Explicitly solvable nonlocal eigenvalue problems and the stability of localized stripes in reaction-diffusion systems

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The transverse stability of localized stripe patterns for certain singularly perturbed two-component reaction-diffusion (RD) systems in the asymptotic limit of a large diffusivity ratio is analyzed. In this semi-strong interaction regime, the crosssectional profile of the stripe is well-approximated by a homoclinic pulse solution of the corresponding 1-D problem. The linear instability of such homoclinic stripes to transverse perturbations is well-known from numerical simulations to be a key mechanism for the creation of localized spot patterns. However, in general, owing to the difficulty in analyzing the associated nonlocal and non self-adjoint spectral problem governing stripe stability for these systems, it has not previously been possible to provide an explicit analytical characterization of these instabilities, including determining the growth rate and the most unstable mode within the band of unstable transverse wavenumbers. Our focus is to show that such an explicit characterization of the transverse instability of a homoclinic stripe is possible for a subclass of RD system for which the analysis of the underlying spectral problem reduces to the study of a rather simple algebraic equation in the eigenvalue parameter. Although our simplified theory for stripe stability can be applied to a class of RD system, it is illustrated only for homoclinic stripe and ring solutions for a subclass of the Gierer-Meinhardt model and for a three-component RD system modeling patterns of criminal activity in urban crime.

Key words: Matched asymptotic expansions, nonlocal eigenvalue problem, stripe, instability band, crime hot spots.

1 Introduction

For certain two-component singularly perturbed reaction-diffusion (RD) systems in 2-D spatial domains, various types of spatially localized patterns consisting of either spots, stripes, mixed spot-stripe patterns, or space-filling curves, have been observed and studied both numerically and analytically. In particular, the Gierer–Meinhardt (GM) activator-inhibitor system modeling biological morphogenesis admits a wide range of spot and stripe patterns (cf. [11], [16], [25]). A more intricate set of spatial-temporal localized patterns, such as self-replicating spots, oscillating spots, and labyrinthine stripe patterns occur for the Gray-Scott (GS) model of theoretical chemistry (cf. [33], [30], [31], [6]). Localized stripe patterns have also been studied in other diverse settings including, a hybrid chemotaxis RD system modeling fish skin patterns on growing domains (cf. [21], [32]), the Swift-Hohenberg model (cf. [12]), a generalized Schnakenberg RD system modeling root hair initiation in the epidermal cells of plants (cf. [4]), and an RD system modeling urban crime (cf. [47]).

For a two-component RD system, a homoclinic stripe occurs when either one or both of the two solution components becomes localized, or concentrates, on a planar curve in the 2-D domain. We shall consider the so-called semi-strong interaction regime that arises when the ratio of the two diffusivities is asymptotically large, so that only one of the two solution components (the fast component) is localized to form a stripe. The cross-sectional profile of the stripe is then closely approximated by a homoclinic pulse solution of the corresponding 1-D fast subsystem. The two simplest types of homoclinic stripe solutions are a stripe of zero curvature, which results when a 1-D homoclinic pulse solution is trivially extended along the mid-line of a rectangular domain, and a homoclinic ring solution, which occurs when a pulse is concentrated on a circular ring that lies concentrically within a disk. The main goal of this paper is to show that, for

a certain sub-class of reaction kinetics in the RD system, it is possible to readily analyze the linear stability of these two simplest types of homoclinic stripe solutions to 2-D transverse perturbations.

In the simpler context of a one-dimensional spatial domain, there has been much work over the past decade in analyzing the existence, stability, and dynamics of steady-state and quasi steady-state state pulse solutions to various RD systems in the semi-strong interaction regime, such as the GM, GS, Schnakenberg, and Brusselator, models (see [5], [7], [8], [13], [14], [17], [27], [30], [34], [39], [40], [42], [43], [44] and the references therein).

There have been relatively fewer studies on the existence and stability of homoclinic stripe and ring solutions in 2-D spatial domains for two component RD systems in the semi-strong interaction regime. The conditions for the existence of steady-state homoclinic ring solutions for the GM model in N-D radially symmetric domains with $N \ge 2$ was investigated in [29]. However, no stability analysis of such solutions was given. In [9] and [19] the stability of a homoclinic stripe solution of zero curvature to transverse perturbations was analyzed for the GM model in a rectangular domain. A similar stability analysis of homoclinic stripe and ring solutions for the GS model was given in [26] and [20]. More recently, in [4] the transverse stability of a homoclinic stripe solution in a rectangular domain for a generalized Schnakenberg-type system was investigated, with applications to root hair formation in plants.

In many semi-strong RD systems, the spatial profile of a pulse is a C^2 smooth homoclinic solution w(y) satisfying

(1.1 a)
$$w'' - w + f(w) = 0, \quad -\infty < y < \infty; \quad w \to 0 \text{ as } |y| \to \infty, \quad w'(0) = 0, \quad w(0) > 0,$$

where f(w) is assumed to satisfy

(1.1 b)
$$f(w)$$
 is C^1 for $w \ge 0$, $f(0) = 0$, $f'(0) < 1$

Upon defining $Q(w) \equiv -w + f(w)$, we assume that Q(w) has the following properties:

(1.1 c)

$$Q(0) = 0, \quad Q'(0) < 0; \quad Q(s) = 0, \quad Q'(s) > 0, \quad \text{for } s > 0, \qquad Q(w) < 0 \quad \text{for } 0 < w < s,$$

$$Q(w) > 0 \quad \text{for } s < w \le w_m, \qquad \int_{0}^{w_m} Q(\eta) \, d\eta = 0.$$

Under these conditions, Theorem 5 of [3] guarantees the existence of a unique homoclinic solution to (1.1 *a*). We remark that the condition w'(0) = 0 in (1.1 *a*) eliminates the translation invariance and ensures that w(y) is an even function. Since f'(0) < 1, we have that $w \sim ae^{-b|y|}$ as $y \to \pm \infty$, where $b \equiv \sqrt{1 - f'(0)}$, for some constant a > 0.

In particular, (1.1 b) and (1.1 c) hold when $f(w) = w^p$ where p > 1. For this nonlinearity the homoclinic is given explicitly by

(1.2)
$$w(y) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \left(\operatorname{sech}\left[\frac{(p-1)y}{2}\right]\right)^{2/(p-1)}$$

We remark that homoclinic solutions can exist for (1.1 *a*) under slightly milder conditions on f(w). In particular, for the choice $f(w) = w \log w$, f is C^1 for w > 0, the conditions in (1.1 *c*) still hold, but $f'(w) \to -\infty$ as $w \to 0^+$. For this choice, it is readily shown that there is a homoclinic solution to (1.1 *a*) given explicitly by $w = e^{3/2}e^{-y^2/4}$, which has a faster decay as $|y| \to \infty$ than does (1.2).

The main technical challenge in the stability analysis of either localized pulses or homoclinic stripes for RD systems in the semi-strong interaction regime is that one must analyze the spectrum of a class of nonlocal eigenvalue problem (NLEP) for a C^2 eigenfunction $\Phi(y)$ of the form

(1.3 a)
$$L_0 \Phi - \chi(\lambda) h(w) \int_{-\infty}^{\infty} g(w) \Phi \, dy = \lambda \Phi, \quad -\infty < y < \infty; \qquad \Phi \to 0 \quad \text{as} \quad |y| \to \infty,$$

where w is the homoclinic satisfying (1.1). Here $\chi(\lambda)$, depending on the eigenvalue parameter, is assumed to be analytic in $\operatorname{Re}(\lambda) \geq 0$, and L_0 is the linearized operator defined by

(1.3 b)
$$L_0 \Phi \equiv \Phi'' - \Phi + f'(w)\Phi$$

In (1.3 a), we assume that g(w) and h(w) are C^2 smooth on w > 0, and that they satisfy

$$g(0) = 0$$
, $g(w) > 0$ for $w > 0$, $g(w) = \mathcal{O}(w^{\alpha_1})$, as $w \to 0^+$; $h(0) = 0$, $h(w) = \mathcal{O}(w^{\alpha_2})$, as $w \to 0^+$,

for some $\alpha_1 > 0$ and $\alpha_2 > 0$. Since the NLEP (1.3 *a*) is non-self-adjoint and non-local, it is difficult to find sufficient conditions that guarantee that all discrete eigenvalues of (1.3 *a*) satisfy $\operatorname{Re}(\lambda) \leq 0$. For simple power nonlinearities where $f(w) = w^p$ with p > 1, $h(w) = w^m$ with m > 1, and $g(w) = w^q$ with $q \geq 2$, there are many rigorous results for the spectrum of (1.3 *a*) for some range of the exponents *p*, *m*, and *q* (see the survey [45]). However, the theory is intricate and still incomplete.

More recently, in Lemma 2.4 of [28], it was shown that if $f(w) = w^p$ and $g(w) = w^{r-1}$ with p = 2r - 3 and r > 2, then any unstable eigenvalue in $\operatorname{Re}(\lambda) > 0$ for (1.3 *a*) must be a root of the equation

(1.5)
$$\lambda = (r^2 - 2r) - \chi(\lambda) \int_{-\infty}^{\infty} h(w) w^{r-1} dy$$

Since h(w) is smooth for w > 0, with $h = O(w^{\alpha_2})$ as $w \to 0$ for some $\alpha_2 > 0$, the exponential decay of w as $|y| \to \infty$ guarantees that the integral in (1.5) is finite. We refer to such NLEP problems as "explicitly solvable". In §2 we first extend the analysis of [28] by deriving a condition between f(w) and g(w) in (1.3) for which the problem of determining any unstable discrete eigenvalues of (1.3 a) is reduced to that of determining the roots to a similar simple and explicit function of the eigenvalue parameter.

In [28], the observation (1.5) was an essential element for providing a comprehensive theory for the stability of a one-pulse solution for a class of 1-D RD system of the form

(1.6)
$$v_t = \varepsilon^2 v_{xx} - v + a(u)v^{2r-3}, \qquad \tau u_t = u_{xx} + (u_b - u) + \frac{1}{\varepsilon}b(u)v^r,$$

for r > 2 on the infinite line. Here u_b is a constant, and the functions a(u) and b(u) satisfy certain mild conditions.

The specific goal of this paper is to extend the 1-D pulse stability analysis of [28] to study the transverse stability of homoclinic stripes and rings for RD systems in the semi-strong interaction regime for which the associated NLEP problem is explicitly solvable. Instead of considering stripe and ring solutions for the general system (1.6) in a 2-D context, for simplicity we will only illustrate our simplified theory for analyzing the transverse stability of a stripe in the context of a subclass of the generalized GM model for which we set r = 3, $a(u) = u^{-q}$, $b(u) = u^{-s}$, $u_b = 0$, with q > 0, $s \ge 0$ and 3q/2 - (s+1) > 0 in (1.6). We remark that most of the previous rigorous results of [9] and [19] for stripe stability do not apply to the range of exponents of the nonlinear terms for our subclass of the generalized GM model.

For this subclass of the generalized GM model, we shown that the associated NLEP characterizing the transverse stability of homoclinic stripe and ring solutions is explicitly solvable in the sense that any unstable eigenvalue of the NLEP satisfies a rather simple and explicit transcendental equation. In this way, in §3 we readily analyze the transverse stability of a homoclinic stripe solution centered along the mid-line of a rectangular domain. In particular, in terms of the wavenumber m of the transverse mode, we show that for all $\tau \ge 0$ there is a band $m_- < m < m_+$ of unstable modes where $m_- = \mathcal{O}(1)$ and $m_+ = \mathcal{O}(\varepsilon^{-1})$ as $\varepsilon \to 0$. For $\tau \ll 1$, both the growth rate and the most unstable mode within this band are identified analytically. Moreover, we characterize precisely the Hopf bifurcation value of τ that occurs for transverse

wavemodes on the range $0 < m < m_{-}$. Such a precise and explicit characterization of transverse instabilities of a stripe solution was not available for the particular GM models studied in [9] and [19] owing to the difficulty in analyzing the underlying NLEP. Similar explicit results for the transverse stability of homoclinic ring solutions for our subclass of the GM model are given in §4. The importance of characterizing transverse instabilities is that these instabilities are known to lead to the disintegration, or breakup, of the homoclinic stripe or ring into localized spots (cf. [9], [19]).

In §5 we extend our simplified theory for stripe stability to the two-component RD system of [37] modeling spatialtemporal patterns of residential burglary. This chemotactic-type system characterizes the evolution of both the "attractiveness" of the environment for burglary and the density of criminals, where the criminals are assumed to undergo a biased random walk, or drift, towards regions of higher attractiveness. For details of the development of this model see [36] and [38]. Previous analytical and numerical studies of this model include a weakly nonlinear analysis of Turing-type patterns in [37], an analysis of the stability of localized hot-spot patterns in [18], and a study of homoclinic snaking behavior in [24]. In §5 we extend this previous work by analyzing the transverse stability of a localized stripe of criminal activity in a 2-D rectangular domain. For our stripe stability analysis we consider the more general 3-component RD system of [35] that models the additional effect of police deployment, and that reduces to the model of [37] in the absence of police. For this problem, we show that the underlying NLEP is explicitly solvable in certain cases. In this way, we are able to explicitly identify a band of unstable transverse wavenumbers, and calculate both the growth rate and most unstable mode within this band. This instability is shown to lead to the breakup of the stripe into a localized hot-spot of criminal activity. Finally, a brief discussion and some open problems are given in §6.

2 Explicitly Solvable Nonlocal Eigenvalue Problems

In this section we introduce a class of problems for which the discrete spectrum of the associated NLEP reduces to the study of a simple algebraic equation for the eigenvalue parameter. We begin by recalling a rigorous result, established in Theorem 5.4 of [2], for the spectrum of the local operator L_0 in (1.3 b) associated with the homoclinic satisfying (1.1).

Lemma 2.1 Assume that f(w) in (1.1 a) satisfies (1.1 b) and (1.1 c), so that (1.1 a) has a homoclinic w. Consider the local eigenvalue problem $L_0\psi = \nu\psi$ on \mathbb{R} for $\psi \in \mathcal{H}^1(\mathbb{R})$. This problem admits the eigenvalues $\nu_0 > 0$ and $\nu_1 = 0$. The eigenvalue ν_0 is simple, and the corresponding eigenfunction ψ_0 has one sign. When $\nu_1 = 0$ we have $\psi_1 = w'$. The continuous spectrum is the portion $Re(\lambda) \leq -1 + f'(0) < 0$ with $Im(\lambda) = 0$ of the negative real axis.

This result, proved in Theorem 5.4 of [2], also establishes that ψ_0 has exponential decay as $|y| \to \infty$. A similar result for some specific choices of f(w) is given in [23]. In addition, we remark that depending on the specific form of f(w), there may be other discrete eigenvalues ν_j for j > 1 satisfying $-1 + f'(0) < \nu_j < 0$. In fact, in Proposition 5.6 of [7] an explicit determination of the discrete spectrum of L_0 was determined for the power law nonlinearity $f = w^p$ for p > 1.

We now introduce a sub-class of (1.3 *a*) for which any unstable discrete eigenvalue of (1.3 *a*) can be determined in terms of the roots of a certain algebraic equation in λ . Suppose that f(w) and g(w) are related in such a way, through the linear operator $L_0 \equiv \frac{d^2}{dy^2} - 1 + f'(w)$, that for some constant $\sigma > 0$, where g(w) satisfies (1.4). This implies that the principal eigenpair ψ_0 and ν_0 in the spectrum of the linearization of the spike solution, determined by f(w), is simply $\psi_0 = g(w) > 0$ and $\nu_0 = \sigma$, i.e. $L_0\psi_0 = \nu_0\psi_0$. Whenever (2.1) holds, the NLEP (1.3 *a*) is explicitly solvable in the following sense:

Principal Result 2.2 Suppose that (1.4) and (2.1) hold, where w = w(y) is the homoclinic of (1.1). Then, any unstable discrete eigenvalue of (1.3 a) must satisfy

(2.2)
$$\lambda = \sigma - \chi(\lambda) \int_{-\infty}^{\infty} g(w)h(w) \, dy$$

where the integral can be evaluated through a simple quadrature as

(2.3)
$$\int_{-\infty}^{\infty} g(w)h(w) \, dy = \sqrt{2} \int_{0}^{w_m} \frac{g(w)h(w)}{\sqrt{V(w)}} \, dw \,, \qquad V(w) \equiv -\int_{0}^{w} \left[-\eta + f(\eta)\right] \, d\eta$$

Here w_m is the amplitude of the pulse defined in (1.1 c).

