (1.12) to obtain

$$v_2(y) \sim \tfrac{1}{2}\, \left[\partial_{x_i} \partial_{x_j} u_0(x_0)\right]\, y_i\, y_j + \partial_{x_i} u_0(x_0)\, B_i\, \gamma\, \delta_{n3} \quad \text{as} \quad |y| \to \infty\,. \tag{2.13}$$

When $u_0(x_0) = 0$, the equation for v_2 is obtained from (1.11) upon setting $v_0 \equiv 0$. The solution to (1.11) and (2.13) can be written in the form

$$v_2(y) = \left[\partial_{x_i} \partial_{x_j} u_0(x_0)\right]\, \hat{v}_{2ij}(y) + \partial_{x_i} u_0(x_0)\, B_i\, \gamma\, \delta_{n3}\, \tilde{v}_2(y)\,. \tag{2.14}$$

Here $\hat{v}_{2ij}(y)$ and $\tilde{v}_2(y)$ are solutions of (1.11) with the following asymptotic forms as $|y| \to \infty$:

$$\hat{v}_{2ij}(y) = \frac{1}{2} y_i y_j + \frac{D_{ij}}{|y|^{n-2}} + \cdots\,, \qquad \tilde{v}_2(y) = 1 + \frac{E}{|y|^{n-2}} + \cdots\,. \tag{2.15}$$

When (2.4), (2.5) and (2.14), (2.15) are used in the matching condition (1.12), the terms of order ϵ^n yield the following condition on u_3 as x tends to x_0:

$$u_3(x) \sim \frac{\partial_{x_i} u_0(x_0)\, B_{ij}\, (x_j - x_{0j})}{|x - x_0|^n} + \frac{\left[\partial_{x_i} \partial_{x_j} u_0(x_0)\right]\, D_{ij}}{|x - x_0|^{n-2}} + \frac{\partial_{x_i} u_0(x_0)\, B_i\, \gamma\, E\, \delta_{n3}}{|x - x_0|^{n-2}}\,. \tag{2.16}$$

Invoking the solvability condition (1.15) on (2.11), (2.16), and using $u_0(x_0) = 0$, we obtain λ_3 as

$$\lambda_3 = -\omega_n\, \partial_{x_i} u_0(x_0)\, B_{ij}\, \partial_{x_j} u_0(x_0)\,. \tag{2.17}$$

Then (2.11) and (2.16) can be solved uniquely for u_3. We summarize our results as follows.

COROLLARY 2. *For dimension $n \geq 3$, assume that $|y|^k V(y) \to 0$ as $|y| \to \infty$ for $k = n+3$ and that $\mu = 0$ is not an eigenvalue of* (1.3). *Assume that $u_0(x_0) = 0$ and that $\lambda(\epsilon)$ is an eigenvalue of* (1.2) *that tends to a simple isolated eigenvalue λ_0 of* (1.1) *as $\epsilon \to 0$. Then, for $\epsilon \ll 1$,*

$$\lambda(\epsilon) = \lambda_0 - \epsilon^n \, \omega_n \, \partial_{x_i} u_0(x_0) \, B_{ij} \, \partial_{x_j} u_0(x_0) + \cdots \,, \tag{2.18}$$

$$u(x,\epsilon) = u_0(x_0) + \epsilon^{n-1} u_2(x) + \epsilon^n u_3(x) + \cdots \,, \qquad |x - x_0| \gg \epsilon \,, \tag{2.19}$$

$$\begin{aligned} u(x,\epsilon) = {} & \epsilon \, [\partial_{x_i} u_0(x_0)] \, \hat{v}_{1i} \left(\frac{x - x_0}{\epsilon} \right) + \epsilon^2 \left[\partial_{x_i} \partial_{x_j} u_0(x_0) \right] \hat{v}_{2ij} \left(\frac{x - x_0}{\epsilon} \right) \\ & + \epsilon^2 \, \partial_{x_i} u_0(x_0) \, B_i \, \gamma \, \delta_{n3} \, \hat{v}_2 \left(\frac{x - x_0}{\epsilon} \right) + \cdots \,, \qquad |x - x_0| = O(\epsilon) \,. \end{aligned} \tag{2.20}$$

3. Deletion of a small subdomain ($n \geq 3$). We now perturb the eigenvalue problem (1.1) by removing from D a small subdomain D_ϵ of "radius" $O(\epsilon)$, centered at some point x_0 in D, and we impose a boundary condition on the resulting hole. The perturbed problem is

$$\triangle u(x,\epsilon) + \Big(\lambda(\epsilon) \, \rho(x) - U(x) \Big) \, u(x,\epsilon) = 0 \,, \qquad x \in D \backslash D_\epsilon \,, \tag{3.1a}$$

$$\partial_n \, u(x,\epsilon) + b(x) \, u(x,\epsilon) = 0 \,, \qquad x \in \partial D \,, \tag{3.1b}$$

$$\epsilon \, \partial_n \, u(x,\epsilon) + \kappa \, u(x,\epsilon) = 0 \,, \qquad x \in \partial D_\epsilon \,, \tag{3.1c}$$

$$\int_{D \backslash D_\epsilon} u^2(x,\epsilon) \, \rho(x) \, dx = 1 \,. \tag{3.1d}$$

Here κ is a constant and $\partial_n u$ is the directional derivative of u along the outward normal to $D \backslash D_\epsilon$. The domain D_ϵ is obtained from a fixed domain D_1 by shrinking the distance from every point of ∂D_1 to x_0 by the factor ϵ. If D has no boundary, (3.1b) is vacuous.

We will now solve (3.1) for $\kappa > 0$. In the outer region away from the hole D_ϵ, we expand u, λ, as in (1.4), (1.5) with $\nu_1(\epsilon) = \epsilon^{n-2}$ and $\nu_2(\epsilon) = \epsilon^{n-1}$ to derive (1.6) for the correction u_1 and (2.1) for the correction u_2. The solvability condition for u_1 is given in (1.15), with an analogous condition holding for u_2. To determine the singular behavior for u_1 and u_2 as x tends to x_0, needed for the solvability condition, we must construct an inner expansion near D_ϵ. In the inner region, we write $y = \epsilon^{-1}(x - x_0)$, $v(y,\epsilon) = u(x_0 + \epsilon y, \epsilon)$ to obtain, in place of (1.7),

$$\triangle_y v - \epsilon^2 \, U(x_0 + \epsilon y) \, v = -\epsilon^2 \, \lambda \, \rho(x_0 + \epsilon y) \, v \,, \quad y \notin D_1, \tag{3.2a}$$

$$\partial_n v + \kappa \, v = 0 \,, \quad y \in \partial D_1 \,. \tag{3.2b}$$

In (3.2) $\triangle_y$ and ∂_n denote derivatives with respect to y and D_1 is the domain D_ϵ in the y variable. Expanding v in powers of ϵ as in (1.8), we obtain the following inner problems (3.3)–(3.5) in place of (1.9)–(1.11):

$$\triangle_y v_0 = 0 \,, \quad y \notin D_1 \,; \quad \partial_n v_0 + \kappa \, v_0 = 0 \,, \quad y \in \partial D_1 \,, \tag{3.3}$$

$$\triangle_y v_1 = 0 \,, \quad y \notin D_1 \,; \quad \partial_n v_1 + \kappa \, v_1 = 0 \,, \quad y \in \partial D_1 \,, \tag{3.4}$$

$$\triangle_y v_2 = \Big(U(x_0) - \lambda_0 \rho(x_0) \Big) v_0 \,, \quad y \notin D_1 \,; \quad \partial_n v_2 + \kappa \, v_2 = 0 \,, \quad y \in \partial D_1 \,. \tag{3.5}$$

The matching condition for the inner and outer solutions is expressed in (1.12).

Rather than repeating the analysis of §§1 and 2, we observe that the calculation parallels that in those sections if we replace the inner problems (1.9)–(1.11) by (3.3)–(3.5). Then the far field forms of the solutions of the inner problems (3.3) and (3.4) are still as given in (1.13), (2.4), and (2.5), with the constants C, C_i, B_i, B_{ij}, and so forth, depending on κ. The constant α in (2.2) is independent of κ. To obtain λ_1, we match the leading terms in (1.12) to obtain the matching condition $v_0(y) \to u_0(x_0)$ as $|y| \to \infty$. Writing $v_0(y) = u_0(x_0)\hat{v}_0(y)$, where $\hat{v}_0$ satisfies (3.3) and $\hat{v}_0(\infty) = 1$, we obtain the far field behavior of $\hat{v}_0$ given in (1.13). Proceeding as in (1.14) and (1.15), we find that λ_1 is given by (1.16) in which $C = C(\kappa)$. Then we can proceed as in (2.1)–(2.6) to derive (2.7) for the second correction to the unperturbed eigenvalue. The results are summarized in the two term expansion for $\lambda(\epsilon)$, and the inner and outer solutions given in (2.8)–(2.10).

In the special case when $u_0(x_0) = 0$, the terms in (2.8) of orders $O(\epsilon^{n-2})$ and $O(\epsilon^{n-1})$ both vanish. To obtain the first nonvanishing correction to λ_0, we proceed as in the derivation of (2.17). All the results of (2.11)–(2.16) apply to the present case if we replace the inner problems (1.9)–(1.11) by (3.3)–(3.5). The constants D_{ij} and E in (2.15) depend on κ whereas the constant γ in (2.12) is independent of κ. The results for this case are then summarized in (2.18)–(2.20).

The conclusion is that Proposition 1 and Corollaries 1 and 2 hold when a hole is deleted from the domain, instead of a potential being added, provided that v_0, v_1, and v_2 are the solutions of (3.3)–(3.5). These results are not useful when $\kappa = 0$ because then the corrections they yield for $\lambda(\epsilon)$ vanish.

