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SUMMING LOGARITHMIC EXPANSIONS FOR SINGULARLY PERTURBED EIGENVALUE PROBLEMS*

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Dedicated to the memory of Farouk Odeh.

Abstract. Strong localized perturbations of linear and nonlinear eigenvalue problems in a bounded two-dimensional domain D are considered. The effects on an eigenvalue λ_0 of the Laplacian, and on the fold point λ_{c0} of a nonlinear eigenvalue problem, of removing a small subdomain D_ϵ, of "radius" ϵ, from D and imposing a condition on the boundary of the resulting hole, are determined. Using the method of matched asymptotic expansions, it is shown that the expansions of the eigenvalues and fold points for these perturbed problems start with infinite series in powers of $(-1/\log[\epsilon d(\kappa)])$. Here $d(\kappa)$ is a constant that depends on the shape of D_ϵ and on the precise form of the boundary condition on the hole. In each case, it is shown that the entire infinite series is contained in the solution of a single related problem that does not involve the size or shape of the hole. This related problem is not stiff and can be solved numerically in a straightforward way. Thus a hybrid asymptotic-numerical method is obtained, which has the effect of summing these infinite logarithmic expansions. Results obtained from the hybrid formulation are shown to be in close agreement with full numerical solutions to the perturbed problems. The hybrid method is then used to determine the absorption time distribution for a particle performing Brownian motion in a domain with reflecting walls containing several small absorbing obstacles.

Key words. eigenvalues, strong localized perturbations, logarithmic expansions, overlapping grids

AMS(MOS) subject classifications. 35P, 41

1. Introduction. We consider strong localized perturbations of certain linear and nonlinear eigenvalue problems in a bounded two-dimensional domain D. The unperturbed linear eigenvalue problem is

$$\triangle u_0 + \lambda_0 \, u_0 = 0\,, \quad x \in D\,; \quad \partial_n u_0 + b\, u_0 = 0\,, \quad x \in \partial D\,; \quad \int_D u_0^2 \, dx = 1\,; \tag{1.1}$$

the unperturbed nonlinear eigenvalue problem is

$$\triangle u_0 + \lambda_0 \, F(u_0) = 0\,, \quad x \in D\,; \quad \partial_n u_0 + b\, u_0 = 0\,, \quad x \in \partial D\,; \quad \int_D u_0^2 \, dx = \alpha > 0\,. \tag{1.2}$$

Here $b > 0$ is a constant and $F(u) > 0$ is a smooth nonlinear source term. We assume that the solutions to (1.2) can be parameterized as $u_0(x,\alpha)$, $\lambda_0(\alpha)$ and that the graph

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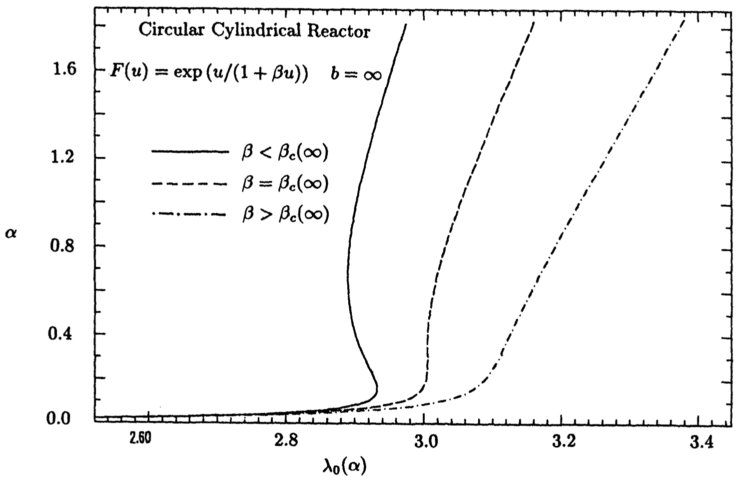

FIG. 1. *Circular cylindrical reactor: Plots of* α *versus* λ_0 *for three values of* β.

of α versus λ_0 is multiple-valued with a simple fold point at $\lambda_{c0} \equiv \lambda_0(\alpha_0)$, where $\lambda_0'(\alpha_0) = 0$ (see Fig. 1). When $F(u) = \exp(u/(1+\beta u))$, with $\beta > 0$, (1.2) is a simple model for the spontaneous combustion of a solid material. In this context u_0 is the dimensionless temperature, β is the dimensionless activation energy parameter characterizing the material, and the critical value λ_{c0} determines the onset of thermal runaway.

These problems are perturbed by deleting from D a small subdomain D_ϵ of "radius" ϵ, centered at a point $x_0 \in D$, and imposing the condition $\epsilon \partial_n u + \kappa u = 0$, with $\kappa > 0$, on the boundary of the resulting hole. Our goal is to determine the change in the simple eigenvalues λ_0 of (1.1) and in the critical value λ_{c0} of (1.2) as a result of the perturbation. In the context of thermal explosion theory, the increase in λ_{c0} determines the effect of a cooling rod of cross section D_ϵ in delaying the onset of thermal runaway.

The perturbed nonlinear eigenvalue problem was considered previously in [6], [9], [11], and [12]. Using the method of matched asymptotic expansions, it was shown in [11] that the expansion of the fold point $\lambda_c(\epsilon)$ for the perturbed problem is given by

$$\lambda_c(\epsilon) \sim \lambda_{c0} + \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right) \frac{2\pi\, u_0(x_0,\alpha_0) u_{0\alpha}(x_0,\alpha_0)}{\int_D u_{0\alpha}(x,\alpha_0) F[u_0(x,\alpha_0)]\, dx} + \cdots \quad \text{as} \quad \epsilon \to 0\,.$$

Here $d(\kappa)$ is a constant depending on κ and on the shape of the cooling rod. A numerical method to determine the coefficient of $(-1/\log[\epsilon d(\kappa)])$ was given in [12]. This result was extended to one higher order in [11] in the special case where $F = e^u$, when D is a circular cylindrical domain containing a cooling rod of arbitrary cross section centered at the origin. Similar results were obtained in [9] for an arbitrary nonlinearity. However, these results did not indicate how to calculate additional terms in the expansion of $\lambda_c(\epsilon)$ in general. In the limiting case of an annular domain with $\kappa = \infty$, for which $d = 1$, a three-term expansion for $\lambda_c(\epsilon)$ was given in [6]. For this geometry, it was shown how, in principle, further terms in the expansion of $\lambda_c(\epsilon)$ can be obtained.

Some results for the perturbed linear eigenvalue problem were obtained previously in [7] and [10] (see also some related earlier work in [8]). Let $\lambda(\epsilon)$ be an eigenvalue of the perturbed problem for which $\lambda(\epsilon)$ tends to a simple eigenvalue λ_0 of (1.1) as $\epsilon \to 0$. Then, when $\kappa = \infty$ and D_ϵ is a circular domain of radius ϵ, it was proved in [7] that

$$\lambda(\epsilon) \sim \lambda_0 + \left(-\frac{1}{\log \epsilon}\right) 2\pi \left[u_0(x_0)\right]^2 + \cdots \quad \text{as} \quad \epsilon \to 0\,.$$

Using the method of matched asymptotic expansions, this result was extended in [10] to allow for $\kappa > 0$ and for an arbitrarily shaped hole. It was also shown in [10] that the expansion of $\lambda(\epsilon)$ starts with an infinite series in powers of $(-1/\log[\epsilon d(\kappa)])$. The coefficient of $(-1/\log[\epsilon d(\kappa)])^j$ was found to be independent of ϵ, of κ, and of the shape of the hole. In a certain limiting case, a few of these coefficients were determined analytically. By extending the results of [11] to infinite logarithmic order, it can be shown that the coefficient of $(-1/\log[\epsilon d(\kappa)])^j$ in the expansion of $\lambda_c(\epsilon)$ is also independent of ϵ, of κ, and of the shape of the hole.

These infinite-order logarithmic expansions for $\lambda(\epsilon)$ and $\lambda_c(\epsilon)$ are, unfortunately, of limited use for practical purposes since each coefficient in these expansions can be obtained only by solving a partial differential equation. In addition, when ϵ is only moderately small, the terms in the series decrease very slowly as j increases, and thus many terms must be calculated to ensure that a truncated series is in close agreement with the exact result.

Instead of evaluating the coefficients in these expansions directly, we will formulate two related problems with solutions that are the sums of the infinite-order logarithmic expansions for the perturbed linear and nonlinear eigenvalue problems, respectively. Each problem depends on the size and shape of the perturbing subdomain only via the product $\epsilon d(\kappa)$. Since these problems are not stiff, they can be solved numerically in a relatively straightforward way. The constant $d(\kappa)$, which characterizes the perturbing subdomain, is computed from the numerical solution to a separate nonstiff canonical problem.

An outline of the paper is as follows. In §2 we give the form of the infinite-order logarithmic expansion for the perturbed eigenvalue $\lambda(\epsilon)$ that was found in [10] in the case of a single hole. We then formulate the related problem whose solution is the sum of the infinite logarithmic expansion. In §3 we extend the results of §2 to the case where several identical holes are removed from D. In §4 we show how to sum the infinite-order logarithmic expansion for the fold point $\lambda_c(\epsilon)$ of the perturbed nonlinear eigenvalue problem. In §5 we sum this infinite logarithmic expansion in a specific case. In §6 we give the numerical method used to determine $d(\kappa)$ and to solve the full problems and the related problem for the linear eigenvalue problem. In §7 we compare the results for $\lambda(\epsilon)$ obtained from the related problem with those results obtained from a full numerical solution to the perturbed linear eigenvalue problem. The comparisons are made for various geometries, several values of κ, and for either one or a few identical holes. A similar comparison is made in §8 for the perturbed nonlinear eigenvalue problem in the case where D is a circular cylindrical domain containing a cooling rod of arbitrary cross section centered at the origin. In §9 we apply the method of §3 to determine the absorption time distribution of a particle performing Brownian motion in a two-dimensional domain, with reflecting walls, that contains several small absorbing obstacles.

2. The infinite logarithmic expansion for the linear eigenvalue problem:

One hole. The perturbed linear eigenvalue problem is

$$\Delta u + \lambda u = 0, \quad x \in D \backslash D_\epsilon ; \quad \int_{D \backslash D_\epsilon} u^2 \, dx = 1, \tag{2.1a}$$

$$\partial_n u + b u = 0, \quad x \in \partial D, \tag{2.1b}$$

$$\epsilon \, \partial_n u + \kappa u = 0, \quad x \in \partial D_\epsilon . \tag{2.1c}$$

Here $\kappa > 0$ is constant, D_ϵ is a small domain of "radius" $O(\epsilon)$ centered at a point x_0 in D, and $\partial_n u$ is the directional derivative of u along the outward normal to $D \backslash D_\epsilon$. We assume that D_ϵ is a scaled version of a fixed domain D_1, which we write as $D_\epsilon = \epsilon D_1$. Let $\lambda(\epsilon)$ be an eigenvalue of (2.1) with eigenfunction $u(x, \epsilon)$. We suppose that $\lambda(\epsilon) \to \lambda_0$ and $u(x, \epsilon) \to u_0(x)$ as $\epsilon \to 0$, where λ_0 is a simple eigenvalue of the unperturbed problem (1.1) and $u_0(x)$ is the corresponding normalized eigenfunction. The presence of the small parameter ϵ in (2.1c) only serves to set the range of κ under consideration. In [10], expansions for $\lambda(\epsilon)$ were constructed for the cases where $\kappa = O(1)$, $\kappa = O(\epsilon)$, and $\kappa = 0$. Since in the latter two cases $\lambda(\epsilon)$ depends algebraically, rather than logarithmically, on ϵ, in §§2 and 3 we will assume that $\kappa > 0$, where κ is independent of ϵ.

Using the method of matched asymptotic expansions, it was shown in [10] that $\lambda(\epsilon)$ has the following asymptotic expansion as $\epsilon \to 0$:

$$\lambda(\epsilon) = \lambda_0 + \left(-\frac{1}{\log \left[\epsilon d(\kappa) \right]} \right) \lambda_1 + \sum_{j=2}^{\infty} \left(-\frac{1}{\log \left[\epsilon d(\kappa) \right]} \right)^j \lambda_j + \cdots . \tag{2.2a}$$

Here, $\lambda_1 = 2\pi \left[u_0(x_0) \right]^2$ and λ_j for $j \geq 2$ is defined by

$$\lambda_j = 2\pi \, c_{j-1} \left[u_0(x_0) \right]^2 - \sum_{i=1}^{j-1} \lambda_{j-i} \, (u_i, u_0) \, , \qquad (u_i, u_0) \equiv \int_D u_i u_0 \, dx \, . \tag{2.2b}$$

The outer expansion away from the hole is

$$u(x, \epsilon) = u_0(x) + \sum_{j=1}^{\infty} \left(-\frac{1}{\log \left[\epsilon d(\kappa) \right]} \right)^j u_j(x) + \cdots , \qquad |x - x_0| \gg O(\epsilon) , \tag{2.3}$$

and the inner expansion near the hole has the form

$$u(x, \epsilon) = u_0(x_0) \, v \left[\epsilon^{-1}(x - x_0) \right] \sum_{j=0}^{\infty} \left(-\frac{1}{\log \left[\epsilon d(\kappa) \right]} \right)^{j+1} c_j + \cdots , \qquad |x - x_0| = O(\epsilon) . \tag{2.4}$$

The outer corrections u_j for $j = 1, 2, \ldots$, written in (2.2b) and (2.3), are the solutions to

$$\Delta u_j + \lambda_0 \, u_j = -\lambda_j \, u_0 - (1 - \delta_{j1}) \sum_{i=1}^{j-1} \lambda_{j-i} \, u_i \, , \quad x \in D \, , \quad x \neq x_0 \, , \tag{2.5a}$$

$$\partial_n u_j + b \, u_j = 0 \, , \qquad x \in \partial D \, , \tag{2.5b}$$

$$u_j(x) \sim u_0(x_0) \left(c_{j-1} \log |x - x_0| + c_j \right) \quad \text{as} \quad x \to x_0 \, , \tag{2.5c}$$

$$\sum_{i=0}^{j} \int_D u_i \, u_{j-i} \, dx = 0 \, . \tag{2.5d}$$

Here δ_{jk} is the Kronecker delta. Upon defining $c_0 = 1$, the constants c_j for $j \geq 1$ are determined from the solution to (2.5).