To establish (2.2) we use Green's identity on g(w) and Φ . Since L_0 is self-adjoint, we use the decay of Φ and g[w(y)] as $|y| \to \infty$, together with (2.1) and (1.3 a), to obtain

$$0 = \int_{-\infty}^{\infty} \left(g(w) L_0 \Phi - \Phi L_0 g(w) \right) \, dy = \left(\chi \int_{-\infty}^{\infty} h(w) g(w) \, dy + \lambda - \sigma \right) \int_{-\infty}^{\infty} g(w) \Phi \, dy \, .$$

Therefore, for eigenfunctions for which $\int_{-\infty}^{\infty} g(w) \Phi dy \neq 0$, we get that (2.2) holds. Next, consider the eigenfunctions for which $\int_{-\infty}^{\infty} g(w) \Phi dy = 0$, which correspond to any discrete eigenvalues of L_0 not equal to ν_0 . In fact, since g(w) is the unique and one-signed principal eigenfunction of L_0 (see (2.1) and Lemma 2.1), and any two eigenfunctions of the selfadjoint operator L_0 must be orthogonal, it follows that these other eigenfunctions must belong to the set of eigenfunctions of L_0 corresponding to the zero eigenvalue and any negative real eigenvalues of L_0 . Therefore, any unstable eigenvalue of (1.3 a) in $\operatorname{Re}(\lambda) > 0$, must be a root of the algebraic equation (2.2). Finally, the integral $\int_{-\infty}^{\infty} g(w)h(w) dy$ is finite owing to the decay of w as $|y| \to \infty$ and the assumed behavior in (1.4) as $w \to 0$ of g(w) and h(w). The result in (2.3) follows from changing variables after determining w'(y) from a first integral of (1.1 a).

We conclude that if (2.1) holds then any discrete eigenvalue of (1.3 a) is either a discrete eigenvalue of L_0 not equal to ν_0 or it solves (2.2). Next, we show that for a given g(w), there is a simple linear first order ODE that determines a reaction dynamics f(w) so that (2.1) holds. For certain g(w), this ODE can be integrated explicitly to identify an explicit closed form expression for f(w). This ODE is characterized as follows.

Principal Result 2.3 Let $\sigma > 0$ and g(w) satisfying (1.4) be given. Then, if (2.1) holds, f(w) satisfies the ODE

(2.4)
$$\left(\frac{f}{gg'}\right)' = \frac{1}{g^2} \left(\left(\frac{\Sigma}{g'}\right)' - 2w \right); \qquad \Sigma(w) \equiv \int_0^w \xi(s) \, ds, \qquad \xi(s) \equiv sg'(s) + (\sigma+1)g(s),$$

with f(0) = 0. In this first order ODE, the primes denote derivatives with respect to w.

To establish this result, we first suppose that (2.1) holds. Then, since $d^2w/dy^2 = w - f(w)$ and $(dw/dy)^2 = w^2 - 2\mathcal{F}(w)$, where $\mathcal{F}(w) \equiv \int_0^w f(s) \, ds$, we calculate that

$$L_0 g = \frac{d^2}{dy^2} [g(w)] - g + f'g = g'' \left(\frac{dw}{dy}\right)^2 + g' \frac{d^2w}{dy^2} - g + f'g = g'' \left(w^2 - 2\mathcal{F}\right) + g' \left(2w - 2f\right) + g'f + gf' - wg' - g ,$$

where $f' \equiv df/dw$, and $g' \equiv dg/dw$. Upon setting $L_0g = \sigma g$, we get

$$[g'(w^2 - 2\mathcal{F})]' + (fg)' = \xi(w) \equiv wg' + (\sigma + 1)g,$$

where primes denote derivatives with respect to w. By integrating this equation, and setting f(0) = g(0) = 0, we get $w^2 - 2\mathcal{F}(w) + fg/g' = \Sigma(w)/g'$, where $\Sigma(w)$ is defined in (2.4). We differentiate this equation with respect to w to get

$$f' + f\left(-\frac{g'}{g} - \frac{g''}{g'}\right) = \left(\left(\frac{\Sigma}{g'}\right)' - 2w\right)\frac{g'}{g},$$

where we used $\mathcal{F}' = f$. Finally, upon multiplying by the integrating factor $(gg')^{-1}$ we obtain (2.4).

Principal Result 2.3 shows that in terms of g(w), the problem for determining f(w) such that (2.1) holds is reduced to a simple quadrature. We now give two examples to illustrate this result.

Example 1: A simple class of explicitly solvable NLEP's is obtained by setting $(\Sigma/g')' = 2w$ in (2.4). This yields that g(w) satisfies Euler's equation $w^2g'' + wg' - (\sigma + 1)g = 0$, which has the bounded solution $g = w^{\sqrt{\sigma+1}}$ for $\sigma + 1 > 0$. Then, (2.4) yields f/(gg') = C, so that $f = Cw^{2\sqrt{\sigma+1}-1}$. We set C = 1 for convenience, and define $\sigma + 1 \equiv q^2$, to obtain the following power-law class of explicitly solvable NLEP's:

(2.5)
$$g = w^q, \qquad f = w^{2q-1}, \qquad L_0 g = (q^2 - 1)g, \qquad \text{for} \quad q > 1.$$

For q > 1, (1.1 c) holds, and so (1.1 a) has a homoclinic. We remark that for the special case q = 3/2, for which $f = w^2$ and $w = (3/2) \operatorname{sech}^2(y/2)$, the spectrum of the linearized problem $L_0 \Phi = \nu \Phi$ has been analyzed in the context of of wave-scattering by a sech² potential well (cf. [22]). It is well-known (cf. [22]) that $L_0 \Phi_0 = (5/4) \Phi_0$ with $\Phi_0 = \operatorname{sech}^3(y/2)$, which agrees with our result $L_0 w^q = (q^2 - 1) w^q$ when q = 3/2.

Example 2: Let g(w) = w and choose $\sigma > 0$. Then, we calculate $\xi(w) \equiv wg' + (\sigma + 1)g = w(\sigma + 2)$, so that $\Sigma(w) = (\sigma + 2)w^2/2$ from (2.4). Therefore, (2.4) gives

$$\left(\frac{f}{w}\right)' = \frac{1}{w^2} \left(\left(\frac{(\sigma+2)w^2}{2}\right)' - 2w \right) = \frac{\sigma}{w},$$

with f(0) = 0. This gives $f = \sigma w \log w$. For this form of f(w), there is a homoclinic satisfying (1.1 *a*) given explicitly by $w = e^{(\sigma+2)/(2\sigma)}e^{-y^2\sigma/4}$. To verify this result we use $w'' - w + \sigma w \log w = 0$ to calculate

$$L_0 w = w'' - w + \sigma f'(w)w = w'' - w + \sigma \left(1 + \log w\right)w = \sigma w$$

In §3–5 below we analyze the stability of localized stripes for several RD systems for which the underlying spectral problem is an explicitly solvable NLEP in the sense of Principal Result 2.2.

3 Stability of a Stripe for the GM Model

Next, we analyze the stability of a stripe in a rectangular domain Ω for a subclass of the GM model given by

(3.1)
$$v_t = \varepsilon^2 \Delta v - v + \frac{v^3}{u^q}; \qquad \tau u_t = \Delta u - u + \frac{v^3}{\varepsilon u^s}; \qquad x \equiv (x_1, x_2) \in \partial \Omega$$

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with homogeneous Neumann conditions $\partial_n u = \partial_n v = 0$ on $\partial \Omega$. In (3.1), the rectangular domain Ω is defined by

(3.2)
$$\Omega \equiv \{ (x_1, x_2) \mid -l < x_1 < l, \quad 0 < x_2 < d \}$$

The GM exponents q and s are assumed to satisfy the standard conditions (cf. [13])

(3.3)
$$q > 0, \quad s \ge 0, \quad \text{with} \quad \zeta \equiv \frac{3q}{2} - (s+1) > 0.$$

In (3.1) we have, without loss of generality, set the diffusivity of u to unity since it can be absorbed into ε , d, and l.

To construct our stripe solution, we must first construct a 1-D pulse solution to (3.1) in the limit $\varepsilon \to 0$, that is independent of x_2 , and is such that v concentrates at $x_1 = 0$. By using the method of matched asymptotic expansions in the limit $\varepsilon \to 0$ such a solution was constructed formally in [13], and the result is given below in (3.4). A mathematically rigorous construction of this 1-D pulse, with the same asymptotic description as in (3.4), is given in Chapter 1 of [46]. A homoclinic stripe solution results from a trivial extension of this 1-D pulse in the transverse direction. The result is summarized in the following formal statement (see Principal Result 2.1 of [19]):

Principal Result 3.1 For $\varepsilon \to 0$, an equilibrium homoclinic stripe solution to (3.1), labeled by $v_e(x_1)$ and $u_e(x_1)$, is given by

(3.4)
$$v_e(x_1) \sim \mathcal{U}_e^{\gamma} w\left(\varepsilon^{-1} x_1\right); \qquad u_e(x_1) \sim \mathcal{U}_e \frac{G_l(x_1)}{G_l(0)}.$$

Here $w(y) = \sqrt{2} \operatorname{sech} y$ is the unique homoclinic solution to

(3.5)
$$w'' - w + w^3 = 0, \qquad -\infty < y < \infty; \qquad w \to 0 \quad as \quad |y| \to \infty,$$

with w(0) > 0 and w'(0) = 0. The constants \mathcal{U}_e, γ , and $G_l(0)$ in (3.4), are defined by

(3.6)
$$\mathcal{U}_e^{\zeta} \equiv \frac{1}{bG_l(0)}; \qquad b \equiv \int_{-\infty}^{\infty} w^3 \, dy = \sqrt{2}\pi; \qquad \gamma \equiv \frac{q}{2}; \qquad G_l(0) = \frac{1}{2} \coth l \,,$$

where ζ is defined in (3.3). The Green's function $G_l(x_1)$ in (3.4) satisfies

(3.7)
$$G_{lx_1x_1} - G_l = -\delta(x_1), \quad |x_1| \le l; \quad G_{lx_1}(\pm l) = 0; \qquad G_l(x_1) = \frac{\cosh\left(l - |x_1|\right)}{2\sinh\left(l\right)}$$

To analyze the stability of the stripe to transverse perturbations, we introduce the perturbation

(3.8)
$$v = v_e + e^{\lambda t + imx_2} \phi(x_1), \qquad u = u_e + e^{\lambda t + imx_2} \eta(x_1), \quad \text{with} \quad m = \frac{k\pi}{d},$$

where k is an integer. The relation $m = k\pi/d$ results from the Neumann conditions on $x_2 = 0, d$ of $\partial\Omega$. In the analysis below we treat m as a continuous variable and determine a band of instability for m. Values of k for which $k\pi/d$ lie within this band represent unstable perturbations. Substituting (3.8) into (3.1), we obtain the eigenvalue problem

(3.9 a)
$$\varepsilon^{2}\phi_{x_{1}x_{1}} - \phi + \frac{3v_{e}^{2}}{u_{e}^{q}}\phi - \frac{qv_{e}^{3}}{u_{e}^{q+1}}\eta = \left(\lambda + \varepsilon^{2}m^{2}\right)\phi, \quad |x_{1}| \leq l; \quad \phi_{x_{1}}(\pm l) = 0,$$

(3.9 b)
$$\eta_{x_1x_1} - \left(1 + \tau \lambda + m^2\right)\eta = -\frac{3v_e^2}{\varepsilon u_e^s}\phi + \frac{sv_e^3}{\varepsilon u_e^{s+1}}\eta, \quad |x_1| \le l; \quad \eta_{x_1}(\pm l) = 0.$$

We remark that since instabilities may occur for high spatial frequencies we must analyze (3.9) for $0 < m \leq \mathcal{O}(\varepsilon^{-1})$.

Next we derive an NLEP governing the stability of the stripe on an $\mathcal{O}(1)$ time-scale. The corresponding localized eigenfunction has the form $\phi(x_1) \sim \Phi\left(\varepsilon^{-1}x_1\right)$, where $\int_{-\infty}^{\infty} w^2 \Phi(y) \, dy \neq 0$. Since unstable eigenfunctions of this type are

found to lead to a disintegration of the stripe into spots, this instability is termed a breakup instability (cf. [19]). Since the NLEP governing breakup instabilities of a homoclinic stripe solution was derived in Appendix A of [19], and summarized in Principal Result 2.2 of [19], we will only briefly highlight its derivation here. For the related problem of analyzing the stability of a 1-D homoclinic pulse, a detailed derivation of an NLEP was given in §3 of [13] and in §2 of [43].

We now outline the derivation of the NLEP. In terms of the inner variable $y = x_1/\varepsilon$, we use $v_e \sim \mathcal{U}_e^{\gamma} w(y)$ and $u_e \sim \mathcal{U}_e$ from (3.4) to obtain from (3.9 a) that $\Phi(y)$ satisfies

(3.10)
$$\Phi'' - \Phi + 3w^2 \Phi - q \mathcal{U}_e^{q/2 - 1} w^3 \eta(0) = \left(\lambda + \varepsilon^2 m^2\right) \Phi, \quad -\infty < y < \infty, \qquad \Phi \to 0 \quad \text{as} \quad |y| \to \infty.$$

Since the equation in (3.9 b) for η is not singularly perturbed, $\eta(0)$ is determined from the outer solution for $\eta(x_1)$. By using (3.4) and (3.6) to calculate the coefficients in (3.9 b) in terms of Dirac masses, we obtain that $\eta(x_1)$ satisfies

(3.11)
$$\eta_{x_1x_1} - \theta_{\lambda}^2 \eta = \left(\frac{s}{G_l(0)}\eta(0) - \frac{3\mathcal{U}_e^{1-\gamma}}{bG_l(0)}\int_{-\infty}^{\infty} w^2 \Phi \, dy\right)\,\delta(x_1)\,,\qquad |x_1| \le l\,;\qquad \eta_{x_1}(\pm l) = 0$$

To solve (3.11), we introduce $G_{\lambda}(x_1)$ satisfying

$$(3.12) G_{\lambda x_1 x_1} - \theta_{\lambda}^2 G_{\lambda} = -\delta(x_1), \quad |x_1| \le l; \qquad G_{\lambda x_1}(\pm l) = 0; \qquad G_{\lambda}(x_1) = \frac{\cosh\left[\theta_{\lambda}\left(l - |x_1|\right)\right]}{2\theta_{\lambda}\sinh\left(\theta_{\lambda}l\right)}.$$

By writing $\eta(x_1)$ in terms of $G_{\lambda}(x_1)$, and then using (3.7) and (3.12), we calculate $\eta(0)$ as

(3.13)
$$\eta(0) = \frac{3\mathcal{U}_e^{1-\gamma}}{b} \int_{-\infty}^{\infty} w^2 \Phi \, dy \left[s + \frac{\theta_\lambda \tanh(\theta_\lambda l)}{\tanh l} \right]^{-1} \,.$$

where $\theta_{\lambda} \equiv \sqrt{1 + m^2 + \tau \lambda}$ is the principal value of the square root. With this choice, $\eta(0)$ is analytic in $\operatorname{Re}(\lambda) \geq 0$ and has a branch cut on the portion $\operatorname{Re}(\lambda) \leq -(1 + m^2)/\tau$, $\operatorname{Im}(\lambda) = 0$ of the negative real axis in the λ -plane. This choice also ensures that $G_{\lambda} \to 0$ as $|x_1| \to \infty$ for the infinite line problem where $l = \infty$.

By substituting (3.13) into (3.10), we obtain the following NLEP governing breakup instabilities of a stripe:

Principal Result 3.2 Let $\varepsilon \to 0$ and suppose that $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$. Then, $\Phi(y)$ satisfies the NLEP

(3.14 a)
$$L_0 \Phi - \chi w^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy = (\lambda + \varepsilon^2 m^2) \Phi \quad -\infty < y < \infty; \quad \Phi \to 0 \quad as \quad |y| \to \infty,$$

(3.14 b)
$$\chi \equiv \frac{3q}{b} \left[s + \frac{\theta_{\lambda} \tanh(\theta_{\lambda}l)}{\tanh l} \right]^{-1}, \quad b \equiv \int_{-\infty}^{\infty} w^3 \, dy = \sqrt{2}\pi, \qquad \theta_{\lambda} \equiv \sqrt{1 + m^2 + \tau\lambda}.$$

Here $\chi = \chi(\lambda)$, and L_0 is the local operator defined by $L_0 \Phi \equiv \Phi'' - \Phi + 3w^2 \Phi$.

In §2 and Appendix A of [19] some rigorous results were obtained for the stability of a stripe for a general GM model under various ranges of the exponents of the nonlinearities in the reaction kinetics. These ranges where the rigorous results of [19] apply do not include the subclass (3.1) of the GM model, which results in the NLEP of (3.14).

We observe that the NLEP of (3.14) is explicitly solvable in the sense of Principal Result 2.2. Since $L_0w^2 = 3w^2$ (see (2.5) of Example 1 of §2), we conclude by replacing λ , σ , g(w), and h(w) in (2.2) with $\lambda + \varepsilon^2 m^2$, 3, w^2 , and w^3 , respectively, that any unstable eigenvalue of (3.14) must satisfy $\lambda + \varepsilon^2 m^2 = 3 - \chi \int_{-\infty}^{\infty} w^5 dy$. Moreover, since the local problem $L_0\psi = \nu\psi$ has no discrete eigenvalues in $-1 < \nu < 0$ (see Proposition 5.6 of [7]), it follows that (3.14) has no other nonzero discrete eigenvalues. By using (3.14 b) for χ and $\int_{-\infty}^{\infty} w^5 dy = 3\pi/\sqrt{2}$, we obtain the following main result:

Principal Result 3.3 Any unstable discrete eigenvalue λ of (3.14) must be a root of the transcendental equation

(3.15)
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9q}{2} \left[s + \frac{\theta_\lambda \tanh(\theta_\lambda l)}{\tanh l} \right]^{-1}, \qquad \theta_\lambda \equiv \sqrt{1 + m^2 + \tau \lambda}$$

where $\theta_{\lambda} \equiv \sqrt{1 + m^2 + \tau \lambda}$ denotes the principal value of the square root.