We now consider the case where $\kappa = 0$. In this case, we have $\hat{v}_0 \equiv 1$, and thus $C(0) = 0$, $C_i(0) = 0$, and so forth. Therefore, it follows from (1.6), (1.14), and (1.16) that $\lambda_1 = 0$ and $u_1 \equiv 0$. In addition, from (2.5) we obtain that $B_i(0) = 0$, and thus from (2.1), (2.6), and (2.7), we conclude that $\lambda_2 = 0$ and $u_2 \equiv 0$. Therefore, in this case, the expansion of $\lambda(\epsilon)$ is $\lambda(\epsilon) = \lambda_0 + \epsilon^n \lambda_3 + \cdots$. The corresponding expansions of the outer and inner solutions are $u = u_0 + \epsilon^n u_3 + \cdots$ and $v = u_0(x_0) + \epsilon v_1 + \epsilon v_2 + \cdots$, respectively. Substituting the outer expansion into (3.1), we find that u_3 satisfies (2.11). Substituting the inner expansion into (3.2) and setting $\kappa = 0$, we obtain

$$\text{(3.6)} \qquad \Delta_y v_1 = 0\,, \quad y \notin D_1\,; \quad \partial_n v_1 = 0\,, \quad y \in \partial D_1\,,$$

$$\text{(3.7)} \qquad \Delta_y v_2 = \Big(U(x_0) - \lambda_0 \rho(x_0)\Big)\, u_0(x_0)\,, \quad y \notin D_1\,; \quad \partial_n v_2 = 0\,, \quad y \in \partial D_1\,.$$

From the matching condition (1.12), we find that $v_1 \sim [\partial_{x_i} u_0(x_0)]\, y_i$ and $v_2 \sim \left[\partial_{x_i}\partial_{x_j} u_0(x_0)\right] y_i y_j/2$ as $y \to \infty$. The solution for v_1 can be written as $v_1 = [\partial_{x_i} u_0(x_0)]\, \hat{v}_{1i}$, where $\hat{v}_{1i}$ is the solution to (3.6) for which $\hat{v}_{1i} - y_i = o(1)$ as $y \to \infty$. Its asymptotic form is given in (2.5) with $B_i(0) = 0$, and thus

$$\text{(3.8)} \quad v_1(y) = [\partial_{x_i} u_0(x_0)]\, \hat{v}_{1i}(y) = [\partial_{x_i} u_0(x_0)] \left(y_i + \frac{B_{ij}(0) y_j}{|y|^n} + \cdots \right) \quad \text{as } y \to \infty\,.$$

To solve (3.7), we write $v_2(y)$ as

$$\text{(3.9)} \qquad v_2(y) = \tfrac{1}{2} \left[\partial_{x_i}\partial_{x_j} u_0(x_0)\right]\, y_i\, y_j + \phi(y)\,.$$

Substituting this form into (3.7) and using (1.1a) evaluated at x_0, we obtain

$$\text{(3.10)} \quad \Delta_y \phi = 0\,, \quad y \notin D_1\,; \qquad \partial_n \phi = -\tfrac{1}{2} \left[\partial_{x_i}\partial_{x_j} u_0(x_0)\right] \partial_n \left(y_i\, y_j\right)\,, \quad y \in \partial D_1\,.$$

Imposing the condition that $\phi \to 0$ as $|y| \to \infty$, we find that $\phi \sim -E/|y|^{n-2}$. To determine E, we apply the divergence theorem to (3.10) in the region outside D_1 to obtain

$$\omega_n \, (n-2) \, E = \frac{1}{2} \int_{\partial D_1} \left[\partial_{x_i} \partial_{x_j} u_0(x_0) \right] \, \partial_n \, (y_i \, y_j) \, ds \, . \tag{3.11}$$

Then, applying the divergence theorem to the interior of D_1 and using (1.1a), we derive

$$E = -\frac{1}{\omega_n \, (n-2)} \int_{D_1} \Delta u_0(x_0) \, dx = \frac{\Big(\lambda_0 \, \rho(x_0) - U(x_0) \Big) \, u_0(x_0) \, V_1}{\omega_n \, (n-2)} \, . \tag{3.12}$$

Here V_1 is the volume of D_1. Therefore, from (3.9) the far field behavior of v_2 is

$$v_2(y) = \frac{1}{2} \left[\partial_{x_i} \partial_{x_j} u_0(x_0) \right] \, y_i \, y_j - \frac{E}{|y|^{n-2}} + \cdots \qquad \text{as} \quad y \to \infty \, . \tag{3.13}$$

Using (3.8) and (3.13) in the matching condition (1.12), we obtain the following singular behavior for u_3 as $x \to x_0$:

$$u_3(x) \sim \left[\partial_{x_i} u_0(x_0) \right] \, \frac{B_{ij}(0) \, (x_j - x_{0j})}{|x - x_0|^n} - \frac{E}{|x - x_0|^{n-2}} \, . \tag{3.14}$$

Imposing the solvability condition on the problem for u_3, defined by (2.11) and (3.14), we determine λ_3 as

$$\lambda_3 = V_1 \Big(\lambda_0 \, \rho(x_0) - U(x_0) \Big) \left[u_0(x_0) \right]^2 - \omega_n \, \partial_{x_i} u_0(x_0) \, B_{ij}(0) \, \partial_{x_j} u_0(x_0) \, . \tag{3.15}$$

Problems (2.11) and (3.14) can then be solved uniquely for u_3. We summarize our results in the following statement.

COROLLARY 3. *For dimension* $n \geq 3$, *assume that* $\kappa = 0$ *in* (3.1c). *Let* $\lambda(\epsilon) \to \lambda_0$ *where* λ_0 *is a simple eigenvalue of* (1.1). *Then, for* $\epsilon \ll 1$,

$$\lambda(\epsilon) = \lambda_0 + \epsilon^n \left[V_1 \Big(\lambda_0 \rho(x_0) - U(x_0) \Big) \left[u_0(x_0) \right]^2 - \omega_n \partial_{x_i} u_0(x_0) B_{ij}(0) \partial_{x_j} u_0(x_0) \right] + \cdots \tag{3.16}$$

$$u(x, \epsilon) = u_0(x) + \epsilon^n \, u_3(x) + \cdots \, , \qquad |x - x_0| \gg O(\epsilon) \, , \tag{3.17}$$

$$u(x, \epsilon) = u_0(x_0) + \epsilon \, \left[\partial_{x_i} u_0(x_0) \right] \, \hat{v}_{1i} \left(\frac{x - x_0}{\epsilon} \right) + \frac{1}{2} \left[\partial_{x_i} \partial_{x_j} u_0(x_0) \right] \, (x_i - x_{0i}) \, (x_j - x_{0j}) + \epsilon^2 \, \phi \left(\frac{x - x_0}{\epsilon} \right) + \cdots \, , \qquad |x - x_0| = O(\epsilon) \, . \tag{3.18}$$

We note from (3.16) *that the coefficient of order* ϵ^n *does not vanish when* $u_0(x_0) = 0$, *and thus* (3.15) *still gives the first nonvanishing correction to the eigenvalue in this case.*

The constants $C(\kappa)$ and $B_{ij}(\kappa)$ appearing in (2.8), (2.18), and (3.16) are known for some special hole geometries. In particular, when D_1 is a sphere of radius a and

$n \geq 3$, then $C(\kappa) = \kappa a^{n-1}/[(n-2)+\kappa a]$. For $n = 3$ and $\kappa = \infty$, the constant $C(\infty)$ appearing in (2.8) is the capacitance of D_1, which is known explicitly for some domains, D_1. When D_1 is a sphere of radius a, we have $C(\infty) = a$; for a circular disk of radius a, $C(\infty) = 2a/\pi$; for an oblate spheroid with semi-axes $a_1 = a_2 > a_3$, $C(\infty) = (a_1^2 - a_3^2)^{1/2}/\cos^{-1}(a_3/a_1)$; for a prolate spheroid with semi-axes $a_1 > a_2 = a_3$, $C(\infty) = (a_1^2 - a_2^2)^{1/2}/\cosh^{-1}(a_1/a_2)$. For a general ellipsoid,

(3.19)
$$C(\infty) = 2\Big(\int_0^\infty d\eta/R(\eta)\Big)^{-1}, \quad \text{where} \quad R(\eta) = (a_1^2+\eta)^{1/2}(a_2^2+\eta)^{1/2}(a_3^2+\eta)^{1/2}.$$

Szegö [8] has shown that the sphere has the smallest capacitance of all domains of the same volume. Thus among all such domains, deleted at x_0, a sphere produces the smallest first-order perturbation of each eigenvalue.

When $\kappa = 0$ and $n = 3$, $B_{ij}(0)$ is called the polarizability tensor of D_1, which can also be found explicitly for various hole geometries. If D_1 is an ellipsoid with semi-axes a_i aligned with the coordinate axes, then

$$B_{ij}(0) = \frac{V_1}{4\pi}(1-n_i)^{-1}\delta_{ij} \qquad \text{where} \quad n_i = \frac{1}{2}a_1a_2a_3\int_0^\infty \frac{d\eta}{R(\eta)\,(a_i^2+\eta)}. \tag{3.20}$$

Here $V_1 = 4\pi a_1 a_2 a_3/3$ is the volume of D_1. If D_1 is a sphere of radius a, then $n_i = \frac{1}{3}$, and thus $B_{ij}(0) = a^3\delta_{ij}/2$. In addition, if $\kappa = \infty$ and D_1 is an ellipsoid with semi-axes a_i, which are aligned with the coordinate axes, then

$$B_{ij}(\infty) = -\frac{V_1}{4\pi\, n_i}\delta_{ij}. \tag{3.21}$$

Finally, if D_1 is a sphere of radius a and $n = 3$, then

$$B_{ij}(\kappa) = a^3\left(\frac{1-\kappa a}{2+\kappa a}\right)\delta_{ij}. \tag{3.22}$$

Thus, in the limiting cases $\kappa \to \infty$ and $\kappa \to 0$, we obtain $B_{ij}(\infty) = -a^3\delta_{ij}$ and $B_{ij}(0) = a^3\delta_{ij}/2$ for a sphere of radius a when $n = 3$.

The expansion (2.8) for $\lambda(\epsilon)$ simplifies when $\partial_{x_i}u_0(x_0) = 0$ or when D_1 has special symmetry. In particular, when D_1 is a sphere then $C_i(\kappa) = B_i(\kappa) = 0$ and thus in this case the term of order ϵ^{n-1} in (2.8) vanishes when $n > 3$. For a sphere with $n = 3$, (2.8) provides an almost explicit two-term expansion for $\lambda(\epsilon)$ in terms of $u_0(x_0)$ and the known quantity $C(\kappa)$, given above. The constant α in (2.8), which can be determined only from the solution to (1.6) and (2.2), cannot be found explicitly for arbitrary domains D and hole locations. It will be determined explicitly in a special case below. Finally, we note that (2.8) can be simplified as $\kappa \to 0$, by using the limiting behaviors $C(\kappa) \sim S_1\kappa/[(n-2)\omega_n]$, $C_i(\kappa) = o(1)$ and $B_i(\kappa) = o(1)$ for $\kappa \ll 1$. Here S_1 is the surface area of ∂D_1. Thus if $\kappa = \epsilon\kappa_0$, with κ_0 independent of ϵ, (2.8) reduces to

$$\lambda(\epsilon) = \lambda_0 + \epsilon^{n-1}S_1\,\kappa_0\,\big[u_0(x_0)\big]^2 + \cdots, \qquad (\kappa = \epsilon\kappa_0). \tag{3.23}$$

3.1. Deletion of a small subdomain ($n = 3$): **Examples.** Now we consider some examples of the theory developed above. For some simple geometries, our asymptotic results will be compared with exact analytical results and with some results obtained previously in certain limiting cases. To construct explicit solutions, we assume below that $\rho \equiv 1$, $U \equiv 0$, and $b =$ constant in (1.1) and (3.1).

If D_ϵ is a hole of arbitrary shape with $\kappa = \infty$ and $n = 3$, the result (1.17) applies with $C(\infty)$ being the capacitance of D_1. This result was proved by Ozawa [5], and some related results were proved earlier by Swanson [7]. When our two-term result (2.8) is applied to a spherical hole of radius ϵ with $\kappa = \infty$ and $n = 3$, then $C = 1$, $C_i = 0$, and $B_i = 0$. This result was also proved by Ozawa [3]. Our Proposition 1 and Corollaries 1–3 extend these results to the case of arbitrary hole shape, arbitrary $\kappa \geq 0$, and vanishing $u_0(x_0)$.

If D is a sphere of radius 1, then the unperturbed eigenfunctions and eigenvalues for (1.1) are known explicitly. We now determine the corrections to the unperturbed eigenvalues corresponding to spherically symmetric eigenfunctions as a result of placing a small hole, centered at some point $r_0 = |x_0| < 1$, inside the unit sphere.

