Asymptotic Methods for PDE Problems in Fluid Mechanics and Related Systems with Strong Localized Perturbations in Two-Dimensional Domains

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CISM Advanced Course; Asymptotic Methods in Fluid Mechanics: Surveys and Recent Advances

Lecture III: A Range of Miscellaneous Problems

Outline of Lecture II

SPECIFIC PROBLEMS CONSIDERED:

1. Slow Viscous Flow Over a Cylinder with Asymmetric Cross-section
2. Linear Biharmonic BVP with Holes (Problem 5)
3. A Nonlinear Biharmonic BVP: Concentration Phenomena
4. Remarks on Low Peclet Number Flow (Problem 6)
5. Remarks on Localized Solutions in Other Contexts

Slow Viscous Flow Over a Cylinder

Consider slow, steady, two-dimensional flow of a viscous incompressible fluid around an infinitely long straight cylinder. The Reynolds number satisfies $\varepsilon \equiv U_\infty L/\mu \ll 1$ where U_∞ is the velocity of the fluid in the x-direction at infinity, μ is the kinematic viscosity, and $2L$ is the diameter of the cross-section of the cylinder.

Assume first that the cross-sectional shape Ω of the cylinder is asymmetric about the direction of the oncoming stream, The dimensionless stream function ψ satisfies

$$\triangle^2\psi + \varepsilon\, J_\rho\left[\psi, \triangle\psi\right] = 0\,, \quad \text{for} \ \ \rho > \rho_b(\theta)\,, \tag{2.1a}$$

$$\psi = \partial_n\psi = 0\,, \quad \text{on} \ \ \rho = \rho_b(\theta)\,, \tag{2.1b}$$

$$\psi \sim y\,, \quad \text{as} \ \ \rho = (x^2+y^2)^{1/2} \to \infty\,. \tag{2.1c}$$

Here J_ρ is the Jacobian defined by $J_\rho\left[a,b\right] \equiv \rho^{-1}\left(\partial_\rho a\, \partial_\theta b - \partial_\theta a\, \partial_\rho b\right)$. The boundary of the cross-section is $\rho = \rho_b(\theta)$ for $-\pi \le \theta \le \pi$.

Slow Viscous Flow: Asymmetric Body I

For an asymmetric body, not aligned with the stream, a similar hybrid method can be formulated:

In the Oseen region we must solve

$$\Delta^2\Psi_H + J_r(\Psi_H, \Delta\Psi_H) = 0\,, \qquad r > 0\,, \tag{2.2a}$$

$$\Psi_H \sim \rho\sin\theta, \qquad r \to \infty, \tag{2.2b}$$

$$\Psi_H \sim \mathbf{A}(\varepsilon)\cdot\left[\mathbf{x} + \nu(\varepsilon)\mathbf{x}\log|\mathbf{x}| + \nu(\varepsilon)\mathbf{M}\mathbf{x}\right]\,, \qquad r \to 0 \tag{2.2c}$$

Notice again that we have a constraint to determine the vector $\mathbf{A}$.

In the Stokes region, the vector function $\boldsymbol{\psi}_c(\rho,\theta) = (\psi_c^x(r,\theta), \psi_c^y(r,\theta))$ is the canonical inner solution satisfying

$$\Delta^2\boldsymbol{\psi}_c = 0, \qquad (\rho,\theta) \notin \Omega_0, \tag{2.3a}$$

$$\boldsymbol{\psi}_c = \frac{\partial\boldsymbol{\psi}_c}{\partial n} = 0, \qquad (\rho,\theta) \in \partial\Omega_0, \tag{2.3b}$$

$$\boldsymbol{\psi}_c \sim \mathbf{y}\log|\mathbf{y}| + \mathbf{M}\mathbf{y}\,, \qquad \rho = |\mathbf{y}| \to \infty\,. \tag{2.3c}$$

- Here $\mathbf{M}$ is a 2×2 matrix that depends on the shape of the body. It can be found analytically for an ellipse at an angle of inclination

Slow Viscous Flow: Asymmetric Body II

The drag and lift coefficients are given in terms of $\mathbf{A}$ by

$$(C_L, -C_D) = -\frac{4\pi}{\varepsilon}[\nu(\varepsilon)\mathbf{A}(\varepsilon) + \cdots], \qquad \nu(\varepsilon) = -1/\log\varepsilon. \tag{2.4}$$

- For an ellipse of semi-axes a and b at an angle of elevation α to the free-stream

$$m_{11} = \frac{(b-a)\cos^2\alpha - b}{a+b} - \log\left(\frac{a+b}{2}\right), \tag{2.5a}$$

$$m_{12} = m_{21} = \frac{(a-b)\sin\alpha\cos\alpha}{a+b}, \tag{2.5b}$$

$$m_{22} = \frac{(a-b)\cos^2\alpha - a}{a+b} - \log\left(\frac{a+b}{2}\right). \tag{2.5c}$$

- For other shapes fast boundary integral methods based on Goursat's complex variable formula can be used (Greengard, Kropinski, Mayo (1996)).

Slow Viscous Flow: Asymmetric Body III

For an ellipse at an angle α of inclination, the leading-order (first term in log expansion) lift coefficient is

$$C_L = \frac{4\pi}{\varepsilon(\log\varepsilon)^2}\frac{a-b}{a+b}\sin\alpha\cos\alpha. \tag{2.6}$$

A two-term expansion for C_L was found by Shintani et al. (1983);

$$(C_L)_S \sim \frac{4\pi}{R(\log R - t_+)(\log R - t_-)}\left(\frac{a-b}{a+b}\right)\sin 2\alpha, \tag{2.7}$$

where

$$t_\pm = -\gamma + 4\log(2) - \log[1+b/a] \pm \frac{1}{2}\left\{1 + 2\left(\frac{a-b}{a+b}\right)\cos 2\alpha + \left(\frac{a-b}{a+b}\right)^2\right\}^{\frac{1}{2}}.$$

Here, $R = 2\varepsilon$ and $\gamma = 0.5772\ldots$ is Euler's constant.

Slow Viscous Flow: Asymmetric Body IV

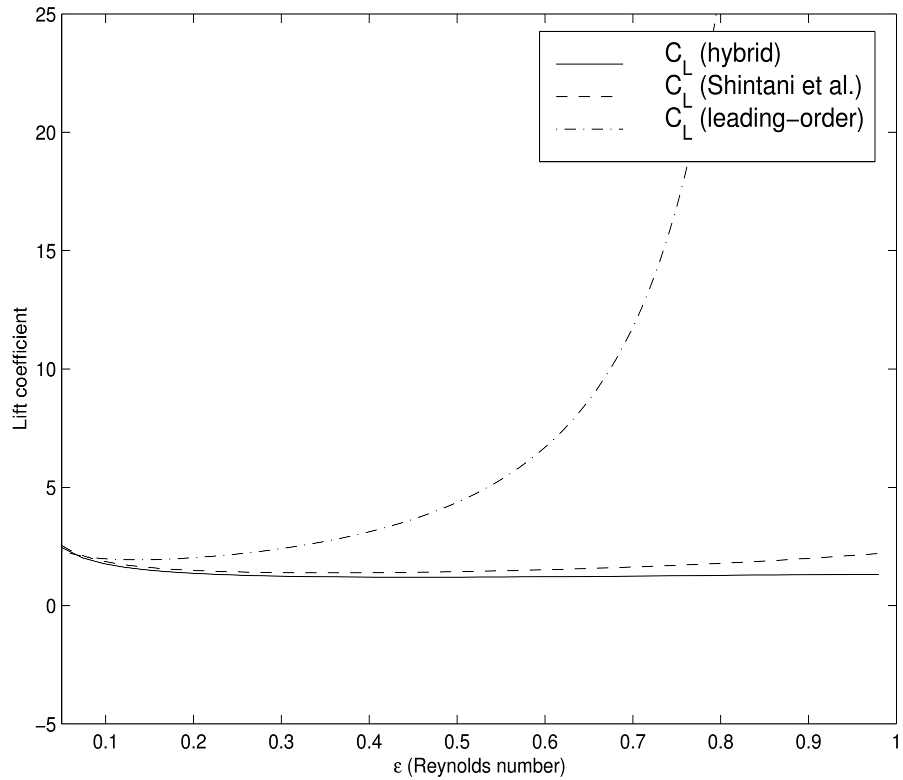


Figure 1: Lift coefficient, C_L, versus Reynolds number, ε, of an elliptic cylinder with major semi-axis $a = 1$ and minor semi-axis $b = 0.5$ at an angle of inclination, $\alpha = \pi/4$, comparing the hybrid results with the leading-order form and the two-term result of Shintani *et al.*