The term $v(y)$ in the inner expansion (2.4) is the solution to the following problem:

$$\Delta_y v = 0\,, \quad y \notin D_1\,; \quad \partial_n v + \kappa v = 0\,, \quad y \in \partial D_1\,, \tag{2.6a}$$

$$v(y) = \log|y| - \log[d(\kappa)] + o(1) \quad \text{as} \quad |y| \to \infty\,. \tag{2.6b}$$

Here $y = \epsilon^{-1}(x - x_0)$, Δ_y and ∂_n denote derivatives with respect to y, and D_1 is the domain D_ϵ in the y variable. When $\kappa > 0$, (2.6b) also defines the constant $d(\kappa)$ uniquely. This constant depends on κ and on the shape of the domain D_1, but it is independent of the orientation of D_1 with respect to the coordinate axes. When $\kappa = \infty$, $d(\infty)$ is called the logarithmic capacitance of D_1.

We note that the functions u_j and the constants c_j defined by (2.5) are independent of ϵ, of κ, and of the shape of the hole. Thus the coefficients λ_j in (2.2b) are also independent of κ and of the shape of the hole. Therefore, if λ_j can be found when D_1 is a circular hole centered at $x_0 \in D$, with $\kappa = \infty$ corresponding to $u = 0$ on ∂D_1, it is then known for a hole of arbitrary shape centered at x_0 for any $\kappa > 0$. In other words, the series in (2.2a) implies that, for every hole of arbitrary shape with $\kappa > 0$, there is a circular hole with $\kappa = \infty$ that gives the same logarithmic corrections to all orders to the unperturbed eigenvalue. This statement no longer holds if the algebraic terms in the expansion of $\lambda(\epsilon)$, which are neglected in (2.2a), are retained.

Let D_1 be a circular domain of radius 1 with $u = 0$ on ∂D_1, in which case $d(\infty) = 1$. For this problem, let us denote by $\lambda^*(\epsilon)$ a function asymptotic to the sum of the terms written explicitly on the right side of (2.2a),

$$\lambda^*(\epsilon) \sim \lambda_0 + \left(-\frac{1}{\log\epsilon}\right) 2\pi\left[u_0(x_0)\right]^2 + \sum_{j=2}^{\infty}\left(-\frac{1}{\log\epsilon}\right)^j \lambda_j\,. \tag{2.7}$$

Then, when D_1 is a hole of arbitrary shape with $\kappa > 0$, the series (2.7) is asymptotic to $\lambda^*[\epsilon d(\kappa)]$.

Let D be a circular cylindrical domain of radius one, with $b = \infty$ corresponding to $u = 0$ on ∂D, which contains a small hole of arbitrary shape centered at the origin. We consider the perturbation of the eigenvalues of (1.1) corresponding to radially symmetric eigenfunctions. For this geometry, the first two correction terms λ_1, λ_2, calculated using (2.2), were given in [10]. To three terms it was found that

$$\lambda(\epsilon) = \lambda_0 + \frac{2}{\left(J_0'(\sqrt{\lambda_0})\right)^2}\left[\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right) + c_1\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^2 + \cdots\right], \tag{2.8a}$$

where

$$c_1 = -\frac{\pi Y_0'(\sqrt{\lambda_0})}{2J_0'(\sqrt{\lambda_0})} + \log(\sqrt{\lambda_0}/2) + \gamma + \frac{\lambda_0^{-1}}{\left(J_0'(\sqrt{\lambda_0})\right)^2}\,. \tag{2.8b}$$

Here λ_0 is a root of $J_0(\sqrt{\lambda_0}) = 0$ and $\gamma = .5772\ldots$ is Euler's constant. In addition, using the exact solution for $\lambda(\epsilon)$, which is available for two concentric circles, it was found that $\lambda^*[\epsilon d(\kappa)]$ is the branch of the following transcendental equation that tends to λ_0 as $\epsilon \to 0$:

$$\frac{J_0(\sqrt{\lambda^*})}{Y_0(\sqrt{\lambda^*})} = \frac{\pi}{2}\left(\log[\epsilon d(\kappa)] + \log(\sqrt{\lambda^*}/2) + \gamma\right)^{-1}\,. \tag{2.9}$$

Expanding λ^* for $\epsilon d(\kappa) \ll 1$, we obtain (2.8). By comparing (2.8) and (2.9) with the exact solution $\lambda(\epsilon)$ for two concentric circles for $d = 1$, it was shown in [10] that (2.8) was rather inaccurate unless ϵ was very small. As expected, $\lambda^*(\epsilon)$ was significantly closer to $\lambda(\epsilon)$ than (2.8) for moderately small values of ϵ. Similar agreement is expected to hold for holes of arbitrary shape. It was not shown in [10] how to obtain (2.9) directly from (2.2), nor was it shown how to obtain $\lambda^*[\epsilon d(\kappa)]$ in any other way for arbitrary domains where an exact solution is not available.

For arbitrary domains and arbitrary hole geometries it is difficult to determine $\lambda^*[\epsilon d(\kappa)]$ by summing (2.7). No explicit formulas for λ_j, when $j \geq 2$, are available. These coefficients must be found by first determining each u_j from the numerical solution to a partial differential equation and then evaluating the inner products in (2.2b) using a quadrature rule.

We now present a simple method to determine $\lambda^*[\epsilon d(\kappa)]$ directly, which avoids having to compute the λ_j. Since the c_j, defined in (2.5c), are independent of κ, of ϵ, and of the shape of the hole, the inner expansion (2.4) has the form

$$u(x,\epsilon) \sim A[\epsilon d(\kappa)]\, v\left[\epsilon^{-1}(x - x_0)\right]\left(-1/\log\left[\epsilon d(\kappa)\right]\right) \quad \text{for} \quad |x - x_0| = O(\epsilon)\,. \tag{2.10}$$

Here $A(z)$ is an unknown function of $z \equiv \epsilon d(\kappa)$, which is to be determined. In the limit $z \to 0$, we have $A(z) \to u_0(x_0)$. With the far field behavior of $v(y)$ given in (2.6b), the far field form of the inner solution (2.10) is given by

$$u(x,\epsilon) \sim A[\epsilon d(\kappa)] U(x; x_0, \epsilon d(\kappa))\,, \quad \text{where} \quad U(x; x_0, \epsilon d(\kappa)) \equiv 1 - \frac{\log|x - x_0|}{\log[\epsilon d(\kappa)]}\,. \tag{2.11}$$

To determine $\lambda^*[\epsilon d(\kappa)]$, we will solve (2.1a), (2.1b) in all of D subject to the condition $u(x,\epsilon) = A[\epsilon d(\kappa)] U(x; x_0, \epsilon d(\kappa)) + o(1)$ as $x \to x_0$. Therefore, we must solve the following related problem. Let $u^*(x; z)$, $\lambda^*(z)$, $A(z)$ be the solution of

$$\Delta u^*(x;z) + \lambda^*(z)\, u^*(x;z) = 0\,, \quad x \in D\,, \quad x \neq x_0\,, \tag{2.12a}$$

$$\partial_n u^*(x;z) + b\, u^*(x;z) = 0\,, \qquad x \in \partial D\,, \tag{2.12b}$$

$$u^*(x;z) = A(z)\left[1 - \frac{\log|x - x_0|}{\log z}\right] + o(1) \quad \text{as} \quad x \to x_0\,, \tag{2.12c}$$

$$\int_D \left[u^*(x;z)\right]^2 dx = 1\,. \tag{2.12d}$$

Here z is a parameter defined by $z = \epsilon d(\kappa)$. If we expanded λ^* and u^* in powers of $-1/\log z$, with $\lambda^*(0) = \lambda_0$, we would recover the terms written explicitly in (2.2a) and (2.3). Thus the solution λ^* to (2.12) is asymptotic to the sum of the series in (2.7).

This method for summing the infinite-order logarithmic expansion for $\lambda(\epsilon)$ involves the numerical solution of the parameterized nonstiff problem (2.12) rather than the sequence of problems (2.5). It may be viewed as a hybrid asymptotic-numerical method in that we have exploited the form of the inner asymptotic expansion and the eigenvalue asymptotic expansion to account for all terms in $\lambda(\epsilon)$ of order $O((-1/\log[\epsilon d(\kappa)])^j)$ for all $j \geq 0$. Our goal below is to compute the solution branch $\lambda^*(z)$ to (2.12), which emanates from a simple eigenvalue of (1.1).

It is preferable for numerical purposes to solve (2.12a)–(2.12c) for u^*, λ^* when $A = 1$ in (2.12c). Then, after computing the solution to (2.12a)–(2.12c) with $A = 1$, we can satisfy the normalization condition (2.12d) by multiplying u^* by an appropriate constant. To compute the solution to (2.12a)–(2.12c) with $A = 1$, it is convenient to remove the singular behavior (2.12c) by introducing the new unknown $\phi(x;z)$ defined by

$$u^*(x;z) = -\frac{\pi}{2\log z} Y_0\left(\sqrt{\lambda^*}|x-x_0|\right) + \phi(x;z)\,. \tag{2.13}$$

Here $Y_0(w)$ is the Bessel function of the second kind of order zero. Substituting (2.13) into (2.12a)–(2.12c) with $A = 1$, and using $Y_0(w) \sim 2\pi^{-1}[\log(w/2)+\gamma] + o(1)$ as $w \to 0$, we obtain

$$\Delta\phi(x;z) + \lambda^*(z)\,\phi(x;z) = 0\,, \qquad x \in D\,, \tag{2.14a}$$

$$\partial_n\phi(x;z) + b\,\phi(x;z) = \frac{\pi}{2\log z}\left[\partial_n + b\right] Y_0\left(\sqrt{\lambda^*}|x-x_0|\right), \qquad x \in \partial D\,, \tag{2.14b}$$

$$\phi(x_0;z) = 1 + \frac{1}{\log z}\left(\log\left(\frac{\sqrt{\lambda^*}}{2}\right) + \gamma\right). \tag{2.14c}$$

Here γ is Euler's constant. The numerical method used to solve (2.14) is given in §6, and some results of the numerical computations are given in §7.

Now let D be a circular domain of radius 1 containing a hole centered at the origin, and assume that $\lambda^*(0) = \lambda_0$, where λ_0 is an eigenvalue of (1.1) corresponding to a radially symmetric eigenfunction. We now show that the solution λ^* to (2.14) reproduces (2.9) exactly. For this geometry, the solution to (2.14) is

$$\phi(x;z) = \frac{\pi}{2\log z}\left[\frac{\sqrt{\lambda^*}\,Y_0'(\sqrt{\lambda^*}) + b\,Y_0(\sqrt{\lambda^*})}{\sqrt{\lambda^*}\,J_0'(\sqrt{\lambda^*}) + b\,J_0(\sqrt{\lambda^*})}\right] J_0(\sqrt{\lambda^*}r)\,, \qquad r = |x|\,, \tag{2.15a}$$

where $\lambda^*(z)$ satisfies the transcendental equation

$$\frac{\sqrt{\lambda^*}\,J_0'(\sqrt{\lambda^*}) + b\,J_0(\sqrt{\lambda^*})}{\sqrt{\lambda^*}\,Y_0'(\sqrt{\lambda^*}) + b\,Y_0(\sqrt{\lambda^*})} = \frac{\pi}{2}\left(\log z + \log(\sqrt{\lambda^*}/2) + \gamma\right)^{-1}. \tag{2.15b}$$

Letting $b \to \infty$ in (2.15b), we recover (2.9).

Finally, to obtain $\lambda^*[\epsilon d(\kappa)]$, we must determine the constant $d(\kappa)$, defined in (2.6b), by solving a separate canonical problem numerically. This problem is described in §6.

3. The infinite logarithmic expansion for the linear eigenvalue problem: N holes. Similar asymptotic expansions to those given in §2 can be written in the case when N holes, each centered at some distinct $x_0^i \in D$ for $i = 1,\ldots,N$, are removed from D. The holes are assumed to be well separated in the sense that $|x_0^j - x_0^i| \gg O(\epsilon)$ for $i \neq j$. We assume that these holes have the same boundary condition and the same basic shape, but we will allow them to have different orientations with respect to the coordinate axes. We also assume that we can write each of the holes D_ϵ^i in terms of scaled domains D^i as $D_\epsilon^i = \epsilon D^i$, where D^i is independent of ϵ. For each i, D^i represents a rotation of a certain basic hole D_1. The perturbed problem is

$$\Delta u + \lambda u = 0\,, \qquad x \in D\backslash \cup_{i=1}^N D_\epsilon^i\,; \qquad \int_{D\backslash\cup_{i=1}^N D_\epsilon^i} u^2\,dx = 1\,, \tag{3.1a}$$

$$\partial_n u + b u = 0\,, \qquad x \in \partial D\,, \tag{3.1b}$$

$$\epsilon\,\partial_n u + \kappa u = 0\,, \quad x \in \partial D_\epsilon^i\,, \quad i = 1,\ldots,N\,. \tag{3.1c}$$

Here $\kappa > 0$ is a constant independent of i.