These eigenfunctions and associated eigenvalues for (1.1) are given by

(3.24)

$$u_0(r) = N_0\, j_0(\sqrt{\lambda_0} r)\,, \quad N_0 = \left(\frac{\lambda_0}{2\pi}\right)^{1/2} \left(1 + \frac{\cos^2(\sqrt{\lambda_0})}{b-1}\right)^{-1/2}, \quad \tan(\sqrt{\lambda_0}) = \frac{\sqrt{\lambda_0}}{1-b},$$

where $j_0(z) \equiv z^{-1} \sin z$. For the perturbed problem (3.1), the first correction to λ_0 was determined above for three ranges of κ. From (2.8), (3.23), and (3.16), we obtain

$$\lambda(\epsilon) = \lambda_0 + 4\pi\epsilon C N_0^2 j_0^2(\sqrt{\lambda_0} r_0), \tag{3.25a}$$

$$-4\pi\epsilon^2 N_0^2 j_0(\sqrt{\lambda_0} r_0)\left[\sqrt{\lambda_0} j_0'(\sqrt{\lambda_0} r_0)\frac{x_{0i}}{r_0}(B_i + C_i) + \alpha j_0(\sqrt{\lambda_0} r_0) C^2\right] + \cdots \quad \kappa > 0,$$

$$\lambda(\epsilon) = \lambda_0 + S_1\, \epsilon^2\, \kappa_0\, N_0^2\, j_0^2(\sqrt{\lambda_0} r_0) + \cdots, \quad \kappa = \epsilon\kappa_0\,, \tag{3.25b}$$

(3.25c)

$$\lambda(\epsilon) = \lambda_0 + \epsilon^3 \lambda_0 N_0^2 \left[V_1 j_0^2(\sqrt{\lambda_0} r_0) - \frac{4\pi}{r_0^2}\left(j_0'(\sqrt{\lambda_0} r_0)\right)^2 x_{0i} B_{ij}(0) x_{0j}\right] + \cdots, \quad \kappa = 0\,.$$

Here S_1 is the surface area of ∂D_1, V_1 is the volume of D_1, and D_1 is the domain D_ϵ magnified by ϵ^{-1}. The constants α, $C(\kappa)$, $C_i(\kappa)$, $B_i(\kappa)$, $B_{ij}(0)$ were defined in (2.2), (1.13), and (2.5). Explicit expressions for $C(\kappa)$, $B_{ij}(0)$ were given above for some hole geometries.

In particular, if D_1 is a sphere of radius a, then $B_{ij}(0) = a^3\delta_{ij}/2$. In this case, (3.25c) becomes

$$\lambda(\epsilon) = \lambda_0 + \frac{4\pi\, a^3\, \epsilon^3}{3}\, \lambda_0\, N_0^2 \left[j_0^2(\sqrt{\lambda_0} r_0) - \frac{3}{2}\left(j_0'(\sqrt{\lambda_0} r_0)\right)^2\right] + \cdots, \quad \kappa = 0\,. \tag{3.26}$$

Suppose that $b = \infty$ in (3.24) so that $u = 0$ on $r = 1$. Then, for the smallest eigenvalue $\lambda_0 = \pi^2$ of (1.1), a numerical calculation shows that the coefficient of order ϵ^3 in (3.26) is positive if $r_0 < .5845$ and negative if $.5845 < r_0 < 1$. A similar result occurs for the nonlinear eigenvalue problems arising in combustion theory, which we considered in [10]. In that context, the change in sign of $\lambda(\epsilon) - \lambda_0$ was shown to have an interesting physical interpretation.

In the special case when $r_0 = 0$ and $\kappa > 0$, it is possible to determine explicitly the constant α appearing in (3.25a). Since $j_0'(0) = 0$, then (3.25a) reduces to

$$\lambda(\epsilon) = \lambda_0 + 4\,\pi\,\epsilon\, C(\kappa)\, N_0^2 - 4\,\pi\,\epsilon^2\, \alpha\, C^2(\kappa)\, N_0^2 + \cdots\,. \tag{3.27}$$

The constant α is found from the solution to (1.6) and (2.2), which become

(3.28)
$$u_{1rr} + \frac{2}{r} u_{1r} + \lambda_0 u_1 = -\lambda_1 u_0 , \qquad \text{in} \quad 0 < r < 1 ,$$
$$u_{1r} + b u_1 = 0 , \qquad \text{on} \quad r = 1 ,$$
$$\int_0^1 u_0 u_1 r^2 \, dr = 0 , \qquad u_1 = -C(\kappa) u_0(0) \Big(r^{-1} + \alpha \Big) + o(1) , \quad \text{as} \quad r \to 0 .$$

The solution to (3.28) is
(3.29a)
$$u_1(r) = N_0 \left[\frac{\lambda_1}{2\lambda_0} \cos(\sqrt{\lambda_0} r) - \frac{C(\kappa)}{r} \cos(\sqrt{\lambda_0} r) - \frac{1}{r\sqrt{\lambda_0}} \left(\alpha C(\kappa) + \frac{\lambda_1}{2\lambda_0} \right) \sin(\sqrt{\lambda_0} r) \right] ,$$

where

(3.29b)
$$\alpha = \frac{\pi N_0^2}{\lambda_0} \left[-3 + 4 \left(\frac{\pi N_0^2}{\lambda_0} - \frac{1}{2} \right) \sin^2(\sqrt{\lambda_0}) \right] .$$

Using (3.29b) in (3.27) gives an explicit two-term expansion for $\lambda(\epsilon)$ when a hole of arbitrary shape is located at the center of a sphere of radius 1. In particular, if $b = \infty$, then from (3.24) we find $\lambda_0 = n^2\pi^2$, $\pi N_0^2/\lambda_0 = \frac{1}{2}$, and thus (3.27) becomes

(3.30)
$$\lambda(\epsilon) = \lambda_0 \Big[1 + 2\, C(\kappa)\, \epsilon + 3\, C^2(\kappa)\, \epsilon^2 + \cdots \Big] .$$

We now compare the asymptotic results (3.25b), (3.25c), and (3.30) with the exact eigenvalues of (3.1) when $b = \infty$ and D_ϵ is a sphere of radius ϵa centered at the origin. For two concentric spheres, the exact eigenvalue relation for (3.1) is

(3.31)
$$j_0(z) = y_0(z) \left[\frac{\epsilon z j_0'(a\epsilon z) - \kappa j_0(a\epsilon z)}{\epsilon z y_0'(a\epsilon z) - \kappa y_0(a\epsilon z)} \right] .$$

Here $z \equiv \sqrt{\lambda}$ and $y_0(z) \equiv z^{-1} \cos z$. Expanding λ in powers of ϵ for $\kappa > 0$, $\kappa = \epsilon\kappa_0$ and $\kappa = 0$, we readily recover (3.25b), (3.25c), and (3.30) with $r_0 = 0$, $C(\kappa) = \kappa a^2/(1 + \kappa a)$, $S_1 = 4\pi a^2$ and $V_1 = 4\pi a^3/3$. In Table 1 we compare the asymptotic result (3.30) with the smallest eigenvalue of (3.1) obtained by solving (3.31) numerically when $\kappa = 1$. In Table 2 we compare our asymptotic result (3.30) with the smallest eigenvalue of (3.1) for $\kappa = \infty$, which is given explicitly by $\lambda(\epsilon) = \pi^2(1-\epsilon)^{-2}$. From these tables, we observe that the three-term expansion (3.30) for the smallest eigenvalue of (3.1), when $a = 1$, is within 3 percent of the exact value even when $\epsilon = \frac{1}{5}$. We would expect similar agreement for the case of a hole of arbitrary shape.

Now, if $\kappa > 0$ and λ_0 is not the minimum eigenvalue of (1.1), then $j_0(\sqrt{\lambda_0} r_0) = 0$ for some r_0. For a deletion centered at a point on such a nodal line, the corrections to λ_0 of orders ϵ and ϵ^2 in (3.25a) both vanish. The first nonzero correction is $O(\epsilon^3)$, which is given in (2.17). From (2.18), the expansion of $\lambda(\epsilon)$ is

(3.32)
$$\lambda(\epsilon) = \lambda_0 - 4\pi\, \epsilon^3 \frac{N_0^2\, \lambda_0}{r_0^2} \left[j_0'(\sqrt{\lambda_0} r_0) \right]^2 x_{0i}\, B_{ij}(\kappa)\, x_{0j} + \cdots .$$

The tensor $B_{ij}(\infty)$ was given explicitly in (3.21) for ellipsoids, and $B_{ij}(\kappa)$ was given in (3.22) when D_1 is a sphere of radius a.

TABLE 1

Concentric spheres: $\kappa = 1.0$, $b = \infty$, *lowest eigenvalue.*

ϵ	λ (3.30) (2 term)	λ (3.30) (3 term)	λ (3.31) (exact)	perc. err., 3 term
0.010	9.9683	9.9690	9.9691	0.001
0.050	10.363	10.382	10.383	0.011
0.100	10.857	10.931	10.946	0.137
0.125	11.103	11.219	11.251	0.284
0.150	11.350	11.517	11.574	0.492
0.175	11.597	11.823	11.919	0.805
0.200	11.844	12.140	12.289	1.213

TABLE 2

Concentric spheres: $\kappa = \infty$, $b = \infty$, *lowest eigenvalue.*

ϵ	λ (3.30) (2 term)	λ (3.30) (3 term)	λ (3.31) (exact)	perc. err., 3 term
0.010	10.067	10.070	10.070	0.001
0.050	10.857	10.931	10.936	0.046
0.100	11.844	12.140	12.185	0.369
0.125	12.337	12.800	12.891	0.706
0.150	12.830	13.497	13.660	1.193
0.175	13.324	14.231	14.501	1.862
0.200	13.817	15.002	15.421	2.717

4. Deletion of a small subdomain ($n = 2$). Now we analyze the perturbed eigenvalue problem (3.1) in two dimensions. Substituting the outer expansion (1.4) and the eigenvalue expansion (1.5) into (3.1), we obtain (1.6) for the outer correction u_1. In the inner region, in place of (1.8), we allow for the more general asymptotic expansion

$$v(y,\epsilon) = \mu_0(\epsilon)\, v_0(y) + \mu_1(\epsilon)\, v_1(y) + \mu_2(\epsilon)\, v_2(y) + \cdots . \tag{4.1}$$

Here the $\mu_i(\epsilon)$ are gauge functions to be determined. Substituting (4.1) into (3.2), we obtain

$$\triangle_y v_0 = 0\,, \quad y \notin D_1\,; \quad \partial_n v_0 + \kappa\, v_0 = 0\,, \quad y \in \partial D_1\,. \tag{4.2}$$

The matching condition, analogous to (1.12), is

$$u_0(x) + \nu_1(\epsilon)\, u_1(x) + \nu_2(\epsilon)\, u_2(x) + \cdots \sim \mu_0(\epsilon)\, v_0(y) + \mu_1(\epsilon) v_1(y) + \mu_2(\epsilon)\, v_2(y) + \cdots . \tag{4.3}$$

We first consider the case where $\kappa > 0$ in (3.1c). In two dimensions, (4.2) has a solution $\hat{v}(y)$ with the asymptotic behavior

$$\hat{v}(y) = \log|y| - \log[d(\kappa)] + \cdots \qquad \text{as } |y| \to \infty\,. \tag{4.4}$$

For $\kappa > 0$, this solution is unique, and the constant $d(\kappa)$ is determined. It is given explicitly below for some simple hole geometries. When $\kappa = \infty$, $d(\infty)$ is called the logarithmic capacitance of D_1. It is well known that a circular domain has the smallest logarithmic capacitance of all domains of the same area (see [1]).