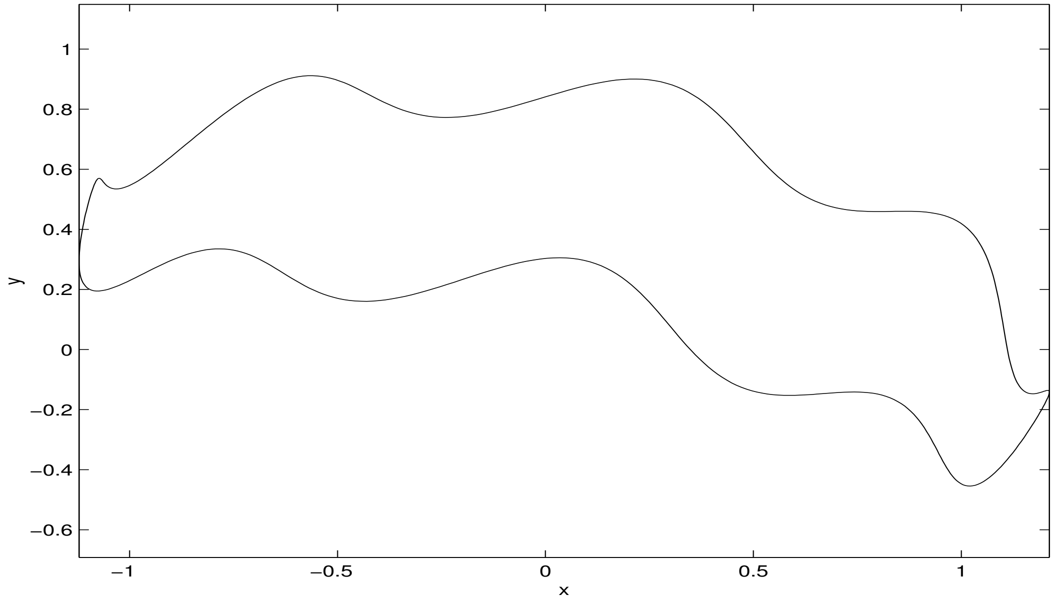
Slow Viscous Flow: Asymmetric Body V

Now apply the hybrid method to a more complicated object. We need only modify the body-shape matrix $\mathbf{M}$ to compute the force coefficients, C_D and C_L. The boundary profile of the object is

$$x = \xi\cos\beta - \eta\sin\beta, \qquad y = \xi\sin\beta + \eta\cos\beta,$$

$$\xi = \frac{a\cos\theta + a\cos\theta}{(a\cos\theta)^2 + (b + a\sin\theta)^2}, \quad \eta = \frac{b + a\sin\theta - (b + a\sin\theta)}{(a\cos\theta)^2 + (b + a\sin\theta)^2} + \frac{b}{3}\cos(N\theta)\,,$$

with $0 \le \theta < 2\pi$. We choose $\beta = -0.3, a = 1.2, b = 0.3, N = 8$.

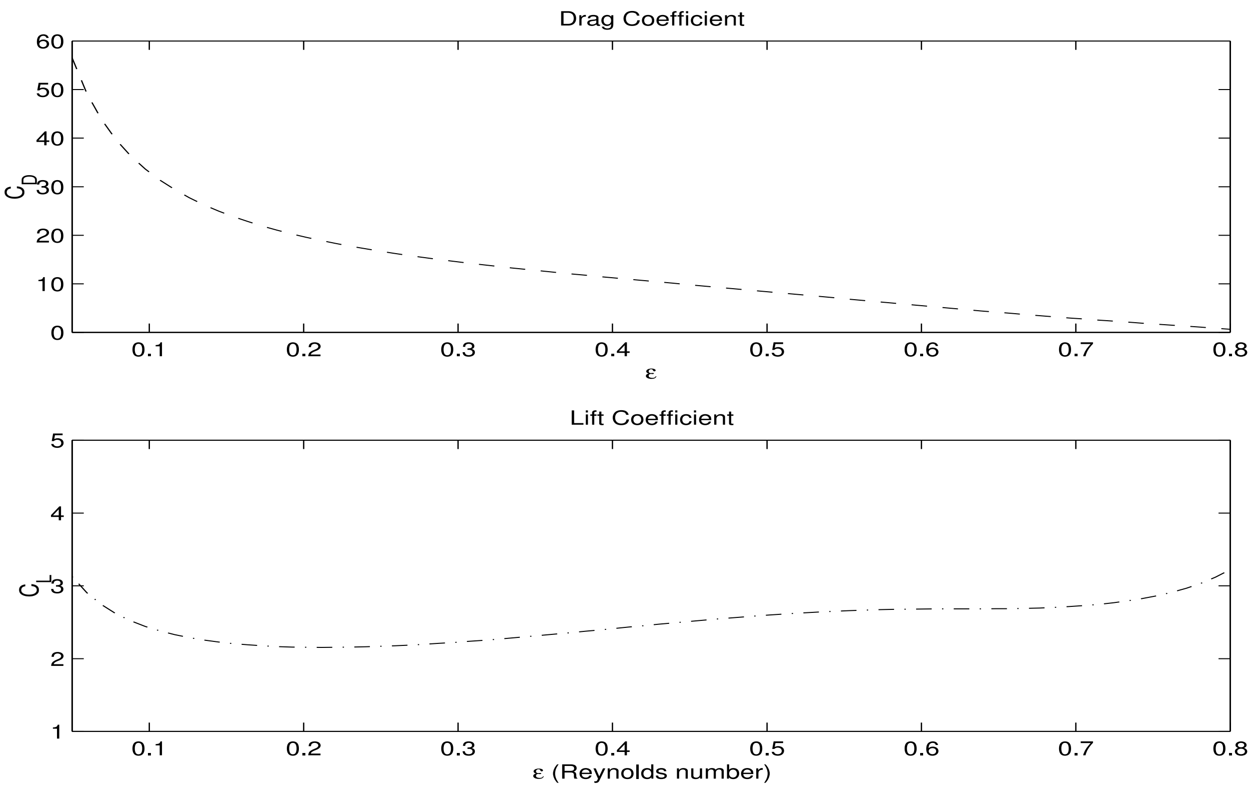


Slow Viscous Flow: Asymmetric Body VI

By using fast boundary integral methods based on Goursat's complex variable formula (Ref: Greengard, Kropinski, Mayo (1996)).

$$\mathbf{M} = \begin{bmatrix} -1.0019045557844 & 0.1550966443197 \\ 0.1550966443197 & -0.5484962829688 \end{bmatrix}.$$

The Drag and Lift coefficients are as shown:



Problem 5 From Notes

Problem 5: Consider the Biharmonic equation in the two-dimensional concentric annulus, formulated as

$$\triangle^2 u = 0 \,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon \,, \tag{5.1a}$$

$$u = f \,, \quad u_r = 0 \,, \quad \text{on} \quad r = 1 \,, \tag{5.1b}$$

$$u = u_r = 0 \,, \quad r = \varepsilon \,. \tag{5.1c}$$

Here Ω is the unit disk centered at the origin, containing a small hole of radius ε centered at $\mathbf{x} = 0$, i.e. $\Omega_\varepsilon = \{\mathbf{x} \mid |\mathbf{x}| \leq \varepsilon\}$. Consider the following two choices for f: **Case I:** $f = 1$. **Case II:** $f = \sin\theta$. For each of these two cases calculate the exact solution, and from it determine an approximation to the solution in the outer region $|\mathbf{x}| \gg \mathcal{O}(\varepsilon)$. Can you re-derive these results from singular perturbation theory in the limit $\varepsilon \to 0$?.