We now proceed to analyze (3.15) for the two cases $\tau = 0$ and $\tau > 0$. For the case where $\tau = 0$, (3.15) reduces to

(3.16)
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9q}{2} \left[s + \frac{\sqrt{1 + m^2} \tanh(\sqrt{1 + m^2}l)}{\tanh l} \right]^{-1}$$

Moreover, if we let $l \to \infty$ in (3.16), so that the effect of the sidewalls is insignificant, we get

(3.17)
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9q}{2\left[s + \sqrt{1 + m^2}\right]}$$

For $\tau = 0$ and $\varepsilon \ll 1$, a simple calculation shows that the instability band for (3.16) is

(3.18)
$$\lambda > 0$$
 for $0 < m_{-} < m < m_{+}$, $m_{+} \sim \frac{\sqrt{3}}{\varepsilon} - \frac{3q \tanh l}{4} + o(1)$,

and where $m_{-} \sim \sqrt{z_{-}^2 - 1}$ when $\varepsilon \ll 1$. Here $z_{-} > 1$ is the unique root of $\kappa(z) = 0$ in z > 1, where

(3.19)
$$\kappa(z) \equiv \frac{z \tanh(zl)}{\tanh l} - (\xi + 1), \qquad \xi \equiv \frac{3q}{2} - (s+1) > 0.$$

Since $\xi > 0$, we have that $\kappa(z)$ satisfies $\kappa(1) < 0$, $\kappa'(z) > 0$ for z > 1 and $\kappa(z) \to +\infty$ as $z \to +\infty$. Hence, there exists a unique root $z_{-} > 1$ to $\kappa(z) = 0$. In contrast, for $0 < m < m_{-}$ or $m > m_{+}$, we have $\lambda < 0$. To determine the mode $m = m_{\text{dom}}$ in $m_{-} < m < m_{+}$, corresponding to the largest growth rate within the instability band, we set $d\lambda/dm = 0$ in (3.16) and solve for m. For $\varepsilon \ll 1$, we readily obtain that

(3.20)
$$m_{\rm dom} \sim \varepsilon^{-2/3} \left[\frac{9q \tanh l}{4}\right]^{1/3} \gg 1 \qquad \text{for } l > 0.$$

For the case $l = \infty$, we obtain that $m_{-} \sim \sqrt{\xi^2 + 2\xi}$ for $\varepsilon \ll 1$, and can readily derive a two-term expansion for m_{dom} as

(3.21)
$$m_{\rm dom} \sim \varepsilon^{-2/3} \left(\frac{9q}{4}\right)^{1/3} - \frac{2s}{3}$$

For $\tau = 0$, $\epsilon = 0.05$, q = 1 and s = 0, in Fig. 1(a) we plot λ versus m for a few values of l. Similar plots are shown in Fig. 1(b) for q = 2. For $l = \infty$ and q = 1, the asymptotic predictions for the edges of the instability band and the most unstable mode are compared with numerical results from (3.16) in the caption of Fig. 1.

The most unstable mode m_{dom} within the instability band can be used to predict the number of localized spots that occur from the transverse (breakup) instability of the stripe. From the form of the perturbation in (3.8), we predict that the stripe will break up into N spots, where N is the closest integer to $m_{\text{dom}}d/(2\pi)$ and d is the width of the rectangle. The asymptotic theory is compared with results from full numerical computations of (3.1) in §3.2.

3.1 Stripe Stability For $\tau > 0$

Next, we consider the case where $\tau > 0$ in (3.15), for which (3.15) is a transcendental equation in λ . For simplicity, we first restrict the analysis to the case of no sidewalls for which $l = \infty$. For $l = \infty$, we obtain from (3.15) that

(3.22)
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9q}{2\left[s + \sqrt{1 + m^2}\sqrt{1 + \hat{\tau}\lambda}\right]}, \qquad \hat{\tau} \equiv \frac{\tau}{1 + m^2}.$$



Figure 1. Plot of λ versus m for $\varepsilon = 0.05$, s = 0 and $\tau = 0$, for several values of l. Left panel: q = 1. Right panel: q = 2. The dotted, solid, and heavy solid curves in the two panels are for l = 0.5, l = 1, and $l = \infty$, respectively. For $l = \infty$ and q = 1, the asymptotic result (3.18) predicts the instability band 1.11 < m < 33.89, while the numerical result from (3.16) yields 1.12 < m < 33.86. Moreover, the most unstable mode from the asymptotic theory (3.20) is $m_{\text{dom}} \sim 9.65$, which compares very favorably with the numerical result of $m_{\text{dom}} \approx 9.60$ from (3.16).

Then, (3.22) is equivalent to finding the roots of $\mathcal{F}(\lambda) = 0$, where

(3.23 a)
$$\mathcal{F}(\lambda) \equiv 2\sqrt{1+\hat{\tau}\lambda} - \mathcal{G}(\lambda), \qquad \mathcal{G}(\lambda) \equiv d_0 - \frac{d_1}{\beta - \lambda}$$

and $d_0 \leq 0, d_1 < 0$, and β are defined by

(3.23 b)
$$d_0 = -\frac{2s}{\sqrt{1+m^2}} \le 0, \qquad d_1 = -\frac{9q}{\sqrt{1+m^2}} < 0, \qquad \beta \equiv 3 - \varepsilon^2 m^2.$$

In (3.23 a), the principal value of the square root is taken so that $\eta(x_1)$ in (3.8) is analytic except on a portion of the negative real axis (see the discussion following (3.13)). An eigenvalue relation similar to (3.23) was derived and studied in [28] for the stability analysis of a one-pulse solution for a class of RD system.

Let J denote the number of roots of (3.23) in $\operatorname{Re}(\lambda) > 0$. To determine J for various parameter ranges of d_0, d_1, β , and $\hat{\tau}$, we use the argument principle. We choose the counterclockwise contour consisting of the imaginary axis $-iR \leq \operatorname{Im}\lambda \leq iR$ and the semi-circle Γ_R , given by $|\lambda| = R > 0$, for $-\pi/2 \leq \arg \lambda \leq \pi/2$. We observe that when $m > \sqrt{3}/\varepsilon$, $\mathcal{G}(\lambda)$ is analytic in $\operatorname{Re}(\lambda) > 0$, and when $0 < m < \sqrt{3}/\varepsilon$, $\mathcal{G}(\lambda)$ has a simple pole in $\operatorname{Re}(\lambda) > 0$ at $\lambda = \beta$. For $\hat{\tau} > 0$, we have that $\mathcal{F}(\lambda) \sim 2\sqrt{\hat{\tau}\lambda}$ as $|\lambda| \to \infty$ on Γ_R , so that the change in the argument of $\mathcal{F}(\lambda)$ over Γ_R as $R \to \infty$ is $\pi/2$. By using the argument principle, together with $\mathcal{F}(\overline{\lambda}) = \overline{\mathcal{F}(\lambda)}$, we obtain for any $\hat{\tau} > 0$ that

(3.24)
$$J = \frac{1}{4} + H(\beta) + \frac{1}{\pi} [\arg \mathcal{F}]_{\Gamma_I} , \text{ where } H(\beta) \equiv \begin{cases} 1 & \text{for } \beta > 0, \\ 0 & \text{for } \beta < 0. \end{cases}$$

Here $[\arg \mathcal{F}]_{\Gamma_I}$ is the change in the argument of \mathcal{F} when the semi-axis $\Gamma_I = i\lambda_I$, $0 \le \lambda_I < \infty$ is traversed downwards. Along the imaginary axis $\lambda = i\lambda_I$ for $\lambda_I > 0$, we can decompose $\mathcal{F}(i\lambda_I) = \mathcal{F}_R(\lambda_I) + i\mathcal{F}_I(\lambda_I)$, where

$$(3.25) \quad \mathcal{F}_R(\lambda_I) \equiv \mathcal{K}_+(\hat{\tau}\lambda_I) - d_0 + \frac{d_1\beta}{\beta^2 + \lambda_I^2}, \qquad \mathcal{F}_I(\lambda_I) \equiv \mathcal{K}_-(\hat{\tau}\lambda_I) + \frac{d_1\lambda_I}{\beta^2 + \lambda_I^2}; \qquad \mathcal{K}_\pm(\zeta) \equiv \sqrt{2} \left[\sqrt{1 + \zeta^2} \pm 1\right]^{1/2}.$$

Since $\hat{\tau} > 0$, $\mathcal{F}_R \sim 2\sqrt{\hat{\tau}\lambda_I}$ and $\mathcal{F}_I \sim 2\sqrt{\hat{\tau}\lambda_I}$ as $\lambda_I \to \infty$, then $\arg(\mathcal{F}(i\lambda_I)) \to \pi/4$ as $\lambda_I \to \infty$. We further calculate that

(3.26)
$$\mathcal{F}_R(0) = 2 - \mathcal{G}(0); \qquad \mathcal{F}_I(0) = 0, \qquad \frac{d}{d\lambda_I} \mathcal{F}_R(\lambda_I) = \hat{\tau} \mathcal{K}'_+(\hat{\tau}\lambda_I) - \frac{2d_1\lambda_I\beta}{(\beta^2 + \lambda_I)^2}$$

Explicitly Solvable Nonlocal Eigenvalue Problems and Stripe Stability

Next, we note that $\mathcal{G}(0) = 2$ when $\lambda = 0$ is a root of $\mathcal{F}(\lambda) = 0$ in (3.23 a). We readily calculate that

(3.27)
(i)
$$\mathcal{G}(0) < 2$$
 when $m_{-} < m < m_{+}$ or $m > \frac{\sqrt{3}}{\varepsilon}$,
(ii) $\mathcal{G}(0) > 2$ when $m_{+} < m < \frac{\sqrt{3}}{\varepsilon}$, or $0 < m < m_{-}$

Here $m_{-} \sim \sqrt{\xi^2 + 2\xi}$ and $m_{+} \sim \sqrt{3}/\varepsilon - 3q/4 + o(1)$ as $\varepsilon \to 0$. Our first result characterizes the range of m for which there are no Hopf bifurcations as $\hat{\tau}$ is varied.

Principal Result 3.4 Let J denote the number of roots in $Re(\lambda) > 0$ of $\mathcal{F}(\lambda) = 0$ in (3.23). Then, for any $\hat{\tau} > 0$,

(3.28)
(I)
$$J = 0$$
 when $m > m_+ \sim \sqrt{3}/\varepsilon - 3q/4 + o(1)$,
(II) $J = 1$ when $m_- < m < m_+$,
(III) $J = 0$ or $J = 2$ when $0 < m < m_- \sim \sqrt{\xi^2 + 2\xi}$.

For (II) the root is located on the positive real axis in the interval $0 < \lambda < \beta$. In (III), there are two positive real roots in $0 < \lambda < \beta$ when $\hat{\tau}$ is sufficiently large. Moreover, for $0 < m < m_{-}$, J = 0 for $0 < \hat{\tau} \ll 1$.

To establish (I) we consider two sub-cases; (Ia) $m > \sqrt{3}/\varepsilon$ and (Ib) $m_+ \sim \sqrt{3}/\varepsilon - 3q/4 + o(1) < m < \sqrt{3}/\varepsilon$.

For (Ia) we have from (3.27) that $\mathcal{F}_R(0) = 2 - \mathcal{G}(0) > 0$. In addition, since $d_0 \leq 0$, $d_1 < 0$, and $\beta < 0$, we have from (3.25) that $\mathcal{F}_R(\lambda_I) > 0$ for all $\lambda_I > 0$ and $\hat{\tau} > 0$. Since $\arg \mathcal{F}(i\lambda_I) \to \pi/4$ as $\lambda_I \to \infty$, it follows that $[\arg \mathcal{F}]_{\Gamma_I} = -\pi/4$. Then, since $\beta < 0$, we conclude from (3.24) that J = 0.

For (Ib) we have from (3.27) that $\mathcal{F}_R(0) = 2 - \mathcal{G}(0) < 0$. Since $\beta > 0$ and $d_1 < 0$, it follows from (3.26) that $\frac{d}{d\lambda_I}\mathcal{F}_R(\lambda_i) > 0$ with $\mathcal{F}_R(\lambda_I) \to +\infty$ as $\lambda_I \to \infty$. It follows that there is a unique root λ_I^* to $\mathcal{F}_R(\lambda_I^*) = 0$. If we can show that $\mathcal{F}_I(\lambda_I^*) < 0$, it would then follow that $[\arg \mathcal{F}]_{\Gamma_I} = -5\pi/4$, and consequently J = 0 from (3.24). To verify this sign of $\mathcal{F}_I(\lambda_I^*)$ we first calculate the root λ_I^* of $\mathcal{F}_R(\lambda_I^*) = 0$. On the range $\sqrt{3}/\varepsilon - 3q/4 + o(1) < m < \sqrt{3}/\varepsilon$, we set $m = \sqrt{3}/\varepsilon + m_1$, where $-3q/4 < m_1 < 0$. This determines β as $\beta = 3 - \varepsilon^2 m^2 \sim -2\sqrt{3}m_1\varepsilon$ for $\varepsilon \ll 1$. For $\varepsilon \to 0$, it is readily shown that the root to $\mathcal{F}_R(\lambda_I^*) = 0$ occurs when $\lambda_I^* \sim \varepsilon \lambda_{I0}$. Upon using (3.25) and (3.23 b) for d_0 and d_1 , we set $\mathcal{F}_R(\lambda_I^*) = 0$ to get

(3.29)
$$\mathcal{K}_{+}(0) + o(1) \sim -\frac{2s\varepsilon}{\sqrt{3}} - \frac{18qm_{1}}{12m_{1}^{2} + \lambda_{I0}^{2}}$$

Upon solving for λ_{I0} we obtain $\lambda_{I0} = \sqrt{-9qm_1 - 12m_1^2}$, which exists for $-3q/4 < m_1 < 0$. Since $d_1 < 0$, $\lambda_I = \mathcal{O}(\varepsilon)$, $\beta \sim 3 - \varepsilon^2 m^2 \sim -2\sqrt{3}m_1\varepsilon$ and $\mathcal{K}_-(0) = 0$, we calculate from (3.25) that

$$\mathcal{F}_{I}(\lambda_{I}^{\star}) \sim \mathcal{K}_{-}(\varepsilon \hat{\tau} \lambda_{I0}) - \frac{9q\lambda_{I0}}{\sqrt{3}\left[12m_{1}^{2} + \lambda_{I0}^{2}\right]} < 0.$$

Thus, $\mathcal{F}_I(\lambda_I^*) < 0$ as claimed, and consequently J = 0.

To establish the second statement (II) in (3.28), we have from (i) of (3.27) that $\mathcal{G}(0) < 2$ when $m_- < m < m_+$. Thus, $\mathcal{F}_R(0) = 2 - \mathcal{G}(0) > 0$. Moreover, since $\frac{d}{d\lambda_I} \mathcal{F}_R(\lambda_I) > 0$ from (3.26), we conclude that $\mathcal{F}_R(\lambda_I) > 0$ for all $\hat{\tau} > 0$ and $\lambda_I > 0$. Then, since $\arg \mathcal{F}(i\lambda_I) \to \pi/4$ as $\lambda_I \to \infty$, it follows that $[\arg \mathcal{F}]_{\Gamma_I} = -\pi/4$, and consequently J = 1 from (3.24). We now show that this root is on the positive real axis in the interval $0 < \lambda < \beta$. To show this, we plot $2\sqrt{1 + \hat{\tau}\lambda}$ and $\mathcal{G}(\lambda)$ from (3.23 *a*) on the real axis $\lambda > 0$. On $0 \le \lambda < \beta$, we have $\mathcal{G}(0) < 2$, $\mathcal{G}'(\lambda) > 0$, $\mathcal{G}''(\lambda) > 0$, and $\mathcal{G}(\lambda) \to +\infty$ as $\lambda \to \beta^-$. Since $2\sqrt{1 + \hat{\tau}\lambda}$ is an increasing concave function it follows that there there is a unique root to $2\sqrt{1 + \hat{\tau}\lambda} = \mathcal{G}(\lambda)$ on $0 < \lambda < \beta$.

Finally, we establish (III) in (3.28). For $0 < m < m_-$, we have $\mathcal{G}(0) > 2$ so that $\mathcal{F}_R(0) = 2 - \mathcal{G}(0) < 0$. Since $d_1 < 0$ and $\beta > 0$, we have $\frac{d}{d\lambda_I} \mathcal{F}_R(\lambda_I) > 0$ from (3.26). Since $\mathcal{F}_R(\lambda_I) \to +\infty$ as $\lambda_I \to \infty$, it follows that there exists a unique root

 $\lambda_I^* > 0$ to $\mathcal{F}_R(\lambda_I^*) = 0$. If $\mathcal{F}_I(\lambda_I^*) > 0$, we obtain $[\arg \mathcal{F}]_{\Gamma_I} = 3\pi/4$, and conclude from (3.24) that J = 2. If $\mathcal{F}_I(\lambda_I^*) < 0$, we obtain $[\arg \mathcal{F}]_{\Gamma_I} = -5\pi/4$, and conclude from (3.24) that J = 0. When $0 < \hat{\tau} \ll 1$, it is readily shown that J = 0. For $\hat{\tau} \gg 1$, there are two real positive roots to $2\sqrt{1+\hat{\tau}\lambda} = \mathcal{G}(\lambda)$ on $0 < \lambda < \beta$. Thus J = 2 for $\hat{\tau} \gg 1$. This completes the derivation of Principal Result 3.4.