The leading terms in (4.3) are $u_0(x_0)$ on the left and $\mu_0(\epsilon)v_0(y)$ on the right. They match if

$$\mu_0(\epsilon) = -\frac{1}{\log[\epsilon d(\kappa)]}, \qquad v_0(y) = u_0(x_0)\,\hat{v}(y). \tag{4.5}$$

Then the right side of (4.3) contains the singular term $(-1/\log[\epsilon d(\kappa)])\,u_0(x_0)\,\log|x - x_0|$ which must match $\nu_1(\epsilon)u_1(x)$. Thus we conclude that

$$\nu_1(\epsilon) = -\frac{1}{\log[\epsilon d(\kappa)]}, \qquad u_1(x) \sim u_0(x_0)\,\log|x - x_0| \qquad \text{as} \quad x \to x_0. \tag{4.6}$$

The solvability condition (1.15) applied to problems (1.6) and (4.6) for u_1 determines λ_1 to be

$$\lambda_1 = 2\pi\left[u_0(x_0)\right]^2. \tag{4.7}$$

Then u_1 is the unique solution to (1.6) with singular behavior (4.6).

We now calculate further terms in the expansion of the eigenvalue. In the outer region, we choose the gauge functions as $\nu_j(\epsilon) = (-1/\log[\epsilon d(\kappa)])^j$ for $j = 1, 2, \ldots$. Then, substituting (1.4) and (1.5) into (3.1), we obtain the following equations for $u_j(x)$:

$$\Delta u_j - U(x)\,u_j + \lambda_0\rho(x)\,u_j = -\lambda_j\,\rho(x)\,u_0 - \sum_{i=1}^{j-1}\lambda_{j-i}\,\rho(x)\,u_i, \quad |x - x_0| \gg O(\epsilon), \tag{4.8a}$$

$$\left[\partial_n + b(x)\right]u_j = 0, \quad x \in \partial D, \tag{4.8b}$$

$$\sum_{i=0}^{j}\int_D u_i\,u_{j-i}\,\rho(x)\,dx = 0. \tag{4.8c}$$

In the inner region, we take $\mu_j(\epsilon) = (-1/\log[\epsilon d(\kappa)])^{j+1}$ for $j = 0, 1, 2, \ldots$. Substituting (4.1) into (3.2), we find that $v_j(y)$ satisfies (4.2) with v_0 replaced by v_j. We allow $v_j(y)$ to grow logarithmically as $|y| \to \infty$, so we write $v_j(y) = a_j u_0(x_0)\hat{v}(y)$, where the a_j are constants to be determined. Here $\hat{v}(y)$ is the solution to (4.2) with far field behavior (4.4). From the leading-order match (4.5), we found that $a_0 = 1$. Writing the right side of the matching condition (4.3) in outer variables, we have

$$v(y,\epsilon) \sim a_0\,u_0(x_0) + \sum_{i=1}^{\infty}\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^i u_0(x_0)\left(a_{i-1}\log|x - x_0| + a_i\right) + \cdots. \tag{4.9}$$

Comparing (4.9) with the left-hand side of (4.3), we find that u_j must have the following singular behavior as $x \to x_0$:

$$u_j(x) \sim a_{j-1}\,u_0(x_0)\,\log|x - x_0| \qquad \text{for} \quad j = 1, 2, \ldots. \tag{4.10}$$

Now u_j is the solution of (4.8) and (4.10). Applying the solvability condition (1.15) to this problem for u_j, we obtain

$$\lambda_j = 2\pi\, a_{j-1} \left[u_0(x_0)\right]^2 - \sum_{i=1}^{j-1} \lambda_{j-i}\,(u_i, u_0) \qquad \text{for} \quad j = 1, 2, \dots . \tag{4.11}$$

Here we have defined the inner product (u, v) by $(u, v) = \int_D uv\rho\, dx$. Substituting (4.11) into the right-hand side of (1.5) gives the expansion for $\lambda(\epsilon)$. To determine the a_j, we must calculate the next term in u_j as $x \to x_0$. From the unique solution to (4.8), (4.10), we obtain

$$u_j(x) \sim u_0(x_0) \left[a_{j-1} \log|x - x_0| + \beta_j\right] \qquad \text{for} \quad j = 1, 2, \dots . \tag{4.12}$$

Here β_j is a constant determined by the solution u_j to (4.8) and (4.10). It depends on the unperturbed solution and the hole location but is independent of the shape of D_1. Then matching the remaining terms of order $(-1/\log[\epsilon d(\kappa)])^j$ in (4.9), we get

$$a_j = \beta_j \qquad \text{for} \quad j = 1, 2, \dots . \tag{4.13}$$

Substituting (4.13) into (4.11), and using $a_0 = 1$ determines λ_j. We summarize our results in the following statement.

PROPOSITION 2. *For dimension $n = 2$, assume that $\kappa > 0$ in* (3.1c). *Let $\lambda(\epsilon) \to \lambda_0$ as $\epsilon \to 0$, where λ_0 is a simple eigenvalue of* (1.1). *Then, for $\epsilon \ll 1$,*

$$\lambda(\epsilon) = \lambda_0 + \left(-\frac{1}{\log[\epsilon d(\kappa)]}\right) 2\pi \left[u_0(x_0)\right]^2 + \sum_{j=2}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^j \lambda_j + \cdots , \tag{4.14a}$$

where

$$\lambda_j = 2\pi\, \beta_{j-1} \left[u_0(x_0)\right]^2 - \sum_{i=1}^{j-1} \lambda_{j-i}\,(u_i, u_0) \qquad \textit{for} \quad j = 2, 3, \dots . \tag{4.14b}$$

Here β_j for $j = 1, 2, \dots$ is found from (4.12). *The outer and inner expansions are given by*

$$u(x, \epsilon) = u_0(x) + \sum_{j=1}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^j u_j(x) + \cdots , \qquad |x - x_0| \gg \epsilon , \tag{4.15}$$

$$u(x, \epsilon) = u_0(x_0)\, \hat{v}\left(\frac{x - x_0}{\epsilon}\right) \sum_{j=0}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^{j+1} \beta_j + \cdots , \qquad |x - x_0| = O(\epsilon) . \tag{4.16}$$

The outer solution u_j satisfies (4.8) *and* (4.10) *with a_{j-1} replaced by β_{j-1}. In* (4.16) *we have labeled $\beta_0 = 1$.*

Similar analyses can be done to treat the cases when $u_0(x_0) = 0$ or when $\kappa = 0$. The main results are summarized below, and the details can be found in the appendices.

COROLLARY 4. *For dimension $n = 2$, assume that $u_0(x_0) = 0$. Let $\lambda(\epsilon) \to \lambda_0$ as $\epsilon \to 0$, where λ_0 is a simple eigenvalue of* (1.1). *Then, for $\epsilon \ll 1$,*

(4.17) $$\lambda(\epsilon) = \lambda_0 + \epsilon^2 \lambda^* + \cdots, \qquad \lambda^* = -2\pi\epsilon^2\, \partial_{x_i} u_0(x_0)\, B_{ij}(\kappa)\, \partial_{x_j} u_0(x_0),$$

(4.18) $$u(x,\epsilon) = u_0(x) + \frac{\epsilon\, g(x)\, \partial_{x_i} u_0(x_0) B_i(\kappa)}{(\log[\epsilon d(\kappa)] + \beta)} + \epsilon^2 u^*(x) + \cdots, \qquad |x - x_0| \gg \epsilon,$$

(4.19)
$$u(x,\epsilon) = \epsilon v_0\left(\frac{x-x_0}{\epsilon}\right) + \frac{\epsilon\, \hat{v}[(x-x_0)/\epsilon]\, \partial_{x_i} u_0(x_0) B_i(\kappa)}{(\log[\epsilon d(\kappa)] + \beta)} + \cdots, \qquad |x - x_0| = O(\epsilon).$$

In (4.17)–(4.19), *the function $v_0(y)$ and the constants $B_i(\kappa)$, $B_{ij}(\kappa)$ satisfy*

(4.20a) $$\triangle_y v_0 = 0, \quad y \notin D_1; \quad \partial_n v_0 + \kappa v_0 = 0, \quad y \in \partial D_1,$$

(4.20b) $$v_0 = [\partial_{x_i} u_0(x_0)]\left(y_i + B_i(\kappa) + \frac{B_{ij}(\kappa)\, y_j}{|y^2|} + \cdots\right), \quad y \to \infty.$$

In (4.18) *the function $u^*(x)$ satisfies*

(4.21a) $$\triangle u^* - U(x)\, u^* + \lambda_0\, \rho(x) u^* = -\lambda^* \rho(x) u_0, \qquad x \neq x_0,$$

(4.21b) $$[\partial_n + b(x)]\, u^* = 0, \quad x \in \partial D; \qquad \int_D u_0(x) u^*(x)\, \rho(x)\, dx = 0,$$

(4.21c) $$u^*(x) \sim [\partial_{x_i} u_0(x_0)]\, \frac{B_{ij}(\kappa)(x_j - x_{0j})}{|x - x_0|^2} \quad \text{as} \quad x \to x_0.$$

The function $g(x)$, which is proportional to the Green's function for (4.21), *satisfies* (4.21a), (4.21b) *with $\lambda^* = 0$, subject to the singular behavior $g(x) \sim \log|x - x_0|$ as $x \to x_0$. The constant β appearing in* (4.18) *and* (4.19) *is defined uniquely by $g(x) - \log|x - x_0| = \beta + o(1)$ as $x \to x_0$. Finally, the function $\hat{v}(y)$ and the constant $d(\kappa)$ are found from the solution to* (4.2) *with far field behavior* (4.4). *The details of this calculation can be found in Appendix* A.

COROLLARY 5. *For dimension $n = 2$, assume that $\kappa = 0$ in* (3.1c). *Let $\lambda(\epsilon) \to \lambda_0$ as $\epsilon \to 0$, where λ_0 is a simple eigenvalue of* (1.1). *Then, for $\epsilon \ll 1$,*

(4.22) $$\lambda(\epsilon) = \lambda_0 + \epsilon^2 \lambda_1 + \cdots, \qquad \lambda_1 = 2\pi\epsilon^2 [E\, u_0(x_0) - \partial_{x_i} u_0(x_0) B_{ij}(0) \partial_{x_j} u_0(x_0)],$$

(4.23) $$u(x,\epsilon) = u_0(x) + \epsilon^2\, u_1(x) + \cdots, \qquad |x - x_0| \gg O(\epsilon),$$

(4.24)
$$u(x,\epsilon) = u_0(x_0) + \epsilon\, [\partial_{x_i} u_0(x_0)]\, \hat{v}_{1i}\left(\frac{x - x_0}{\epsilon}\right) + \frac{1}{2}\left[\partial_{x_i}\partial_{x_j} u_0(x_0)\right] (x_i - x_{0i})(x_j - x_{0j})$$
$$+ (\epsilon^2 \log\epsilon)\, E + \epsilon^2\, \phi\left(\frac{x - x_0}{\epsilon}\right) + \cdots, \qquad |x - x_0| = O(\epsilon).$$

In (4.22) *the constants $B_{ij}(0)$ are found from* (4.20) *with $\kappa = 0$. The constant E is given in terms of the area A_1 of D_1 by*

(4.25) $$E = \frac{A_1}{2\pi}\left(\lambda_0\, \rho(x_0) - U(x_0)\right) u_0(x_0).$$

In (4.23) *the function* u_1 *satisfies* (4.21) *with* u^*, λ^* *replaced by* u_1, λ_1 *and with* (4.21c) *replaced by*

$$u_1(x) \sim [\partial_{x_i} u_0(x_0)] \frac{B_{ij}(0)(x_j - x_{0j})}{|x - x_0|^2} + E \log |x - x_0| \quad as \quad x \to x_0 . \tag{4.26}$$

In (4.24) *the function* $\hat{v}_{1i}(y)$ *is the solution to* (4.20a), *with* $\kappa = 0$, *having the far field behavior* $\hat{v}_{1i}(y) \sim y_i + B_{ij}(0)y_j/|y|^2 + \cdots$. *Finally, the function* $\phi(y)$ *appearing in* (4.24) *is the solution to* (3.10) *with the far field form* $\phi(y) \sim E \log |y|$. *The details of this calculation can be found in Appendix* B.