Remark 1: The leading-order outer problem for Case I is different from what you might expect.

Remark 2: For Case 2 one can sum an infinite logarithmic expansion in a similar way as for slow viscous flow. The result can then be verified from the exact solution.

Solution to Problem 5 From Notes: I

Solution:

Case I: We consider the perturbed problem

$$\triangle^2 u = 0\,, \quad \varepsilon < r < 1\,, \tag{5.2a}$$

$$u = 1\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.2b}$$

$$u = u_r = 0\,, \quad \text{on} \quad r = \varepsilon\,. \tag{5.2c}$$

We first find the exact solution of this problem and expand it for $\varepsilon \to 0$.

Since the radially symmetric solutions are linear combinations of $\{r^2, r^2 \log r, \log r, 1\}$, the solution to (5.2a,b) is

$$u = A\left(r^2 - 1\right) + Br^2 \log r - (2A + B)\log r + 1\,, \tag{5.3}$$

for any constants A and B. Then, imposing that $u = u_r = 0$ on $r = \varepsilon$, we get two equations for A and B:

$$2A\left(1 - \varepsilon^2\right) A + B\left(1 - \varepsilon^2 - 2\varepsilon^2 \log \varepsilon\right) = 0\,, \tag{5.4a}$$

$$A\left(1 + 2\log\varepsilon - \varepsilon^2\right) + B\left(1 - \varepsilon^2\right) \log \varepsilon = 1\,. \tag{5.4b}$$

Solution to Problem 5 From Notes: II

Equation (5.4a) gives

$$A = -\frac{B}{2}\left(1 - \frac{2\varepsilon^2 \log \varepsilon}{1-\varepsilon^2}\right) . \tag{5.5}$$

Upon substituting this into (5.4b), we obtain that B satisfies

$$-\frac{B}{2}\left(1 - \frac{2\varepsilon^2 \log \varepsilon}{1-\varepsilon^2}\right)\left(1 + \frac{2\log \varepsilon}{1-\varepsilon^2}\right) + B\log\varepsilon = \frac{1}{1-\varepsilon^2} \tag{5.6a}$$

which reduces after some algebra to

$$-\frac{B}{2} + 2\varepsilon^2 (\log\varepsilon)^2 B \sim 1 + \mathcal{O}(\varepsilon^2) . \tag{5.6b}$$

This determines B, while (5.5) determines A. Therefore,

$$B \sim -2 - 8\varepsilon^2 \left(\log\varepsilon\right)^2 , \qquad A \sim 1 + 4\varepsilon^2 \left(\log\varepsilon\right)^2 . \tag{5.7}$$

Solution to Problem 5 From Notes: III

Upon substituting (5.7) into (5.3), we obtain the following two-term expansion in the outer region $r \gg \mathcal{O}(\varepsilon)$:

$$u \sim u_0(r) + \varepsilon^2 (\log \varepsilon)^2 u_1(r) + \cdots \,, \tag{5.8a}$$

where $u_0(r)$ and $u_1(r)$ are defined by

$$u_0(r) = r^2 - 2r^2 \log r \,, \qquad u_1 = 4\left(r^2 - 1\right) - 8r^2 \log r \,. \tag{5.8b}$$

It is interesting to note that the leading-order outer solution $u_0(r)$ is not a C^2 smooth function as $r \to 0$, but that it does satisfy the point constraint $u_0(0) = 0$.

Hence, in the limit of small hole radius the ε-dependent solution does not tend to the unperturbed solution in the absence of the hole. This unperturbed solution would have $B = 0$ and $A = 0$ in (5.3), and consequently $u = 1$ in the outer region.

Solution to Problem 5 From Notes: IV

Next, we show how to recover (5.8) from a matched asymptotic expansion analysis. In the outer region we expand the solution as

$$u \sim w_0 + \sigma w_1 + \cdots \,, \tag{5.9}$$

where $\sigma \ll 1$ is an unknown gauge function, and where w_0 satisfies:

$$\triangle^2 w_0 = 0\,, \quad 0 < r < 1\,; \quad w_0(1) = 1\,, \quad w_{0r}(1) = 0\,, \quad w_0(0) = 0\,. \tag{5.10}$$

The solution is readily calculated as

$$w_0 = r^2 - 2r^2 \log r\,. \tag{5.11}$$

The problem for w_1 is

$$\triangle^2 w_1 = 0\,, \quad 0 < r < 1\,; \quad w_1(1) = w_{1r}(1) = 0\,. \tag{5.12}$$

Its solution is given in terms of unknown coefficients α_1 and β_1 as

$$w_1 = \alpha_1 \left(r^2 - 1\right) + \beta_1 r^2 \log r - (2\alpha_1 + \beta_1) \log r\,. \tag{5.13}$$

The behavior of w_1 as $r \to 0$, as found below by matching to the inner solution, will determine α_1 and β_1.

Solution to Problem 5 From Notes: V

In the inner region we set $r = \varepsilon\rho$ and obtain from (5.11) that the terms of order $\mathcal{O}(\varepsilon^2 \log \varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ will be generated in the inner region.

Therefore, this suggests that in the inner region we expand the solution as

$$v(\rho) = \left(\varepsilon^2 \log \varepsilon\right) v_0(\rho) + \varepsilon^2 v_1(\rho) + \cdots \,. \tag{5.14}$$

The functions v_0 and v_1 must satisfy $v_j(1) = v_{j\rho}(1) = 0$. Therefore, we obtain for $j = 0, 1$ that

$$v_j = A_j\left(\rho^2 - 1\right) + B_j \rho^2 \log \rho - (2A_j + B_j) \log \rho \,. \tag{5.15}$$

We substitute (5.15) into (5.14), and write the resulting expression in terms of the outer variable $r = \varepsilon\rho$.

