Asymptotic Methods for PDE Problems in Fluid Mechanics and Related Systems with Strong Localized Perturbations in Two-Dimensional Domains

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CISM Workshop; Asymptotic Methods in Fluid Mechanics: Surveys and Recent Advances

Lecture II: Nonlinearity: A Nonlinear Elliptic Problem and Slow Viscous Flow

Outline of Lecture II

TWO SPECIFIC PROBLEMS CONSIDERED:

1. A Nonlinear Eigenvalue Problem with a Hole
2. Slow Viscous Flow Over a Cylinder

Nonlinear Elliptic Problem with a Hole I

Consider the following nonlinear elliptic problem with smooth nonlinearity in a bounded two-dimensional domain with a small hole

$$\triangle w + F(w) = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon\,, \tag{1.1a}$$

$$\partial_n w + b(w - w_b) = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{1.1b}$$

$$w = \alpha\,, \quad \mathbf{x} \in \partial\Omega_\varepsilon\,. \tag{1.1c}$$

Here α is constant, $b > 0$, and Ω_ε is a small hole of radius $\mathcal{O}(\varepsilon)$ with $\Omega_\varepsilon \to \mathbf{x}_0 \in \Omega$ uniformly as $\varepsilon \to 0$.

- Applications to steady-state combustion theory where $F(w)$ is an exponential function.
- The primary difference between the linear problem and the unperturbed problem corresponding to (1.1) is that, depending on the precise nature of the nonlinearity $F(w)$, the unperturbed problem may have no solution, a unique solution, or multiple solutions. We shall assume that the unperturbed problem has at least one solution, and we will focus on determining how a specific solution to this problem is perturbed by the presence of the subdomain Ω_ε.

Nonlinear Elliptic Problem with a Hole II

In the outer region we expand w as

$$w = W_0(\mathbf{x};\nu) + \sigma W_1 + \cdots \,, \tag{1.2}$$

where $\sigma \ll \nu^k$ for any $k > 0$. The leading-order term $W_0(\mathbf{x};\nu)$ satisfies

$$\triangle W_0 + F(W_0) = 0\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\}\,, \tag{1.3a}$$

$$\partial_n W_0 + b(W_0 - w_b) = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{1.3b}$$

$$W_0 \quad \text{is singular as } \mathbf{x} \to \mathbf{x}_0\,. \tag{1.3c}$$

The analysis of the solution in the inner region is the same as for the linear problems since the effect of the nonlinear term in the inner region is $\mathcal{O}(\varepsilon^2)$, which is transcendentally small compared to the logarithmic terms.

Hence, we require that W_0 has the following singular behavior as $\mathbf{x} \to \mathbf{x}_0$

$$W_0 = \alpha + \gamma + \gamma\nu \log|\mathbf{x} - \mathbf{x}_0| + o(1)\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0\,. \tag{1.4}$$

Here $\gamma = \gamma(\nu)$ is to be found and ν is defined in terms of the logarithmic capacitance d by $\nu = -1/\log(\varepsilon d)$. Notice that the regular part of the singularity structure is prescribed.

Nonlinear Elliptic Problem with a Hole III

We suppose that for some range of the parameter S we can find a solution to (1.3) with the singular behavior

$$W_0 \sim S \log |\mathbf{x} - \mathbf{x}_0| \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.5}$$

Then, in terms of this solution we define the regular part $R = R(S; \mathbf{x}_0)$ of this Coulomb singularity by

$$R(S; \mathbf{x}_0) = \lim_{\mathbf{x} \to \mathbf{x}_0} \left(W_0 - S \log |\mathbf{x} - \mathbf{x}_0| \right) \,. \tag{1.6a}$$

In general R is a nonlinear function of S at each $\mathbf{x}_0$. Therefore, we have

$$W_0 \sim S \log |\mathbf{x} - \mathbf{x}_0| + R(S; \mathbf{x}_0) + o(1) \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0 \,. \tag{1.6b}$$

Equating (1.6b) to (1.4) we get

$$S = \nu \gamma \,, \qquad R = \alpha + \gamma \,, \qquad \nu = -1/\log(\varepsilon d) \,. \tag{1.7}$$

For fixed εd and α, these relations are two nonlinear algebraic equations for the two unknowns S and γ.

Nonlinear Elliptic Problem with a Hole IV

Alternatively, we can view these relations as providing a parametric representation of the desired curve $\gamma = \gamma(\nu)$ in the form $\nu = \nu(S)$ and $\gamma = \gamma(S)$, where

$$\gamma = R(S; \mathbf{x}_0) - \alpha \,, \qquad \nu = \frac{S}{R(S; \mathbf{x}_0) - \alpha} \,. \tag{1.8}$$

The equation for ν in (1.8) is an implicit equation determining S in terms of ε from $\nu = -1/\log(\varepsilon d)$.