The constant $d(\kappa)$ defined in (4.4) and the tensor $B_{ij}(\kappa)$ defined in (4.20b) are known explicitly for some hole geometries. When D_1 is a circle of radius a, then

$$d(\kappa) = a \exp(-1/\kappa a) , \qquad B_{ij}(\kappa) = a^2 \left(\frac{1 - \kappa a}{1 + \kappa a} \right) \delta_{ij} . \tag{4.27a}$$

If D_1 is an ellipse with semi-axes a_1 and a_2, aligned with the coordinate axes, then (4.27b)

$$d(\infty) = (a_1 + a_2)/2 , \qquad B_{ij}(\infty) = -(a_1 + a_2)^2 \delta_{ij}/4 , \qquad B_{ij}(0) = (a_1 + a_2)^2 \delta_{ij}/4 .$$

Finally, we note that (4.14a) can be simplified when $\kappa \to 0$. By using the divergence theorem and the definition of $d(\kappa)$ given in (4.4), we find that

$$\log[d(\kappa)] \sim -\frac{2\pi}{L_1 \kappa} \qquad \text{as} \quad \kappa \to 0 . \tag{4.28}$$

Here L_1 is the length of ∂D_1. Thus, if $\kappa = \epsilon \kappa_0$, with κ_0 independent of ϵ, a two-term expansion for $\lambda(\epsilon)$, obtained by using (4.28) in (4.14a), is given by

$$\lambda(\epsilon) = \lambda_0 + \epsilon L_1 \kappa_0 \left[u_0(x_0)\right]^2 + \cdots , \qquad (\kappa = \epsilon \kappa_0) . \tag{4.29}$$

4.1. Deletion of a small subdomain ($n = 2$)**: Examples.** Now we consider some examples of the theory developed above, and we compare our results with some previous results. We take $\rho \equiv 1$, $U \equiv 0$, and b constant in (1.1) and (3.1).

If D_ϵ is a circle of radius ϵ, then, from (4.14a), (4.22), (4.25), and (4.27a), we obtain

$$\lambda(\epsilon) = \lambda_0 + (-\frac{1}{\log \epsilon}) 2\pi \left[u_0(x_0)\right]^2 + \cdots , \quad \kappa = \infty , \tag{4.30a}$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon^2 \Big(\pi \lambda_0 \left[u_0(x_0)\right]^2 - 2\pi |\nabla u_0(x_0)|^2\Big) + \cdots , \quad \kappa = 0. \tag{4.30b}$$

These results were proved by Ozawa [6], [4]. See also Swanson [7]. Our Proposition 2 and Corollaries 4 and 5 extend these results to the case of arbitrary hole shape, arbitrary $\kappa \geq 0$, vanishing $u_0(x_0)$, and in the case where $\kappa > 0$ to further terms.

To illustrate the theory, we let D be a circular cylindrical domain of radius 1. The normalized radially symmetric eigenfunctions and associated eigenvalues for (1.1) are (4.31)

$$u_0(r) = N_0 \, J_0(\sqrt{\lambda_0} r) , \qquad N_0 = \left(1 + \frac{\lambda_0}{b^2}\right)^{-1/2} \frac{\pi^{-1/2}}{J_0'(\sqrt{\lambda_0})} , \qquad \frac{J_0'(\sqrt{\lambda_0})}{J_0(\sqrt{\lambda_0})} = -\frac{b}{\sqrt{\lambda_0}} .$$

We now determine the correction to λ_0 as a result of placing a small hole, centered at some point $r_0 = |x_0| < 1$. From (4.14a), (4.14b), (4.29), and (4.22), we obtain
(4.32a)

$$\lambda(\epsilon) = \lambda_0 + 2\pi N_0^2\, J_0^2(\sqrt{\lambda_0} r_0)\left[\left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right) + \beta_1\left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right)^2 + \cdots\right], \quad \kappa > 0,$$

(4.32b)
$$\lambda(\epsilon) = \lambda_0 + L_1\,\epsilon\,\kappa_0\, N_0^2\, J_0^2(\sqrt{\lambda_0} r_0) + \cdots, \quad \kappa = \epsilon\kappa_0,$$

(4.32c)
$$\lambda(\epsilon) = \lambda_0 + \epsilon^2\lambda_0 N_0^2\left[A_1 J_0^2(\sqrt{\lambda_0} r_0) - \frac{2\pi}{r_0^2}\left(J_0'(\sqrt{\lambda_0} r_0)\right)^2 x_{0i}\, B_{ij}(0)\, x_{0j}\right] + \cdots, \kappa = 0.$$

The constant β_1 in (4.32a) is determined from the solution to (4.8) and (4.12). Explicit expressions for $d(\kappa)$ and $B_{ij}(0)$ were given in (4.27) for some simple hole geometries.

In particular, if D_1 is a circle of radius a, then $B_{ij}(0) = a^2\delta_{ij}$. In this case, (4.32c) becomes

$$\lambda(\epsilon) = \lambda_0 + \pi\, a^2\,\epsilon^2\lambda_0\, N_0^2\left(J_0^2(\sqrt{\lambda_0} r_0) - 2\left[J_0'(\sqrt{\lambda_0} r_0)\right]^2\right) + \cdots, \quad \kappa = 0. \tag{4.33}$$

If $b = \infty$, then, for the smallest eigenvalue of (1.1), a numerical calculation shows that the coefficient of order ϵ^2 in (4.33) is positive if $r_0 < .2008$ and negative if $.2008 < r_0 < 1$.

In the special case when $r_0 = 0$ and $\kappa > 0$, it is possible to determine explicitly the constant β_1 appearing in (4.32a). This constant is determined from the solution to (4.8) and (4.12), which become

$$\begin{gathered} u_{1rr} + \frac{1}{r}u_{1r} + \lambda_0\, u_1 = -\lambda_1 u_0 \quad \text{in} \quad 0 < r < 1, \\ u_{1r} + b\, u_1 = 0 \quad \text{on} \quad r = 1, \\ \int_0^1 u_0\, u_1\, r\, dr = 0, \qquad u_1 \sim u_0(0)\left(\log r + \beta_1\right) \quad \text{as} \quad r \to 0. \end{gathered} \tag{4.34}$$

The solution to (4.34) is
(4.35a)

$$u_1(r) = N_0\left[\frac{\pi N_0^2}{\sqrt{\lambda_0}}\, r\, J_0'(\sqrt{\lambda_0} r) + \frac{\pi}{2}\, Y_0(\sqrt{\lambda_0} r) + \left(\beta_1 - \log(\sqrt{\lambda_0}/2) - \gamma\right) J_0(\sqrt{\lambda_0} r)\right],$$

where

$$\beta_1 = -\frac{\pi}{2}\,\frac{\left(Y_0'(\sqrt{\lambda_0}) - \sqrt{\lambda_0}\, Y_0(\sqrt{\lambda_0})/b\right)}{J_0'(\sqrt{\lambda_0})\,(1 + \lambda_0/b^2)} + \log\left(\sqrt{\lambda_0}/2\right) + \gamma + \frac{(1 + \lambda_0/b^2)^{-2}}{\lambda_0\left[J_0'(\sqrt{\lambda_0})\right]^2}. \tag{4.35b}$$

Here $\gamma = .5772\ldots$ is Euler's constant. Using (4.35b) in (4.32a), and setting $r_0 = 0$, we obtain an explicit three-term expansion for $\lambda(\epsilon)$ when a hole of arbitrary shape is located at the center of a circular cylinder of radius one.

We now compare the asymptotic results (4.32) with the exact eigenvalues of (3.1) when $b = \infty$ and D_ϵ is a circle of radius ϵ centered at the origin. In this case, we find

TABLE 3

Concentric circles: $\kappa = 0$, $b = \infty$, *lowest eigenvalue.*

ϵ	λ (4.32c)	λ (4.36) (exact)	perc. err.
0.010	5.785	5.785	.0001
0.050	5.837	5.836	.009
0.100	5.998	5.993	.076
0.150	6.266	6.254	.199
0.175	6.440	6.424	.252
0.200	6.641	6.624	.270
0.225	6.869	6.854	.228

TABLE 4

Concentric circles: $\kappa = \infty$, $b = \infty$, *lowest eigenvalue.*

ϵ	λ (4.32a) (2 term)	λ (4.32a) (3 term)	λ (4.36) (exact)	perc. err., 3 term
0.001	6.857	7.027	7.048	0.294
0.005	7.184	7.473	7.518	0.605
0.010	7.395	7.777	7.845	0.872
0.025	7.795	8.391	8.519	1.510
0.050	8.260	9.163	9.391	2.419
0.075	8.648	9.856	10.189	3.263
0.100	9.006	10.535	10.982	4.075
0.125	9.352	11.226	11.800	4.864

from (4.27a) that $d(\kappa) = \exp(-1/\kappa)$. For two concentric circles, the exact eigenvalue relation is

$$J_0(z) = Y_0(z)\left[\frac{\epsilon\, z\, J_0'(\epsilon z) - \kappa\, J_0(\epsilon z)}{\epsilon\, z\, Y_0'(\epsilon z) - \kappa\, Y_0(\epsilon z)}\right], \tag{4.36}$$

where $z = \sqrt{\lambda}$. Expanding $\lambda(\epsilon)$ for $\epsilon \ll 1$ in (4.36), we readily recover (4.32).

In Table 3 we compare the expansion (4.32c) for the smallest eigenvalue of (3.1), when $b = \infty$, with the corresponding exact result obtained from the numerical solution to (4.36). The asymptotic result is found to be within 1 percent of the exact result even when $\epsilon \approx \frac{1}{4}$. In Table 4 we compare our three-term result (4.32a), (4.35b) for the smallest eigenvalue of (3.1), in the cases where $b = \infty$ and $\kappa = \infty$, against the exact result obtained from the numerical solution to (4.36). In this case the constant β_1 is found to be $\beta_1 = 1.092$. The asymptotic and exact results agree to within 5 percent for $\epsilon < .125$.