Solution to Problem 5 From Notes: VI

A short calculation gives that the far-field behavior of (5.14) is

$$v \sim -(\log\varepsilon)^2 B_0 r^2 + (\log\varepsilon)\left[(A_0 - B_1)r^2 + B_0 r^2 \log r\right] + A_1 r^2 + B_1 r^2 \log r + 2A_0\varepsilon^2 (\log\varepsilon)^2 + \mathcal{O}(\varepsilon^2 \log\varepsilon)\,. \tag{5.16}$$

In contrast, the two-term outer solution from (5.9), (5.11) and (5.13) is

$$u \sim r^2 - 2r^2 \log r + \sigma\left[\alpha_1\left(r^2 - 1\right) + \beta_1 r^2 \log r - (2\alpha_1 + \beta_1)\log r\right] + \cdots\,. \tag{5.17}$$

Upon comparing (5.16) with (5.17), we conclude that

$$B_0 = 0\,, \quad B_1 = A_0\,, \quad A_1 = 1\,, \quad B_1 = -2\,, \quad \sigma = \varepsilon^2 (\log\varepsilon)^2\,. \tag{5.18}$$

The constant term $-4\varepsilon^2(\log\varepsilon)^2$ on the right-hand side of (5.16) is unmatched. Consequently, w_1 is bounded as $r \to 0$ and has the point value $w_1(0) = -4$. Thus, $2\alpha_1 + \beta_1 = 0$ and $\alpha_1 = 4$ in (5.17). This gives $\beta_1 = -8$, and specifies the second-order term (reproducing the exact solution) as

$$w_1 = 4\left(r^2 - 1\right) - 8r^2 \log r\,. \tag{5.19}$$

Solution to Problem 5 From Notes: VII

Remark: It is impossible to match to an outer solution u_0 that does not satisfy the point constraint $u_0(0) = 0$. In addition, we further remark that point constraints are possible with the Biharmomic operator, since the free-space Green's function has singularity $\mathcal{O}\left(|\mathbf{x} - \mathbf{x}_0|^2 \log |\mathbf{x} - \mathbf{x}_0|\right)$ as $\mathbf{x} \to \mathbf{x}_0$. However, with a point constraint we will not have C^2 smoothness.

Satisfying point constraints with the biharmonic operator is the basis of what is known as Biharmonic interpolation.

Solution to Problem 5 From Notes: VIII

Case II: Next, we consider the perturbed problem

$$\triangle^2 u = 0\,, \quad \varepsilon < r < 1\,, \tag{5.20a}$$

$$u = \sin\theta\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.20b}$$

$$u = u_r = 0\,, \quad \text{on} \quad r = \varepsilon\,. \tag{5.20c}$$

We first find the exact solution of (5.20) and expand it for $\varepsilon \to 0$. Since the solutions to (5.20) proportional to $\sin\theta$ are linear combinations of $\{r^3, r\log r, r, r^{-1}\}\sin\theta$, the solution to (5.20a,b) is

$$u = \left(Ar^3 + Br\log r + \left(-2A + \frac{1}{2} - \frac{B}{2}\right) r + \left(\frac{1}{2} + A + \frac{B}{2}\right)\frac{1}{r}\right)\sin\theta\,, \tag{5.21}$$

for any A and B. Then, imposing that $u = u_r = 0$ on $r = \varepsilon$, we get

$$A\varepsilon^3 + B\varepsilon\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right)\varepsilon + \left(\frac{1}{2} + A + \frac{B}{2}\right)\varepsilon^{-1} = 0\,, \tag{5.22a}$$

$$3A\varepsilon^2 + B + B\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right) - \left(\frac{1}{2} + A + \frac{B}{2}\right)\varepsilon^{-2} = 0\,. \tag{5.22b}$$

Solution to Problem 5 From Notes: IX

By comparing the $\mathcal{O}(\varepsilon^{-1})$ and $\mathcal{O}(\varepsilon^{-2})$ terms in (5.22), it follows that

$$\frac{1}{2} + A + \frac{B}{2} = \kappa\varepsilon^2 \,, \tag{5.23}$$

where κ is an $\mathcal{O}(1)$ constant to be found. Substituting (5.23) into (5.22), and neglecting the higher order $A\varepsilon^3$ and $3A\varepsilon^2$ terms in (5.22), we get

$$B\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right) \approx -\kappa \,, \quad B + B\log\varepsilon + \left(-2A + \frac{1}{2} - \frac{B}{2}\right) \approx \kappa \,. \tag{5.24}$$

Add the two equations to eliminate κ, to get

$$B + 2B\log\varepsilon + (-4A + 1 - B) = 0 \,. \tag{5.25}$$

From (5.25), together with $A \sim -(1+B)/2$ from (5.23), we obtain that

$$B \sim \frac{3\nu}{2-\nu} \,, \qquad A = 1 - \frac{3}{2-\nu} \,, \quad \text{where} \quad \nu \equiv \frac{-1}{\log\left[\varepsilon e^{1/2}\right]} \,. \tag{5.26}$$

Solution to Problem 5 From Notes: X

Finally, substituting (5.26) into (5.21), we obtain that the outer solution has the asymptotics

$$u \sim \left((1-\tilde{A})r^3 + \nu \tilde{A} r \log r + \tilde{A} r \right) \sin\theta \,, \quad r \gg \mathcal{O}(\varepsilon) \,. \tag{5.27a}$$

where $\tilde{A}$ is defined by

$$\tilde{A} \equiv \frac{3}{2-\nu} \,, \qquad \nu \equiv \frac{-1}{\log\left[\varepsilon e^{1/2}\right]} \,. \tag{5.27b}$$

We remark that (5.27) is an infinite-order logarithmic series approximation to the exact solution. However, it does not contain transcendentally small terms of algebraic order in ε as $\varepsilon \to 0$.

Notice again the loss of smoothness, this time proportional to a directional derivative of the free-space Green's function.

Solution to Problem 5 From Notes: XI

Next, we show how to derive (5.27) by employing the hybrid formulation used to treat the slow viscous flow problem.

In order to sum the infinite logarithmic series we formulate a hybrid method with a singularity structure. In the inner region, with inner variable $\rho \equiv \varepsilon^{-1} r$, we look for an inner (Stokes) solution in the form

$$v(\rho, \theta) = u(\varepsilon\rho, \theta) \sim \varepsilon \nu \tilde{A}(\nu) \left(\rho \log \rho - \frac{\rho}{2} + \frac{1}{2\rho} \right) \sin\theta \, . \tag{5.28}$$

Here $\nu \equiv -1/\log\left[\varepsilon e^{1/2}\right]$ and $\tilde{A} \equiv \tilde{A}(\nu)$ is a function of ν to be found. The extra factor of ε in (5.28) is needed since the solution in the outer region is not algebraically large as $\varepsilon \to 0$.

Now letting $\rho \to \infty$, and writing (5.28) in terms of the outer variable $r = \varepsilon\rho$, we obtain that the far-field form of (5.28) is

$$v \sim \left(\tilde{A}\nu r \log r + \tilde{A} r \right) \sin\theta \, . \tag{5.29}$$

Solution to Problem 5 From Notes: XII

Therefore, the hybrid solution w_H to (5.20) that sums all the logarithmic terms must satisfy

$$\triangle^2 w_H = 0\,, \quad 0 < r < 1\,, \tag{5.30a}$$

$$w_H = \sin\theta\,, \quad w_{Hr} = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.30b}$$

$$w_H \sim \left(\tilde{A}\nu r \log r + \tilde{A} r\right)\sin\theta\,, \quad \text{as} \quad r \to 0\,. \tag{5.30c}$$

Note: a singularity structure with regular and singular parts specified

The solution to (5.30a,b) in terms of unknown constants α and β is

$$w_H = \left(\alpha r^3 + \beta r \log r + \left(-2\alpha + \frac{1}{2} - \frac{\beta}{2}\right) r + \left(\frac{1}{2} + \alpha + \frac{\beta}{2}\right)\frac{1}{r}\right)\sin\theta\,. \tag{5.31}$$

The condition (5.30c) then yields three equations for α, β, and $\tilde{A}$:

$$\beta = \tilde{A}\nu\,, \qquad -2\alpha + \frac{1}{2} - \frac{\beta}{2} = \tilde{A}\,, \qquad \frac{1}{2} + \alpha + \frac{\beta}{2} = 0\,, \tag{5.32}$$

Solution to Problem 5 From Notes: XIII

We solve to obtain

$$\beta = \tilde{A}\nu \,, \qquad \tilde{A} = \frac{3}{2-\nu} \,, \qquad \alpha = 1 - \tilde{A} \,. \tag{5.33}$$

Upon substituting (5.33) into (5.31), we obtain that this agrees with the asymptotics of the exact solution.