The key result (II) of Principal Result 3.4 shows that there is a unique unstable real eigenvalue for any $\hat{\tau} > 0$ in the breakup instability band $m_{-} < m < m_{+}$. Since eigenvalues cannot cross through the origin $\lambda = 0$ as $\hat{\tau}$ is varied, Property (III) of Principal Result 3.4 proves the existence of a Hopf bifurcation value $\hat{\tau}_{H}$ of $\hat{\tau}$ when $0 < m < m_{-}$.

We now show that $\hat{\tau}_H$ is unique and determine it explicitly. To do so, we first set $\mathcal{F}_R = \mathcal{F}_I = 0$ in (3.25) to get

(3.30)
$$\sqrt{2} \left[\sqrt{a}+1\right]^{1/2} = d_0 - \frac{d_1\beta}{\zeta}, \qquad \sqrt{2} \left[\sqrt{a}-1\right]^{1/2} = -\frac{d_1\lambda_I}{\zeta}; \qquad a \equiv 1 + \hat{\tau}^2 \lambda_I^2, \qquad \zeta \equiv \beta^2 + \lambda_I^2.$$

Upon dividing these two equations we get

(3.31)
$$\sqrt{a} + 1 = \hat{\tau}A$$
, where $A \equiv \beta - \frac{d_0\zeta}{d_1}$

Since the first equation of (3.30) can be written as $\sqrt{2} \left[\sqrt{a}+1\right]^{1/2} = -d_1 A/\zeta$, we obtain from using (3.31) that $\sqrt{2\hat{\tau}}A^{1/2} = -d_1 A/\zeta$. Upon solving for $\hat{\tau}$, and recalling that $A \equiv \beta - d_0 \zeta/d_1$, we obtain

(3.32)
$$\hat{\tau} = \frac{d_1^2}{2\zeta^2} A = \frac{d_1^2}{2} \left(\frac{\beta}{\zeta^2} - \frac{d_0}{\zeta d_1} \right) \,,$$

which determines $\hat{\tau}$ in terms of ζ . To determine ζ we square and add the two expressions in (3.30). This yields that $4\sqrt{a} = (d_0 - d_1\beta/\zeta)^2 + d_1^2\lambda_I^2/\zeta^2$. Then, by using $\sqrt{a} = -1 + \hat{\tau}A$, $\lambda_I^2 = \zeta - \beta^2$, and (3.32) for $\hat{\tau}$ in terms of A, we get that $-4 + 2A^2d_1^2/\zeta^2 = d_1^2A^2/\zeta^2 + d_1^2(\zeta - \beta^2)/\zeta^2$. Finally, we solve for A^2 and recall that $A = \beta - d_0\zeta/d_1$ to obtain $\zeta - \beta^2 + 4\zeta^2/d_1^2 = (\beta - d_0\zeta/d_1)^2$. By rewriting this expression we get that $\zeta > \beta^2$ must be a root of the quadratic equation

(3.33)
$$\mathcal{M}(\zeta) \equiv \left(d_0^2 - 4\right)\zeta^2 - \left(d_1^2 + 2\beta d_0 d_1\right)\zeta + 2\beta^2 d_1^2 = 0.$$

By analyzing the roots of $\mathcal{M}(\xi) = 0$ we can characterize the Hopf bifurcation value of $\hat{\tau}$ on the range $0 < m < m_{-}$:

Principal Result 3.5 Suppose that $0 < m < m_{-} = \sqrt{\xi^2 + 2\xi}$ where $\xi = 3q/2 - (s+1) > 0$. Then, there exists a unique value $\hat{\tau}_H = \hat{\tau}_H(m) > 0$ of $\hat{\tau}$ for which $\lambda = i\lambda_I$ is a root of $\mathcal{F}(i\lambda_I) = 0$ in (3.23). This yields a Hopf bifurcation value $\tau_H = (1 + m^2)\hat{\tau}_H$ for the equilibrium stripe solution of (3.1). The Hopf bifurcation point $\hat{\tau}_H$ and λ_{IH} is given by

(3.34)
$$\hat{\tau}_H = \frac{d_1^2}{2\zeta^2} A, \qquad \lambda_{IH} = \sqrt{\zeta - \beta^2}, \qquad where \qquad A \equiv \beta - \frac{d_0\zeta}{d_1}.$$

Here ζ is the smallest root of the quadratic equation (3.33) on the interval $\zeta > \beta^2$, given explicitly by

(3.35)
$$\zeta = \frac{(2d_0d_1\beta + d_1^2)}{2(d_0^2 - 4)} + \frac{1}{2(4 - d_0^2)}\sqrt{(2d_0d_1\beta + d_1^2)^2 - 8(d_0^2 - 4)\beta^2d_1^2}, \quad \text{when} \quad d_0 \neq -2,$$
$$\zeta = \frac{2\beta^2d_1}{d_1 - 4\beta}, \quad \text{when} \quad d_0 = -2.$$

Here $\beta = 3 - \varepsilon^2 m^2$, $d_0 = -2s/\sqrt{1+m^2} \le 0$, and $d_1 = -9q/\sqrt{1+m^2} < 0$. When s = 0, for which $d_0 = 0$, we have

(3.36)
$$\hat{\tau}_H = \frac{1}{\beta} \left[1 + \frac{c^2}{2} + c\sqrt{1 + \frac{c^2}{4}} \right], \qquad c \equiv -\frac{d_1}{2\beta\sqrt{2}} = -\frac{9q}{2\sqrt{2(1 + m^2)(3 - \varepsilon^2 m^2)}}.$$

The derivation of this result consists of examining the roots of $\mathcal{M}(\zeta) = 0$ for three cases: Case 1: $-2 < d_0 < 0$; Case 2:

 $d_0 = -2$; Case 3: $d_0 < -2$. For $0 < m < m_-$, we have $\mathcal{G}(0) > 2$ and $d_1 < 0$ from (3.27). For each case, (3.33) yields

(3.37)
$$\mathcal{M}(\beta^2) = -4\beta^4 + \beta^4 \left(\frac{d_1^2}{\beta^2} - \frac{2d_0d_1}{\beta} + d_0^2\right) = \beta^4 \left([\mathcal{G}(0)]^2 - 4\right).$$

Since $\mathcal{G}(0) > 2$, we conclude that $\mathcal{M}(\beta^2) > 0$.

For Case I where $-2 < d_0 < 0$, we have that $\mathcal{M}(\zeta) \to -\infty$ as $\zeta \to \pm \infty$. Therefore, from the intermediate value theorem there exists a unique root ζ_- to $\mathcal{M}(\zeta) = 0$ in $\beta^2 < \zeta < \infty$, while the other root is in $-\infty < \zeta < \beta^2$. Since $\zeta = \beta^2 + \lambda_I^2$, the relevant root is ζ_- . At this root we must show that $A = \beta - d_0\zeta_-/d_1 > 0$, so that $\hat{\tau}_H > 0$ from (3.34). For $-2 < d_0 < 0$, we use $\mathcal{G}(0) > 2$ to obtain $d_1/\beta < d_0 - 2$. Therefore, since $d_0 < 0$ and $\zeta_- > 0$, we obtain that $-d_0\zeta_-/d_1 > -d_0\zeta_-/(\beta(d_0 - 2))$. From this inequality we calculate that

$$A = \beta - \frac{d_0\zeta_-}{d_1} > \beta - \frac{d_0\zeta_-}{\beta(d_0 - 2)} = \frac{d_0(\zeta_- - \beta^2) + 2\beta^2}{\beta(2 - d_0)} > 0,$$

since $-2 < d_0 < 0$ and $\zeta_- > \beta^2$. Therefore, when $-2 < d_0 < 0$, (3.33) has a unique root in $\zeta > \beta^2$, for which $\hat{\tau}_H > 0$ in (3.34). This root is given by the first expression in (3.35).

Next, consider Case II where $d_0 = -2$, for which $\mathcal{M}(\zeta) = -(d_1^2 - 4\beta d_1)\zeta + 2\beta^2 d_1^2$. Since $d_1 < 0$, then $\mathcal{M}(\zeta) \to -\infty$ as $\zeta \to +\infty$, while $\mathcal{M}(\beta^2) > 0$ from (3.37). Therefore, the unique root $\zeta = 2\beta^2 d_1/(d_1 - 4\beta)$ to $\mathcal{M}(\zeta) = 0$ is in $\zeta > \beta^2$. At this root we calculate A as

$$A = \beta - \frac{d_0 \zeta}{d_1} = \beta + \frac{2\zeta}{d_1} = \beta + \frac{4\beta^2}{d_1 - 4\beta} = \frac{\beta d_1}{d_1 - 4\beta} > 0,$$

since $d_1 < 0$. Thus, $\hat{\tau}_H > 0$ in (3.34).

Finally, we consider Case III where $d_0 < -2$. This case is more intricate since $\mathcal{M}(\zeta) \to +\infty$ as $\zeta \to +\infty$ and $\mathcal{M}(\beta^2) > 0$. Therefore, the behaviour of the roots of $\mathcal{M}(\xi) = 0$ in $\zeta > \beta^2$ is not immediately clear. To analyze these roots it is convenient to define $\zeta_c \equiv d_1\beta/d_0$ to be the value of ζ for which A = 0. For $\zeta < \zeta_c$ we have A > 0, while for $\zeta > \zeta_c$, we have A < 0. Moreover, $\zeta_c > \beta^2$ since $d_0 - d_1/\beta > 2$. We now calculate $\mathcal{M}(\zeta_c)$ as

(3.38)
$$\mathcal{M}(\zeta_c) = \left(d_0^2 - 4\right) \frac{d_1^2 \beta^2}{d_0^2} - \left(d_1^2 + 2d_0 d_1 \beta\right) \frac{d_1 \beta}{d_0} + 2\beta^2 d_1^2 = \frac{d_1^2 \beta^2}{d_0^2} \left[d_0^2 - 4 - \frac{d_1 d_0}{\beta}\right]$$

Since $d_0 - d_1/\beta > 2$ and $d_0 < 0$, we have $-d_0 d_1/\beta < d_0 (2 - d_0)$. By using this estimate in (3.38), we obtain

$$\mathcal{M}(\zeta_c) < \frac{d_1^2 \beta^2}{d_0^2} \left[d_0^2 - 4 + 2d_0 - d_0^2 \right] = \frac{d_1^2 \beta^2}{d_0^2} \left[2d_0 - 4 \right]$$

Thus, since $d_0 < 0$, we have $\mathcal{M}(\zeta_c) < 0$. By the intermediate value theorem, it follows that $\mathcal{M}(\zeta) = 0$ must have two real roots ζ_{\pm} , which satisfy $\beta^2 < \zeta_- < \zeta_c$ and $\zeta_c < \zeta_+$. However, since A > 0 for $\zeta = \zeta_- < \zeta_c$ and A < 0 for $\zeta = \zeta_+ > \zeta_c$, only the smaller of the two roots yields a $\hat{\tau}_H > 0$ from (3.34). Therefore, the smaller root ζ_- gives the Hopf bifurcation, and this root determines λ_I as $\lambda_I = \sqrt{\zeta_- - \beta^2}$. In this way we obtain the first of (3.35).

When $d_0 = 0$, we readily calculate from (3.34) and (3.35) that $\mathcal{M}(\zeta) = 0$ has a unique root in $\zeta > \beta^2$ given by

(3.39)
$$\zeta = c_0 + \sqrt{c_1}, \qquad c_0 = -\frac{d_1^2}{8}, \qquad c_1 = \frac{d_1^4}{64} + \frac{\beta^2 d_1^2}{2}$$

Then, from (3.32) with $d_0 = 0$, we get

$$\hat{\tau} = \frac{d_1^2 \beta}{2} \left(\frac{1}{c_0 + \sqrt{c_1}} \right)^2 = \frac{d_1^2 \beta}{2(c_0^2 - c_1)} \left(c_0^2 + c_1 - 2c_0 \sqrt{c_1} \right) \,.$$

Since $c_0^2 - c_1 = -\beta^2 d_1^2/2$, we obtain the following expression which is equivalent to (3.36):

$$\hat{\tau} = \frac{d_1^2 \beta}{2} \left(\frac{2}{\beta^2 d_1^2}\right)^2 \left[\frac{d_1^4}{32} + \frac{\beta^2 d_1^2}{2} + \frac{d_1^2}{4} \sqrt{\frac{d_1^4}{64} + \frac{\beta^2 d_1^2}{2}}\right] = \frac{1}{\beta} \left[\frac{d_1^2}{16\beta^2} + 1 - \frac{d_1}{2\beta\sqrt{2}} \sqrt{1 + \frac{d_1^2}{32\beta^2}}\right].$$

This completes the derivation of Principal Result 3.5.



Figure 2. Plot of λ versus *m* in the instability band computed numerically from (3.22) for the infinite-line problem with $l = \infty$, $\varepsilon = 0.05$, $\tau = 2$, and s = 0, with q = 1 (top curve) and q = 2 (bottom curve).

In Fig. 2 we illustrate (II) of Principal Result 3.4. For $l = \infty$, $\varepsilon = 0.05$, s = 0, $\tau = 2$, and for both q = 1 and q = 2, we plot the real unstable eigenvalue λ , computed from (3.22), within the instability band $m_{-} < m < m_{+}$. Since $0 < \lambda < 3 - \varepsilon^2 m^2$, we conclude on the range $\mathcal{O}(1) \ll m = \mathcal{O}(\varepsilon^{-1})$ that $\sqrt{1 + m^2 + \tau \lambda} \approx \sqrt{1 + m^2}$ when $\tau = \mathcal{O}(1)$. Therefore, except near the lower threshold m_{-} where $m = \mathcal{O}(1)$, changing τ by $\mathcal{O}(1)$ has a rather negligible effect on the dispersion relation. This is evident by comparing the curves in Fig. 2 and Fig. 1.



Figure 3. Plot of τ_H (left panel) and the Hopf bifurcation frequency λ_{IH} (right panel) versus m when $0 < m < m_{-}$ for the infinite line problem where $l = \infty$, as computed from (3.34) and (3.35) of Principal Result 3.5. The parameter values are $\varepsilon = 0.05$, s = 0, while the solid and heavy solid curves are for q = 1 and q = 2, respectively.

From Fig. 2, we observe that as $m \to m_{-}$ from above, we have $\lambda \to 0$ when q = 2 but $\lambda \to \lambda_{0} > 0$ when q = 1. This result is readily explained from Fig. 3(a) where the Hopf bifurcation threshold τ_{H} for τ is plotted on $0 < m < m_{-}$ for q = 1 and q = 2. The corresponding Hopf bifurcation frequency λ_{IH} is plotted in Fig. 3(b). In Fig. 3(a), we observe for q = 1 that the value $\tau_{H} \approx 1.5$ when $m = m_{-} \approx 1.12$ is below the value $\tau = 2$ used in Fig. 2. In addition, we observe that the stripe is unstable for any m in $0 < m < m_{-}$ when $\tau = 2$. Alternatively, for q = 2, we have from Fig. 3(a) that $\tau_{H} \approx 6.12$ when $m = m_{-} \approx 2.85$. From Fig. 3(b) we conclude that $\lambda_{IH} \to 0^{+}$ as $m \to m_{-}$ from below. From Fig. 3(a) we conclude that $\tau = 2 < \tau_{H}$ for any m in $0 < m < m_{-}$. Thus, the stripe is stable for all modes in $0 < m < m_{-}$ when $\tau = 2$.