We now compare our asymptotic results for some higher eigenvalues of (3.1) for two concentric cylinders. Specifically, we consider $b = \infty$ and we label the unperturbed eigenvalues by $\sqrt{\lambda_{0n}} = z_{0n}^2$, where $J_0(z_{0n}) = 0$ for $n = 1, 2, \ldots$. In Table 5 we compare the three-term result (4.32a), (4.35b) for these higher eigenvalues, when $b = \infty$, $\kappa = \infty$, and $\epsilon = .03$, against the exact results obtained from (4.36). The agreement between

TABLE 5

Concentric circles: $\kappa = \infty$, $b = \infty$, $\epsilon = .03$, *higher eigenvalues*.

n	λ (4.32a) (2 term)	λ (4.32a) (3 term)	λ (4.36) (exact)	perc. err., 3 term
1	7.899	8.559	8.707	1.701
5	2.363×10^2	2.464×10^2	2.572×10^2	4.199
9	7.805×10^2	8.032×10^2	8.433×10^2	4.758
13	1.641×10^3	1.677×10^3	1.766×10^3	5.006
17	2.816×10^3	2.868×10^3	3.024×10^3	5.150

the asymptotic and exact results for $\lambda(\epsilon)$ deteriorates as n increases. This is not surprising since the validity of our asymptotic theory is restricted to the case where $\epsilon^2\lambda_0 = o(1)$, and so our result is not uniform in n as $n \to \infty$. Although our asymptotic results have only been compared to exact results for concentric cylinders, we anticipate a similar agreement for other values of κ, b, hole geometries, and hole locations.

We now show how, for $r_0 = 0$, we can get a better determination of the perturbed eigenvalue than that given by (4.32a). For simplicity, we take $b = \infty$. Then, upon neglecting terms of order ϵ in (4.36) and setting $\kappa = \infty$, we obtain,

$$\frac{J_0(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} \sim \frac{\pi}{2}\left[\log\epsilon + \log(\sqrt{\lambda}/2) + \gamma\right]^{-1}. \tag{4.37}$$

Let $\lambda = \lambda^*(\epsilon)$ be the solution branch to this transcendental equation emanating from an eigenvalue λ_0 of the unperturbed problem. If we were to expand $\lambda^*(\epsilon)$ in a series in powers of $(-1/\log\epsilon)$, we would recover (4.14a), (4.14b) with coefficients λ_1, λ_2, ... for two concentric circles. For a hole of arbitrary shape with any $\kappa \neq 0$, located at $r_0 = 0$, the series (4.14a) expresses λ in powers of $\left(-1/\log\left[\epsilon d(\kappa)\right]\right)$. The coefficients λ_j are independent of κ and of the shape of the hole. Therefore the sum of the series is just $\lambda^*\left[\epsilon d(\kappa)\right]$. Thus, for a cylinder of radius one with $b = \infty$, which contains a hole of arbitrary shape located at $r_0 = 0$, we have

$$\lambda(\epsilon) = \lambda^*\left[\epsilon d(\kappa)\right] + O\left(\epsilon/\log\epsilon\right). \tag{4.38}$$

In Table 6 we compare (4.38) with the exact solution from (4.36) for two concentric circles with $\kappa = \infty$ for which $d(\infty) = 1$. The comparison is made for the lowest eigenvalue of (1.1). In this table, we also give the result for the renormalized series $\lambda(\epsilon) = \lambda_0 - 2\pi N_0^2\left(\log\left[\epsilon d(\kappa)\right] + \beta_1\right)^{-1}$, which is based on the first three terms in (4.32a) when $b = \infty$ and $r_0 = 0$. From this table, we observe that (4.38) agrees significantly better with the exact results than either (4.32a) or the renormalized series. A more general procedure to sum logarithmic expansions resulting from singularly perturbed eigenvalue problems is given in [12].

Now, if $\kappa > 0$ and λ_0 is not the minimum eigenvalue of (1.1), then $J_0(\sqrt{\lambda_0}r_0) = 0$ for some r_0. The expansion for $\lambda(\epsilon)$ in this case is given in (4.17), which becomes

$$\lambda(\epsilon) = \lambda_0 - 2\pi\,\epsilon^2\frac{N_0^2\lambda_0}{r_0^2}\left[J_0'(\sqrt{\lambda_0}r_0)\right]^2 x_{0i}\,B_{ij}(\kappa)\,x_{0j} + \cdots. \tag{4.39}$$

The tensor $B_{ij}(\infty)$ was given in (4.27b) for elliptical holes aligned with the coordinate axes, and $B_{ij}(\kappa)$ was given in (4.27a) for circular holes.

TABLE 6

Concentric circles: $\kappa = \infty$, $b = \infty$, *lowest eigenvalue.*

ϵ	λ (4.32a) (3 term)	λ (renormalized)	λ (4.38) (transc.)	λ (4.36) (exact)
0.025	8.391	8.644	8.517	8.519
0.050	9.163	9.686	9.381	9.391
0.075	9.856	10.742	10.155	10.189
0.100	10.535	11.921	10.899	10.982
0.125	11.226	13.307	11.625	11.800
0.150	11.953	15.012	12.337	12.660
0.175	12.715	17.196	13.027	13.577

5. Perturbations of boundary conditions ($n = 2, 3$). Next, we consider a strong perturbation of the boundary condition (1.1b) within a sphere of radius ϵ centered at a point x_0 on ∂D. For simplicity, we consider the case with b constant in (1.1b) and we assume that this constant is changed to $\epsilon^{-1}\kappa$ within that sphere. The perturbed problem is

$$\Delta u(x,\epsilon) + \Big(\lambda(\epsilon)\,\rho(x) - U(x)\Big)\, u(x,\epsilon) = 0\,, \qquad x \in D\,, \tag{5.1a}$$

$$\partial_n\, u(x,\epsilon) + b\, u(x,\epsilon) = 0\,, \qquad x \in \partial D\,, \tag{5.1b}$$

$$\epsilon\, \partial_n\, u(x,\epsilon) + \kappa\, u(x,\epsilon) = 0\,, \qquad x \in \partial D_\epsilon\,, \tag{5.1c}$$

$$\int_D u^2(x,\epsilon)\,\rho(x)\,dx = 1\,. \tag{5.1d}$$

Here κ is constant, ∂D_ϵ is that part of the boundary within the sphere of radius ϵ on which b has been changed, and ∂D is the rest of the boundary of D.

We proceed to solve this problem for $\epsilon \ll 1$ by the method of §2–4. In particular, if $n = 3$ the equations of §3 still apply if we modify the inner problems so that they are solved in the half-space bounded by the tangent plane to ∂D_ϵ at x_0. The solvability condition (1.15) also still applies if we now note that ∂D_σ denotes only that part of ∂D_σ lying in D and that part tends to a hemisphere as σ tends to zero. As a consequence, 4π must be replaced by 2π when we adapt the results in §3 with $n = 3$ to (5.1). Corresponding changes must be made in the two-dimensional case.

5.1. The two-dimensional case. In the neighborhood of x_0, we introduce orthogonal curvilinear coordinates (s, n) with origin at x_0, where s is arclength along ∂D and $-n$ is the distance from x to ∂D. Then (5.1) transforms exactly to

$$\begin{aligned} &u_{nn} + \frac{1}{p+n} u_n + \frac{1}{(1+p^{-1}n)^2} u_{ss} + \Big(\lambda\,\rho(x) - U(x)\Big)\, u = 0\,, \qquad x \in D\,, \\ &\partial_n u + b\, u = 0, \quad \text{on } n = 0\,, \qquad |s| > \epsilon\,, \\ &\epsilon\, \partial_n u + \kappa\, u = 0, \quad \text{on } n = 0\,, \qquad |s| < \epsilon\,, \end{aligned} \tag{5.2}$$

where p is the radius of curvature of ∂D. Here u and x are to be expressed in terms of s and n.

In the *inner region*, we introduce $\xi = \epsilon^{-1}s$, $\eta = \epsilon^{-1}n$, $v(\xi, \eta, \epsilon) = u(s, n, \epsilon)$ and we expand $v = \mu_0(\epsilon)v_0 + \mu_1(\epsilon)v_1 + \cdots$. From (5.2), we obtain to leading order

$$\begin{aligned} &v_{0\xi\xi} + v_{0\eta\eta} = 0\,, \quad \eta < 0 \\ &\partial_\eta v_0 = 0 \quad (b < \infty) \quad \text{or} \quad v_0 = 0 \quad (b = \infty)\,, \quad \eta = 0\,, \quad |\xi| > 1 \\ &\partial_\eta v_0 + \kappa v_0 = 0\,, \quad \eta = 0\,, \quad |\xi| < 1\,. \end{aligned} \tag{5.3}$$

The matching condition for the inner and outer solutions, analogous to (1.12), is

$$u_0(0,0) + n\,\partial_n u_0(0,0) + s\,\partial_s u_0(0,0) + \nu_1(\epsilon)u_1(s,n) + \cdots \sim \mu_0(\epsilon)v_0(y) + \mu_1(\epsilon)v_1(y) + \cdots\,. \tag{5.4}$$

We now determine the first correction to λ_0 for three different ranges of b and κ.

If $b < \infty$ and $\kappa \neq 0$, (5.3) has a solution with the asymptotic form

$$v_0(y) = u_0(0,0)\left(\log|y| - \log d(\kappa) + \cdots\right) \quad \text{as} \quad |y| = (\xi^2 + \eta^2)^{1/2} \to \infty.$$

If $\kappa = \infty$, then $d(\infty) = \frac{1}{2}$. Then the first term on the right-hand side of (5.4) matches $u_0(0,0)$ if $\mu_0(\epsilon) = \left(-1/\log\left[\epsilon d(\kappa)\right]\right)$. Since the second and third terms on the left-hand side of (5.4) are $O(\epsilon)$, we must next match the fourth term on the left-hand side of (5.4) to the rest of $v_0(y)$, which results in

$$\nu_1(\epsilon) = \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right), \qquad u_1(s,n) \sim \frac{u_0(0,0)}{2}\log\left(n^2 + s^2\right) \qquad \text{as} \quad n^2 + s^2 \to 0. \tag{5.5}$$

Using (5.5) in (1.15) determines λ_1. Then from (1.5) we obtain, for $b < \infty$ and $\kappa \neq 0$,

$$\lambda(\epsilon) = \lambda_0 + \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right)\pi\left[u_0(x_0)\right]^2 + \cdots\,. \tag{5.6}$$

If $b = \infty$ and $\kappa \neq \infty$, then $u_0(0,0) = 0$, $\partial_s u_0(0,0) = 0$, and thus the correction to λ_0 in (5.6) vanishes. In this case, the matching condition (5.4) requires that $\mu_0(\epsilon) = \epsilon$ and $v_0(\xi,\eta) \sim \eta\,\partial_n u_0(0,0)$ as $\eta \to -\infty$. Then there is a solution to (5.3) that has the asymptotic form

$$v_0(\xi,\eta) = \left[\partial_n u_0(0,0)\right]\left(\eta + \frac{e(\kappa)\,\eta}{\xi^2 + \eta^2} + \cdots\right) \qquad \text{as} \quad \xi^2 + \eta^2 \to \infty\,, \tag{5.7}$$

for some constant $e(\kappa)$. Matching v_0 to the term $\nu_1(\epsilon)u_1$ in (5.4) yields

$$\nu_1(\epsilon) = \epsilon^2\,, \qquad u_1(s,n) \sim \left[\partial_n u_0(0,0)\right]\frac{e(\kappa)\,\eta}{n^2 + s^2} \qquad \text{as} \qquad n^2 + s^2 \to 0\,. \tag{5.8}$$

Using (5.8) in the solvability condition (1.15) determines λ_1, and then from (1.5) we obtain, for $b = \infty$ and $\kappa \neq \infty$,

$$\lambda(\epsilon) = \lambda_0 - \pi\,\epsilon^2\,e(\kappa)\left[\partial_n u_0(x_0)\right]^2 + \cdots\,. \tag{5.9}$$

When $\kappa = 0$, the exact solution to the problem for v_0 is

$$v_0(\xi,\eta) = \left[\partial_n u_0(0,0)\right]\left(\eta - \int_0^\infty \frac{J_1(\mu)}{\mu}\,e^{\mu\eta}\cos(\mu\xi)\,d\mu\right)\,, \qquad (b = \infty\,,\ \kappa = 0)\,. \tag{5.10}$$

Here $J_1(\mu)$ is the Bessel function of order 1. Upon expanding the integral in (5.10) for $\xi^2+\eta^2\to\infty$ and comparing the result with (5.7), we find that $e(0)=\frac{1}{2}$.