For this problem, the outer expansion for u is given in (2.3), for $|x - x_0^i| \gg O(\epsilon)$ and $i = 1, \ldots, N$. Each outer correction u_j now satisfies (2.5) for $x \neq x_0^i$, where (2.5c) is replaced by

$$u_j(x) \sim u_0(x_0^i)\left(c_{j-1}^i \log|x - x_0^i| + c_j^i\right) \quad \text{as } x \to x_0^i\,. \tag{3.2}$$

Upon defining $c_0^i = 1$ for $i = 1, \ldots, N$, the constants c_j^i for $j \geq 1$ and $i = 1, \ldots, N$ are uniquely determined by the u_j. In terms of c_j^i and u_j, the infinite-order logarithmic expansion for $\lambda(\epsilon)$ is given by (2.2a), where λ_j is now defined by

(3.3)
$$\lambda_1 = 2\pi \sum_{i=1}^{N} \left[u_0(x_0^i)\right]^2, \quad \lambda_j = 2\pi \sum_{i=1}^{N} c_{j-1}^i \left[u_0(x_0^i)\right]^2 - \sum_{n=1}^{j-1} \lambda_{j-n}\,(u_n, u_0)\,, \quad j \geq 2\,.$$

The first correction term λ_1 represents a superposition of the corrections due to N isolated holes. The interaction of the holes and their resulting correction to λ_0 appears only in the higher-order terms λ_j for $j \geq 2$. We denote by $\lambda^*(\epsilon)$ the infinite-order logarithmic expansion for $\lambda(\epsilon)$ when each D^i is a circle of radius one with $u = 0$ ($\kappa = \infty$) on the boundary of each hole,

$$\lambda^*(\epsilon) \sim \lambda_0 + \left(-\frac{1}{\log\epsilon}\right) 2\pi \sum_{i=1}^{N} \left[u_0(x_0^i)\right]^2 + \sum_{j=2}^{\infty} \left(-\frac{1}{\log\epsilon}\right)^j \lambda_j\,. \tag{3.4}$$

All the λ_j are independent of κ and of the shape of the hole. When all the holes have the same $d(\kappa) \neq 1$, the sum of the series corresponding to (3.4) is simply $\lambda^*[\epsilon d(\kappa)]$.

We now construct N inner expansions each of the form (2.4) with $y^i = \epsilon^{-1}(x - x_0^i)$. For each $i = 1, \ldots, N$, we find

(3.5)
$$u(x, \epsilon) = u_0(x_0^i)\, v^i\left[\epsilon^{-1}(x - x_0^i)\right] \sum_{j=0}^{\infty} \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right)^{j+1} c_j^i + \cdots\,, \qquad |x - x_0^i| = O(\epsilon)\,,$$

where $v^i(y^i)$ is the solution to

$$\Delta_y v^i = 0\,, \quad y^i \notin D^i\,; \quad \partial_n v^i + \kappa v^i = 0\,, \quad y^i \in \partial D^i\,, \tag{3.6a}$$

$$v^i(y^i) = \log|y^i| - \log\left[d(\kappa)\right] + \frac{P_j^i(\kappa)\, y_j^i}{|y^i|^2} + \cdots \quad \text{as } |y^i| \to \infty\,. \tag{3.6b}$$

The P_j^i are constants determined by the solution to (3.6). Since the holes D^i are rotations of a basic hole D_1, $d(\kappa)$ is independent of i. However, the extra term in the expansion of v^i written in (3.6b), representing the dipole contribution to the far field, does depend on the orientation of the ith hole.

We now present a simple method to determine $\lambda^*[\epsilon d(\kappa)]$ directly for any finite number of identical holes. Since the c_j^i, defined in (3.2), are independent of κ and of the shape of the identical holes, then each inner expansion (3.5) has the form

$$u(x, \epsilon) \sim A_i[\epsilon d(\kappa)]\, v^i\left[\epsilon^{-1}(x - x_0^i)\right]\left(-1/\log\left[\epsilon d(\kappa)\right]\right) \quad \text{for} \quad |x - x_0^i| = O(\epsilon)\,. \tag{3.7}$$

Here the $A_i(z)$, which are parameterized by $z \equiv \epsilon d(\kappa)$, are unknown functions to be determined. Using the first two terms from (3.6b) in (3.7), the far-field behavior of the inner expansion (3.7) is

(3.8)
$$u(x,\epsilon) \sim A_i[\epsilon d(\kappa)]\, U(x;x_0^i,\epsilon d(\kappa)), \quad \text{where} \quad U(x;x_0^i,\epsilon d(\kappa)) \equiv 1 - \frac{\log|x-x_0^i|}{\log[\epsilon d(\kappa)]}.$$

To determine $\lambda^*[\epsilon d(\kappa)]$, we will solve (3.1a), (3.1b) in all of D subject to the conditions $u(x,\epsilon) = A_i[\epsilon d(\kappa)]\, U(x;x_0^i,\epsilon d(\kappa)) + o(1)$ as $x \to x_0^i$. Therefore, we must solve the following related problem. Let $u^*(x;z)$, $\lambda^*(z)$, $A_i(z)$ be the solution of

$$\Delta u^*(x;z) + \lambda^*(z)\, u^*(x;z) = 0, \quad x \in D, \quad x \neq x_0^i \tag{3.9a}$$

$$\partial_n u^*(x;z) + b\, u^*(x;z) = 0, \quad x \in \partial D, \tag{3.9b}$$

$$u^*(x;z) = A_i(z)\left[1 - \frac{\log|x-x_0^i|}{\log z}\right] + o(1) \quad \text{as} \quad x \to x_0^i, \tag{3.9c}$$

$$\int_D [u^*(x;z)]^2\, dx = 1. \tag{3.9d}$$

Here z is a parameter defined by $z = \epsilon d(\kappa)$. Expanding λ^* in powers of $-1/\log z$, we obtain (3.4) with ϵ replaced by z.

It is preferable for numerical purposes to replace the normalization condition (3.9d) with the condition $A_1(z) \equiv 1$. Then (3.9d) can be satisfied, after computing the solution, by multiplying u^* by an appropriate constant. To remove the singular behavior (3.9c), we introduce the new variable $\phi(x;z)$ defined by

$$u^*(x;z) = -\frac{\pi}{2\log z}\sum_{i=1}^{N} A_i\, Y_0\left(\sqrt{\lambda^*}|x-x_0^i|\right) + \phi(x;z). \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain

$$\Delta\phi(x;z) + \lambda^*(z)\,\phi(x;z) = 0, \qquad x \in D, \tag{3.11a}$$

$$\partial_n \phi(x;z) + b\,\phi(x;z) = \frac{\pi}{2\log z}\sum_{i=1}^{N} A_i\,[\partial_n + b] Y_0\left(\sqrt{\lambda^*}|x-x_0^i|\right), \qquad x \in \partial D, \tag{3.11b}$$

(3.11c)
$$\phi(x_0^j;z) = \frac{\pi}{2\log z}\sum_{\substack{i=1\\ i\neq j}}^{N} A_i Y_0\left(\sqrt{\lambda^*}|x_0^j-x_0^i|\right) + A_j\left[1 + \frac{1}{\log z}\left(\log\left(\frac{\sqrt{\lambda^*}}{2}\right) + \gamma\right)\right].$$

Here γ is Euler's constant, and, in solving (3.11), we will set $A_1(z) = 1$. As $z \to 0$, we have $A_i(z) \sim u_0(x_0^i)/u_0(x_0^1)$ and $\lambda^*(z) \sim \lambda_0 - \lambda_1/\log z$, where λ_1 is given in (3.3). The numerical methods used to solve (3.11) and to compute the constant $d(\kappa)$ are given in §6. Some results of the numerical computations are given in §7.

To calculate terms in $\lambda(\epsilon)$ beyond $\lambda^*[\epsilon d(\kappa)]$, we must include the effect of the dipole term in (3.6b), and we must also retain gradient terms in the expansion of $u_0(x)$ as $x \to x_0$. By including both of these effects, we show that

$$\lambda(\epsilon) = \lambda^*[\epsilon d(\kappa)] + O(\epsilon/\log[\epsilon d(\kappa)]). \tag{3.12}$$

In [13] the form of this transcendentally small correction and further corrections are calculated explicitly.

4. The infinite logarithmic expansion: Nonlinear eigenvalue problem. We now consider the perturbed nonlinear eigenvalue problem

$$\Delta u + \lambda F(u) = 0\,, \qquad x \in D\backslash D_\epsilon\,, \tag{4.1a}$$

$$\partial_n u + b u = 0\,, \qquad x \in \partial D\,, \tag{4.1b}$$

$$\epsilon\, \partial_n u + \kappa u = 0\,, \qquad x \in \partial D_\epsilon\,, \tag{4.1c}$$

$$\alpha = \int_{D\backslash D_\epsilon} u^2\, dx\,. \tag{4.1d}$$

Here $\kappa > 0$ is constant, and we again assume that the perturbing domain D_ϵ, which is centered at some $x_0 \in D$, can be written in terms of a fixed domain D_1 as $D_\epsilon = \epsilon D_1$.

Following [11], we seek the solution to (4.1) in the parametric form $u(x, \alpha, \epsilon)$, $\lambda(\alpha, \epsilon)$, and we expand $\lambda(\alpha, \epsilon)$ as

$$\lambda(\alpha,\epsilon) = \lambda_0(\alpha) + \nu(\epsilon)\,\lambda_1(\alpha) + \nu^2(\epsilon)\,\lambda_2(\alpha) + \nu^3(\epsilon)\,\lambda_3(\alpha) + \cdots\,. \tag{4.2}$$

The corrections to the unperturbed fold position $\lambda_{c0} \equiv \lambda_0(\alpha_0)$ are found by solving $\lambda_\alpha(\alpha_c(\epsilon), \epsilon) = 0$ for $\alpha_c(\epsilon)$, which is expanded as $\alpha_c(\epsilon) = \alpha_0 + \nu(\epsilon)\alpha_1 + \nu^2(\epsilon)\alpha_2 + \cdots$. The α_i are determined by setting the coefficients of $\nu^j(\epsilon)$ to zero in the expansion of $\lambda_\alpha(\alpha_c(\epsilon), \epsilon)$. Then we find that the expansion of the fold point for (4.1) has the form

$$\lambda_c(\epsilon) \equiv \lambda(\alpha_c(\epsilon), \epsilon) = \lambda_{c0} + \sum_{i=1}^{\infty} \nu^i(\epsilon)\,\lambda_{ci} + \cdots\,, \tag{4.3}$$

where the first few coefficients λ_{ci} are defined by

$$\lambda_{c1} = \lambda_1(\alpha_0)\,, \qquad \lambda_{c2} = \lambda_2(\alpha_0) - \frac{[\lambda_1'(\alpha_0)]^2}{2\lambda_0''(\alpha_0)}\,. \tag{4.4}$$

In general, the coefficient λ_{ci} depends on $\lambda_i(\alpha_0)$ and on the derivatives $\lambda_j^k(\alpha_0)$ where $0 \le j \le i-1$, $0 \le k \le i-j$. Using singular perturbation methods similar to those used to treat the linear eigenvalue problem, the inner and outer solutions for (4.1) can be constructed, and we find that $\nu(\epsilon) = (-1/\log[\epsilon d(\kappa)])$. Using a solvability condition, the coefficients λ_{ci} in the fold point expansion can, in principle, be found.

As an example, let D be a circular cylindrical reactor of radius 1 and assume that the nonlinear heating term is given by $F(u) = \exp(u/(1+\beta u))$. For this geometry, the graphs of α versus λ_0, which exhibits the solution multiplicity for the unperturbed problem (1.2) when $b = \infty$, are shown in Fig. 1. It is well known that solution multiplicity occurs only when $\beta < \beta_c(b)$. This curve $\beta_c(b)$ has been computed in [2], and, in particular, it was found that $\beta_c(\infty) = .2421$.

Suppose that a cooling rod of arbitrary cross section D_ϵ is located at the center of the circular cylindrical reactor. Then, when $\beta = 0$, a three-term expansion to the fold point was given in [11]. With $b = \infty$, it was found using (4.3) and (4.4) that

$$\lambda_c(\epsilon) = 2 + 2.772\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right) + 2.960\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^2 + \cdots\,. \tag{4.5}$$

The coefficients of $(-1/\log[\epsilon d(\kappa)])$ and $(-1/\log[\epsilon d(\kappa)])^2$ in this expansion are correct only to the number of significant digits shown. (We remark that the definition of $d(\kappa)$ introduced in [11] differs slightly from that used in (2.6b).) Using a numerical scheme to evaluate λ_{c1} and λ_{c2} in (4.3), results similar to (4.5) were obtained in [12] and [9] for the case of an arbitrarily placed cooling rod with $\beta < \beta_c$. In the limiting case of an annular domain with $\kappa = \infty$ on ∂D_1, for which $d = 1$, these results are in agreement with those obtained in [6] using a different method. When compared against the exact result for $\lambda_c(\epsilon)$ in an annular domain with $\beta = 0$, the three-term expansion (4.5) (with $d = 1$) was found in [6] and [11] to be in only fair agreement with $\lambda_c(\epsilon)$ unless ϵ is very small.