This simple example of Case II has shown explicitly, without numerical methods, that the hybrid asymptotic numerical method for summing infinite logarithmic expansions agrees with the results that can be obtained from the exact solution.

Linear Biharmonic BVP I

Consider the deflection of a plate with N holes that is subject to a loading:

$$\triangle^2 u = F(\mathbf{x}) \,, \quad \mathbf{x} \in \Omega \backslash \Omega_p \qquad \Omega_p \equiv \bigcup_{j=1}^{N} \Omega_{\varepsilon_j} \,, \tag{6.1a}$$

$$u = \partial_n u = 0, \quad \mathbf{x} \in \partial\Omega \,. \tag{6.1b}$$

$$u = \partial_n u = 0, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j} \,, \;\; j = 1, \ldots, N \,. \tag{6.1c}$$

Let $u_p(\mathbf{x})$ solve the unperturbed problem

$$\triangle^2 u_p = F(\mathbf{x}) \,, \quad \mathbf{x} \in \Omega \,; \qquad u_p = \partial_n u_p = 0 \,, \quad \mathbf{x} \in \partial\Omega \,. \tag{6.2}$$

We look for a two-term asymptotic solution in the form

$$u = u_0 + \sigma u_1 + \cdots \,, \tag{6.3}$$

where we must impose that u_0 satisfy the point constraints $u_0(\mathbf{x}_j) = 0$ for $j = 1, \ldots, N$. The leading-order solution u_0 has the form

$$u_0 = u_p + \sum_{i=1}^{N} A_i G(\mathbf{x}; \mathbf{x}_i) \,. \tag{6.4}$$

Linear Biharmonic BVP II

The coefficients A_i are determined from the Biharmonic interpolation equations

$$\sum_{i=1}^{N} A_i G(\mathbf{x}_j; \mathbf{x}_i) = -u_p(\mathbf{x}_j)\,. \tag{6.5}$$

Here $G(\mathbf{x};\boldsymbol{\xi})$ is the Biharmonic Green's function satisfying

$$\triangle^2 G = \delta(\mathbf{x}-\boldsymbol{\xi})\,, \quad \mathbf{x}\in\Omega\,; \qquad G = \partial_n G = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{6.6}$$

Then, $G(\mathbf{x};\boldsymbol{\xi})$ can be written in terms of a singular and regular part as

$$G(\mathbf{x};\boldsymbol{\xi}) = \frac{1}{8\pi}|\mathbf{x}-\boldsymbol{\xi}|^2 \log|\mathbf{x}-\boldsymbol{\xi}| + R(\mathbf{x};\boldsymbol{\xi})\,. \tag{6.7}$$

For the unit disk $|\mathbf{x}| = r$ with $r<1$ with $\boldsymbol{\xi} = \mathbf{0}$, then

$$G(\mathbf{x};0) = \frac{1}{8\pi} r^2 \log r - \frac{1}{16\pi}(r^2-1)\,. \tag{6.8}$$

Expanding the outer solution u_0 as $\mathbf{x}\to\mathbf{x}_j$ yields

$$u_0 + \sigma u_1 \sim \mathbf{a}_j\cdot(\mathbf{x}-\mathbf{x}_j) + \cdots + \sigma u_1 + \cdots\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_j \tag{6.9}$$

Linear Biharmonic BVP III

In the j^{th} inner region we write $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)$ and get Stokes equation. The inner solution has the form

$$v = \nu \mathbf{a}_j \cdot \boldsymbol{\psi}_c + \cdots \tag{6.10}$$

where $\boldsymbol{\psi}_c$ is the vector Stokes solution for low Re flow

$$\Delta^2 \boldsymbol{\psi}_c = 0, \qquad (\rho, \theta) \notin \Omega_j, \tag{6.11a}$$

$$\boldsymbol{\psi}_c = \frac{\partial \boldsymbol{\psi}_c}{\partial n} = 0, \qquad (\rho, \theta) \in \partial\Omega_j, \tag{6.11b}$$

$$\boldsymbol{\psi}_c \sim \mathbf{y} \log |\mathbf{y}| + \mathbf{M}_j \mathbf{y}\,, \qquad \rho = |\mathbf{y}| \to \infty\,. \tag{6.11c}$$

Writing the far-field form for v in outer variables, and choosing $\nu = -1/\log\varepsilon$, we get

$$v \sim \mathbf{a}_j \cdot (\mathbf{x} - \mathbf{x}_j) + \nu \left[\mathbf{a}_j \cdot (\mathbf{x} - \mathbf{x}_j) \log |\mathbf{x} - \mathbf{x}_j| + \mathbf{a}_j \cdot \mathcal{M}_j (\mathbf{x} - \mathbf{x}_j)\right] \tag{6.12}$$

Therefore, $\sigma = \nu = -1/\log\varepsilon$ and we cn find a problem for u_1 etc....
Remark: This problem is essentially Case II and we can formulate a problem to sum the infinite logarithmic expansions etc..

Problem 6 From Notes: I

Problem 6: *Consider the following convection-diffusion equation for $T(\mathbf{X})$, with $\mathbf{X}=(X_1,X_2)$ posed outside two circular disks Ω_j for $j=1,2$ of a common radius a, and with a center-to-center separation $2L$ between the two disks:*

$$\kappa \triangle T = \mathbf{U}\cdot\nabla T\,, \quad \mathbf{X}\in\mathbb{R}^2 \backslash \cup_{j=1}^{2}\,\Omega_j\,, \tag{7.1a}$$

$$T = T_j\,, \quad \mathbf{X}\in\partial\Omega_j\,, \quad j=1,2\,, \tag{7.1b}$$

$$T \sim T_\infty\,, \quad |\mathbf{X}|\to\infty\,. \tag{7.1c}$$

Here $\kappa>0$ is constant, T_j for $j=1,2$ and T_∞ are constants, and $\mathbf{U}=\mathbf{U}(\mathbf{X})$ is a given bounded flow field with $\mathbf{U}(\mathbf{X})\to(\mathbf{U}_\infty,\mathbf{0})$ as $|\mathbf{X}|\to\infty$, where U_∞ is constant.

- Non-dimensionalize (7. 1) in terms of U_∞ and the length-scale $\gamma=\kappa/U_\infty$ to derive a convection-diffusion equation outside of two circular disks of radii $\varepsilon\equiv U_\infty a/\kappa$, with inter-disk separation $2L\varepsilon/a$. Here ε is the Peclet number.

Problem 6 From Notes: II

- In the low Peclet number limit $\varepsilon \to 0$ show how a hybrid asymptotic-numerical solution can be implemented to sum the infinite logarithmic expansions for two different distinguished limits: **Case 1:** $L/a = \mathcal{O}(1)$. **Case 2:** $L/a = \mathcal{O}(\varepsilon^{-1})$.
- For a uniform flow with $\mathbf{U} = (U_\infty, 0)$ for $\mathbf{X} \in \mathbb{R}^2$, determine the required Green's function and its regular part.