If $b<\infty$ and $\kappa=\epsilon\kappa_0$, then, in general, $u_0(0,0)\neq 0$. In this case, we take $\mu_0(\epsilon)=1$ and $\mu_1(\epsilon)=\epsilon$ and we expand $v=u_0(0,0)+\epsilon v_1+\cdots$. Then from (5.2), we find that v_1 solves

(5.11)
$$v_{1\xi\xi}+v_{1\eta\eta}=0\,,\quad \eta<0$$
$$\partial_\eta v_1=-b\,u_0(0,0)\,,\quad |\xi|>1\,,\qquad \partial_\eta v_1=-\kappa_0\,u_0(0,0)\,,\quad |\xi|<1\quad \text{on }\ \eta=0\,.$$

By using the divergence theorem, it follows that the solution to (5.11) has the asymptotic form

$$v_1(\xi,\eta)\sim -b\,\eta\,u_0(0,0)-\frac{1}{\pi}(b-\kappa_0)\,u_0(0,0)\log(\xi^2+\eta^2)+\xi\,\partial_s u_0(0,0)+\cdots\,. \tag{5.12}$$

We now write (5.12) in outer variables and note that $\partial_n u_0(0,0)+bu_0(0,0)=0$. Then we find that the far field behavior of the inner solution is given by

(5.13)
$$v\sim u_0(0,0)+n\,\partial_n u_0(0,0)+s\,\partial_s u_0(0,0)+\frac{\epsilon}{\pi}(b-\kappa_0)u_0(0,0)\big(2\log\epsilon-\log(n^2+s^2)\big)+\cdots\,.$$

Comparing (5.13) with (5.4) shows that

$$\nu_1(\epsilon)=\epsilon\,,\qquad u_1(s,n)\sim-\frac{1}{\pi}(b-\kappa_0)u_0(0,0)\,\log(n^2+s^2)\quad \text{as}\quad n^2+s^2\to 0\,. \tag{5.14}$$

The term of order $\epsilon\log\epsilon$ in (5.13) is unmatched thus far, which shows that we must include a term $(\epsilon\log\epsilon)\hat{v}(y)$ in the inner expansion. Since $\hat{v}$ satisfies Laplace's equation with $\partial_n\hat{v}=0$ on $\eta=0$, we can take $\hat{v}=-2(b-\kappa_0)u_0(0,0)/\pi$ to exactly cancel the unmatched term in (5.13). Using (5.14) in (1.15) determines λ_1. Then from (1.5) we obtain, for $b=\infty$ and $\kappa=\epsilon\kappa_0$,

$$\lambda(\epsilon)=\lambda_0+2\,\epsilon\,(\kappa_0-b)\,\big[u_0(x_0)\big]^2+\cdots\,. \tag{5.15}$$

We note that (5.15) can be derived by regular perturbation theory by writing (5.1) as

$$\Delta u+\Big(\lambda_0\,\rho(x)-U(x)\Big)\,u=\big(\lambda_0-\lambda\big)\,\rho(x)\,u\,,\qquad x\in D\,, \tag{5.16a}$$

$$\partial_n\,u+b\,u=\big(b-\kappa_0\big)\,u\,I_\epsilon\,,\qquad x\in\partial D\cup\partial D_\epsilon\,. \tag{5.16b}$$

Here λ_0 is an eigenvalue of (1.1), and I_ϵ is the indicator function, which is defined to be unity if $x\in\partial D_\epsilon$ and zero otherwise. Applying Green's identity to (1.1) and (5.16), we derive

$$\big(\lambda-\lambda_0\big)\int_D u_0\,u\,\rho(x)\,dx=\big(\kappa_0-b\big)\int_{\partial D_\epsilon}u_0\,u\,ds\,. \tag{5.17}$$

Since u differs from u_0 by an amount of order ϵ even in the vicinity of x_0, we can replace u by u_0 in (5.17). Then, since u_0 is normalized and the length of ∂D_ϵ is 2, (5.17) reduces to (5.15).

5.2. The three-dimensional case. The procedure followed in the two-dimensional case can be adapted to treat the three-dimensional case. Since the analysis is similar, we will omit the details. We will give expressions for the first correction to λ_0 for the same ranges of b and κ considered in the two-dimensional case.

If $b < \infty$ and $\kappa \neq 0$, the expansion for $\lambda(\epsilon)$ is given by

$$\lambda(\epsilon) = \lambda_0 + 2\,\epsilon\,\pi\,C(\kappa)\,\big[u_0(x_0)\big]^2 + \cdots . \tag{5.18}$$

The constant $C(\kappa)$ is determined from the solution to the following leading-order inner problem:

(5.19)
$$v_{0\xi_1\xi_1} + v_{0\xi_2\xi_2} + v_{0\eta\eta} = 0\,, \quad \eta < 0$$
$$\partial_\eta v_0 = 0 \quad (\xi_1,\xi_2) \notin \partial D_1\,, \quad \partial_\eta v_0 + \kappa v_0 = 0 \quad (\xi_1,\xi_2) \in \partial D_1 \quad \text{on } \eta = 0\,,$$
$$v_0 \sim u_0(x_0)\Big[1 - C(\kappa)/|y| + \cdots\Big] \qquad \text{as} \quad |y| = (\xi_1^2 + \xi_2^2 + \eta^2)^{1/2} \to \infty\,.$$

Here ∂D_1 denotes the perturbing patch ∂D_ϵ magnified by ϵ^{-1}. If ∂D_1 is a circular patch of radius 1 and $\kappa = \infty$, then the exact solution to (5.19) is

$$v_0 = u_0(x_0)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\mu}{\mu}\, e^{\mu\eta}\, J_0(\mu\rho)\, d\mu\right], \qquad \rho = (\xi_1^2 + \xi_2^2)^{1/2}\,. \tag{5.20}$$

By expanding the integral in (5.20) asymptotically, we obtain $C(\infty) = 2/\pi$, which is the capacitance of a circular disk of radius 1.

If $b = \infty$ and $\kappa \neq \infty$, the expansion for $\lambda(\epsilon)$ is given by

$$\lambda(\epsilon) = \lambda_0 - 2\,\pi\,\epsilon^3\, e(\kappa)\,\big[\partial_n u_0(x_0)\big]^2 + \cdots . \tag{5.21}$$

The constant $e(\kappa)$, representing one element of the polarizability tensor, is determined from the following leading-order inner problem:

(5.22)
$$v_{0\xi_1\xi_1} + v_{0\xi_2\xi_2} + v_{0\eta\eta} = 0\,, \quad \eta < 0$$
$$v_0 = 0 \quad (\xi_1,\xi_2) \notin \partial D_1\,, \quad \partial_\eta v_0 + \kappa v_0 = 0 \quad (\xi_1,\xi_2) \in \partial D_1 \quad \text{on } \eta = 0\,,$$
$$v_0 \sim [\partial_n u_0(x_0)]\left(\eta + \frac{e(\kappa)\,\eta}{|y|^3} + \cdots\right) \quad \text{as} \quad |y| = (\xi_1^2 + \xi_2^2 + \eta^2)^{1/2} \to \infty\,.$$

If ∂D_1 is a circular patch of radius 1 and $\kappa = 0$, then the exact solution to (5.22) is

$$v_0 = \partial_n u_0(x_0)\left[\eta - \frac{2}{\pi}\int_0^\infty j_1(\mu)\, e^{\mu\eta}\; J_0(\mu\rho)\, d\mu\right], \qquad \rho = (\xi_1^2 + \xi_2^2)^{1/2}\,. \tag{5.23}$$

Here $j_1(\mu)$ is the spherical Bessel function of order 1. By expanding the integral in (5.23) asymptotically, we obtain $e(0) = 2/3\pi$.

If $b < \infty$ and $\kappa = \epsilon\,\kappa_0$, the expansion for $\lambda(\epsilon)$ is given by

$$\lambda(\epsilon) = \lambda_0 + A_1\,\epsilon^2\,(\kappa_0 - b)\,\big[u_0(x_0)\big]^2 + \cdots . \tag{5.24}$$

Here A_1 is the area of the scaled patch ∂D_1.

6. Exit time distribution. We now give an application of the results in §5. Suppose that a particle starts from y at time zero and performs a Brownian motion in a three-dimensional domain D with a reflecting wall ∂D perforated by N small holes, each of radius ϵ, with the ith hole centered at x_0^i. We denote by $p(x,y,t,\epsilon)$ the probability density that the particle is at x at time t. Then, assuming a constant diffusion coefficient $\kappa > 0$, p satisfies

$$p_t = k \triangle_x p\,, \quad x \in D\,, \tag{6.1a}$$

$$\partial_\nu p = 0\,, \quad x \in \partial D\,; \qquad p = 0\,, \quad x \in \partial D_{\epsilon_i} \quad \text{for} \quad i = 1,\ldots,N\,, \tag{6.1b}$$

$$p = \delta(x-y), \qquad t = 0\,. \tag{6.1c}$$

The solution of (6.1) is

$$p(x,y,t,\epsilon) = \sum_{n=1}^{\infty} \exp[-\lambda_n(\epsilon)\, k\, t]\, u_n(x,\epsilon)\, u_n(y,\epsilon)\,. \tag{6.2}$$

Here λ_n is the nth eigenvalue and u_n the corresponding normalized eigenfunction of the problem

$$\triangle_x u_n(x,\epsilon) = -\lambda_n(\epsilon)\, u_n(x,\epsilon)\,, \quad x \in D\,; \qquad \int_D u_n^2(x,\epsilon)\, dx = 1\,, \tag{6.3}$$

with the boundary conditions (6.1b). The probability $P(y,t,\epsilon)$ that the particle is in D at time t is given by

$$P(y,t,\epsilon) = \int_D p(x,y,t,\epsilon)\, dx = \sum_{n=1}^{\infty} \exp[-\lambda_n(\epsilon)\, k\, t]\, u_n(y,\epsilon) \int_D u_n(x,\epsilon)\, dx\,. \tag{6.4}$$

If the initial position y is uniformly distributed over D, then we multiply (6.2) by V^{-1}, where V is the volume of D, and integrate it with respect to y to get

$$p_0(x,t,\epsilon) = V^{-1} \sum_{n=1}^{\infty} \exp[-\lambda_n(\epsilon)\, k\, t]\, u_n(x,\epsilon) \int_D u_n(y,\epsilon)\, dy\,. \tag{6.5}$$

The probability $P_0(t,\epsilon)$ that the particle is in D at time t is

$$P_0(t,\epsilon) = \int_D p_0(x,t,\epsilon)\, dx = V^{-1} \sum_{n=1}^{\infty} \exp[-\lambda_n(\epsilon)\, k\, t] \left(\int_D u_n(x,\epsilon)\, dx \right)^2 . \tag{6.6}$$