To improve the agreement between the asymptotic and exact results for $\lambda_c(\epsilon)$ when ϵ is only moderately small, it is necessary to retain further correction terms λ_{ci} in (4.3). We shall instead proceed by a method related to that given in §2, which accounts for all the logarithmic corrections to λ_{c0} and thus, in effect, sums the series written explicitly in (4.3). This method is based on the same observation made in §2. As outlined in Appendix A, it can be shown that the fold point corrections λ_{ci} in (4.3) are independent of ϵ, of κ, and the of shape of the cross section of the cooling rod D_1. The reason why the infinite logarithmic expansions for the linear and nonlinear eigenvalue problems have the same form is because their inner expansions have the same form through all logarithmic terms. Therefore, let $\lambda_c^*(\epsilon)$ denote the sum of the terms written explicitly in (4.3) when D_1 has a circular cross section of radius 1 with $u = 0$ $(\kappa = \infty)$ on ∂D_1,

$$\lambda_c^*(\epsilon) = \lambda_{c0} + \sum_{i=1}^{\infty} \left(-\frac{1}{\log \epsilon}\right)^i \lambda_{ci} \,. \tag{4.6}$$

Then when a cooling rod of arbitrary cross section with $\kappa \neq \infty$ is centered at the same location, the expansion of the fold point for (4.1) is given by

$$\lambda_c(\epsilon) = \lambda_c^*[\epsilon d(\kappa)] + O(\epsilon/\log[\epsilon d(\kappa)]) \,. \tag{4.7}$$

This result is analogous to (3.12) obtained earlier for the linear eigenvalue problem.

We now show how to determine $\lambda_c^*[\epsilon d(\kappa)]$ for the case of a single cooling rod centered at some $x_0 \in D$. This method is very similar to that for the linear eigenvalue problem in the case of one hole. In analogy with (2.10), the inner expansion has the form

$$u(x,\alpha,\epsilon) \sim A(\alpha,z)\, v[\epsilon^{-1}(x-x_0)]\,(-1/\log z) \quad \text{for} \quad |x-x_0| = O(\epsilon) \,.$$

Here $z \equiv \epsilon d(\kappa)$ and $A(\alpha, z)$, which satisfies $A(\alpha,z) \sim u_0(x_0,\alpha)$ as $z \to 0$, is to be determined. The function $v(y)$ is the solution of (2.6). Therefore, in analogy with (2.12), let $u^*(x; z, \alpha)$, $\lambda^*(\alpha, z)$, $A(\alpha, z)$, be the solution to the related problem

$$\triangle u^* + \lambda^* F(u^*) = 0 \,, \quad x \in D \,, \quad x \neq x_0 \,, \tag{4.8a}$$

$$\partial_n u^* + b u^* = 0 \,, \quad x \in \partial D \,, \tag{4.8b}$$

$$u^* = A\left[1 - \frac{\log|x-x_0|}{\log z}\right] + o(1) \quad \text{as} \quad x \to x_0 \,, \tag{4.8c}$$

$$\alpha = \int_D (u^*)^2 \, dx \,. \tag{4.8d}$$

To determine $\lambda_c^*(z)$, we need only determine the first fold point for (4.8). After removing the logarithmic singularity in (4.8c), a standard numerical continuation scheme on a relatively coarse mesh can be used to locate this fold point as a function of z.

5. Summing the fold point expansion in a specific case. We now determine $\lambda_c^*(z)$ when D is a circular cylindrical domain of radius 1 containing a cooling rod of arbitrary cross section centered at the origin. This special case is relevant for combustion theory in that the onset of thermal runaway is delayed more by centering the cooling rod at the origin than at any other location.

For this geometry, (4.8) becomes

$$u^{*''} + \frac{1}{r}u^{*'} + \lambda^* F(u^*) = 0\,, \qquad 0 < r < 1\,, \tag{5.1a}$$

$$u^{*'} + b\,u^* = 0 \quad \text{on } r = 1\,, \tag{5.1b}$$

$$u^* = A\left(1 - \frac{\log r}{\log z}\right) + o(1) \quad \text{as } r \to 0\,, \tag{5.1c}$$

$$\int_0^1 (u^*)^2\, r\, dr = \frac{\alpha}{2\pi}\,. \tag{5.1d}$$

By varying α we can solve (5.1), at each fixed z, for $u^*(r,\alpha,z)$, $\lambda^*(\alpha,z)$, $A(\alpha,z)$ and thus determine the entire response curve α versus λ^*. As $z \to 0$, this curve approximates the response curve of the full problem (4.1) to within terms of $O(\epsilon/\log[\epsilon d(\kappa)])$.

If we only want to determine the location of the first fold point $\lambda_c^*(z)$, it is more convenient to parameterize (5.1) by A rather than α. In addition, to remove the logarithmic singularity in (5.1c), we introduce a new variable v defined by

$$u^* = \frac{A}{\log z}\left(b^{-1} - \log r\right) + v\,. \tag{5.2}$$

Then from (5.1a)–(5.1c) we find that $v(r,A,z)$ and $\lambda^*(A,z)$ satisfy

$$v'' + \frac{1}{r}v' + \lambda^* F^{*0} = 0\,, \qquad 0 < r < 1\,, \tag{5.3a}$$

$$v' + b\,v = 0 \quad \text{on } r = 1\,, \tag{5.3b}$$

$$v = A\left(1 - \frac{b^{-1}}{\log z}\right) \quad \text{and} \quad v' = 0 \quad \text{at } r = 0\,. \tag{5.3c}$$

Here we have defined F^{*0} by

$$F^{*0} \equiv F\left[v + A(b^{-1} - \log r)/\log z\right]\,, \tag{5.4}$$

where $F(\sigma) = \exp\left(\sigma/(1+\beta\sigma)\right)$.

In the case where $\beta = 0$ so that $F = e^u$, we can determine $\lambda^*(A,z)$ and $\lambda_c^*(z)$ analytically from (5.3). When $\beta = 0$, (5.3a) becomes

$$v'' + \frac{1}{r}v' + \lambda^*\, r^{-A/\log z}\, e^{Ab^{-1}/\log z}\, e^{v} = 0\,. \tag{5.5}$$

In the new variables $w(s) = v[r(s)]$, and $s = r^{1-A/2\log z}$, (5.5), (5.3b), (5.3c) become

$$\begin{aligned} &w'' + \frac{1}{s}w' + \tilde{\lambda}\, e^{w} = 0\,, \qquad 0 < s < 1\,, \\ &w' + \tilde{b}\,w = 0 \quad \text{on } s = 1\,, \\ &w = \tilde{A} \quad \text{and} \quad w' = 0\,, \quad \text{at } s = 0\,. \end{aligned} \tag{5.6}$$

Here $\tilde{\lambda}$, $\tilde{b}$, and $\tilde{A}$ are defined by
(5.7)

$$\tilde{\lambda} \equiv \lambda^* \, e^{Ab^{-1}/\log z} \left(1 - \frac{A}{2\log z}\right)^{-2}, \quad \tilde{A} = A\left(1 - \frac{b^{-1}}{\log z}\right), \quad \tilde{b} = b\left(1 - \frac{A}{2\log z}\right)^{-1}.$$

The well-known solution to (5.6) is

$$w(s) = 2\log\left(\frac{1+\rho}{1+\rho\, s^2}\right) + \frac{4\rho}{\tilde{b}\,(1+\rho)}, \qquad \tilde{\lambda} = \frac{8\rho}{(1+\rho)^2} \exp\left(-\frac{4\rho}{\tilde{b}\,(1+\rho)}\right), \tag{5.8}$$

where $\rho > 0$ is the unique root of

$$\tilde{A} = 2\log(1+\rho) + \frac{4\rho}{\tilde{b}\,(1+\rho)}. \tag{5.9}$$

Substituting (5.7) into (5.8) and (5.9), we obtain the following implicit relation for $\lambda^*(A,z)$:

$$\lambda^*(A,z) = 8e^{-A}\left(1 - \frac{A}{2\log z}\right)^2 \rho\,, \qquad (\beta = 0). \tag{5.10a}$$

Here $\rho = \rho(A,z) > 0$ is the unique root of

$$A = \left[1 + \frac{b^{-1}}{\log z}\left(\frac{\rho-1}{\rho+1}\right)\right]^{-1} \left(2\log(1+\rho) + \frac{4\rho}{b\,(1+\rho)}\right). \tag{5.10b}$$

The fold point for (5.1) is given by $\lambda_c^*(z) = \lambda^*(A_c(z), z)$, where $A_c(z)$ is the solution of the transcendental equation $\lambda_A^*(A_c(z), z) = 0$. Therefore, for this geometry and with $\beta = 0$, (4.7) becomes

$$\lambda_c(\epsilon) = \lambda^*\left(A_c[\epsilon d(\kappa)], \epsilon d(\kappa)\right) + O\left(\epsilon/\log[\epsilon d(\kappa)]\right), \qquad (\beta = 0). \tag{5.11}$$

When $\beta = 0$ and $b = \infty$, the results (5.10) and (5.11) can be simplified. As $b \to \infty$, we obtain from (5.10b) that $A \to 2\log(1+\rho)$, and thus (5.10a) reduces to

$$\lambda^*(A,z) = 8\left(e^{-A/2} - e^{-A}\right)\left(1 - \frac{A}{2\log z}\right)^2, \qquad (\beta = 0\,,\, b = \infty). \tag{5.12}$$

By setting $\lambda_A^* = 0$, we find that $A_c(z) > 0$ is the unique solution of

$$\frac{1}{\log z} = \left[\frac{2 - 2\,e^{A_c/2}}{e^{A_c/2} - 2} + \frac{A_c}{2}\right]^{-1}. \tag{5.13}$$

Thus in this special case, (5.11) becomes
(5.14)

$$\lambda_c(\epsilon) = 8\left(e^{-A_c/2} - e^{-A_c}\right)\left(1 - \frac{A_c}{2\log[\epsilon d(\kappa)]}\right)^2 + O\left(\epsilon/\log\epsilon\right), \qquad (\beta = 0\,,\, b = \infty).$$

To show that (5.14) agrees with (4.5), we first expand the solution A_c to (5.13) in a series in powers of $-1/\log z$. Substituting the resulting expansion into (5.14), we obtain

$$\lambda_c(\epsilon) = 2 + 4\log 2\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right) + 2\left(1 + (\log 2)^2\right)\left(-\frac{1}{\log[\epsilon d(\kappa)]}\right)^2 + \cdots, \tag{5.15}$$

which reproduces (4.5).

The results (5.10) and (5.11) can also be simplified when $b = 0$ so that the outer boundary is perfectly insulating. In this case, solutions exist only when $\lambda^* \ll 1$, and from (5.10) we obtain

$$\lambda^*(A, z) = \frac{8A}{(A - 4\log z)} \left(1 - \frac{A}{2\log z}\right)^2 e^{-A}\,, \qquad (\beta = 0\,,\, b = 0)\,. \tag{5.16}$$

The fold point is located at $\lambda^*(A_c(z), z)$ where $A_c(z)$ satisfies $\lambda_A^*(A_c(z), z) = 0$. A simple calculation gives

$$\lambda_c(\epsilon) = 2\left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right) e^{-1} + \frac{3}{2}\left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right)^2 e^{-1} + \cdots\,. \tag{5.17}$$

This two-term expansion agrees with that found in [12].

In the case of an annular domain $\epsilon < r < 1$, for which $d(\kappa) = e^{-1/\kappa}$, an exact solution to the perturbed problem (4.1) with $\beta = 0$ was found in [11]. This solution is given by

$$u = -2\log r + \log\left(\frac{2c^2\lambda^{-1}\rho\, r^c}{(1 + \rho\, r^c)^2}\right)\,,$$

where $\lambda(\rho, \epsilon)$, $c(\rho, \epsilon)$ are the solutions of

$$\log\left(\frac{2c^2\lambda^{-1}\rho}{(1+\rho)^2}\right) = 2b^{-1} - cb^{-1}\frac{(1-\rho)}{1+\rho}\,, \tag{5.18a}$$

$$\log\left(\frac{2c^2\lambda^{-1}\rho}{(1+\rho\,\epsilon^c)^2}\right) = (2-c)\log\left(\epsilon e^{-1/\kappa}\right) - \frac{2\rho\,\kappa^{-1} c\,\epsilon^c}{1+\rho\,\epsilon^c}\,. \tag{5.18b}$$

From (5.18b), we have that $c \to 2$ as $\epsilon \to 0$. Then by discarding the ϵ^c terms in (5.18), and, upon defining $A = \log(2c^2\lambda^{-1}\rho)$, we can easily recover (5.10) derived above.

When $\beta > 0$, (5.3) cannot be transformed to the unperturbed problem by a suitable change of variables nor is there an explicit analytical solution to the perturbed problem in an annular domain. Thus we must determine $\lambda_c^*(z)$ numerically. To do so, we first differentiate (5.3) with respect to A to obtain the variational problem

$$v_A'' + \frac{1}{r}v_A' + \lambda^* F_u^{*0}\, v_A = -\lambda_A^*\, F^{*0} + \frac{(\log r - b^{-1})}{\log z}\,\lambda^* F_u^{*0}\,, \qquad 0 < r < 1\,, \tag{5.19a}$$

$$v_A' + b\,v_A = 0 \quad \text{on } r = 1\,, \tag{5.19b}$$

$$v_A = 1 - \frac{b^{-1}}{\log z} \quad \text{and} \quad v_A' = 0 \quad \text{at } r = 0\,. \tag{5.19c}$$

Here we have defined F_u^{*0}

$$F_u^{*0} \equiv F_u\left[v + A(b^{-1} - \log r)/\log z\right]\,. \tag{5.20}$$

For fixed A, z, we use the collocation package COLYSS [1] to solve the extended system (5.3), (5.19) for v, v_A, λ^*, and λ_A^*. Starting from $A \approx .05$ and choosing a small value of z, we continue in A until we detect the first sign change in λ_A^*. The critical value of A, denoted by $A_c(z)$, at which λ_A^* vanishes is found using Newton's method. After finding $A_c(z)$, the location of the first fold point is given by $\lambda_c^*(z) = \lambda^*(A_c(z), z)$.