Therefore, we can analytically sum all of the logarithmic terms in the expansion of the solution to (1.1) provided that we compute the solution to (1.3), with singular behavior (1.5), and then identify $R(S; \mathbf{x}_0)$ from (1.6a). In general this must be done numerically.

However, we now illustrate the method with an example where $R(S; \mathbf{x}_0)$ can be calculated analytically.

Nonlinear Elliptic Problem with a Hole V

Example: Let Ω be the unit disk, and take $b = \infty$, $w_b = 0$, $F(w) = e^w$, and assume that Ω_ε is an arbitrarily-shaped hole centered at the origin. Then, (1.3) and (1.5) reduce to a radially symmetric problem for $W_0(r)$, given by

$$W_0'' + \frac{1}{r} W_0' + e^{W_0} = 0 \,, \quad 0 < r \leq 1 \,; \quad W_0 = 0 \,, \quad \text{on} \quad r = 1 \,, \qquad (1.9a)$$

$$W_0 \sim S \log r \,, \quad \text{as} \quad r \to 0 \,, \qquad (1.9b)$$

where $r = |\mathbf{x}|$. This problem (1.9) can be solved analytically by first introducing the new variables v and η defined by

$$v = W_0 - S \log r \,, \qquad \eta = r^{1+S/2} \,. \qquad (1.10)$$

When $S > -2$, we then obtain that $v = v(\eta)$ is smooth and satisfies

$$v'' + \frac{1}{\eta} v' + \left(1 + \frac{S}{2}\right)^{-2} e^v = 0 \,, \quad 0 \leq \eta \leq 1 \,; \quad v = 0 \,, \quad \text{on} \quad \eta = 1 \,. \qquad (1.11)$$

Nonlinear Elliptic Problem with a Hole VI

The solution to (1.11), given in parametric form with $\rho = \rho(S)$, is

$$v(\eta) = 2\log\left(\frac{1+\rho}{1+\rho\eta^2}\right), \qquad \left(1+\frac{S}{2}\right)^{-2} = \frac{8\rho}{(1+\rho)^2}. \tag{1.12}$$

The maximum of the right-hand side of (1.12) is 2 and occurs when $\rho = 1$. Therefore, for there to be a solution to (1.11) we require that $(1+S/2)^2 > 1/2$, which yields that $S > \sqrt{2} - 2$.

When $S > \sqrt{2} - 2$, (1.12) has two roots for ρ, and hence (1.11) has two solutions. Consider the smaller root, which we label by $\rho_-(S)$. Then, we calculate that

$$\rho_-(S) = (S+1)(S+3) - (S+2)\left[(S+2)^2 - 2\right]^{1/2}. \tag{1.13}$$

Setting $\eta = 0$ in (1.12), and using (1.10), we compare with (1.6a) to conclude that $R(S;\mathbf{0}) = v(0)$, which yields

$$R(S;\mathbf{0}) = 2\log(1+S/2) + \log\left[8\rho_-(S)\right]. \tag{1.14}$$

Slow Viscous Flow Over a Cylinder

Consider slow, steady, two-dimensional flow of a viscous incompressible fluid around an infinitely long straight cylinder. The Reynolds number satisfies $\varepsilon \equiv U_\infty L/\mu \ll 1$ where U_∞ is the velocity of the fluid in the x-direction at infinity, μ is the kinematic viscosity, and $2L$ is the diameter of the cross-section of the cylinder.

Assume first that the cross-sectional shape Ω of the cylinder is symmetric about the direction of the oncoming stream, but otherwise is arbitrary. In terms of polar coordinates centered inside the body, it follows from the NS equations that the dimensionless stream function ψ satisfies

$$\triangle^2\psi + \varepsilon\, J_\rho\,[\psi, \triangle\psi] = 0\,, \quad \text{for} \quad \rho > \rho_b(\theta)\,, \tag{2.1a}$$

$$\psi = \partial_n\psi = 0\,, \quad \text{on} \quad \rho = \rho_b(\theta)\,, \tag{2.1b}$$

$$\psi \sim y\,, \quad \text{as} \quad \rho = (x^2+y^2)^{1/2} \to \infty\,. \tag{2.1c}$$

Here J_ρ is the Jacobian defined by $J_\rho\,[a,b] \equiv \rho^{-1}\,(\partial_\rho a\,\partial_\theta b - \partial_\theta a\,\partial_\rho b)$. The boundary of the scaled cross-section is denoted by $\rho = \rho_b(\theta)$ for $-\pi \le \theta \le \pi$, and the symmetry condition $\rho_b(\theta) = \rho_b(-\theta)$ is assumed to hold.