We denote by λ_{n0} and u_{n0} the nth eigenvalue and normalized eigenfunction corresponding to $\epsilon = 0$. To determine the correction to λ_{n0}, we must construct an inner expansion near each x_0^i, and one outer expansion valid away from the perforations. A straightforward generalization of the results of §5 shows that

$$\lambda_n(\epsilon) = \lambda_{n0} + 2\pi\epsilon \sum_{i=1}^{N} C^i(\infty) \left[u_{n0}(x_0^i)\right]^2 + \cdots\,, \tag{6.7}$$

$$u_n(x,\epsilon) = u_{n0}(x) + \epsilon\, u_{n1}(x) + \cdots\,, \qquad |x - x_0^i| \gg O(\epsilon)\,, \quad i = 1,\ldots,N\,, \tag{6.8}$$

$$u_n(x,\epsilon) = v_0^i\left(\frac{x - x_0^i}{\epsilon}\right) + \cdots\,, \qquad |x - x_0^i| = O(\epsilon)\,, \quad i = 1,\ldots,N\,. \tag{6.9}$$

In (6.7), $C^i(\infty)$ is the "capacitance" of the ith hole, defined in (5.19). In the outer expansion (6.8), the correction term $u_{n1}(x)$ satisfies

$$\text{(6.10a)}\qquad \Delta_x u_{n1} + \lambda_{n0}\, u_{n1} = -\lambda_{n1}\, u_{n0}\,, \quad x \in D\,; \quad \partial_\nu u_{n1} = 0\,, \quad x \in \partial D\,,$$

$$\text{(6.10b)}\qquad u_{n1}(x) \sim -u_{n0}(x_0^i)\, C^i(\infty)/|x - x_0^i| \quad \text{as} \quad x \to x_0^i\,, \quad i = 1, \ldots, N\,.$$

The expansion (6.9) is the inner expansion of $u_n(x,\epsilon)$ near x_0^i. In terms of a local curvilinear coordinate system near x_0^i, each term v_0^i satisifes (5.19).

The first eigenvalue in the absence of the holes is $\lambda_{10} = 0$ and the corresponding normalized eigenfunction is $u_{10}(x) = V^{-1/2}$. Since the u_{n0} are orthogonal, we have $\int_D u_{n0}(x)\, dx = 0$ for $n \geq 2$. We now use this fact and (6.8) to get

$$\text{(6.11)}\qquad \int_D u_n(x,\epsilon)\, dx = V^{1/2}\delta_{n1} + \epsilon \int_D u_{n1}(x)\, dx + o(\epsilon)\,.$$

Upon using (6.7) and (6.11), the sums in (6.5) and (6.6) collapse to

(6.12)

$$P_0(t,\epsilon) = \exp\left[-\frac{2\pi\epsilon\, k\, t}{V} \sum_{i=1}^{N} C^i(\infty)\right] \left(1 + 2\,\epsilon\, V^{-1/2} \int_D u_{11}(x)\, dx + o(\epsilon)\right)\,,$$

$$\text{(6.13)}\qquad p_0(x,t,\epsilon) = V^{-1} \exp\left[-\frac{2\pi\epsilon\, k\, t}{V} \sum_{i=1}^{N} C^i(\infty)\right] \Big(1 + O(\epsilon)\Big)\,.$$

For $t \gg O(-\log\epsilon)$, (6.5) gives the more refined result

$$\text{(6.14)}\qquad p_0(x,t,\epsilon) = V^{-1} \exp\left[-\frac{2\pi\epsilon\, k\, t}{V} \sum_{i=1}^{N} C^i(\infty)\right] \cdot \left(1 + \epsilon\, V^{1/2}\, u_{11}(x) + \epsilon\, V^{-1/2} \int_D u_{11}(y)\, dy + o(\epsilon)\right)\,.$$

If each hole is circular and has radius ϵ, then $C^i(\infty) = 2/\pi$ for $i = 1, \ldots, N$, and we can replace the exponential in (6.13) and (6.14) by $\exp\,(-4\,\epsilon\, kt\, N/V)$.

A similar analysis can be done for the corresponding two-dimensional case. If each absorbing segment ∂D_{ϵ_i} has length 2ϵ, then, upon using (5.6) with $d(\infty) = \frac{1}{2}$, we have in place of (6.12)

$$\text{(6.15)}\qquad P_0(t,\epsilon) = \exp\left[-\frac{\pi N\, k\, t}{V}\, \nu(\epsilon)\right] \left(1 + 2\,\nu(\epsilon)\, V^{-1/2} \int_D u_{11}(x)\, dx + o[\nu(\epsilon)]\right)\,.$$

Here $\nu(\epsilon) \equiv \left[-\log(\epsilon/2)\right]^{-1}$, and V is the area of D. The outer solution is $u_{10} + \nu(\epsilon) u_{11} + \cdots$, where u_{11} satisfies (6.10) with $n = 1$ and with (6.10b) replaced by

$$\text{(6.16)}\qquad u_{11}(x) \sim V^{-1/2} \log|x - x_0^i| \quad \text{as} \quad x \to x_0^i\,, \quad i = 1, \ldots, N\,.$$

Appendix A. Two dimensions ($u_0(x_0) = 0$). In the inner region, we expand v as

$$\text{(A.1)}\qquad v(y,\epsilon) = \epsilon\, v_0(y) + \epsilon \sum_{i=1}^{\infty} \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right)^{i} v_i(y) + \cdots\,.$$

Substituting (A.1) into (3.2) and using the matching condition (4.3) with $\mu_i(\epsilon) = \epsilon\big(-1/\log[\epsilon d(\kappa)]\big)^i$ for $i \geq 0$, we find that v_0 satisfies (4.20a) with $v_0(y) \sim [\partial_{x_i} u_0(x_0)]\, y_i$ as $y \to \infty$. Further terms in the far field form of v_0 are given in (4.20b). Using (4.20b) in (A.1), we find that the term on the right-hand side of (4.3) that must be matched next is the constant term $\epsilon B_i(\kappa)\partial_{x_i} u_0(x_0)$. This term cannot be matched by the outer solution. To match this term, and similar terms appearing at higher order, we take $v_i(y) = a_i \hat{v}(y)$, for $i > 1$, where the a_i are to be determined. Here $\hat{v}(y)$ is the solution to (4.2) with far field form (4.4). We now choose $a_1 = -\partial_{x_i} u_0(x_0) B_i(\kappa)$ to cancel the constant term of order ϵ generated by v_0. Then, writing the far field form of (A.1) in outer variables, we obtain

$$\text{(A.2)}\qquad \begin{aligned} v \sim \partial_{x_i} u_0(x_0)(x_i - x_{0i}) &+ \epsilon \sum_{i=1}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]} \right)^i \Big(a_i \log|x - x_0| + a_{i+1} \Big) \\ &+ \epsilon^2 \partial_{x_i} u_0(x_0) \frac{B_{ij}(\kappa)(x_j - x_{0j})}{|x - x_0|^2} + \cdots . \end{aligned}$$

The form (A.2) suggests that we take the outer and eigenvalue expansions as

$$\text{(A.3a)}\qquad u(x,\epsilon) = u_0(x) + \epsilon \sum_{i=1}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]} \right)^i u_i(x) + \epsilon^2 u^*(x) + \cdots ,$$

$$\text{(A.3b)}\qquad \lambda(\epsilon) = \lambda_0 + \epsilon \sum_{i=1}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]} \right)^i \lambda_i + \epsilon^2 \lambda^* + \cdots .$$

Substituting (A.3a), (A.3b) into (3.1), we find that u^*, λ^* satisfy (4.21), while u_i, λ_i, for $i \geq 1$, satisfy (1.6) with u_1, λ_1 replaced by u_i, λ_i. Using the matching condition (4.3), we find that $u_i(x) \sim a_i \log|x - x_0| + a_{i+1}$ as $x \to x_0$. Since $u_0(x_0) = 0$, the solvability condition (1.15) applied to u_i yields $\lambda_i = 0$ for $i \geq 1$. Consequently, u_i can be written as $u_i(x) = a_i g(x)$, where $g(x)$ is the unique solution to

$$\text{(A.4a)}\qquad \Delta g - U(x) g + \lambda_0 \rho(x) g = 0\,, \qquad x \neq x_0$$

$$\text{(A.4b)}\qquad [\partial_n + b(x)]\, g = 0\,, \quad x \in \partial D\,; \qquad \int_D u_0(x) g(x) \rho(x)\, dx = 0\,,$$

$$\text{(A.4c)}\qquad g(x) = \log|x - x_0| + \beta + o(1) \quad \text{as} \quad x \to x_0\,.$$

The constant β is determined from the solution to (A.4).

Since $u_i(x) \sim a_i \log|x - x_0| + a_i\beta$ as $x \to x_0$, we conclude that $a_{i+1} = \beta a_i$ for $i \geq 1$. Then, recalling that $a_1 = -\partial_{x_j} u_0(x_0) B_j(\kappa)$, we have

$$\text{(A.5)}\qquad a_i = -\beta^{i-1} \left[\partial_{x_j} u_0(x_0) B_j(\kappa) \right] \quad \text{for} \quad i = 1, 2, \ldots .$$

Now substituting $v_i(y) = a_i \hat{v}(y)$, $u_i(x) = a_i g(x)$ into (A.1) and (A.3a), we obtain two geometric series, which can be summed explictly when $|\beta/\log[\epsilon d(\kappa)]| < 1$. Finally, applying the solvability condition (1.15) to the problem for u^* determines λ^*. The results are summarized in Corollary 4.

Appendix B. Two dimensions ($\kappa = 0$). In the inner region, we expand v as $v = u_0(x_0) + \epsilon v_1 + \epsilon^2 v_2 + \cdots$. Substituting this expansion into (3.2), we find that v_1 and v_2 satisfy (3.6) and (3.7). The far field form for v_1 is given in (3.8) with

$n = 2$. To obtain v_2, we proceed as in (3.9) and (3.10), where we now require that $\phi(y) \sim E \log|y|$ as $y \to \infty$. Instead of (3.12), we find that E is given by (4.25). Thus from (3.9) the far field form of $v_2(y)$, in analogy with (3.13), is

$$\text{(B.1)} \qquad v_2(y) = \frac{1}{2}\left[\partial_{x_i}\partial_{x_j} u_0(x_0)\right] y_i y_j + E \log|y| + \cdots \qquad \text{as} \quad y \to \infty .$$

In the outer region, we expand $u = u_0 + \epsilon^2 u_1 + \cdots$ and $\lambda(\epsilon) = \lambda_0 + \epsilon^2 \lambda_1 + \cdots$. Substituting these expansions into (3.2) and using the matching condition (4.3), we find that u_1 satisfies (1.6) with

$$\text{(B.2)} \qquad u_1 \sim \left[\partial_{x_i} u_0(x_0)\right] \frac{B_{ij}(0)(x_j - x_{0j})}{|x - x_0|^2} + E \log|x - x_0| .$$

Imposing the solvability condition (1.15) then determines λ_1. The term $-E(\epsilon^2 \log \epsilon)$ is unmatched so far. Thus we must insert a switchback term of the form $(\epsilon^2 \log \epsilon)E$ into the inner expansion. The results are summarized in Corollary 5.

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