Then, by varying z the curve, $\lambda_c^*(z)$ is obtained. Numerical results for $\lambda_c^*(z)$ for various activation energy parameters β and Biot numbers b are given in §8. The analytical results derived above when $\beta = 0$ provide a partial check on those results.

6. The numerical method. To confirm the asymptotic results for the perturbed linear and nonlinear eigenvalue problems, we compute the solutions to the full problems (2.1), (3.1), and (4.1) in some domains for various values of the parameters. We also calculate the numerical solutions to two nonstiff auxiliary problems: the related problem (3.9), which determines $\lambda^*(z)$ for the linear eigenvalue problem, and the problem for the constant $d(\kappa)$, given in (2.6).

We use finite-difference methods to compute solutions to the linear and nonlinear eigenvalue problems and the auxiliary problems. We use the method of composite overlapping grids [3] and the program CMPGRD [4] to create grids for the regions of interest. A composite overlapping grid consists of a set of component grids that cover a domain and overlap where they meet. Interpolation is performed at the overlapping boundaries to match the solution across different components. Overlapping grids provide a flexible way to create smooth grids for regions with complicated geometries.

The equations to be solved are discretized to fourth-order accuracy. For each component grid, there is a mapping from the unit square onto the subregion covered by the component grid. The equations are transformed to the coordinates of the unit square and discretized using standard central finite differences. The system of equations, including the interpolation equations, is solved with either a direct sparse matrix solver or an iterative sparse matrix solver. For the perturbed linear eigenvalue problems (2.1), (3.1), we use inverse iteration to compute the eigenvalues. For the perturbed nonlinear eigenvalue problem (4.1), we compute the solution curve as a function of λ using pseudo-arclength continuation. This method is also used to compute the branch $\lambda^*(z)$ as a function of z for the related problem (3.9) after first transforming it to (3.11).

The computational results presented are thought to be correct to the number of digits given (rounded). To estimate the error in a computation the problem is solved on a sequence of finer and finer grids. Usually, at least three grids are used. Only those digits that do not change as the grid is refined are assumed to be correct. As additional confirmation of the numerical results, we have been able to obtain several significant digits in the exact result for $\lambda(\epsilon)$ and $\lambda_c(\epsilon)$ in the case of an annular domain. Therefore, we believe that any difference between the asymptotic and numerical results in §§7 and 8 are not due to the discretization error arising when solving the full perturbed problems numerically. Instead these comparisons give a true indication of the behavior of the hybrid method in approximating the exact result for $\lambda(\epsilon)$ and $\lambda_c(\epsilon)$. It should be noted that the full problems become progressively harder to solve as $\epsilon \to 0$ since the solution varies more rapidly near the hole. In this limit, a very fine grid is required to resolve the solution. The related problems are relatively easy to solve since the solutions remain smooth.

6.1. Computation of the logarithmic capacitance $d(\kappa)$. For holes of arbitrary shape, we compute the logarithmic capacitance $d(\kappa)$, defined in (2.6), numerically. In (2.6), ∂_n is the inward normal derivative to D_1 and D_1 contains the origin $y = 0$. To remove the logarithmic behavior at infinity, it is convenient to introduce a new variable $\phi(y)$ by $\phi(y) = v(y) - \log|y|$. Then from (2.6), $\phi(y)$ satisfies

$$\Delta\phi = 0\,, \quad y \notin D_1\,; \quad \partial_n\phi + \kappa\,\phi = -\left(\partial_n + \kappa\right)\log|y|\,, \quad y \in \partial D_1\,,$$
$$\phi = -\log[d(\kappa)] + o(1) \quad \text{as } |y| \to \infty\,. \tag{6.1}$$

TABLE 1(a)

Logarithmic capacitance $d(\kappa)$ for the elliptical domain (6.4).

κ	$a = 1.0$	$a = 2.0$	$a = 2.5$
∞	1.000	1.250	1.450
1	0.368	0.562	0.724

For computational convenience, we restrict ourselves to the class of domains D_1 that are star-shaped, $D_1 = \{y : r = |y| \leq g(\theta)\}$. Here $g(\theta) > 0$ is a smooth 2π-periodic function and θ is the polar angle. Let D_1' be the domain that is the image of the region exterior to D_1 under the mapping $\rho = 1/r$, $D_1' = \{y : \rho = |y| \leq 1/g(\theta)\}$. Then with $w(\rho, \theta) = \phi(r, \theta)$, (6.1) becomes

$$\begin{aligned} &\Delta w = 0 \quad \text{in } 0 \leq \rho \leq 1/g(\theta)\,, \\ &w_\rho + g'\, w_\theta + \kappa\, g(g^2 + g'^2)^{1/2}\, w = g - \kappa\, g(g^2 + g'^2)^{1/2} \log(g) \quad \text{on } \rho = 1/g(\theta)\,. \end{aligned} \tag{6.2}$$

The logarithmic capacitance $d(\kappa)$ is then given by

$$d(\kappa) = \exp\left[-w(0, \theta)\right]. \tag{6.3}$$

Thus the original problem (2.6) has been transformed to a problem on the bounded domain D_1', which should be easier to solve numerically. From (6.2), (6.3), we note that $d(\kappa)$ is invariant under rotations of D_1. Note that the transformation $\rho = 1/r$ is most appropriate when the shape of the domain boundary is close to a circle. For elongated domains, (6.2) becomes difficult to solve since $\partial D_1'$ begins to develop cusps.

To compute $d(\kappa)$, we solve (6.2) numerically using overlapping grids and finite-difference methods. We consider two types of domains, an ellipse with semi-axes a, a^{-1},

$$g(\theta) = a\, \left[1 + (a^4 - 1)\, \sin^2(\theta)\right]^{-1/2}\,, \tag{6.4}$$

and a region with $2n$ "leaves"

$$g(\theta) = \sigma + \frac{4}{3\sqrt{2}} \left[(3 - \sigma^2)^{1/2} - \sqrt{2}\,\sigma\right] \sin^2(n\theta)\,. \tag{6.5}$$

Here $a > 0$, $0 < \sigma \leq 1$, and n is a positive integer. In both cases, the area of D_1 is π.

In Table 1(a) we give some numerical results for $d(\kappa)$ for the elliptical domains (6.4). Some numerical results for $d(\kappa)$ for the "leaf" type domains (6.5) are given in Table 1(b). There are some special cases for which $d(\kappa)$ is known analytically. When $\kappa = \infty$ and D_1 is an ellipse with semi-axes a and a^{-1}, then $d(\infty) = \frac{1}{2}(a + a^{-1})$. In addition, when D_1 is a circle of radius 1, then $d(\kappa) = \exp(-1/\kappa)$. As with the other numerical results, $d(\kappa)$ is computed using a sequence of finer and finer grids. The values for $d(\kappa)$ given are thought to be correct to the number of digits shown.

7. Linear eigenvalues: Comparison of hybrid method with numerical results. For each of the examples below, we let D be a circular cylindrical domain of radius 1 with $u = 0$ ($b = \infty$) on ∂D. We only consider the change in the lowest eigenvalue of (1.1) due to cutting out N identical holes, each centered at some x_0^i. Thus the

TABLE 1(b)

Logarithmic capacitance $d(\kappa)$, for the "leaf" domain (6.5) with $n = 2$ and $n = 3$.

κ	n	$\sigma = .25$	$\sigma = .75$
∞	2	1.267	1.068
∞	3	1.332	1.096
1	2	0.634	.4417
1	3	0.723	.4962

relevant solution to the unperturbed problem (1.1) is $u_0 = \pi^{-1/2} J_0(\sqrt{\lambda_0} r)/J_0'(\sqrt{\lambda_0})$, where $r = |x|$ and λ_0 is the first root of $J_0(\sqrt{\lambda_0}) = 0$. The two-term expansion for $\lambda(\epsilon)$ is

$$(7.1)\quad \lambda(\epsilon) = \lambda_0 + \left(-\frac{1}{\log\left[\epsilon d(\kappa)\right]}\right) \frac{2}{\left(J_0'(\sqrt{\lambda_0})\right)^2} \sum_{i=1}^{N} J_0^2(\sqrt{\lambda_0}|x_0^i|) + \cdots \quad \text{as} \quad \epsilon \to 0\,.$$

In the case of a single hole centered at the origin, a three-term expansion for $\lambda(\epsilon)$ was given in (2.8). For the examples below, we compare the results for $\lambda(\epsilon)$ obtained from the full problems (2.1) or (3.1), the related problem (3.9) obtained from the hybrid method, and either the two-term expansion (7.1) or the three-term expansion (2.8).

We now consider the case when a single hole, centered at the origin, is removed from D. For this geometry, the solution $\lambda^*\left[\epsilon d(\kappa)\right]$ to the related problem is obtained by solving (2.9) numerically. We first let D_1 be the elliptical domain (6.4) with $a = 2$. Then from Table 1(a) we find that $d(\infty) = 1.25$ and $d(1) \approx .562$. The results for $\lambda(\epsilon)$ obtained from the three-term expansion (2.8), from the related problem (2.9), and from the full problem (2.1) are shown in Table 2(a) for $\kappa = \infty$ and $\kappa = 1$. From this table, we observe that when $\kappa = \infty$, the hybrid method is within 2.5 percent of the numerical result even for $\epsilon = .15$. When $\kappa = 1$, the agreement between the hybrid method and the numerical result for $\lambda(\epsilon)$ is also very close for $\epsilon = .15$. Similar agreement between the hybrid method and the numerical results in the case of the "leaf" domain (6.5), with $n = 2$ and $\sigma = .75$, are shown in Table 2(b). Although the three-term expansion for $\lambda(\epsilon)$ given in these tables is in fair agreement with the numerical results, the hybrid method gives a significantly better determination of $\lambda(\epsilon)$ at a cost of solving the transcendental equation (2.9) numerically. In Fig. 2, we plot the curves $\lambda^*\left[\epsilon d(\kappa)\right]$ versus ϵ obtained from the hybrid method for both the elliptical and "leaf-type" domains. The numerical results for $\lambda(\epsilon)$, at selected values of ϵ, are also shown in this figure. In Fig. 3, we show the mesh used for the numerical solution to the full problem (2.1) for the "leaf" domain.

We now consider the case where an off-centered elliptical hole with $a = 2$ located at $(.25, 0)$ is removed from D. When $\kappa = \infty$, the results for $\lambda(\epsilon)$ obtained from the related problem (2.12), from the two-term expansion (7.1), and from the full problem (2.1), are shown in Table 3(a). Although the hybrid method does not agree as closely with the numerical results as in the case when the hole is centered at the origin, the hybrid method is still within 4.5 percent of the numerical result for $\epsilon = .10$. For this value of ϵ, the error of the two-term expansion is over 10 percent.

In Table 3(b), we give the values of $\lambda(\epsilon)$ obtained from the hybrid method (3.9), from the two-term expansion (7.1), and, from the full problem (3.1), when either two

TABLE 2(a)

Linear eigenvalue: $b = \infty$, *ellipse* $a = 2$, $d(\infty) = 1.250$, $d(1) = .562$, *ellipse centered at* $x_0 = (0,0)$.

κ	ϵ	$\lambda(\epsilon)$ numerical	$\lambda^*[\epsilon d(\kappa)]$	Three-term result (2.8)
∞	0.025	8.75	8.7497	8.5992
∞	0.050	9.785	9.7741	9.5140
∞	0.100	11.726	11.626	11.226
∞	0.150	13.698	13.363	13.109
1	0.025	8.07	8.054	7.9686
1	0.050	8.66	8.634	8.4959
1	0.100	9.60	9.578	9.3389
1	0.150	10.43	10.435	10.108

TABLE 2(b)

Linear eigenvalue: $b = \infty$, *leaf* $n = 2$, $\sigma = .75$, $d(\infty) = 1.068$, $d(1) = .4417$, *leaf centered at* $x_0 = (0,0)$

κ	ϵ	$\lambda(\epsilon)$ numerical	$\lambda^*[\epsilon d(\kappa)]$	Three-term result (2.8)
∞	0.025	8.58	8.583	8.4488
∞	0.050	9.50	9.490	9.2601
∞	0.100	11.20	11.098	10.721
∞	0.150	13.02	12.621	12.252
1	0.025	7.93	7.902	7.8292
1	0.050	8.46	8.403	8.2870
1	0.100	9.30	9.192	8.9947
1	0.150	10.08	9.890	9.6174

or five circular holes, with $\kappa = \infty$, are removed from D. For these examples, we have $d(\infty) = 1$. The precise locations of the holes are given in Table 3(b). From this table, we observe that the hybrid method is in close agreement with the numerical results for $\lambda(\epsilon)$ for both $N = 2$ and $N = 5$. In Fig. 4, we have plotted the curves $\lambda^*(\epsilon)$ versus ϵ for $N = 2$ and $N = 5$. The numerical results for $\lambda(\epsilon)$, at selected values of ϵ, are also shown in this figure. From Table 3(b), we note that even for the relatively small value $\epsilon = .025$, the two-term expansion is in very poor agreement with the numerical results. The reason for this poor agreement is clear from Fig. 5. It shows a contour plot and a surface plot of the eigenfunction obtained from the numerical solution to (3.1) when $N = 5$ and $\epsilon = .025$. This figure indicates that there is a significant interaction between the holes, which is not taken into account by (7.1). The composite mesh used to solve (3.1) is also shown in Fig. 5.