Slow Viscous Flow: Qualitative History

- For $\varepsilon \to 0$, the method of matched asymptotic expansions was developed and used systematically by Kaplun (1957) and by Proudman and Pearson (1957) to resolve the well-known Stokes paradox, and to calculate asymptotically the stream function in both the Stokes region, which is near the body, and in the Oseen region, which is far from the body.
- For $\varepsilon \to 0$, the asymptotic expansion for the drag coefficient C_D of a circular cylindrical body starts with $C_D \sim 4\pi\varepsilon^{-1}F(\varepsilon)$, where $F(\varepsilon)$ is an infinite series in powers of $1/\log\varepsilon$. The coefficients in this series are to be determined from the solutions to certain forced Oseen problems.
- For a cylinder of arbitrary cross-section, Kaplun showed that $C_D \sim 4\pi\varepsilon^{-1}F(\varepsilon d_f)$, where d_f is an 'effective' radius of the cylinder (Kaplun's equivalence principle).
- Owing to the complexity of the analytical calculations, only the first three coefficients in $F(\varepsilon)$ were derived in Kaplun. However, as a result of the slow decay of $1/\log\varepsilon$ with decreasing values of ε, the resulting three-term truncated series for C_D agrees rather poorly with the experimental results of Tritton unless ε is very small.

Slow Viscous Flow III

As a result of such issues, the problem of slow viscous flow around a cylinder has served as a paradigm for problems where matched asymptotic analysis fails to be of much practical use, unless ε is tiny.

Outline: Conventional Analysis In the Stokes (inner) region where $\rho = \mathcal{O}(1)$, ψ has an infinite logarithmic expansion in terms of unknown a_j:

$$\psi_s(\rho,\theta) = \sum_{j=1}^{\infty} \nu^j a_j \psi_c(\rho,\theta) + \cdots \,. \tag{2.2}$$

Here $\nu = \nu(\varepsilon d_f) \equiv -1/\log\left(\varepsilon d_f e^{1/2}\right)$, where d_f is a shape-parameter specified below. Moreover, $\psi_c(\rho,\theta)$ is the solution to the following canonical (inner) Stokes problem:

$$\triangle^2 \psi_c = 0 \,, \quad \text{for} \quad \rho > \rho_b(\theta) \,; \qquad \psi_c(\rho,\theta) = -\psi_c(\rho,-\theta) \,, \tag{2.3a}$$

$$\psi_c = 0 \quad \text{and} \quad \partial_n \psi_c = 0 \,, \quad \text{on} \quad \rho = \rho_b(\theta) \,, \tag{2.3b}$$

$$\psi_c \sim \left(\rho \log \rho - \rho \log \left[d_f e^{1/2}\right]\right) \sin\theta \,, \quad \text{as} \quad \rho \to \infty \,. \tag{2.3c}$$

The constant d_f, which depends on the specific shape of the body, is determined uniquely by the solution to (2.3).

Slow Viscous Flow IV

Then, by using (2.3c), we obtain the following far-field behavior of the Stokes expansion in (2.2):

$$\psi_s(\rho,\theta) \sim \sum_{j=1}^{\infty} \nu^j \, a_j \left(\log\rho - \log\left[d_f e^{1/2} \right] \right) \rho \sin\theta \,, \quad \text{as} \quad \rho \to \infty \,. \tag{2.4}$$

In Oseen region, defined for $\rho = \mathcal{O}(\varepsilon^{-1})$, the body has "radius" $\mathcal{O}(\varepsilon)$. In this region, define r by $r = \varepsilon\rho$ with $r = \mathcal{O}(1)$ and Ψ by $\Psi = \varepsilon\psi$, and then re-write the far-field behavior (2.4) of the Stokes solution in terms of r and Ψ. This yields,

$$\Psi \sim \left(a_1 r \sin\theta + \sum_{j=1}^{\infty} \nu^j \, \left[a_j \log r + a_{j+1} \right] r \sin\theta \right) \,. \tag{2.5}$$

This expression (2.5) yields a singularity structure for the outer Oseen solution as $r \to 0$.