Remark: For Case 1, we require an explicit formula for the logarithmic capacitance, d, of two disks of a common radius, a, and with a center-to-center separation of $2l$. The result is

$$\log d = \log(2\beta) - \frac{\xi_c}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\xi_c}}{m\cosh(m\xi_c)}\,, \tag{7.2}$$

where β and ξ_c are determined in terms of a and l by

$$\beta = \sqrt{l^2 - a^2}\,; \qquad \xi_c = \log\left[\frac{l}{a} + \sqrt{\left(\frac{l}{a}\right)^2 - 1}\right]\,. \tag{7.3}$$

Solution to Problem 6 From Notes: I

Solution:
We introduce the dimensionless variables $\mathbf{x}$, $\mathbf{u}(\mathbf{x})$, and $w(\mathbf{x})$ by

$$\mathbf{x} = \mathbf{X}/\gamma\,, \quad T = T_\infty w\,, \quad u(\mathbf{x}) = \mathbf{U}(\gamma\mathbf{x})/U_\infty\,, \quad \gamma \equiv \kappa/U_\infty\,. \tag{7.4}$$

We define the dimensionless centers of the two circular disks by $\mathbf{x}_j$ for $j = 1, 2$, and their constant boundary temperatures α_j for $j = 1, 2$, by

$$\mathbf{x}_j = \mathbf{X}_j/\gamma\,, \qquad \alpha_j = w_j/T_\infty\,, \qquad j = 1, 2\,. \tag{7.5}$$

Then, (7. 1) transforms in dimensionless form to

$$\triangle w = \mathbf{u}\cdot\nabla w\,, \qquad \mathbf{x} \in \mathbb{R}^2 \backslash \cup_{j=1}^2 D_{\varepsilon j}\,, \tag{7.6a}$$
$$w = \alpha_j\,, \qquad \mathbf{x} \in \partial D_{\varepsilon j}\,, \quad j = 1, 2\,, \tag{7.6b}$$
$$w \sim 1\,, \qquad |\mathbf{x}| \to \infty\,. \tag{7.6c}$$

Here $D_{\varepsilon j} = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_j| \le \varepsilon\}$ is the circular disk of radius ε centered at $\mathbf{x}_j$. The center-to-center separation is

$$|\mathbf{x}_2 - \mathbf{x}_1| = 2l\varepsilon\,, \qquad l \equiv L/a\,. \tag{7.7}$$

The dimensionless flow has limiting behavior $\mathbf{u} \sim (1, 0)$ as $|\mathbf{x}| \to \infty$.

Solution to Problem 6 From Notes: II

Case 1: Assume that $l = \mathcal{O}(1)$ as $\varepsilon \to 0$, so that $|\mathbf{x}_2 - \mathbf{x}_1| = \mathcal{O}(\varepsilon)$. This is the case where the bodies are close together; it leads to a new type of inner problem.

Assume WLOG that $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}$. Introduce the inner variables

$$\mathbf{y} = \varepsilon^{-1}\mathbf{x}\,, \qquad v(\mathbf{y}) = w(\varepsilon\mathbf{y})\,. \tag{7.8}$$

Then, (7.6a,b) transforms to

$$\triangle_{\mathbf{y}} v = \varepsilon \mathbf{u}_0 \cdot \nabla_{\mathbf{y}} v\,, \qquad \mathbf{y} \in \mathbb{R}^2 \setminus \cup_{j=1}^{2} D_j\,, \tag{7.9a}$$

$$v = \alpha_j\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1, 2\,, \tag{7.9b}$$

Here $D_j = \{\mathbf{y} \mid |\mathbf{y} - \mathbf{y}_j| \leq 1\}$ is the circular disk centered at $\mathbf{y}_j = \mathbf{x}_j/\varepsilon$ of radius one, and $\mathbf{u}_0 \equiv \mathbf{u}(\mathbf{0})$. The inter-disk separation is

$$|\mathbf{y}_2 - \mathbf{y}_1| = 2l\,. \tag{7.10}$$

Look for a solution to (7.9) in the form

$$v = v_0 + \nu A v_c\,, \tag{7.11}$$

where $\nu = \mathcal{O}(-1/\log\varepsilon)$ and $A = A(\nu)$ is to be found.

Solution to Problem 6 From Notes: III

Here v_0 is the solution to

$$\triangle_{\mathbf{y}} v_0 = 0\,, \qquad \mathbf{y} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_j\,, \tag{7.12a}$$

$$v_0 = \alpha_j\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1, 2\,, \tag{7.12b}$$

$$v_0 \ \text{ bounded as } |\mathbf{y}| \to \infty\,. \tag{7.12c}$$

Moreover, $v_c(\mathbf{y})$ is the solution to

$$\triangle_{\mathbf{y}} v_c = 0\,, \qquad \mathbf{y} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} D_j\,, \tag{7.13a}$$

$$v_c = 0\,, \qquad \mathbf{y} \in \partial D_j\,, \quad j = 1, 2\,, \tag{7.13b}$$

$$v_c \sim \log|\mathbf{y}|\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{7.13c}$$

Since D_j for $j = 1, 2$ are non-overlapping circular disks, (7.12) can be solved explicitly using conformal mapping. This gives

$$v_0 \sim v_{0\infty} + o(1)\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{7.14}$$

When $\alpha_1 = \alpha_2 = \alpha_c$, then clearly $v_{0\infty} = \alpha_1$.

Solution to Problem 6 From Notes: IV

Next, we solve (7.13) exactly by introducing bipolar coordinates to get

$$v_c(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1)\,, \quad |\mathbf{y}| \to \infty\,, \tag{7.15}$$

where d is given by setting $a = 1$ in (7.2) and (7.3).

Upon substituting (7.14) and (7.15) into (7.11), the far-field behavior of v gives the required singularity structure for the outer hybrid solution V_0 as

$$V_0 \sim v_{0\infty} + A + \nu A \log|\mathbf{x}|\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{0}\,; \quad \nu \equiv \frac{-1}{\log(\varepsilon d)}\,. \tag{7.16}$$

Therefore, to sum the logarithmic expansion we must solve

$$\triangle V_0 = \mathbf{u} \cdot \nabla V_0\,, \qquad \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}\,; \qquad V_0 \sim 1\,, \quad |\mathbf{x}| \to \infty\,, \tag{7.17}$$

with singularity structure (7.16) as $\mathbf{x} \to \mathbf{0}$.

Remark: In this analysis we have neglected the transcendentally small $\mathcal{O}(\varepsilon)$ term in (7.9), representing a weak drift in the inner region.

Solution to Problem 6 From Notes: V

To solve for V_0 we use Green's function $G(\mathbf{x};\boldsymbol{\xi})$ satisfying

$$\triangle G = \mathbf{u}\cdot\nabla G - \delta(\mathbf{x}-\boldsymbol{\xi})\,, \quad \mathbf{x}\in\mathbb{R}^2\,, \tag{7.18a}$$

$$G(\mathbf{x};\boldsymbol{\xi}) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}| + R(\boldsymbol{\xi};\boldsymbol{\xi}) + o(1)\,, \quad \mathbf{x}\to\boldsymbol{\xi}\,, \tag{7.18b}$$

with $G(\mathbf{x};\boldsymbol{\xi})\to 0$ as $|\mathbf{x}|\to\infty$. Here $R(\boldsymbol{\xi};\boldsymbol{\xi})$ is the regular part of G.

The solution to (7.17) with singular behavior $V_0 \sim \nu A\log|\mathbf{x}|$ as $\mathbf{x}\to\mathbf{0}$ is

$$V_0 = 1 - 2\pi\nu A G(\mathbf{x};\mathbf{0})\,. \tag{7.19}$$

By expanding (7.17) as $\mathbf{x}\to 0$, and equating the regular part of the resulting expression with that in (7.16), we determine $A(\nu)$ as

$$A = \frac{1-v_{0\infty}}{1+2\pi\nu R_{00}}\,, \qquad \nu\equiv\frac{-1}{\log(\varepsilon d)}\,, \qquad R_{00}\equiv R(\mathbf{0};\mathbf{0})\,. \tag{7.20}$$

The outer and inner solutions are then given in terms of A. Finally, one can calculate the Nusselt number, representing the average heat flux across the bodies etc...