The comparisons above reveal several factors that determine the accuracy of the hybrid method in approximating $\lambda(\epsilon)$. One such factor is the location of the holes in the domain. In the case of a single hole, the accuracy is greatest when the hole is centered at a point where the gradient of the underlying unperturbed eigenfunction vanishes. As shown in [13], this is because the first term beyond the logarithmic expansion of order $O(\epsilon/\log[\epsilon d(\kappa)])$ is proportional to this gradient. Although this term is transcendentally small in comparison with λ^*, it can be quantitatively significant

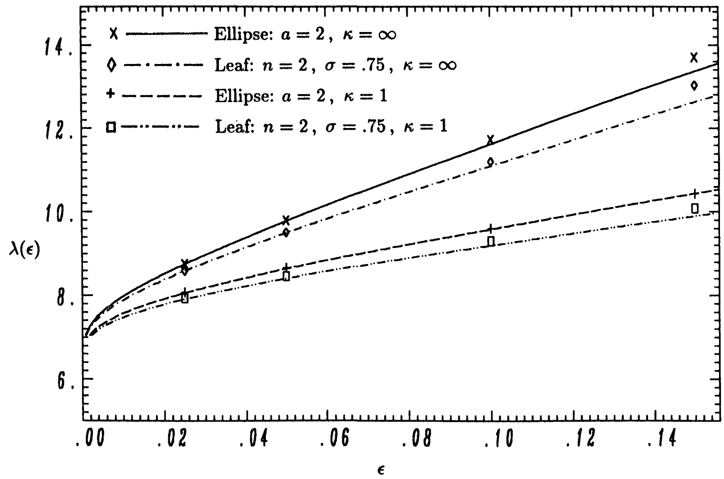

FIG. 2. *Linear eigenvalue problem: The first eigenvalue for a circular cylindrical domain containing an elliptical or "leaf"-shaped hole centered at the origin. The curves are the solutions* $\lambda^*[\epsilon d(\kappa)]$ *of the related problem and the labeled points are the results from the full problem.*

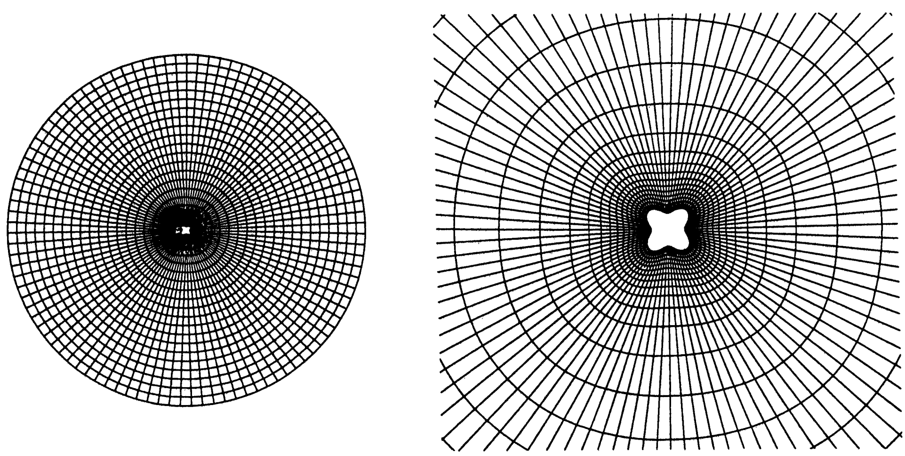

FIG. 3. *The composite mesh used for determining the first eigenvalue for a circular cylindrical domain containing a "leaf"-shaped hole centered at the origin.*

for moderate values of ϵ, especially for holes centered near points where the magnitude of the gradient of the unperturbed eigenfunction is large. The results in Tables 2(a) and 3(a) for the ellipse with $a = 2$ and $\kappa = \infty$ support this conclusion.

Another factor that determines the accuracy of the hybrid method in approximating $\lambda(\epsilon)$ is the shape of the hole. In the case of a single hole, the accuracy is greatest for a circular hole. This is because the first transcendentally small term in the expansion of $\lambda(\epsilon)$ vanishes, for any $\kappa > 0$, for a circular hole.

The results in Tables 2 and 3 also suggest that the accuracy of the hybrid method

TABLE 3(a)

Linear eigenvalue: $b = \infty$, *ellipse* $a = 2$, $d(\infty) = 1.25$, *ellipse centered at* $x_0 = (0.25, 0.0)$.

ϵ	$\lambda(\epsilon)$ numerical	$\lambda^*[\epsilon d(\infty)]$	Two-term result (7.1)
0.025	7.93	7.91	7.563
0.050	8.56	8.48	8.008
0.100	9.76	9.33	8.749

TABLE 3(b).

Linear eigenvalue: $b = \infty$, *circular holes of radius* ϵ *with* $d(\infty) = 1.0$. *Two holes*: *Centered at* $x_0^1 = (0, 0)$, $x_0^2 = (.50, 0.0)$; *five holes*: *Centered at* $x_0^1 = (0, 0)$, $x_0^i = (\pm.50, \pm.50)$, $i = 2, \ldots, 5$.

No. of holes	ϵ	$\lambda(\epsilon)$ numerical	$\lambda^*[\epsilon d(\infty)]$	Two-term result (7.1)
2	0.025	9.49	9.48	8.697
2	0.050	10.50	10.42	9.372
2	0.100	12.16	11.91	10.452
5	0.025	15.74	15.7	11.406
5	0.050	20.0	19.8	12.707
5	0.100	28.1	27.1	14.792

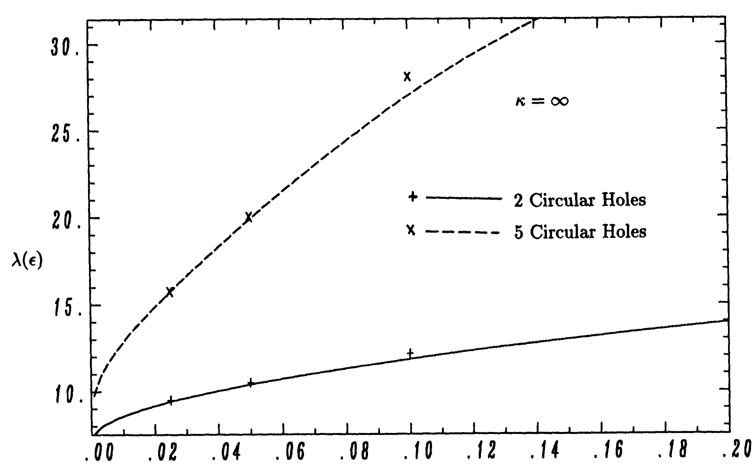

FIG. 4. *Linear eigenvalue problem*: *The first eigenvalue for a circular cylindrical domain containing either two or five circular holes with* $\kappa = \infty$. *The curves are the solutions* $\lambda^*(\epsilon)$ *of the related problem and the labeled points are the results from the full problem.*

should increase as $d(\kappa)$ decreases. Therefore, since $d(\kappa)$ is a decreasing function of κ for a fixed-domain shape D_1, the agreement between the hybrid method and the numerical results for $\lambda(\epsilon)$ should improve as κ decreases.

Finally, we recall from [5] that for all domains D_1 of the same cross-sectional area,

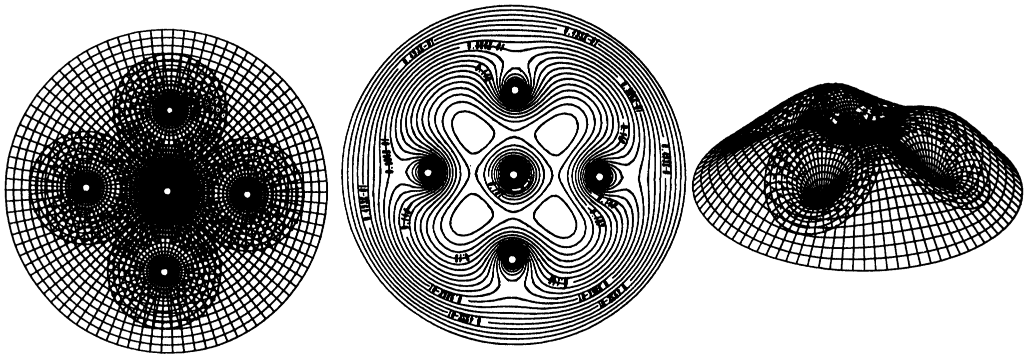

FIG. 5. *Contour and surface plot of the eigenfunction corresponding to the first eigenvalue for a circular cylindrical domain containing five circular holes. The composite mesh used to solve the full problem is also shown.*

which are centered at x_0, the logarithmic capacitance $d(\infty)$ is minimized for a circular domain. Therefore, since numerical evidence indicates that $\lambda^*(z)$ is an increasing function of z, the sum of the series λ^*, and hence the correction to λ_0, is minimized for a circular domain.

8. Nonlinear eigenvalues: Comparison of hybrid method with numerical results. Let D be a circular cylindrical domain of radius 1 containing a cooling rod of arbitrary cross section centered at the origin. We will focus on determining the location of the first fold point $\lambda_c(\epsilon)$ for (4.1) when $F(u) = \exp(u/(1+\beta u))$ (see Fig. 1).

When $\beta = 0$ and $b = \infty$, we will compare the results for $\lambda_c(\epsilon)$ obtained from the full problem (4.1), from the related problem (5.14), and from the three-term expansion (5.15). When $0 < \beta < \beta_c(\infty)$, the results for $\lambda_c(\epsilon)$ obtained from (4.1) will be compared against corresponding results obtained from the hybrid formulation (5.3), (5.19).

In Fig. 6 we plot the curves $\lambda_c^*(z)$ versus z, obtained from the related problem (5.11), for several values of b and for $\beta = 0$. By determining $z = \epsilon d(\kappa)$, we can use these universal curves to determine the location of the first fold point for various shapes and sizes of the cross section D_ϵ of the cooling rod. From the slope of these curves, we observe that reactors with a larger Biot number b on their outer boundary are stabilized more by the presence of a cooling rod than are reactors with a smaller Biot number.

When $\beta = 0$ and $b = \infty$, the solution $\lambda_c^*[\epsilon d(\kappa)]$ to the related problem is found from (5.13) and (5.14). We now compare the asymptotic and numerical results for $\lambda_c(\epsilon)$ for both elliptical and "leaf-type" domains. We first consider the "leaf" domain (6.5) with $n = 2$ and $\sigma = .75$, for which $d(\infty) \approx 1.068$ and $d(1) \approx .4417$. For the case where $\kappa = \infty$, in Table 4(a) we compare the results for $\lambda_c(\epsilon)$ obtained from the three-term expansion (5.15), from the related problem (5.14), and the full problem (4.1). Similar results are shown in Table 5(a) when $\kappa = 1$. From these tables, we

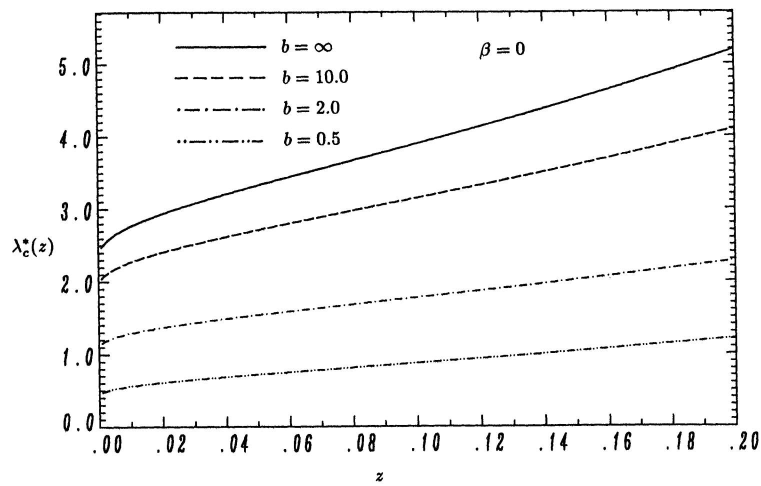

FIG. 6. *Nonlinear eigenvalue problem: The universal curves* $\lambda_c^*(z)$*, for various* b *with* $\beta = 0$*, for a circular cylindrical domain containing a hole of arbitrary shape centered at the origin.*

TABLE 4(a)

Nonlinear eigenvalue: $\beta = 0$, $b = \infty$, $\kappa = \infty$, *leaf* $n = 2$, $\sigma = .75$, $d(\infty) = 1.068$.

ϵ	$\lambda_c(\epsilon)$ numerical	$\lambda_c^*[\epsilon d(\infty)]$	Three-term result (5.15)
0.025	3.030	3.029	2.9908
0.050	3.360	3.361	3.2912
0.100	3.969	3.976	3.8313
0.150	4.620	4.638	4.3968

TABLE 4(b)

Nonlinear eigenvalue: $\beta = 0$, $b = \infty$, $\kappa = \infty$, *ellipse* $a = 2$, $a = 2.5$.

a	ϵ	$\lambda_c(\epsilon)$ numerical	$\lambda_c^*[\epsilon d(\infty)]$	Three-term result (5.15)
2	0.025	3.09	3.0902	3.0465
2	0.050	3.46	3.4663	3.3852
2	0.100	4.15	4.1924	4.0181
2	0.150	4.83	5.0138	4.7129
2.5	0.025	3.15	3.1542	3.1049
2.5	0.050	3.57	3.5808	3.4865
2.5	0.100	4.35	4.4406	4.2299
2.5	0.150	5.02	5.4664	5.0897

observe that the values for λ_c^* obtained from the related problem are within 1 percent of the numerical results for $\lambda_c(\epsilon)$ even when $\epsilon = .15$.