Finally, we briefly consider the case where l is finite and $\tau > 0$. Then, the roots λ to (3.15) are equivalently the roots of $\mathcal{F}_l(\lambda) = 0$, where

(3.40 a)
$$\mathcal{F}_{l}(\lambda) \equiv 2\sqrt{1+\hat{\tau}\lambda} \left(\frac{\tanh\left[l\sqrt{1+m^{2}}\sqrt{1+\hat{\tau}\lambda}\right]}{\tanh\left[l\sqrt{1+m^{2}}\right]} \right) - \mathcal{G}_{l}(\lambda), \qquad \mathcal{G}_{l}(\lambda) \equiv d_{0l} - \frac{d_{1l}}{\beta - \lambda},$$

and $d_{0l} \leq 0, d_{1l} < 0$, and β are defined by

$$(3.40 b) d_{0l} = -\frac{2s}{\sqrt{1+m^2}} \frac{\tanh l}{\tanh \left[l\sqrt{1+m^2}\right]} \le 0, d_1 = -\frac{9q}{\sqrt{1+m^2}} \frac{\tanh l}{\tanh \left[l\sqrt{1+m^2}\right]} < 0, \beta \equiv 3 - \varepsilon^2 m^2.$$

Let m_{-} and m_{+} be the two values of m for which $\mathcal{G}_{l}(0) = 2$. For these values, $\lambda = 0$ is a root of $\mathcal{F}_{l}(0) = 0$ in (3.40 *a*). For $\varepsilon \to 0$, we readily calculate that

(3.41)
$$m_+ \sim \frac{\sqrt{3}}{\varepsilon} - \frac{3q \tanh l}{4} + o(1), \qquad m_- \sim \sqrt{z_-^2 - 1}$$

where $z_{-} > 1$ is the unique root of $\kappa(z) = 0$ in z > 1, as defined in (3.19). With these definitions of m_{-} and m_{+} we can use (3.40 *a*) for $\mathcal{G}_{l}(\lambda)$ to derive the same result (3.27) for $\mathcal{G}_{l}(0)$ for the different ranges of m. The second key result concerns the real part $\mathcal{F}_{Rl}(\lambda_{I}) \equiv \operatorname{Re}[\mathcal{F}_{l}(i\lambda_{I})]$, given by

(3.42)
$$\mathcal{F}_{Rl}(\lambda_I) = \mathcal{C}_{Rl}(\lambda_I) - d_0 + \frac{d_1\beta}{\beta^2 + \lambda_I^2}, \qquad \mathcal{C}_{Rl}(\lambda_I) \equiv \operatorname{Re}\left(2\sqrt{1 + i\hat{\tau}\lambda_I} \frac{\tanh\left[l\sqrt{1 + m^2}\sqrt{1 + i\hat{\tau}\lambda_I}\right]}{\tanh\left[l\sqrt{1 + m^2}\right]}\right).$$

In §3 of [43], it was shown that $C_{Rl}(\lambda_I)$ is a monotone increasing function of λ_I , and as such since $d_1 < 0$ we have that $d\mathcal{F}_{Rl}(\lambda_I)/d\lambda_I > 0$ for λ_I . For the case $l = \infty$, this key monotonicity property was established previously in (3.26).

Given that the key properties (3.27) for $\mathcal{G}_l(0)$ (with the new definitions of m_{\pm}) and $d\mathcal{F}_R(\lambda_I)/d\lambda_I > 0$ for $\lambda_I > 0$ both still hold, it is easy to show that Principal Result 3.4 for the case $l = \infty$ still applies to the case of finite l. The details of the derivation are left to the reader. As such, from (II) of Principal Result 3.4, for any l there is a unique unstable eigenvalue for any $\hat{\tau} > 0$ when m lies within the instability band $m_- < m < m_+$. This eigenvalue is on the positive real axis in $0 < \lambda < \beta$.

We remark that Principal Result 3.5 for the range $0 < m < m_{-}$ does not apply to the case where l is finite. For this range of m, there are two eigenvalues on the positive real axis in $0 < \lambda < \beta$ when $\hat{\tau} \gg 1$, and none when $\hat{\tau} = 0$, and that eigenvalues cannot enter $\operatorname{Re}(\lambda) > 0$ through the origin $\lambda = 0$ as $\hat{\tau}$ is varied. As such, by the continuity of the eigenvalue path as $\hat{\tau}$ is varied, there must exist a Hopf bifurcation value $\hat{\tau}$ for each m on $0 < m < m_{-}$. However, it is an open problem to prove that $\hat{\tau}$ is uniquely determined.

3.2 Numerical Validation of the Stability Theory

Next, we demonstrate the breakup instability phenomena by performing some numerical experiments on

(3.43)
$$v_t = \varepsilon_0^2 \Delta v - v + \frac{v^3}{u}, \qquad \tau u_t = \mathcal{D} \Delta u - u + \frac{v^3}{\varepsilon_0},$$

in the rectangular domain 0 < x < 1, $0 < y < d_0$ with homogeneous Neumann conditions. Upon introducing the new variables $(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{u})$ by $\tilde{x} = 2lx - l$, $\tilde{y} = 2ly$, $\tilde{v} = v/(2l)$, and $\tilde{u} = u/(4l^2)$, and defining ε , l, and d by

(3.44)
$$\varepsilon = (2l)\varepsilon_0, \qquad d = 2ld_0, \qquad \text{where} \qquad l = \frac{1}{2\sqrt{\mathcal{D}}},$$

we obtain that (3.43) transforms to the original system (3.1) with q = 1, s = 0, in the domain $|\tilde{x}| \leq l$ and $0 \leq \tilde{y} \leq d$.

We solve (3.43) numerically on a uniformly spaced $N \times N$ grid with enough resolution to spatially resolve the narrow spatial scale near the stripe. In our computations we take N = 300. The system is solved using centred finite differences in space and the stiff solver ode15s with Jacobian in Matlab for the time-stepping. As initial data for (3.43) we use the solution in Principal Result 3.1, when written in terms of the transformed variables of (3.43), in which we add a random perturbation sampled uniformly on $[-\delta, \delta]$ with $\delta = 0.001$.

We undertake three numerical experiments on (3.43). For experiment 1 we take $\varepsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 2$.



Figure 4. Experiment 1: Contour plot of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 2$. This corresponds to $\varepsilon = 0.05$, l = 1/2, $\tau = 0.1$, and d = 2 in (3.1).

For experiment 2 we take $\varepsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 3$, and for experiment 3 we take $\varepsilon_0 = 0.05$, $\mathcal{D} = 0.1$, $\tau = 0.1$, and $d_0 = 2$. Experiments 1 and 2 correspond to $\varepsilon = 0.05$ and l = 1/2 for (3.1). From the dotted dispersion relation in the left panel of Fig. 1 we observe that there is a large band of unstable modes with roughly comparable growth rates near the dominant mode $m_{\text{dom}} \approx 7.42$. In particular, for the parameters from experiment 1, the band 5 < m < 14 of modes have a growth rate within 95% of that for the maximal mode. We therefore expect a strong interference with several modes which will inhibit an accurate prediction of the number of spots. Furthermore, since the random perturbation does not favour positive or negative amplitudes, there will be a phase correction to consider depending on the initial random perturbation. To alleviate these issues, we compute the discrete Fourier transform of the difference between the computed solution vat a given time and its steady-state stripe profile along the mid-line x = 1/2. This plot indicates which discrete Fourier modes, α , have the largest amplitude contribution. In order to test our spot pattern predictions from the asymptotic theory, we consider a solution composed of the inverse discrete Fourier transform of the mode with the largest amplitude so that we can artificially remove the interference from the other large eigenvalue modes.

In Fig. 4 we plot the results from experiment 1 at four different times t = 0 (a), t = 2.64 (b), t = 3.16 (c), and t = 5 (d). In Fig. 5 we plot the discrete Fourier transform results at the same times. We refer to the term "dominant modes" as any modes that have an amplitude within 95% of the maximum mode amplitude. At the bottom of each figure we plot an inverse Fourier transform solution of the most dominant mode which removes the interference of other modes and gives the best prediction of spot breakup. The wave modes α in the Fourier transform diagram are the discrete Fourier wave modes and do not correspond to either m or the integer k modes in the actual perturbation (3.8). From Fig. 4b at t = 2.64 we observe that several spot structures have emerged from the initial stripe, but with no clear pattern owing to



Figure 5. Experiment 1: Discrete Fourier transform of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 2$. The upper left plot shows the amplitudes from the Fourier transform while the upper right plot displays the phase. Dominant modes are defined as any modes that have an amplitude within 95% of the largest amplitude mode. The bottom graphic in each panel shows an inverse Fourier transform of a solution comprised of only the most dominant mode.

the large number of modes near the dominant mode that have comparable growth rates. This is shown through the Fourier transform in Fig. 5b. By t = 3.16, the two-spot pattern has begun to emerge, as predicted by the dispersion relation in Fig. 1. As t increases further, secondary instabilities occur and by t = 5 we are left with a single spot solution.

Fig. 6 and Fig. 7 show the spatial and Fourier results for experiment 2 at times t = 0 (a), t = 2.42 (b), t = 2.64 (c), and t = 5 (d). The difference between experiment 2 and experiment 1 is that the width d_0 of the domain is increased to $d_0 = 3$. While the dispersion relation as a function of m is unchanged by altering d_0 and leaving l fixed, the actual integer modes, k (and similarly the discrete Fourier modes, α), get scaled by a factor of d_0 . For both experiments 1 and 2 the band 5 < m < 14 contains eigenvalues that are 95% that of the maximal mode amplitude. For $d_0 = 2$ as in experiment 1 this corresponds to 3 < k < 8 while in experiment 2 with $d_0 = 3$ this corresponds to 4 < k < 13. The large clustering is then dispersed over a wider range and therefore there will be less competition between adjacent modes. This is noticed most evidently in the Fourier transform of the solution in Fig. 7. In experiment 1 there was a competition between Fourier modes $\alpha = 3$ and $\alpha = 5$, whereas in experiment 2 there is a competition between modes $\alpha = 5$ and $\alpha = 8$. The usefulness



Figure 6. Experiment 2: Contour plot of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 3$. This corresponds to $\varepsilon = 0.05$, l = 1/2, $\tau = 0.1$, and d = 2 in (3.1).

of looking at the Fourier transform is most evident here because it clearly outlines that the most dominant mode produces the predicted four spot pattern starting at t = 2.42 and continuing until secondary instabilities have dominated by t = 5. Conversely, the interference from the competing modes makes it hard to resolve a four spot pattern in Fig. 6b.

Fig. 8 and Fig. 9 show the spatial and Fourier results for experiment 3 at times t = 0 (a), t = 2.64 (b), t = 4.34 (c), and t = 5 (d). This experiment resets $d_0 = 2$ but uses $\mathcal{D} = 0.1$, so that $\varepsilon = 0.05\sqrt{10} \approx 0.1581$, $l = \sqrt{10}/2 \approx 1.58$, $\tau = 0.1$, and $d = 2\sqrt{10} \approx 6.32$ in (3.1). By computing the dispersion curve from (3.15) we obtain that the most unstable mode has decreased to $m_{\text{dom}} \approx 4.24$. Moreover, for this parameter set, the band of unstable eigenvalues has been narrowed and there is a smaller range of modes that are near 95% of the maximal value. This reduced clustering means that the dominant wave mode should emerge very prominently with little competition from other modes. This is evident in Fig. 8c, which confirms the theoretical prediction that the stripe breaks up into a four-spot pattern. A further effect of increasing l is that the magnitude of the unstable eigenvalues decreases, so that a breakup instability takes longer to form. For experiments 1 and 2, breakup instabilities had been initiated by t = 2.64. However, for experiment 3 there is no evidence of a breakup instabilities at this time (see Fig. 8b). Both the reduced competition and longer time for instability initiation are also evident through the Fourier transform results in Fig. 9. At the end time t = 5 we observe from Fig. 9d that we are still well within the linear stability regime and, as such, secondary instabilities have not yet occurred.

4 Existence and Stability of a Ring for the GM Model

In this section we analyze the stability of a ring solution inside a disk Ω for a subclass of the GM model given by

(4.1)
$$v_t = \varepsilon^2 \Delta v - v + \frac{v^3}{u^q}; \qquad \tau u_t = \Delta u - u + \frac{v^3}{\varepsilon u^s}; \qquad x \in \Omega,$$



Figure 7. Experiment 2: Discrete Fourier transform of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 1$, $\tau = 0.1$, and $d_0 = 3$. Upper left and right plots are the amplitudes and phase, respectively, from the Fourier transform. The bottom graphic in each panel shows an inverse Fourier transform of a solution comprised of only the most dominant mode.

with $\partial_n u = \partial_n v = 0$ on $\partial\Omega$. Here q and s satisfy (3.3), and $\Omega \equiv \{(r, \theta) \mid 0 \le r \le l, 0 \le \theta < 2\pi\}$ with r = |x|.

A ring solution is one for which v concentrates on a circle $r = r_0 > 0$ concentric with Ω . A steady-state ring solution is a ring solution for which r_0 is not arbitrary, but satisfies an equilibrium constraint (see (4.13) below). In our analysis below we will construct a ring solution, determine the equilibrium constraint, and then analyze the stability of the ring solution to instabilities that lead to the breakup of the ring into spots. For the exponent set (3.3), the stability analysis is rather simple since the associated NLEP is explicitly solvable.

We first construct the ring solution by the method of matched asymptotic expansions. A construction based on a Lyapunov-Schmidt reduction was given in [29]. In the inner region near the ring at $r = r_0$, we introduce

(4.2)
$$v = V(y) = V_0 + \varepsilon V_1 + \cdots, \quad u = U(y) = U_0 + \varepsilon U_1 + \cdots, \quad y = \varepsilon^{-1}(r - r_0).$$

We substitute (4.2) into the steady-state problem for (4.1), and collect powers of ε . To leading order we get

$$V_0 = U_0^{\gamma} w(y) \,, \qquad \gamma \equiv q/2 \,,$$



Figure 8. Experiment 3: Contour plot of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 0.1$, $\tau = 0.1$, and $d_0 = 2$. This corresponds to $\varepsilon = 0.05\sqrt{10} \approx 0.1581$, $l = \sqrt{10}/2 \approx 1.58$, $\tau = 0.1$, and $d = 2\sqrt{10} \approx 6.32$ in (3.1).

where $w(y) = \sqrt{2} \operatorname{sech} y$ satisfies (3.5), and U_0 is an unknown constant. At next order, we find that V_1 and U_1 satisfy

(4.3)
$$LV_1 \equiv V_1'' - V_1 + 3w^2 V_1 = -\frac{V_0'}{r_0} + \frac{qV_0^3}{U_0^{q+1}} U_1, \qquad U_1'' = -\frac{V_0^3}{U_0^s}, \qquad -\infty < y < \infty$$

Upon integrating the equation for U_1 over $-\infty < y < \infty$, we get

(4.4)
$$U_1'(+\infty) - U_1'(-\infty) = -U_0^{(3q/2)-s} \int_{-\infty}^{\infty} w^3 \, dy$$

Since $LV'_0 = 0$, the solvability condition for the V_1 equation in (4.3) yields

(4.5)
$$\frac{1}{r_0} \int_{-\infty}^{\infty} (V_0')^2 \, dy = \frac{q}{U_0^{q+1}} \int_{-\infty}^{\infty} V_0^3 V_0' U_1 \, dy$$

Upon substituting $V_0 = U_0^{\gamma} w$ into (4.5), and noting that U_1'' is even, we obtain after two integration by parts that

(4.6)
$$\frac{1}{r_0} = -\frac{qI}{8U_0} \left[U_1'(+\infty) + U_1'(-\infty) \right] = -\frac{q}{2U_0} \left[U_1'(+\infty) + U_1'(-\infty) \right] ,$$

where we have used that $I \equiv \int_{-\infty}^{\infty} w^4 \, dy / \int_{-\infty}^{\infty} (w')^2 \, dy = 4$. After calculating $U'_1(\pm \infty)$ below, (4.6) yields a transcendental equation, referred to as the equilibrium constraint, for the equilibrium ring radius r_0 .

To determine $U'_1(\pm \infty)$ we must consider the outer region. In this region v is transcendentally small and $u \sim u_0$, where

(4.7)
$$u_0'' + \frac{1}{r}u_0' - u_0 = 0, \qquad 0 < r < r_0, \quad r_0 < r < l; \qquad u_0'(l) = 0, \quad u_0(0) \text{ finite}$$

The matching conditions between the inner and outer solution are that $u_0(r_0) = U_0$, $u'_0(r_0^+) = U'_1(+\infty)$, and $u'_0(r_0^-) = U_0(r_0^-)$



Figure 9. Experiment 3: Discrete Fourier transform of the solution v to (3.43) at four times with parameter values $\epsilon_0 = 0.05$, $\mathcal{D} = 0.1$, $\tau = 0.1$, and $d_0 = 2$. Upper left and right plots are the amplitudes and phase, respectively, from the Fourier transform. The bottom graphic in each panel shows an inverse Fourier transform of a solution comprised of only the most dominant mode.