Solution to Problem 6 From Notes: VI

Case 2: Assume $l = \mathcal{O}(\varepsilon^{-1})$ as $\varepsilon \to 0$, and define $l = l_0/\varepsilon$ with $l_0 = \mathcal{O}(1)$, so that $|\mathbf{x}_2 - \mathbf{x}_1| = 2l_0$.

This is the case where the small disks of radius ε are separated by $\mathcal{O}(1)$ distances in (7.6).

There are now two distinct inner regions; one near $\mathbf{x}_1$ and the other at an $\mathcal{O}(1)$ distance away centered at $\mathbf{x}_2$. Since each separated disk is a circle of radius ε, it has a logarithmic capacitance $d = 1$.

Therefore, the infinite-logarithmic series approximation $V_0(\mathbf{x};\nu)$ to the outer solution satisfies

$$\triangle V_0 = \mathbf{u} \cdot \nabla V_0 \,, \qquad \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\} \,; \qquad V_0 \sim 1 \,, \quad |\mathbf{x}| \to \infty \,, \tag{7.21a}$$

$$V_0 \sim \alpha_j + A_j + \nu A_j \log |\mathbf{x} - \mathbf{x}_j| \,, \quad \nu \equiv \frac{-1}{\log \varepsilon} \,. \tag{7.21b}$$

The solution to (7.21) is given explicitly by

$$V_0 = 1 - 2\pi\nu \sum_{i=1}^{2} A_i G(\mathbf{x};\mathbf{x}_i) \,. \tag{7.22}$$

Solution to Problem 6 From Notes: VII

Let $\mathbf{x} \to \mathbf{x}_j$ for $j = 1, 2$ in (7.22) and equate the nonsingular part of the resulting expression with the regular part of the singularity structure in (7.21b) This yields a 2×2 system for A_1 and A_2:

$$A_1 \left(1 + 2\pi\nu R_{11}\right) + 2\pi\nu A_2 G_{12} = 1 - \alpha_1 \, ; \tag{7.23a}$$

$$A_2 \left(1 + 2\pi\nu R_{22}\right) + 2\pi\nu A_1 G_{21} = 1 - \alpha_2 \, . \tag{7.23b}$$

where $G_{ij} = G(\mathbf{x}_j; \mathbf{x}_i)$ and $R_{jj} = R(\mathbf{x}_j; \mathbf{x}_j)$ are the Green's function and its regular part.

For uniform flow where $\mathbf{u} = (1, 0)$, then

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{2\pi} \exp\left[\frac{x_1 - \xi_1}{2}\right] K_0\left(|\mathbf{x} - \boldsymbol{\xi}|\right) \, , R(\boldsymbol{\xi}, \boldsymbol{\xi}) = \frac{1}{2\pi}\left(\log 2 - \gamma_e\right) \, . \tag{7.24}$$

A similar result for G and R can be found for a shear flow etc..

These results for G and its regular part can be used in the results of either Case I or Case II.

Other Problems: Ostwald Ripening I

A similar hybrid method can be used for some time-dependent problems with localized solutions:

Ostwald Ripening: The diffusive evolution of small particles during the late stage coarsening of a first order phase transformation. The chemical potential $u(\mathbf{x};\varepsilon)$, satisfies

$$\begin{aligned}
\Delta u = 0, &\quad \text{in domain D, outside of N particles} \\
u = H, &\quad \text{on i-th particle boundary, } \partial \mathrm{D}_i^{\varepsilon},\ i = 1, \ldots, \mathrm{N} \\
\frac{\partial u}{\partial n} = 0, &\quad \text{on domain boundary} \\
V = -\left[\!\left[\frac{\partial u}{\partial n} \right]\!\right], &\quad \text{on i-th particle boundary, } \partial \mathrm{D}_i^{\varepsilon},\ i = 1, \ldots, \mathrm{N}
\end{aligned}$$

- “Small area fraction”: N particles of size $O(\varepsilon)$ a distance $O(1)$ apart
- H is curvature; V is normal velocity of interface (such that $V > 0$ for a shrinking particle); $[\![\cdot]\!]$ denotes the jump in the bracketed quantity
- Previous 2D studies (unbounded domain): Voorhees et al. 1988, Zhu et al. 1996, Levitan & Domany 1998, ..
- What is the effect of: boundary of the domain, particle interaction?

Other Problems: Ostwald Ripening II

- Radii $r_i(t)$ and centers $\boldsymbol{\xi}_i$ evolve in time
- Define local radius $\rho_i = |\mathbf{x} - \boldsymbol{\xi}_i|/\varepsilon = r_i/\varepsilon$
- For circular particles, curvature of ith particle is $1/(\varepsilon\rho_i)$
- Normal velocity of interface, $V = -dr_i/dt = -d(\varepsilon\rho_i)/dt$

Can write problem for concentration $u(\mathbf{x};\varepsilon)$ as

$$\Delta u = 0, \quad \mathbf{x} \in \Omega \backslash \{\text{outside disks}\}\,; \quad \frac{\partial u}{\partial n} = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{7.25a}$$

$$u = \frac{1}{\varepsilon\rho_i}\,, \quad \frac{d\rho_i}{dt} = \frac{1}{\varepsilon^2}\frac{\partial u}{\partial \rho}\,, \quad \text{on i-th particle boundary} \tag{7.25b}$$

Use hybrid method to derive ODE's for the centers and radii of the particles.

Too Late: N. Alikakos, G. Fusco, G. Karali, *Ostwald Ripening in Two Dimensions: The Rigorous Derivation of the Equations from the Mullins-Sekerka Dynamics*, Journ. Diff. Eq., **205**(1), (2004), pp. 1–49.

Largely Open: Study Ostwald Ripening and Migration Phenomena of Small Droplets in Fourth Order Fluid Film Models using Hybrid method (Glasner, SIAM 2008) (Ref: F. Otto, D. Slepcev, etc..)

Other Problems: Spot Patterns in RD: I

Schnakenburg Model: 2-D domain Ω with $\partial_n u = \partial_n v = 0$ on $\partial\Omega$:

$$v_t = \varepsilon^2 \Delta v - v + uv^2 \,, \qquad \varepsilon^2 u_t = D\Delta u + a - \varepsilon^{-2} uv^2 \,.$$

Here $0 < \varepsilon \ll 1$, with $D > 0$, and $a > 0$, are parameters.

- **Spot pattern:** since $\varepsilon \ll 1$, v can concentrate at discrete points in Ω. **Semi-strong Regime:** $D = O(1)$ so that u is global.**Weak Interaction Regime:** $D = O(\varepsilon^2)$ so that u is localized.. We assume **semi-strong.**
- **Physical Experiments of Spot-Splitting:** The ferrocyanide-iodate-sulphite reaction (Swinney et al, Nature, 1994), the chloride-dioxide-malonic acid reaction (De Kepper et al, J. Phys. Chem, 1998), and certain semiconductor gas discharge systems (Purwins et al, Phys. Lett. A, 2001)
- **Numerical Results of Spot-Splitting in 2-D:** Many studies (Pearson, Nishiura, Muratov, Maini) for related models, i.e. the Gray-Scott (GS) model

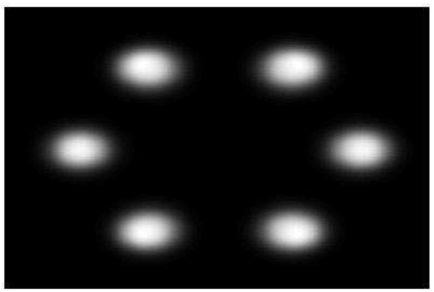
$$v_t = \varepsilon^2 \Delta v - v + Auv^2 \,, \qquad \tau u_t = D\Delta u + (1-u) - uv^2 \,.$$

Other Problems: Spot Patterns in RD: II

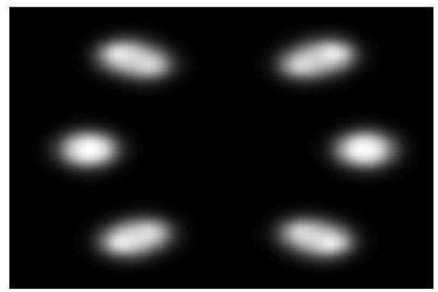
Ref: Kolokolnikov, Ward, Wei, J. Nonl. Sci., V. 19, No. 1, (2009), p. 1–56.