TABLE 5(a)

Nonlinear eigenvalue: $\beta = 0$, $b = \infty$, $\kappa = 1$, *leaf* $n = 2$, $\sigma = .75$, $d(1) = .4417$.

ϵ	$\lambda_c(\epsilon)$ numerical	$\lambda_c^*[\epsilon d(1)]$	Three-term result (5.15)
0.025	2.782	2.782	2.7611
0.050	2.965	2.964	2.9308
0.100	3.259	3.252	3.1930
0.150	3.532	3.509	3.4234

TABLE 5(b)

Nonlinear eigenvalue: $\beta = 0$, $b = \infty$, $\kappa = 1$, *ellipse* $a = 2.5$, $d(1) = .724$.

ϵ	$\lambda_c(\epsilon)$ numerical	$\lambda_c^*[\epsilon d(1)]$	Three-term result (5.15)
0.025	2.91	2.904	2.8751
0.050	3.16	3.154	3.1043
0.100	3.57	3.580	3.4855
0.150	3.87	3.997	3.8496

We now consider the elliptical domain (6.4) with either $a = 2$ or $a = 2.5$. The relevant values of $d(\kappa)$ are given in Table 1(a). For the case where $\kappa = \infty$, in Table 4(b) we compare the results for $\lambda_c(\epsilon)$ obtained from the three-term expansion (5.15), the hybrid method (5.14), and the full problem (4.1). Similar results are shown in Table 5(b) when $\kappa = 1$. From these tables, we now observe that the hybrid method agrees very well with the numerical results only for $\epsilon \leq .10$. The relatively poor performance of the hybrid method for the elliptical domain when $\epsilon = .15$ may be due to the fact that $\epsilon d(\kappa)$ is significantly larger than for the leaf domain. In fact, when $\epsilon = .15$, the three-term expansion is in closer agreement to the numerical results than is the hybrid method, although this is probably fortuitous. In Fig. 7, we plot the curves $\lambda_c^*[\epsilon d(\infty)]$ versus ϵ obtained from the hybrid method (5.14) for both the elliptical and "leaf-type" domains. The selected results for $\lambda_c(\epsilon)$ obtained from the numerical solution to (4.1) are also shown. Similar results for the case where $\kappa = 1$ are displayed in Fig. 8.

To illustrate the rather large effect on $\lambda_c(\epsilon)$ that can result from changing the shape of the cross section of the cooling rod, we now compare the curves $\lambda_c^*(z)$ versus z obtained from the hybrid method (5.14) for two classes of "leaf-type" domains. When $\kappa = 1$ and $\kappa = \infty$, these curves are shown in Fig. 9 for both $\sigma = .25$ and $\sigma = .75$. From this figure, we note that a significant delay in the onset of thermal runaway can result from taking $\sigma = .25$ rather than $\sigma = .75$. Although the area of D_1 for these two values of σ is the same, the domain with $\sigma = .25$ has more pronounced leaves and, consequently, is better able to remove heat than the domain with $\sigma = .75$. By comparing the values of $d(\kappa)$ in Table 1(b) at a fixed value of σ, we note that the onset of thermal runaway will be only marginally delayed by increasing the number of leaves by two.

When $\beta > 0$, the curves $\lambda_c^*(z)$ for the hybrid method (5.3), (5.19) are obtained using the numerical method given in §5. Using this method, in Fig. 10 we plot the

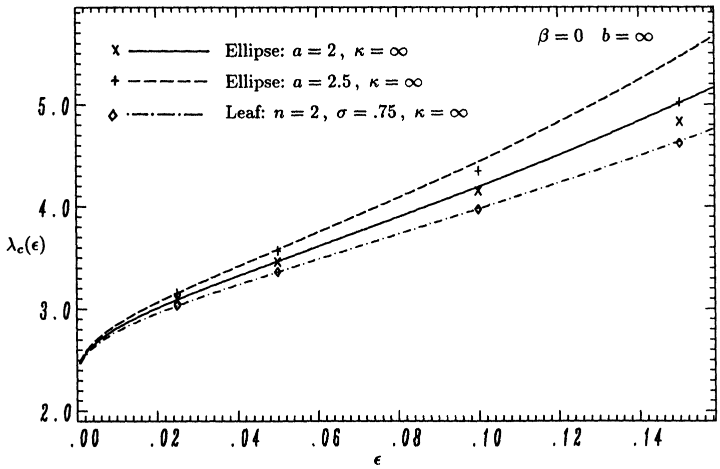

FIG. 7. *Nonlinear eigenvalue problem: The first fold point for a circular cylindrical reactor, with $\beta = 0$ and $b = \infty$, containing an elliptical or "leaf"-shaped hole, with $\kappa = \infty$, centered at the origin. The curves are the solutions $\lambda^*[\epsilon d(\infty)]$ of the related problem and the labeled points are the results from the full problem.*

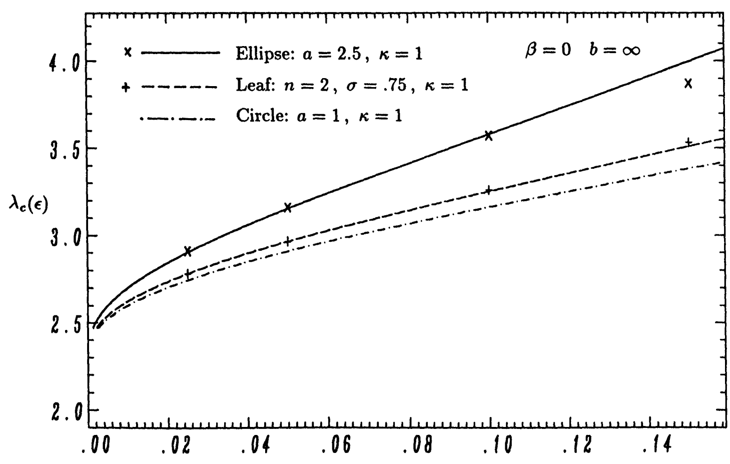

FIG. 8. *Nonlinear eigenvalue problem: The first fold point for a circular cylindrical reactor, with $\beta = 0$ and $b = \infty$, containing an elliptical or "leaf"-shaped hole, with $\kappa = 1$, centered at the origin. The curves are the solutions $\lambda^*[\epsilon d(\infty)]$ of the related problem and the labeled points are the results from the full problem.*

universal curves $\lambda_c^*(z)$, when $b = \infty$, for several values of β. In Table 6, for both elliptical and "leaf-type" domains, we compare, at $\beta = .15$, the predictions for $\lambda_c(\epsilon)$ obtained from the hybrid method against corresponding numerical results for $\lambda_c(\epsilon)$ obtained from the full problem (4.1). These results are also plotted in Fig. 11. We observe that the behavior of the hybrid method in determining the value of $\lambda_c(\epsilon)$ is

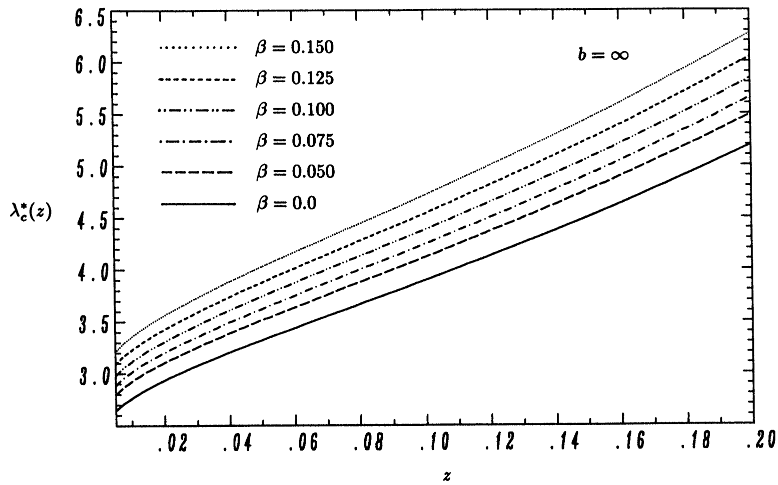

FIG. 9. *Nonlinear eigenvalue problem: The universal curves* $\lambda_c^*(z)$, *for* $b=\infty$ *and* $\beta=0$, *for two types of "leaf"-shaped domains centered at the origin of a circular cylindrical reactor.*

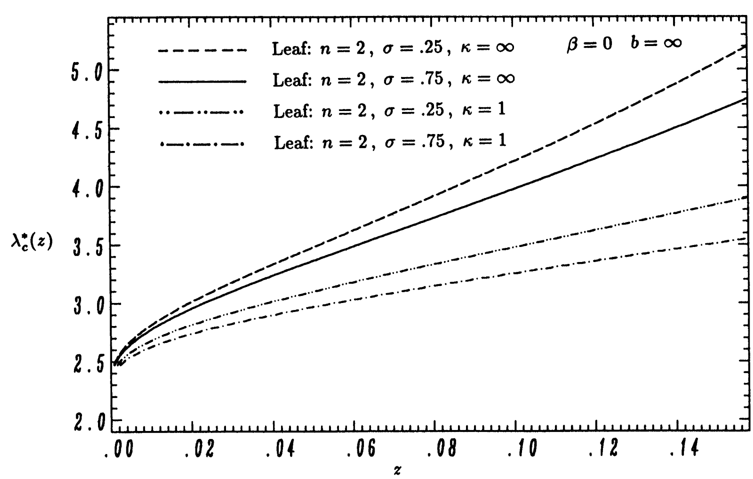

FIG. 10. *Nonlinear eigenvalue problem: The universal curves* $\lambda_c^*(z)$, *for various* β *with* $b=\infty$, *for a circular cylindrical domain containing a hole of arbitrary shape centered at the origin.*

very similar to that in the case $\beta = 0$. We note that, although the hybrid result $\lambda_c^*(z)$ can only be obtained numerically, it is much easier to solve the system of boundary value problems (5.3) and (5.19), than it is to determine $\lambda_c(\epsilon)$ from the numerical solution to the full problem (4.1).

9. Absorption time distribution for Brownian motion in a domain with small holes. As an application of the preceding results, we now consider the Brownian motion of a particle in a two-dimensional domain D, with reflecting walls, containing N small absorbing obstacles D_ϵ^i, each centered at some $x_0^i \in D$. We have

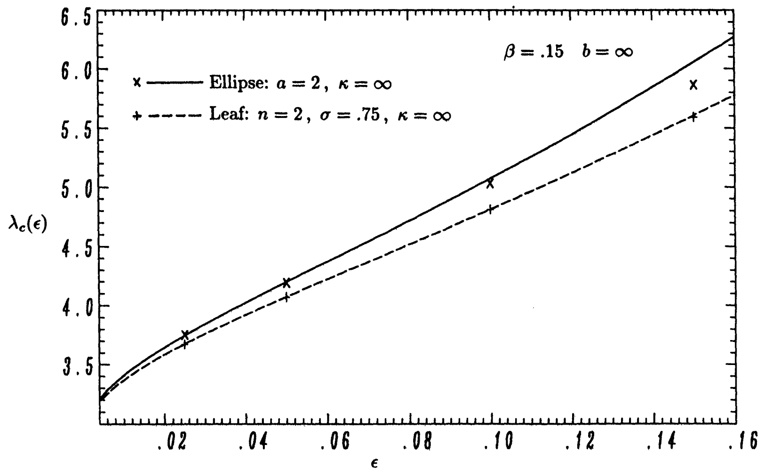

FIG. 11. *Nonlinear eigenvalue problem: The first fold point for a circular cylindrical reactor, with $\beta = .15$ and $b = \infty$, containing an elliptical or "leaf"-shaped hole, with $\kappa = \infty$, centered at the origin. The curves are the solutions $\lambda^*[\epsilon d(\infty)]$ of the related problem and the labeled points are the results from the full problem.*

TABLE 6

Nonlinear eigenvalue: $\beta = .15$, $b = \infty$, $\kappa = \infty$, ellipse $a = 2$; leaf $n = 2$, $\sigma = .75$.

ϵ	$\lambda_c^*[\epsilon d(\infty)]$ ellipse	$\lambda_c(\epsilon)$ ellipse	$\lambda_c^*[\epsilon d(\infty)]$ leaf	$\lambda_c(\epsilon)$ leaf
0.025	3.7459	3.75	3.673	3.67
0.050	4.1987	4.19	4.072	4.07
0.100	5.0710	5.03	4.811	4.81
0.150	6.0531	5.86	5.604	5.59

considered a similar problem before in [10], with absorbing patches on the wall, using only the first term in the expansion of $\lambda(\epsilon)$.

Suppose that the particle starts from some point $\xi \in D \backslash \cup_{i=1}^{N} D_\epsilon^i$ at time zero. Then the probability density $p(x, \xi, t, \epsilon)$ that the particle is at x at time t satisfies

$$p_t = \nu \triangle_x p, \qquad x \in D \backslash \cup_{i=1}^{N} D_\epsilon^i, \tag{9.1a}$$

$$\partial_n p = 0, \quad x \in \partial D; \quad \epsilon \partial_n p + \kappa p = 0, \quad x \in \partial D_\epsilon^i, \quad i = 1, \ldots, N, \tag{9.1b}$$

$$p = \delta(x - \xi), \qquad t = 0. \tag{9.1c}$$

Here ∂_n is the outward normal derivative, while ν and κ are positive constants with κ independent of i. We assume that $D_\epsilon^i = \epsilon D^i$, where for each i the scaled domain D^i represents a rotation of a certain domain D_1.