 $U'_1(-\infty)$. Therefore, from (4.4), we have that u_0 must satisfy

(4.8)
$$u_0(r_0) = U_0, \qquad u'_0(r_0^+) - u'_0(r_0^-) = -U_0^{(3q/2)-s} \int_{-\infty}^{\infty} w^3 \, dy$$

The solution to (4.7) with $u_0(r_0) = U_0$ is

$$(4.9) u_0 = \frac{U_0}{\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)} \begin{cases} \mathcal{G}_1(r)\mathcal{G}_2(r_0), & 0 < r < r_0, \\ \mathcal{G}_1(r_0)\mathcal{G}_2(r), & r_0 < r < l, \end{cases}; \mathcal{G}_1(r) \equiv I_0(r), \mathcal{G}_2(r) \equiv \frac{K_1(l)}{I_1(l)}I_0(r) + K_0(r) = \frac{K_1(l)}{I_1(l)}I_0(r) + K_0(r) = \frac{K_1(l)}{I_1(l)}I_0(r) + \frac{K_1(l)}{I_1(l)}I_0(r) = \frac{K_1(l)}{I_1(l)}I_0(r) + \frac{K_1(l)}{I_1(l)}I_0(r) = \frac{K_1(l)}{I_1(l)}I_0(r) = \frac{K_1(l)}{I_1(l)}I_0(r) + \frac{K_1(l)}{I_1(l)}I_0(r) = \frac{K_1(l)}{I_1(l)}$$

Here $I_{\nu}(z)$ and $K_{\nu}(z)$ are the modified Bessel functions of the first and second kinds of order ν . Upon enforcing the jump condition in (4.8), we obtain, in terms of the Wronskian $W(a, b) \equiv ab' - a'b$, that U_0 satisfies

(4.10)
$$U_0^{\xi} = \frac{W(\mathcal{G}_2, \mathcal{G}_1)}{\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)\int_{-\infty}^{\infty} w^3 \, dy}, \qquad \xi \equiv \frac{3q}{2} - (s+1) > 0.$$

Since $W(\mathcal{G}_2, \mathcal{G}_1) = W(K_0, I_0) = 1/r_0$ and $\int_{-\infty}^{\infty} w^3 dy = \sqrt{2\pi}$, (4.10) determines U_0 as

(4.11)
$$U_0 = \left[\sqrt{2\pi}r_0\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)\right]^{-\xi}$$

Next, we calculate $u'_0(r_0^{\pm})$ from (4.9) and use the matching condition to obtain

(4.12)
$$u_0'(r_0^+) = \frac{U_0 \mathcal{G}_1'(r_0) \mathcal{G}_2(r_0)}{\mathcal{G}_1(r_0) \mathcal{G}_2(r_0)} = U_1'(+\infty), \qquad u_0'(r_0^-) = \frac{U_0 \mathcal{G}_1(r_0) \mathcal{G}_2'(r_0)}{\mathcal{G}_1(r_0) \mathcal{G}_2(r_0)} = U_1'(-\infty).$$

Finally, upon substituting (4.12) into (4.6), we obtain that an equilibrium ring radius must be a root of the transcendental equation $\mathcal{H}(r_0) = 0$, where

(4.13)
$$\mathcal{H}(r_0) \equiv \frac{1}{r_0} + \frac{q}{2} \left(\frac{\mathcal{G}'_1(r_0)}{\mathcal{G}_1(r_0)} + \frac{\mathcal{G}'_2(r_0)}{\mathcal{G}_2(r_0)} \right)$$



Figure 10. Plot of $\mathcal{H}(r_0)$ versus r_0/l , defined in (4.13) for q = 1 (left panel) and for q = 2 (right panel). In the left panel the curves from left to right near $r_0/l = 0$ are for l = 20, l = 10, l = 4, and l = 1. In the right panel the curves from left to right near $r_0/l = 0$ are for l = 10, l = 5, l = 3.622, and l = 2. For q = 1, there are no roots to $\mathcal{H}(r_0) = 0$. For q = 2, there are two ring equilibria for $l > l_c \approx 3.622$ created in a saddle-node bifurcation at $l = l_c$.

The final step in the construction of the equilibrium ring solution is to investigate the conditions for which (4.13) has roots. This requires a numerical computation of $\mathcal{H}(r_0)$. For several values of l, in Fig. 10(a) and Fig. 10(b) we plot $\mathcal{H}(r_0)$ versus r_0/l for q = 1 and q = 2, respectively. Our numerical results show that there are no equilibrium ring radii on $0 < r_0 < l$ when q = 1. For q = 2, there are two equilibria when $l > l_c \approx 3.622$ and none when $l < l_c$. These equilibria are created in a saddle-node bifurcation when $l = l_c$. In Fig. 11 we plot the two equilibrium ring radii versus l for $l > l_c$ when q = 2, as obtained by numerically computing the roots of $\mathcal{H}(r_0) = 0$ from (4.13).



Figure 11. Plot of the equilibrium ring radii versus l for $l > l_c \approx 3.622$ when q = 2, where l_c is the saddle-node point.

We remark that (4.13) was also derived in [29] in a more general context. However, no NLEP analysis of the stability of

the ring solution to non-radially symmetric perturbations was given in [29]. In the next subsection, we derive this NLEP, show that it is explicitly solvable, and obtain a simple transcendental equation for any unstable eigenvalue. In this way, a band of unstable angular modes m is readily identified.

4.1 The Stability of a Ring Solution for the GM Model

Let v_e and u_e denote the ring solution with ring radius r_0 . To examine the stability of this solution to breakup into spots on an $\mathcal{O}(1)$ time-scale we let r, θ denote polar coordinates and we introduce the perturbation

(4.14)
$$v = v_e + e^{\lambda t + im\theta}\phi(r), \qquad u = u_e + e^{\lambda t + im\theta}\eta(r)$$

To enforce 2π periodicity in θ we take m > 0 to be an integer. Upon substituting (4.14) into (4.1), we obtain that

(4.15)
$$\varepsilon^{2} \left(\phi'' + \frac{1}{r} \phi' \right) - \left(1 + \frac{\varepsilon^{2} m^{2}}{r^{2}} \right) \phi + \frac{3v_{e}^{2}}{u_{e}^{q}} \phi - \frac{qv_{e}^{3}}{u_{e}^{q+1}} \eta = \lambda \phi ,$$
$$\left(\eta'' + \frac{1}{r} \eta' \right) - \left(1 + \tau \lambda + \frac{m^{2}}{r^{2}} \right) \eta = -\frac{1}{\varepsilon} \left(\frac{3v_{e}^{2} \phi}{u_{e}^{s}} - \frac{sv_{e}^{3}}{u_{e}^{s+1}} \eta \right) ,$$

with $\phi'(l) = \eta'(l) = 0$ and ϕ , η bounded as $r \to 0$.

In the inner region where $y = \varepsilon^{-1}(r - r_0)$ with $|y| = \mathcal{O}(1)$, we use $u_e \sim U_0$, $v_e \sim U_0^{q/2} w$, and expand $\phi = \Phi(y) + \cdots$ and $\eta = N_0 + \varepsilon N_1 + \cdots$, where N_0 is a constant to be determined. From (4.15), we obtain that

(4.16)
$$\Phi'' - \Phi + 3w^2 \Phi - q U_0^{q/2 - 1} w^3 N_0 = \left(\lambda + \frac{\varepsilon m^2}{r_0^2}\right) \Phi, \qquad -\infty < y < \infty$$

Upon integrating the resulting equation for N_1 we get

(4.17)
$$N_1'(+\infty) - N_1'(-\infty) = -3U_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi \, dy + sU_0^{\xi} \left(\int_{-\infty}^{\infty} w^3 \, dy \right) N_0 \, .$$

In the outer region, $0 < r < r_0$ and $r_0 < r < l$, we expand $\eta = \eta_0 + \cdots$, to obtain from (4.15) that η_0 satisfies

(4.18)
$$\eta_0'' + \frac{1}{r}\eta_0' - (1+\tau\lambda)\eta_0 - \frac{m^2}{r^2}\eta_0 = 0, \qquad 0 < r < r_0 \quad \text{or} \quad r_0 < r < l; \qquad \eta_0'(l) = 0, \quad \eta_0(0) \text{ finite} = 0,$$

The matching conditions between the inner and outer solution are that $\eta_0(r_0) = N_0$, $\eta'_0(r_0^+) = N'_1(+\infty)$, and $\eta'_0(r_0^-) = N'_1(-\infty)$. Thus, from (4.17), we conclude that

(4.19)
$$\eta_0(r_0) = N_0, \qquad \eta'_0(r_0^+) - \eta'_0(r_0^-) = -3U_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi \, dy + sU_0^{\xi} \left(\int_{-\infty}^{\infty} w^3 \, dy \right) N_0.$$

The solution to (4.18) with $\eta_0(r_0) = N_0$ is

(4.20 a)
$$\eta_0 = \frac{N_0}{\mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r_0)} \begin{cases} \mathcal{G}_{1m}(\theta_\lambda r)\mathcal{G}_{2m}(\theta_\lambda r_0), & 0 < r < r_0, \\ \mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r), & r_0 < r < l, \end{cases}$$

where we have defined

(4.20 b)
$$\mathcal{G}_{1m}(\theta_{\lambda}r) \equiv I_m(\theta_{\lambda}r), \qquad \mathcal{G}_{2m}(\theta_{\lambda}r) \equiv K_m(\theta_{\lambda}r) - \frac{K'_m(\theta_{\lambda}l)}{I'_m(\theta_{\lambda}l)} I_m(\theta_{\lambda}r), \qquad \theta_{\lambda} \equiv \sqrt{1+\tau\lambda}$$

Here $I_m(z)$ and $K_m(z)$ are the two modified Bessel functions of order m, and again we have chosen the principal branch of $\sqrt{1 + \tau \lambda}$ (see the discussion following (3.13)). Upon enforcing the jump condition in (4.19), we obtain an algebraic

equation for N_0 , with solution

(4.21)
$$N_0 = \eta_0(r_0) = 3U_0^{q-s} \int_{-\infty}^{\infty} w^2 \Phi \, dy \left[sU_0^{\xi} \int_{-\infty}^{\infty} w^3 \, dy + \frac{W(\mathcal{G}_{2m}, \mathcal{G}_{1m})}{\mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r_0)} \right]^{-1}$$

The Wronskian in (4.21) is evaluated as $W(\mathcal{G}_{2m}, \mathcal{G}_{1m}) = \theta_{\lambda} W[K_m(\theta_{\lambda}r_0), I_m(\theta_{\lambda}r_0)] = 1/r_0$. Then, by substituting (4.21) into (4.16), and using (4.10) for U_0^{ξ} , we obtain an NLEP for Φ , which is summarized formally as follows:

Principal Result 4.1 Let $\varepsilon \to 0$ and consider a ring solution for (4.1) where the GM exponent set satisfies (3.3). Let $r_0 > 0$ be the radius of the ring that is concentric with the disk 0 < r < l and satisfies $r_0 < l$. Then, the stability of this ring solution on an $\mathcal{O}(1)$ time-scale is determined by the spectrum of the NLEP

(4.22 a)

$$L_0 \Phi - \chi w^3 \int_{-\infty}^{\infty} w^2 \Phi \, dy = \left(\lambda + \frac{\varepsilon^2 m^2}{r_0^2}\right) \Phi \quad -\infty < y < \infty; \quad \Phi \to 0 \quad as \quad |y| \to \infty,$$

$$\chi \equiv \frac{3q}{\int_{-\infty}^{\infty} w^3 \, dy} \left[s + \frac{\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)}{\mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r_0)}\right]^{-1},$$

where \mathcal{G}_1 , \mathcal{G}_2 and \mathcal{G}_{1m} , \mathcal{G}_{2m} are defined in terms of r_0 , $\tau\lambda$, and m by (4.9) and (4.20 b), respectively. In (4.22 a), $L_0\Phi \equiv \Phi'' - \Phi + 3w^2\Phi$. Any unstable eigenvalue of (4.22 a) is a root of the transcendental equation $\mathcal{R}(\lambda) = 0$, where

(4.22 b)
$$\mathcal{R}(\lambda) \equiv \frac{2\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)}{\mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r_0)} - \mathcal{G}_{\star}(\lambda), \qquad \mathcal{G}_{\star}(\lambda) \equiv -2s - \frac{9q}{\lambda - \beta}, \qquad \beta \equiv 3 - \frac{\varepsilon^2 m^2}{r_0^2}.$$



Figure 12. Plot of λ versus *m* within the instability band, as computed from (4.23), for the two equilibrium ring solutions with $r_0 \approx 2.56$ and $r_0 \approx 1.08$ that exist when l = 5 and q = 2 (see Fig. 11). The other parameter values are $\varepsilon = 0.05$ and s = 0. Left panel: $\tau = 0$. The heavy solid and solid curves are for the larger and smaller equilibrium ring radii, respectively. Right panel: the curves in the left panel are compared with the dispersion relations (dotted curves) obtained from changing $\tau = 0$ to $\tau = 4$. The curves in the left panel are rather insensitive to changes in τ .

To derive (4.22 *b*) from (4.22 *a*) we simply note that (4.22 *a*) is explicitly solvable and use Principal Result 2.2 together with $\int_{-\infty}^{\infty} w^5 dy = \left(\int_{-\infty}^{\infty} w^3 dy\right) (3/2)$. The roots of (4.22 *b*) can be equivalently written as

(4.23)
$$\lambda = 3 - \frac{\varepsilon^2 m^2}{r_0^2} - \frac{9q}{2} \left[s + \frac{\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)}{\mathcal{G}_{1m}(\theta_\lambda r_0)\mathcal{G}_{2m}(\theta_\lambda r_0)} \right]^{-1}$$

We remark that (4.23) becomes an explicit expression for λ when $\tau = 0$ since $\theta_{\lambda} = 1$. By using well-known asymptotics for $K_m(z)$ and $I_m(z)$ for large orders m, it follows for $\varepsilon \to 0$ and $\tau = 0$ that $\lambda < 0$ when $m > m_+ \sim \sqrt{3}r_0/\varepsilon$. For $\tau = 0$,

 $\varepsilon = 0.05$, and s = 0, in Fig. 12 we use (4.23) to plot λ versus m within the instability band for the two equilibrium ring solutions with radii $r_0 \approx 2.56$ and $r_0 \approx 1.08$ that exist when l = 5 and q = 2 (see Fig. 11). From the left panel of Fig. 12, we observe that the maximum growth rates for the two ring solutions are roughly the same, but that there are fewer unstable modes for the solution with the smaller ring radius. In the right panel of Fig. 12 we show the marginal effect on the dispersion curves of changing τ from $\tau = 0$ (solid curves) to $\tau = 4$ (dotted curves). The dispersion curves for $\tau = 4$ were obtained by using Newton's method on the transcendental equation (4.23) for λ .

Finally, we briefly outline the computer-assisted derivation using (4.22 b) that for any $\tau > 0$ there is a unique unstable eigenvalue in $\operatorname{Re}(\lambda) > 0$ for any m in some band $0 < m_- < m < m_+$. To determine the edges m_- and m_+ of the band we set $\lambda = 0$ in (4.23). To determine the upper band edge $m_+ \gg 1$ as $\varepsilon \to 0$, we use well-known large order expansions of $K_m(z)$ and $I_m(z)$, to derive from (4.20 b) that for $m \to \infty$,

(4.24)
$$\mathcal{G}_{1m}(r_0)\mathcal{G}_{2m}(r_0) \sim \frac{1}{2m} \left[1 + e^{-2m(l-r_0)} \right] \sim \frac{1}{2m}, \quad \text{for} \quad r_0 < l$$

We then substitute this expression into (4.23), where we set $\lambda = 0$, to obtain for $m \gg 1$ that

$$3 - \frac{\varepsilon^2 m^2}{r_0^2} \sim \frac{9q}{2} \left(\frac{1}{s + 2m\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)} \right) \sim \frac{9q}{4m\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)} + \mathcal{O}(m^{-2}).$$

Upon solving for m, we calculate for $\varepsilon \to 0$ that

(4.25)
$$m_{+} \sim \frac{\sqrt{3}r_{0}}{\varepsilon} - \frac{3q}{8\mathcal{G}_{1}(r_{0})\mathcal{G}_{2}(r_{0})} + o(1), \quad \text{as} \quad \varepsilon \to 0.$$

In contrast, for $\varepsilon \to 0$, the lower edge m_{-} of the band, with $m_{-} = \mathcal{O}(1)$, is a root of $\mathcal{M}(m) = 0$, where

(4.26)
$$\mathcal{M}(m) \equiv \frac{2\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)}{\mathcal{G}_{1m}(r_0)\mathcal{G}_{2m}(r_0)} + (2s - 3q)$$

We calculate $\mathcal{M}(0) = 2(1 + s - 3q/2) < 0$ from (3.3). In addition, since $\mathcal{G}_{1m}(r_0)\mathcal{G}_{2m}(r_0)$ is a monotone decreasing function of m with asymptotics $\mathcal{G}_{1m}(r_0)\mathcal{G}_{2m}(r_0) \sim 1/(2m)$ as $m \to \infty$ (see (4.24)), it follows that $\mathcal{M}(m)$ is monotone increasing in m with $\mathcal{M}(m) \to +\infty$ as $m \to \infty$. Therefore, there exists a unique root m_- to (4.26).