- **Slow Dynamics:** a DAE system for the evolution of K spots.
- **Spot-Splitting Instability** peanut-splitting and the splitting direction.

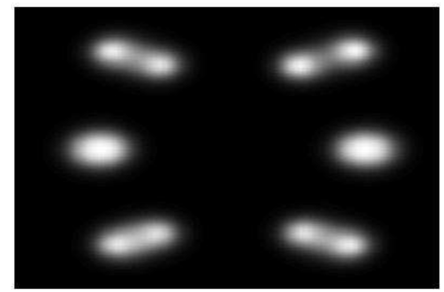
Example: $\Omega = [0,1]^2$, $\varepsilon = 0.02$, $a = 51$, $D = 0.1$.



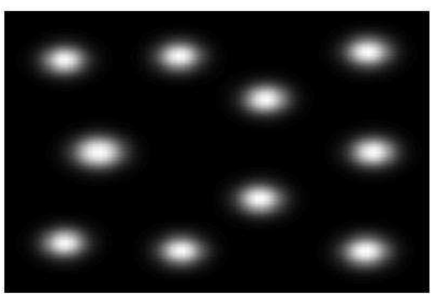
$t = 4.0$



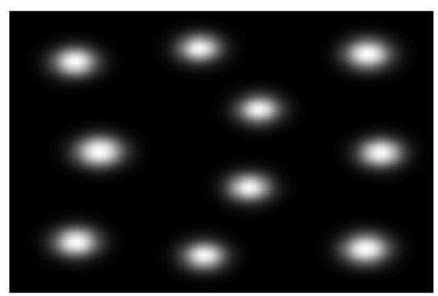
$t = 25.5$



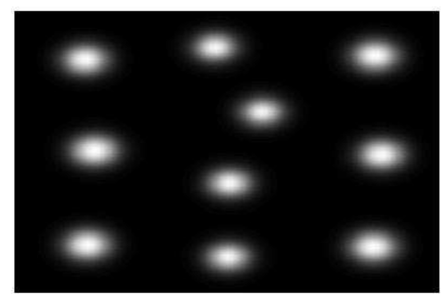
$t = 40.3.$



$t = 280.3$



$t = 460.3$



$t = 940.3.$

Other Problems: Spot Patterns in RD: III

Construction of a One-Spot Pattern by Singular Perturbation Techniques:

Inner Region: near the spot location $x_0 \in \Omega$ introduce $\mathcal{V}(\mathbf{y})$ and $\mathcal{U}(\mathbf{y})$ by

$$u = \frac{1}{\sqrt{D}}\mathcal{U}\,, \quad v = \sqrt{D}\mathcal{V}\,, \quad \mathbf{y} = \varepsilon^{-1}(\mathbf{x} - x_0)\,, \quad x_0 = x_0(\varepsilon^2 t)\,.$$

To leading order, $\mathcal{U} \sim U(\rho)$ and $\mathcal{V} \sim V(\rho)$ (radially symmetric) with $\rho = |\mathbf{y}|$.

This yields the **coupled core problem** with $U'(0) = V'(0) = 0$, where:

$$V_{\rho\rho} + \frac{1}{\rho}V_\rho - V + UV^2 = 0\,, \quad U_{\rho\rho} + \frac{1}{\rho}U_\rho - UV^2 = 0\,, \qquad 0 < \rho < \infty\,,$$
$$V \to 0\,, \qquad U \sim S\log\rho + \chi(S) + o(1)\,, \quad \text{as} \quad \rho \to \infty\,.$$

- Here $S > 0$ is called the “source strength” and is a parameter to be determined upon matching to an outer solution.
- The nonlinear function $\chi(S)$ must be computed numerically.

Other Problems: Spot Patterns in RD: IV

Outer Region: $v \ll 1$ and $\varepsilon^{-2}uv^2 \to 2\pi\sqrt{D}S\delta(\mathbf{x}-x_0)$. Hence,

$$\Delta u = -\frac{a}{D} + \frac{2\pi}{\sqrt{D}} S\,\delta(\mathbf{x}-x_0)\,, \quad \mathbf{x}\in\Omega\,;\partial_n u = 0\,, \quad \mathbf{x}\in\partial\Omega\,,$$

$$u \sim \frac{1}{\sqrt{D}}\left[S\log|\mathbf{x}-x_0| + \chi(S) + \frac{S}{\nu}\right] \quad \text{as} \quad \mathbf{x}\to x_0\,, \quad \nu\equiv -1/\log\varepsilon\,.$$

Key Point: the regular part of this singularity structure is **specified** and was obtained from matching to the **inner core solution.**

- Divergence theorem yields S (and inner core solution U and V) as

$$S = \frac{a|\Omega|}{2\pi\sqrt{D}}\,.$$

- The outer solution is given uniquely in terms of the Neumann G-function

$$u(\mathbf{x}) = -\frac{2\pi}{\sqrt{D}}\left(SG(\mathbf{x};x_0)+u_c\right)\,,$$

where $\quad S + 2\pi\nu S R(x_0;x_0) + \nu\chi(S) = -2\pi\nu u_c\,, \qquad \nu\equiv -1/\log\varepsilon\,.$

Other Problems: Spot Patterns in RD: V

Collective Coordinates: S_j, x_j, for $j = 1, \ldots, K$.

Principal Result: (DAE System): *For "frozen" spot locations $\mathbf{x}_j$, the source strengths S_j and u_c satisfy the nonlinear algebraic system*

$$S_j + 2\pi\nu \left(S_j R_{j,j} + \sum_{\substack{j=1 \\ j \neq i}}^{N} S_i G_{j,i} \right) + \nu \chi(S_j) = -2\pi\nu u_c \,, \quad j = 1, \ldots, K \,,$$

$$\sum_{j=1}^{K} S_j = \frac{a|\Omega|}{2\pi\sqrt{D}} \,, \qquad \nu \equiv \frac{-1}{\log \varepsilon} \,.$$

The slow dynamics of the spots with speed $O(\varepsilon^2)$ satisfies

$$x_j' \sim -2\pi\varepsilon^2 \gamma(S_j) \left(S_j \nabla R(\mathbf{x}_j; \mathbf{x}_j) + \sum_{\substack{j=1 \\ j \neq i}}^{N} S_i \nabla G(\mathbf{x}_j; \mathbf{x}_i) \right) \,, \quad j = 1, \ldots, K \,.$$

Here $G_{j,i} \equiv G(\mathbf{x}_j; \mathbf{x}_i)$ and $R_{j,j} \equiv R(\mathbf{x}_j; \mathbf{x}_j)$ (Neumann G-function).

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Our papers available at: http://www.math.ubc.ca/ ward/prepr.html

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