The solution to (9.1) is

$$p(x, \xi, t, \epsilon) = \sum_{m=1}^{\infty} \exp\left[-\lambda_m(\epsilon)\, \nu\, t\right] u_m(x, \epsilon)\, u_m(\xi, \epsilon). \tag{9.2}$$

Here λ_m is the mth eigenvalue and u_m the corresponding normalized eigenfunction of (3.1). Let A_ϵ denote the area of $D \backslash \cup_{i=1}^{N} D_\epsilon^i$. We now assume that the initial point ξ is uniformly distributed over $D \backslash \cup_{i=1}^{N} D_\epsilon^i$. Then we multiply (9.2) by A_ϵ^{-1} and integrate it with respect to ξ and x to obtain the following expression for the probability $P_0(t,\epsilon)$ that the particle is in $D \backslash \cup_{i=1}^{N} D_\epsilon^i$ at time t:

$$P_0(t,\epsilon) = A_\epsilon^{-1} \sum_{m=1}^{\infty} \exp\left(-\lambda_m(\epsilon)\, \nu\, t\right) \left(\int_{D \backslash \cup_{i=1}^{N} D_\epsilon^i} u_m(x,\epsilon)\, dx \right)^2 . \tag{9.3}$$

We denote by λ_{m0} and u_{m0} the mth eigenvalue and normalized eigenfunction of the unperturbed problem (1.1) with $b = 0$. The first eigenvalue of this problem is $\lambda_{10} = 0$ and the corresponding normalized eigenfunction is $u_{10}(x) = A_0^{-1/2}$. Since the perturbed eigenfunction satisfies $u_m(x,\epsilon) = u_{m0}(x) + O(-1/\log[\epsilon d(\kappa)])$ for $|x - x_0^i| \gg O(\epsilon)$, we have by orthogonality that

$$\int_{D \backslash \cup_{i=1}^{N} D_\epsilon^i} u_m(x,\epsilon)\, dx = A_0^{1/2} \delta_{m1} + O(-1/\log[\epsilon d(\kappa)])\,. \tag{9.4}$$

Substituting (9.4) into (9.3), we obtain

$$P_0(t,\epsilon) = \exp\left(-\lambda_1(\epsilon)\, \nu\, t\right)\, \left(1 + O(-1/\log[\epsilon d(\kappa)])\right)\,. \tag{9.5}$$

To obtain $P_0(t,\epsilon)$ for $\epsilon \ll 1$, we must determine $\lambda_1(\epsilon)$ for $\epsilon \ll 1$. The leading-order expansion for $\lambda_1(\epsilon)$, obtained from (3.3), is

$$\lambda_1(\epsilon) \sim \frac{2\pi N}{A_0} \left(-1/\log[\epsilon d(\kappa)]\right) \quad \text{as} \quad \epsilon \to 0\,. \tag{9.6}$$

With this approximation for $\lambda_1(\epsilon)$, we find that $P_0(t,\epsilon)$ depends only on the area A_0 of D and is independent of the location of the absorbing obstacles within D. To obtain a better estimate of the decay rate $\lambda_1(\epsilon)$, which accounts for all the logarithmic corrections to $\lambda_{10} = 0$, we use the hybrid method of §3. Let $\lambda^*(z)$ be the solution to (3.9), with $b = 0$, for which $\lambda^*(z) \to 0$ as $z \to 0$. Then, from (3.12) we have

$$\lambda_1(\epsilon) = \lambda^*[\epsilon d(\kappa)] + O(\epsilon / \log[\epsilon d(\kappa)])\,, \tag{9.7}$$

and thus (9.5) becomes

$$P_0(t,\epsilon) = \exp\left(-\lambda^*[\epsilon d(\kappa)]\, \nu\, t\right)\, \left[1 + O(-1/\log[\epsilon d(\kappa)])\right]\,. \tag{9.8}$$

We now compute $\lambda^*[\epsilon d(\kappa)]$ from the hybrid method (3.9) for the four examples described below.

First, we let D be a circle of radius unity containing four small obstacles centered at $(\pm j/4, \pm j/4)$. For the first three examples, we will choose j to be either 1, 2, or 3. For the last example, we let D be the square $0 \le x_1 \le \sqrt{\pi}$, $0 \le x_2 \le \sqrt{\pi}$ containing four small obstacles located at $(\sqrt{\pi}/4, \sqrt{\pi}/4)$, $(\sqrt{\pi}/4, 3\sqrt{\pi}/4)$, $(3\sqrt{\pi}/4, \sqrt{\pi}/4)$, and $(3\sqrt{\pi}/4, 3\sqrt{\pi}/4)$. Since $A_0 = \pi$ and $N = 4$, the leading-order expansion (9.6) yields the same absorption time distribution for these four examples. Although we can treat

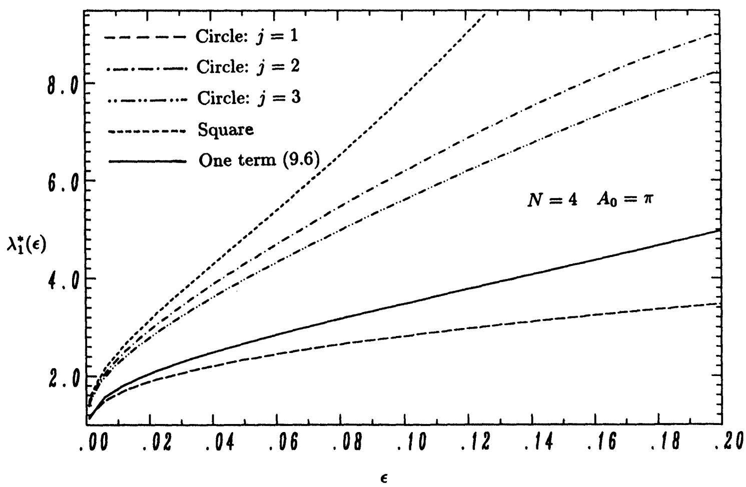

FIG. 12. *Absorption time distribution problem*: *The curves* $\lambda_1^*(\epsilon)$ *versus* ϵ *are the solutions to the related problem approximating the first eigenvalue for a circular or square domain containing four circular perfectly absorbing obstacles.*

holes of arbitrary shape, for illustration purposes we take $\kappa = \infty$ and we choose each D^i to be a circle of radius 1 so that $d = 1$.

In Fig. 12 we plot the results for $\lambda_1^*(\epsilon)$, obtained from the hybrid method, for each of the four examples. In this figure, we have also plotted the leading-order result (9.6). By comparing $\lambda_1^*(\epsilon)$ for the three examples pertaining to the circular domain, we conclude that $P_0(t,\epsilon)$ depends significantly on the location of the absorbing obstacles within D when ϵ is only moderately small. The results for $\lambda_1^*(\epsilon)$ for $j = 1$, $j = 2$, and $j = 3$ indicate that the decay rate $\lambda_1^*(\epsilon)$ is small when the obstacles are clustered near the origin ($j = 1$) or are near the outer boundary ($j = 3$). This conclusion is certainly reasonable on physical grounds in that when the obstacles are clustered together or are near the outer boundary, the obstacles will have a smaller effective exposed surface with which to trap a wandering particle. Thus as a function of j, for $0 < j < 4$, we expect that $\lambda_1^*(\epsilon)$ is concave and is maximized when the particles are roughly midway between the origin and the outer boundary. In addition, by comparing the results for $\lambda_1^*(\epsilon)$ for the circular domain with those for the square domain, we observe that $P_0(t,\epsilon)$ depends not only on the area A_0 of D but also depends significantly on the shape of the domain D. These examples show that when ϵ is only moderately small, $P_0(t,\epsilon)$ is influenced significantly by the shape of D and the location of the absorbing obstacles within D. The hybrid method for computing the decay rate $\lambda_1(\epsilon)$ has been able to retain these two qualitative features for the absorption time distribution $P_0(t,\epsilon)$ which is lacking when (9.6) is used. We remark that, although we have not compared the results from the hybrid method with corresponding numerical results from the full problem, we anticipate agreement like that for the examples in §7.

Although we have only determined the absorption time distribution in the case where there are four small obstacles in D, with only a modest increase in computational effort, we can determine this distribution when a moderately large number of absorbing obstacles are in D. We also note that by using the solution u^* to (3.9), the result (9.8) can be improved. Instead of using the leading-order result (9.4) in (9.3), $P_0(t,\epsilon)$

can be determined to within terms of $O(\epsilon)$ by substituting $u_m(x,\epsilon) = u_m^*(x;z) + O(\epsilon/\log[\epsilon d(\kappa)])$, for $|x - x_0^i| \gg O(\epsilon)$, directly in (9.3). Here $u_m^*(x;z)$, $\lambda_m^*(z)$ is the solution to the related problem (3.9) for which $\lambda_m^*(z) \to \lambda_{m0}$ as $z \to 0$, and u_m^* is normalized by $\int_D (u_m^*)^2\, dx = 1$.

Appendix A. The infinite logarithmic expansion for the fold point. In the outer region, away from the cooling rod, $u(x,\alpha,\epsilon)$ is expanded as

$$u(x,\alpha,\epsilon) = u_0(x,\alpha) + \sum_{j=1}^{\infty} \nu^j(\epsilon)\, u_j(x,\alpha) + \cdots\,. \tag{A.1}$$

Substituting (A.1) and (4.2) into (4.1a), (4.1b), (4.1d), and collecting terms of order $\nu^j(\epsilon)$, we find that u_j for $j \geq 1$ satisfies

$$\triangle u_j + \lambda_0\, F_u^0\, u_j = -\lambda_j\, F^0 + R_j\,, \quad x \neq x_0\,, \quad x \in D\,, \tag{A.2a}$$

$$\partial_n u_j + b\, u_j = 0\,, \quad x \in \partial D\,, \quad \sum_{i=0}^{j} \int_D u_i\, u_{j-i}\, dx = 0\,. \tag{A.2b}$$

Here $F^0 \equiv F\,(u_0(x,\alpha))$, $F_u^0 \equiv F_u\,(u_0(x,\alpha))$, and the forcing term R_j depends on the solutions u_k, λ_k for $0 \leq k \leq j-1$ found at earlier stages. The as yet unknown behavior of u_j as $x \to x_0$ is found below.

The inner expansion is found to have the following form analogous to (2.4):

$$u(x,\alpha,\epsilon) = u_0(x_0,\alpha)\, v\,[\epsilon^{-1}(x - x_0)] \sum_{j=0}^{\infty} \left(-\frac{1}{\log[\epsilon d(\kappa)]} \right)^{j+1} c_j(\alpha) + \cdots\,, \quad |x - x_0| = O(\epsilon)\,. \tag{A.3}$$

Here $v(y)$ is the solution of (2.6). Matching (A.3) and (A.1) we find that $\nu(\epsilon) = (-1/\log[\epsilon d(\kappa)])$. In addition, we find that the constants $c_j(\alpha)$, with $c_0(\alpha) = 1$, are found recursively from the unique solution to (A.2) with singular behavior

$$u_j(x,\alpha) \sim u_0(x_0,\alpha)\ [c_{j-1}(\alpha)\, \log|x - x_0| + c_j(\alpha)] \quad \text{as } x \to x_0 \quad \text{for} \quad j = 1, 2, \ldots\,. \tag{A.4}$$

With $c_{j-1}(\alpha)$ known, the problem (A.2), (A.4) can be solved uniquely for $\lambda_j(\alpha)$ and $u_j(x,\alpha)$. In terms of this solution, $c_j(\alpha)$ can be found from (A.4). Then $c_j(\alpha)$ determines the strength of the logarithmic singularity for u_{j+1}.

At the simple fold point α_0, $\lambda_0(\alpha_0)$, the inhomogeneous terms in (A.2) must satisfy one solvability condition, and $\lambda_j(\alpha_0)$ decouples from u_j. To derive this condition, we differentiate (1.2) with respect to α to obtain an equation for $u_{0\alpha}$. Then applying Green's Theorem to $u_{0\alpha}$ and (A.2), and using the singularity behavior (A.4), we derive

$$\lambda_j(\alpha) = \left[(R_j, u_{0\alpha}) + \lambda_0'\, (F^0, u_j) + 2\pi\, c_{j-1}(\alpha)\, u_0(x_0,\alpha)\, u_{0\alpha}(x_0,\alpha) \right] / \,(F^0, u_{0\alpha})\,. \tag{A.5}$$

Evaluating (A.5) at $\alpha = \alpha_0$ determines $\lambda_j(\alpha_0)$ as

$$\lambda_j(\alpha_0) = [(R_j, u_{0\alpha}) + 2\pi\, c_{j-1}(\alpha_0)\, u_0(x_0,\alpha_0)\, u_{0\alpha}(x_0,\alpha_0)] \,/\, (F^0, u_{0\alpha})\,. \tag{A.6}$$

To obtain the derivatives $\lambda_j^k(\alpha_0)$ needed to determine the fold point corrections in (4.3), we can, in principle, differentiate (A.5) k times and then evaluate the resulting expressions at $\alpha = \alpha_0$. By carrying out this procedure, it follows that the fold point corrections λ_{ci} in the infinite logarithmic series (4.3) will be independent of ϵ, of κ, and of the shape of the hole.

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