A winding number argument, which relies on a numerical computation, is then used to show that there is a unique root to (4.22 b) in $\operatorname{Re}(\lambda) > 0$ for any m on the range $m_- < m < m_+$. Proceeding as in §3.1, we choose the counterclockwise contour consisting of the imaginary axis $-iR \leq \operatorname{Im}\lambda \leq iR$ and the semi-circle Γ_R , given by $|\lambda| = R > 0$, for $-\pi/2 \leq \arg\lambda \leq \pi/2$. On the range $m_- < m < m_+$, $\mathcal{R}(\lambda)$ is analytic in $\operatorname{Re}(\lambda) > 0$ except at the simple pole $\lambda = \beta$. By using the large argument expansions of $K_m(z)$ and $I_m(z)$, valid for $\operatorname{Re}(z) > 0$, we readily derive that $\mathcal{R}(\lambda) \sim 2r_0\sqrt{\tau\lambda}\mathcal{G}_1(r_0)\mathcal{G}_2(r_0)$ as $|\lambda| \to \infty$ on Γ_R . Therefore, the change in the argument of $\mathcal{R}(\lambda)$ over Γ_R as $R \to \infty$ is $\pi/2$. By using the argument principle, together with $\mathcal{R}(\overline{\lambda}) = \overline{\mathcal{R}(\lambda)}$, we obtain for any $\tau > 0$ that the number J of unstable eigenvalues in $\operatorname{Re}(\lambda) > 0$ is

(4.27)
$$J = \frac{5}{4} + \frac{1}{\pi} \left[\arg \mathcal{R} \right]_{\Gamma_I} \,.$$

Here $[\arg \mathcal{F}]_{\Gamma_I}$ is the change in the argument of \mathcal{F} when the semi-axis $\Gamma_I = i\lambda_I$, $0 \leq \lambda_I < \infty$ is traversed downwards. To calculate $[\arg \mathcal{R}]_{\Gamma_I}$, we decompose $\mathcal{R}(i\lambda_I) = \mathcal{R}_R(\lambda_I) + i\mathcal{R}_I(\lambda_I)$. On the range $m_- < m < m_+$, we have $\mathcal{R}_R(0) > 0$ and $\mathcal{R}_I(0) = 0$. Moreover, we have $\mathcal{R}(i\lambda_I) \sim 2r_0 e^{\pi i/4} \sqrt{\tau \lambda_I} \mathcal{G}_1(r_0) \mathcal{G}_2(r_0)$ as $\lambda_I \to \infty$. Therefore, it follows that $[\arg \mathcal{R}]_{\Gamma_I} = -\pi/4$, and consequently J = 1 from (4.27), if one can guarantee that $\mathcal{R}_R(\lambda_I) > 0$ for all $\lambda_I > 0$. The analytical verification of this inequality is difficult, as it requires detailed computations of global properties of modified Bessel functions of complex arguments. However, a simple numerical computation of these Bessel functions shows that this inequality is indeed satisfied. We conclude that, for any $\tau > 0$, there exists a unique unstable eigenvalue on the positive real axis for the ring solution when m lies within the band $m_- < m < m_+$.

5 Hot Spot Patterns of Urban Crime: The Stability of a Stripe

The urban crime model of [35] that incorporates the effect of police has the form

(5.1 a) $A_t = \varepsilon^2 \Delta A - A + \rho A + \alpha, \quad x \in \Omega; \qquad \partial_n A = 0, \quad x \equiv (x_1, x_2) \in \partial\Omega,$

(5.1 b)
$$\rho_t = D\nabla \cdot \left(A^2 \nabla \left(\rho/A^2\right)\right) - \rho A + \gamma - \alpha - U, \quad x \in \Omega; \qquad \partial_n \rho = 0, \quad x \in \partial \Omega$$

(5.1 c)
$$\tau_u U_t = D\nabla \cdot (A^q \nabla (U/A^q)), \quad x \in \Omega; \qquad \partial_n U = 0, \quad x \in \partial\Omega,$$

where Ω is the rectangular domain of (3.2). Here A measures the attractiveness to burglary, while ρ and U denote the densities of criminals and police, respectively. The diffusivity of criminals and police is D and D/τ_u , respectively. We will assume that $\tau_u > 0$ with $\tau_u = \mathcal{O}(1)$. In (5.1), the constants $\alpha > 0$ and $\gamma - \alpha > 0$ model the baseline attractiveness and the background rate of criminal re-introduction after a burglary, respectively. The drift, or convection term, in (5.1 b) and (5.1 c) model biased random walk of the criminals and police towards regions of higher attractiveness, respectively, with the parameter q > 0 measuring the strength of this drift for the police. For q = 2, the police mimic their drift response to that of the criminals. Further details of the model, as well as related models for including the effect of police, are given in [36]–[38], [15], and [47]. In the construction of the steady-state stripe solution below, we will need to assume that q > 1. For the case q = 3, it will be shown that the NLEP governing the transverse stability of this stripe is explicitly solvable.

We will analyze the transverse stability properties of a steady-state stripe solution for (5.1) for the regime $\mathcal{O}(1) \ll D \ll \mathcal{O}(\varepsilon^{-2})$. As motivated by the scalings in [18], we introduce the new variables v, D_0 , and u by

(5.2)
$$\rho = \varepsilon^2 v A^2, \qquad U = u A^q, \qquad D = D_0 / \varepsilon^2$$

where we will assume that $D_0 \gg \mathcal{O}(\varepsilon^2)$ so that $D \gg 1$. In terms of these new variables, (5.1) becomes

(5.3 a)
$$A_t = \varepsilon^2 \Delta A - A + \varepsilon^2 v A^3 + \alpha, \quad x \in \Omega; \qquad \partial_n A = 0, \quad x \equiv (x_1, x_2) \in \partial\Omega,$$

(5.3 b)
$$\varepsilon^{2} \left(vA^{2} \right)_{t} = D_{0} \nabla \cdot \left(A^{2} \nabla v \right) - \varepsilon^{2} vA^{3} + \gamma - \alpha - uA^{q}, \quad x \in \Omega; \qquad \partial_{n} v = 0, \quad x \in \partial\Omega;$$

(5.3 c)
$$\varepsilon^{2} \tau_{u} \left(A^{q} u \right)_{t} = D_{0} \nabla \cdot \left(A^{q} \nabla u \right), \quad x \in \Omega; \qquad \partial_{n} u = 0, \quad x \in \partial \Omega.$$

We now construct a steady-state stripe solution consisting of a localized region of high attractiveness, which we center along the mid-line $x_1 = 0$ of the rectangle. To do so, we simply construct a steady-state 1-D pulse $A = A(x_1)$, $v = v(x_1)$, and $u = u(x_1)$, and extend it trivially in the x_2 direction. Since the total number of police is conserved due to (5.1 c), we have $\int_{\Omega} U(x,t) dx = \mathcal{U}$ for all time, where $\mathcal{U} > 0$ is the initial number of police deployed. As such, we have for the steady-state stripe solution that $\int_{-l}^{l} U(x_1) dx_1 = U_0 \equiv \mathcal{U}/d$, where d is the width of Ω . It follows from (5.3 c) that the steady-state 1-D solution $u(x_1)$ is a constant given by

(5.4)
$$u(x_1) = U_0 / \int_{-l}^{l} \left[A(x_1) \right]^q \, dx_1 \, ,$$

and that the steady-state 1-D problem for $A(x_1)$ and $v(x_1)$, from (5.3 a) and (5.3 b), is

(5.5 a)
$$\varepsilon^2 A_{x_1 x_1} - A + \varepsilon^2 v A^3 + \alpha = 0, \quad |x_1| \le l; \quad A_{x_1}(\pm l) = 0,$$

(5.5 b)
$$D_0 \left(A^2 v_{x_1} \right)_{x_1} - \varepsilon^2 v A^3 + \gamma - \alpha - \frac{U_0 A^q}{\int_{-l}^{l} A^q \, dx_1} = 0, \qquad |x_1| \le l \,; \qquad v_{x_1}(\pm l) = 0,$$

In the inner region $|x_1| \leq \mathcal{O}(\varepsilon)$ near the pulse we set $y = x_1/\varepsilon$ and expand $A = A_0/\varepsilon + \cdots$ and $v = v_0 + \cdots$ as in [18].

We readily obtain that v_0 is a constant and $A_{0yy} - A_0 + v_0 A_0^3 = 0$. This yields the leading order inner solution

(5.6)
$$A(x_1) \sim \frac{1}{\varepsilon \sqrt{v_0}} w \left(x_1 / \varepsilon \right) , \qquad v \sim v_0 ,$$

where $w(y) = \sqrt{2} \operatorname{sech} y$ satisfies (3.5), and where the constant v_0 is to be determined. To determine the constant v_0 , we integrate (5.5 b) over $|x_1| \leq l$ to get $-\varepsilon^2 \int_{-l}^{l} v A^3 dx_1 + 2l(\gamma - \alpha) - U_0 = 0$. Since $A = \mathcal{O}(\varepsilon^{-1})$ in the inner region, while $A = \mathcal{O}(1)$ in the outer region, the integral above can be calculated asymptotically as $\varepsilon^2 \int_{-l}^{l} v A^3 dx_1 \sim v_0^{-1/2} \int_{-\infty}^{\infty} w^3 dy = v_0^{-1/2} \sqrt{2\pi}$. In this way, provided that $0 < U_0 < 2l(\gamma - \alpha)$, we calculate v_0 as

(5.7)
$$v_0 = \frac{2\pi^2}{\left[2l(\gamma - \alpha) - U_0\right]^2}$$

Therefore a stripe solution exists only if the total police deployment U_0 per cross-sectional area satisfies $U_0 < 2l(\gamma - \alpha)$. In the inner region, the police concentration $U(x_1)$, given by $U(x_1) = uA^q$, becomes

(5.8)
$$U = \frac{U_0 A^q}{\int_{-l}^{l} A^q \, dx_1} \sim \frac{U_0}{\varepsilon} \frac{w^q}{\int_{-\infty}^{\infty} w^q \, dy}, \qquad |x_1| \le \mathcal{O}(\varepsilon)$$

In the outer region, we have from (5.5 *a*) that $A_{out} = \alpha + \mathcal{O}(\varepsilon^2)$. To determine the leading-order outer problem for v, we first need to estimate the integral $\int_{-l}^{l} A^q dx_1$ in (5.5 *b*). Since $A = \mathcal{O}(\varepsilon^{-1})$ in the inner region $|x_1| \leq \mathcal{O}(\varepsilon)$, while $A = \mathcal{O}(1)$ in the outer region $\mathcal{O}(\varepsilon) \ll |x_1| \leq 1$, it follows that when q > 1 the contribution to the integral $\int_{-l}^{l} A^q dx_1$ from the inner region is dominant, with the estimate $\int_{-l}^{l} A^q dx_1 = \mathcal{O}(\varepsilon^{1-q}) \gg 1$. We will henceforth assume that q > 1, so that the nonlocal term in (5.5 *b*) can be neglected to leading-order in the outer region. Then, from (5.5 *b*) we obtain that the leading-order outer problem for v is $v \sim \tilde{v}_0 + o(1)$, where \tilde{v}_0 satisfies

(5.9)
$$\tilde{v}_{0x_1x_1} = -\frac{(\gamma - \alpha)}{D_0\alpha^2}, \quad 0 < |x_1| < l; \quad \tilde{v}_{0x_1}(\pm l) = 0, \quad \tilde{v}_0(0) = v_0.$$

This yields the leading-order outer solution for \tilde{v}_0 as given below in (5.12).

Finally, we calculate u. Since q > 1, we use $A \sim \varepsilon^{-1} w / \sqrt{v_0}$ to estimate the integral in (5.4). This yields that

(5.10)
$$u = \frac{U_0}{\int_{-l}^{l} A^q \, dx_1} \sim \varepsilon^{q-1} \tilde{u}_e, \quad \text{where} \quad \tilde{u}_e \equiv \frac{U_0 v_0^{q/2}}{\int_{-\infty}^{\infty} w^q \, dy}$$

We summarize our result in the following statement.

Principal Result 5.1 For $\varepsilon \ll 1$, $D \gg 1$, and $U_0 < 2l(\gamma - \alpha)$, the steady-state pulse solution for (5.3) is given to leading order in the inner region by

(5.11)
$$A(x_1) \sim \frac{1}{\varepsilon \sqrt{v_0}} w(x_1/\varepsilon) , \qquad v \sim v_0 , \qquad U(x_1) \sim \frac{U_0}{\varepsilon} \frac{w^q}{\int_{-\infty}^{\infty} w^q \, dy} , \qquad |x_1| \le \mathcal{O}(\varepsilon) ,$$

where $v_0 \equiv 2\pi^2 \left[2l(\gamma - \alpha) - U_0\right]^{-2}$, $U_0 = \mathcal{U}/d$, and $w = \sqrt{2} \operatorname{sech}(x_1/\varepsilon)$. In the outer region, $\mathcal{O}(\varepsilon) \ll |x_1| \le l$, we have

(5.12)
$$A \sim \alpha, \quad v \sim \frac{(\gamma - \alpha)}{2D_0 \alpha^2} \left[l^2 - (l - |x_1|)^2 \right] + v_0, \quad U \sim \varepsilon^{q-1} U_0 \alpha^q \frac{v_0^{q/2}}{\int_{-\infty}^{\infty} w^q \, dy}.$$

The criminal density in the inner and outer regions, as obtained from (5.2), is

(5.13)
$$\rho(x_1) \sim \left[w\left(x_1/\varepsilon\right)\right]^2, \quad |x_1| = \mathcal{O}(\varepsilon); \qquad \rho(x_1) \sim \varepsilon^2 \alpha^2 \left[v_0 + \frac{(\gamma - \alpha)}{2D_0 \alpha^2} \left(l^2 - (l - |x_1|)^2\right)\right], \quad \mathcal{O}(\varepsilon) \ll |x_1| \le l$$

In Fig. 13(a) we plot the asymptotic results for A and ρ in the inner region, as obtained from Principal Result 5.1, for



Figure 13. Plot of the steady-state pulse from Principal Result 5.1 in the inner region for $\varepsilon = 0.05$, l = 1.0, $\gamma = 2$, and $\alpha = 1$. Left panel: The attractiveness $A \sim w(x_1/\varepsilon)/(\varepsilon\sqrt{v_0}) + \alpha$ (heavy solid curve) and criminal density $\rho \sim w(x_1/\varepsilon)$ (solid curve) Right panel: The police density U from (5.12) for q = 2 (heavy solid curve) and for q = 3 (solid curve).

 $\varepsilon = 0.05$, l = 1, $\gamma = 2$, and $\alpha = 1$. These results are independent of q. In Fig. 13(b) we plot the corresponding police density $U(x_1)$, as given in (5.11), for q = 2 and q = 3. The police density for q = 3 is slightly more narrow and has a larger peak than for q = 2.

5.1 The Stability of a Stripe

Next, we will derive an NLEP governing the stability of the stripe solution to transverse perturbations that lead to the breakup of the stripe into localized hot spots. Let A_e , v_e , and u_e denote the steady-state solution constructed in the previous subsection and summarized in Principal Result 5.1. We then extend it trivially in the x_2 direction to make a stripe. To determine the stability of this stripe with respect to transverse perturbations we introduce

(5.14)
$$A = A_e(x_1) + e^{\lambda t + imx_2} \phi(x_1), \qquad v = v_e(x_1) + e^{\lambda t + imx_2} \varepsilon \psi(x_1), \qquad u = u_e + e^{\lambda t + imx_2} \varepsilon^q \eta(x_1).$$

Here $m = k\pi/d$ where d is the width of the rectangle and k > 0 is an integer. The relative sizes in ε in (5.14) are such that ϕ , ψ , and η are all $\mathcal{O}(1)$ in the inner region. Upon substituting (5.14) into (5.3), we obtain on $|x_1| \leq l$ that

(5.15 a)
$$\varepsilon^2 \phi_{x_1 x_1} - (1 + \varepsilon^2 m^2)\phi + 3\varepsilon^2 v_e A_e^2 \phi + \varepsilon^3 A_e^3 \psi = \lambda \phi,$$

$$(5.15 b) \quad D_0 \left[\varepsilon A_e^2 \psi_{x_1} + 2A_e v_{ex_1} \phi \right]_{x_1} - \varepsilon m^2 D_0 A_e^2 \psi - 3\varepsilon^2 v_e A_e^2 \phi - \varepsilon^3 \psi A_e^3 - \varepsilon^q A_e^q \eta - q A_e^{q-1} u_e \phi = \lambda \varepsilon^2 \left(\varepsilon A_e^2 \psi + 2A_e v_e \phi \right) \,,$$

$$(5.15 c) D_0 \left[\varepsilon^q A_e^q \eta_{x_1} + q A_e^{q-1} u_{ex_1} \phi \right]_{x_1} - \varepsilon^q m^2 D_0 A_e^q \eta = \varepsilon^2 \tau_u \lambda \left(\varepsilon^q A_e^q \eta + q A_e^{q-1} u_e \phi \right) \,.$$

In the analysis of (5.15) we must allow for spatial perturbations of high frequency as $\varepsilon \to 0$. As such, we consider the range $0 < m \leq \mathcal{O}(\varepsilon^{-1})$. Below, we show that the upper stability threshold occurs when $m = \mathcal{O}(\varepsilon^{-1})$.

In (5.15 *b*) and (5.15 *c*), we note that $u_{ex_1} = 0$ and $u_e \sim \varepsilon^{q-1}\tilde{u}_e$, where \tilde{u}_e is given in (5.10). In the outer region where $A_e \sim \alpha$ we obtain from (5.15 *a*) that $\phi_{out} = \mathcal{O}(\varepsilon^3 \psi_{out})$. when $0 < m \leq \mathcal{O}(\varepsilon^{-1})$. Next, we estimate the terms in (5.15 *b*) in the outer region. We obtain from (5.10), and our estimate of ϕ_{out} , that $qA_e^{q-1}u_e\phi = \mathcal{O}(\varepsilon^{q+2}\psi_{out})$. Moreover, since q > 1, we have $\varepsilon^q A_e^q \eta \ll \mathcal{O}(\varepsilon)$. In this way, we obtain in the outer region that (5.15 *b*) reduces to

(5.16)
$$\psi_{x_1x_1} - m^2 \psi = 0, \qquad \mathcal{O}(\varepsilon) < |x_1| \le l; \qquad \psi_{x_1}(\pm l) = 0.$$

Similarly, since $\tau_u = \mathcal{O}(1)$ and $u_{ex_1} = 0$, we obtain from (5.15 c) that to leading order

(5.17)
$$\eta_{x_1x_1} - m^2 \eta = 0, \qquad \mathcal{O}(\varepsilon) < |x_1| \le l; \qquad \eta_{x_1}(\pm l) = 0.$$

In the inner region, we look for a localized eigenfunction for ϕ in the form $\phi = \Phi(x_1/\varepsilon)$. Since the equations for ψ and

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 η are not singularly perturbed, we obtain that $\psi \sim \psi(0)$ and $\eta \sim \eta(0)$ to leading order in the inner region. Then, since $A_e \sim \varepsilon^{-1} w / \sqrt{v_0}$ and $v_e \sim v_0$ in the inner region, as obtained from (5.11), we find from (5.15 *a*) that $\Phi(y)$ satisfies

(5.18)
$$\Phi'' - \Phi + 3w^2 \Phi + \frac{1}{v_0^{3/2}} w^3 \psi(0) = \left(\lambda + \varepsilon^2 m^2\right) \Phi, \qquad -\infty < y < \infty$$

Next, we derive the jump conditions for η and ψ across x = 0. To do so, we introduce an intermediate length-scale δ with $\mathcal{O}(\varepsilon) \ll \delta \ll 1$ and integrate (5.15 b) from $-\delta < x_1 < \delta$ and use $A_e \sim \alpha$ at $x_1 = \pm \delta$. This yields that

(5.19 *a*)
$$e_0 \left[\psi_{x_1}\right]_0 = e_1 \psi(0) + e_2 \eta(0) + e_3 \eta(0) +$$

where we have defined $[\psi_{x_1}]_0 \equiv \psi_{x_1}(0^+) - \psi_{x_1}(0^-)$. Here e_j for $j = 0, \ldots, 3$ are defined by

(5.19 b)

$$e_{0} = D_{0}\alpha^{2}, \qquad e_{1} = \frac{D_{0}m^{2}}{\varepsilon v_{0}} \int_{-\infty}^{\infty} w^{2} \, dy + \frac{1}{v_{0}^{3/2}} \int_{-\infty}^{\infty} w^{3} \, dy,$$

$$e_{2} = \frac{1}{v_{0}^{q/2}} \int_{-\infty}^{\infty} w^{q} \, dy, \qquad e_{3} = 3 \int_{-\infty}^{\infty} w^{2} \Phi \, dy + \frac{q\tilde{u}_{e}}{v_{0}^{(q-1)/2}} \int_{-\infty}^{\infty} w^{q-1} \Phi \, dy.$$

In a similar way, we integrate (5.15 c) across $-\delta < x_1 < \delta$ and use $A_e \sim \alpha$ at $x_1 = \pm \delta$. This yields that

(5.20 *a*)
$$f_0 [\eta_{x_1}]_0 = f_1 \eta(0) + f_2,$$

where f_j for $j = 0, \ldots, 2$ are defined by

(5.20 b)
$$f_0 = D_0 \alpha^q, \qquad f_1 = \frac{D_0 m^2 \varepsilon^{1-q}}{v_0^{q/2}} \int_{-\infty}^{\infty} w^q \, dy + \frac{\varepsilon^{3-q} \tau_u \lambda}{v_0^{q/2}} \int_{-\infty}^{\infty} w^q \, dy, \qquad f_2 = \frac{\varepsilon^{3-q} q \tilde{u}_e \tau_u \lambda}{v_0^{(q-1)/2}} \int_{-\infty}^{\infty} w^{q-1} \Phi \, dy.$$

In (5.19 b) and (5.20 b), \tilde{u}_e is given in (5.10).

Next, we must solve for $\psi(x_1)$ and $\eta(x_1)$ from the solution to (5.16) and (5.17) subject to the jump conditions (5.19) and (5.20), and the boundary conditions $\psi_{x_1}(\pm l) = \eta_{x_1}(\pm l) = 0$. From this solution, we calculate $\psi(0)$, which then determines the NLEP for $\Phi(y)$ from (5.18). To solve for η , we introduce $G_m(x_1)$ satisfying

(5.21)
$$G_{mx_1x_1} - m^2 G_m = -\delta(x_1), \quad |x_1| \le l; \qquad G_{mx_1}(\pm l) = 0; \qquad G_m(x_1) = \frac{\cosh\left[m(l-|x_1|)\right]}{2m\sinh(ml)},$$

for m > 0. In terms of $G_m(x_1)$, and using $[G_{mx_1}]_0 = -1$, the solution to (5.17) with (5.20) is

(5.22)
$$\eta(x_1) = \eta(0) \frac{G_m(x_1)}{G_m(0)}, \qquad \eta(0) = -\frac{f_2}{f_1 + f_0/G_m(0)}$$

Similarly, for m > 0, the solution to (5.16) subject to (5.19) is

(5.23)
$$\psi(x_1) = \psi(0) \frac{G_m(x_1)}{G_m(0)}, \qquad \psi(0) = -\frac{e_2 \eta(0) + e_3}{e_1 + e_0/G_m(0)}$$

We estimate the asymptotic order of the terms in (5.20 b) as $f_0/G_m(0) = m \tanh(ml)\mathcal{O}(1)$, $f_1 = m^2 \varepsilon^{1-q} \mathcal{O}(1) + \varepsilon^{3-q} \tau_u \mathcal{O}(1)$, and $f_2 = \varepsilon^{3-q} \tau_u \mathcal{O}(1)$. As such, when $\tau_u = \mathcal{O}(1)$ and q > 1, we conclude for any m > 0 with $m \gg \mathcal{O}(\varepsilon)$ that

$$f_1 + \frac{f_0}{G_m(0)} \sim \frac{D_0 m^2 \varepsilon^{1-q}}{v_0^{q/2}} \int_{-\infty}^{\infty} w^q \, dy \,, \qquad \eta(0) = -\frac{f_2}{f_1 + f_0/G_m(0)} \sim \frac{\mathcal{O}(\varepsilon^{3-q})}{\mathcal{O}(\varepsilon^{1-q}m^2)} = \mathcal{O}(\varepsilon^2/m^2) \ll 1 \,.$$

Since $\eta(0) \ll 1$ when q > 1, $\tau_u = \mathcal{O}(1)$, and $m \gg \mathcal{O}(\varepsilon)$, we conclude from (5.23) that, in this parameter regime,

(5.24 a)
$$\psi(0) \sim -\frac{e_3}{e_1 + e_0/G_m(0)}$$

Upon using (5.10) for \tilde{u}_e , the coefficients in (5.24 a) are

(5.24 b)
$$e_3 = 3 \int_{-\infty}^{\infty} w^2 \Phi \, dy + q v_0^{1/2} U_0 \frac{\int_{-\infty}^{\infty} w^{q-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^q \, dy}, \quad e_1 = \frac{D_0 m^2}{\varepsilon v_0} \int_{-\infty}^{\infty} w^2 \, dy + \frac{1}{v_0^{3/2}} \int_{-\infty}^{\infty} w^3 \, dy, \quad e_0 = D_0 \alpha^2 \, dy$$

Upon substituting (5.24 *a*) into (5.18), and by using (5.7) for v_0 together with $\int_{-\infty}^{\infty} w^2 dy = 4$ and $\int_{-\infty}^{\infty} w^3 dy = \sqrt{2\pi}$, we obtain the following NLEP with two nonlocal terms:

(5.25 a)
$$L_0\Phi - \frac{\chi_0 w^3}{\int_{-\infty}^{\infty} w^3 dy} \left(3 \int_{-\infty}^{\infty} w^2 \Phi \, dy + q v_0^{1/2} U_0 \frac{\int_{-\infty}^{\infty} w^{q-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^q \, dy} \right) = \left(\lambda + \varepsilon^2 m^2 \right) \Phi$$

(5.25 b)
$$\chi_0 \equiv \left(1 + \frac{4D_0 m^2 \varepsilon^{-1}}{[2l(\gamma - \alpha) - U_0]} + \frac{4D_0 \alpha^2 \pi^2 m \tanh(ml)}{[2l(\gamma - \alpha) - U_0]^3}\right)^{-1}$$

The analysis of the spectrum of (5.25) is more challenging than the NLEP's of §3 and §4 owing to the presence of the two nonlocal terms. In our analysis below, we will focus on the special case q = 3 for which this NLEP reduces to

(5.26)
$$L_0 \Phi - \chi w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \left(\lambda + \varepsilon^2 m^2\right) \Phi, \qquad \chi \equiv \chi_0 \left[\frac{6l(\gamma - \alpha)}{2l(\gamma - \alpha) - U_0}\right],$$

which is explicitly solvable. Here χ_0 is defined by (5.25 b). It is an open problem to analyze (5.25) for arbitrary q > 1.

The NLEP (5.26) for q = 3 is a special case of the class of explicitly solvable NLEP's of Principal Result 2.2. Upon replacing λ , σ , g(w), and h(w) in (2.2) with $\lambda + \varepsilon^2 m^2$, 3, $w^2 / \int_{-\infty}^{\infty} w^3 dy$, and w^3 , we obtain the following result for the spectrum of (5.26):

Principal Result 5.2 Let $\varepsilon \to 0$, q = 3, $\tau_u = \mathcal{O}(1)$, $U_0 < 2l(\gamma - \alpha)$, with m > 0 and $m \gg \mathcal{O}(\varepsilon)$. Then, the transverse stability of a stripe solution for (5.1) on an $\mathcal{O}(1)$ time-scale is determined by the sign of the discrete eigenvalue

(5.27)
$$\lambda = 3 - \varepsilon^2 m^2 - \frac{9l(\gamma - \alpha)}{[2l(\gamma - \alpha) - U_0]} \left[1 + \frac{4D_0 m^2 \varepsilon^{-1}}{[2l(\gamma - \alpha) - U_0]} + \frac{4D_0 \alpha^2 \pi^2 m \tanh(ml)}{[2l(\gamma - \alpha) - U_0]^3} \right]^{-1}$$

To determine the edges of the instability band for a stripe, we set $\lambda = 0$ in (5.27) and solve for m. For $\varepsilon \ll 1$, the upper edge m_+ of the instability band is $m_+ \sim \sqrt{3}/\varepsilon$, with $\lambda < 0$ for $m > m_+$. In contrast, the lower edge m_- of the instability band satisfies $m_- \sim \varepsilon^{1/2} m_{0-}$ where m_{0-} satisfies

$$3 \sim \frac{9l(\gamma - \alpha)}{[2l(\gamma - \alpha) - U_0]} \left(1 + \frac{4D_0m_0^2}{[2l(\gamma - \alpha) - U_0]} \right)^{-1}.$$

Upon solving for m_{0-} , we conclude for $\varepsilon \ll 1$ that $\lambda > 0$ when

(5.28)
$$\varepsilon^{1/2}m_{0-} < m < \frac{\sqrt{3}}{\varepsilon}, \qquad m_{0-} \equiv \sqrt{\frac{l(\gamma - \alpha) + U_0}{4D_0}}$$

We remark that the lower $\mathcal{O}(\varepsilon^{1/2})$ edge of the band is consistent with the assumption $m \gg \mathcal{O}(\varepsilon)$ used to derive (5.27). In addition, we note that the lower edge of the band increases with the level U_0 of police effort. This shows that as U_0 increases, less transverse modes become unstable.

Finally, we estimate the mode $m_{\rm dom}$ within the instability band that has the largest growth rate. To do so, we set

 $d\lambda/dm = 0$ in (5.27), and obtain that $m_{\rm dom}$ is the root of

$$2\varepsilon^2 m \sim \frac{9l(\gamma - \alpha)}{[2l(\gamma - \alpha) - U_0]} \left(\frac{4D_0 m^2 \varepsilon^{-1}}{[2l(\gamma - \alpha) - U_0]} + \cdots\right)^{-2} \left(\frac{8D_0 m}{\varepsilon [2l(\gamma - \alpha) - U_0]} + \cdots\right)$$

For $\varepsilon \ll 1$, this reduces to $16D_0m^4 \sim 36l\varepsilon^{-1}(\gamma - \alpha)$. For $\varepsilon \ll 1$, this yields the most unstable mode

(5.29)
$$m_{\rm dom} \sim \varepsilon^{-1/4} \left[\frac{9}{4D_0} l(\gamma - \alpha) \right]^{1/4}$$

which is independent of U_0 . We predict that a stripe for the RD crime model on a domain of width d will break up into N localized hot-spots, where N is the closest integer to $m_{\text{dom}}d/2\pi$.

In Fig. 14(a) we use (5.27) to plot λ versus m for the parameter set $\varepsilon = 0.05$, $D_0 = 1$, $\gamma = 2$, $\alpha = 1$, l = 1, and $U_0 = 1$. In the caption of the figure, the asymptotic predictions for the edges of the instability band, as obtained from (5.28), are compared with results from (5.27). From (5.29), the asymptotic prediction for the most unstable mode is $m_{\text{dom}} \sim 2.59$, which compares well with the numerically computed result $m_{\text{dom}} \approx 2.44$ as computed from (5.27). In Fig. 14(b) we compare λ versus m near the lower threshold m_{-} for $U_0 = 1$ and for $U_0 = 1.5$.



Figure 14. Left panel: Plot of λ versus m, as given in (5.27), for $\varepsilon = 0.05$, $D_0 = 1$, $\gamma = 2$, $\alpha = 1$, l = 1, and $U_0 = 1$. The asymptotic prediction as $\varepsilon \to 0$ for the instability band from (5.28) is 0.158 < m < 34.64. The corresponding numerical result is 0.131 < m < 34.56. Right panel: plot of λ versus m near the lower threshold m_- for $U_0 = 1$ (solid curve) and $U_0 = 1.5$ (heavy solid curve). The lower edge of the instability band increases as U_0 increases.

As compared with the most unstable mode associated with the dispersion relation for the GM stripe or ring solutions, as studied in §3 and §4, the most unstable mode m_{dom} for the crime problem has a significantly smaller value. This suggests that a stripe for the crime problem will break up into significantly fewer spots than that for the GM model. For the parameter values in Fig. 14(a) suppose that d = 2 so that $m_{\text{dom}} d/2\pi = 2.59/\pi \approx 0.82$. This suggests that a stripe on a square domain of side-length two should break up into only one spot. To validate this claim we computed full numerical solutions to (5.3) for the parameter set $\varepsilon = 0.05$, $D_0 = 1$, $\gamma = 2$, $\alpha = 1$, $U_0 = 1$, and l = 1. The computations were done using the adaptive grid finite difference solver VLUGR [1]. The initial conditions were taken to be the steady-state stripe solution of Principal Result 5.1 in which the attractiveness field is perturbed by 1% uniformly distributed random noise near the stripe location. The results for A at different times, as shown in the gray-scale plot of Fig. 15, confirm the theoretical prediction that the stripe breaks up into only one spot.

6 Discussion

We have given an explicit analytical characterization of the instability of either homoclinic stripes or rings to transverse perturbations for some RD problems for which the study of the associated NLEP can be reduced to the analysis of some



Figure 15. Gray-scale plot of the attractiveness A, as computed numerically from (5.3) in a square domain of side-length 2 for the parameter set $\varepsilon = 0.05$, $D_0 = 1$, $\gamma = 2$, $\alpha = 1$, and $U_0 = 1$. The initial condition was the steady-state solution of Principal Result 5.1 in which A is perturbed by 1% uniformly distributed random noise near the stripe location. Left: t = 0.01. Left Middle: t = 3.21. Right Middle: t = 3.41. Right: t = 3.61. The stripe breaks up into a solitary spot.

simple algebraic equations in the eigenvalue parameter. A similar simplified stripe stability analysis can also be done for the class (1.6) of RD system.

We remark that RD systems of the form (1.6), for which the associated NLEP is particularly tractable, can also be used as a starting point to investigate qualitatively new problems in the stability theory of pulses. In this general direction, [41] provides an analysis of delayed bifurcations and instabilities of 1-D pulse patterns due to slowly varying control parameters for some specific RD systems of the class (1.6). Moreover, in [10] an analysis of pulse stability in the presence of a time delay in the reaction-kinetics for a subclass of the GM model was given.

We conclude with a brief discussion of some open problems. One problem is to analyze the linear instability of a steady-state ring solution in a concentric sphere for our subclass of the GM model, for which the associated NLEP is explicitly solvable. For this 3-D problem the transverse perturbation can be written in terms of spherical harmonics. Only the steady-state problem was investigated previously in [29]. A second problem is to give a more comprehensive analysis of (2.4) of Principal Result 2.3 in order to identify further simple, but non power-law, nonlinearities that lead to explicitly solvable NLEPs. This would then lead to a generalization of the class (1.6) of RD systems for which the associated NLEP is explicitly solvable. A third open problem is to analyze the stability of a stripe pattern for the crime model (5.1) for the general case where $q \neq 3$ and q > 1. In this case, the analysis of the NLEP is challenging owing to the presence of two nonlocal terms. Finally, for RD systems associated with explicitly solvable NLEP's, it would be interesting to develop a weakly nonlinear theory for the instability of homoclinic stripes to transverse perturbations and to show analytically that this instability is in fact subcritical and leads to the formation of localized spots.

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