Slow Viscous Flow V

Next, we expand the Oseen Ψ in an infinite logarithmic series as

$$\Psi(r,\theta) = r\sin\theta + \nu\Psi_1(r,\theta) + \sum_{j=2}^{\infty} \nu^j \Psi_j(r,\theta) + \cdots . \qquad (2.6)$$

From the NS equation, and by matching to the singularity behavior of as $r \to 0$, we find that $a_1 = 1$ and that Ψ_1 and Ψ_j for $j \geq 2$ satisfy the forced Oseen problems on $0 < r < \infty$;

$$L_{0s}\Psi_1 \equiv \triangle^2\Psi_1 + \left(r^{-1}\sin\theta\,\partial_\theta - \cos\theta\,\partial_r\right)\,\triangle\Psi_1 = 0\,, \qquad (2.7a)$$

$$\Psi_1 \sim (\log r + a_2)\,r\sin\theta\,, \quad \text{as} \quad r \to 0\,; \quad \partial_r\Psi_1 \to 0\,, \quad \text{as} \quad r \to \infty\,, \qquad (2.7b)$$

$$L_{0s}\Psi_j = -\sum_{k=1}^{j-1} J_r\left[\Psi_k, \triangle\Psi_{j-k}\right]\,, \qquad (2.7c)$$

$$\Psi_j \sim (a_j \log r + a_{j+1})\,r\sin\theta\,, \quad \text{as} \quad r \to 0\,; \quad \partial_r\Psi_j \to 0\,, \quad \text{as} \quad r \to \infty\,. \qquad (2.7d)$$

Here L_{0s} is referred to as the linearized Oseen operator and Ψ_1 is the linearized Oseen solution.

Slow Viscous Flow VI

- The a_j, with $a_1 = 1$, are found recursively from forced Oseen problems
- The first two coefficients from Kaplun are

$$a_2 = \gamma_e - \log 4 - 1 \approx -1.8091\,,$$

$$a_3 - a_2^2 = -\int_0^\infty \left[r^{-1} I_1(2r) + 1 - 4K_1(r)I_1(r)\right] K_0(r)K_1(r)\,dr \approx -0.8669\,.$$

- The drag coefficient for a cylinder of arbitrary cross-section is given in terms of the coefficients a_j by

$$C_D \sim 4\pi\varepsilon^{-1}\nu(\varepsilon d_f)\left(\sum_{j=0}^{\infty} a_{j+1}\nu^j(\varepsilon d_f) + \cdots\right)\,, \quad \nu(z) \equiv -\frac{1}{\log\left[z e^{1/2}\right]}\,. \tag{2.9}$$

- Kaplun's three-term approximation for C_D results from using the known formulae for a_j for $j = 1, 2, 3$;

$$C_D \sim \frac{4\pi}{\varepsilon}\hat{\nu}(\varepsilon d_f)\left[1 - 0.8669\,\hat{\nu}^2(\varepsilon d_f)\right]\,, \quad \hat{\nu}(z) \equiv \left[\log\left(3.7027/z\right)\right]^{-1}\,.$$

Slow Viscous Flow: Hybrid I

We now formulate a hybrid method for summing the infinite logarithmic series for the drag coefficient. Let $A^\star(z)$ denote a function which is asymptotic to the sum of the terms written explicitly in (2.2):

$$A^\star(z) \sim \sum_{j=1}^{\infty} \nu^{j-1}(z)\, a_j\,, \qquad z \equiv \varepsilon d_f\,. \tag{2.10}$$

Then, the Stokes expansion is asymptotic to

$$\psi_s(\rho,\theta) = \nu(z) A^\star(z) \psi_c(\rho,\theta) + \cdots\,, \qquad z = \varepsilon d_f\,. \tag{2.11}$$

We substitute the far-field behavior of ψ_c from (2.3c), and write the resulting expression in terms of the Oseen variables $r = \varepsilon\rho$ and $\Psi = \varepsilon\psi$.

This leads to a required singularity structure for the outer solution:

$$\Psi \sim A^\star(z)\,[1 + \nu(z)\,\log r]\, r \sin\theta\,, \quad \text{as} \quad r \to 0\,; \quad z = \varepsilon d_f\,. \tag{2.12}$$

Notice that both the singular and the regular part of the singularity structure is prescribed

Slow Viscous Flow: Hybrid II

In the outer or Oseen region we introduce the parameter-dependent auxilliary streamfunction $\Psi_H \equiv \Psi_H(r,\theta;S)$, satisfying

$$\triangle^2\Psi_H + J_r\left[\Psi_H, \triangle\Psi_H\right] = 0\,, \quad r > 0\,, \tag{2.13a}$$

$$\Psi_H(r,\theta;S) = -\Psi_H(r,-\theta;S)\,, \tag{2.13b}$$

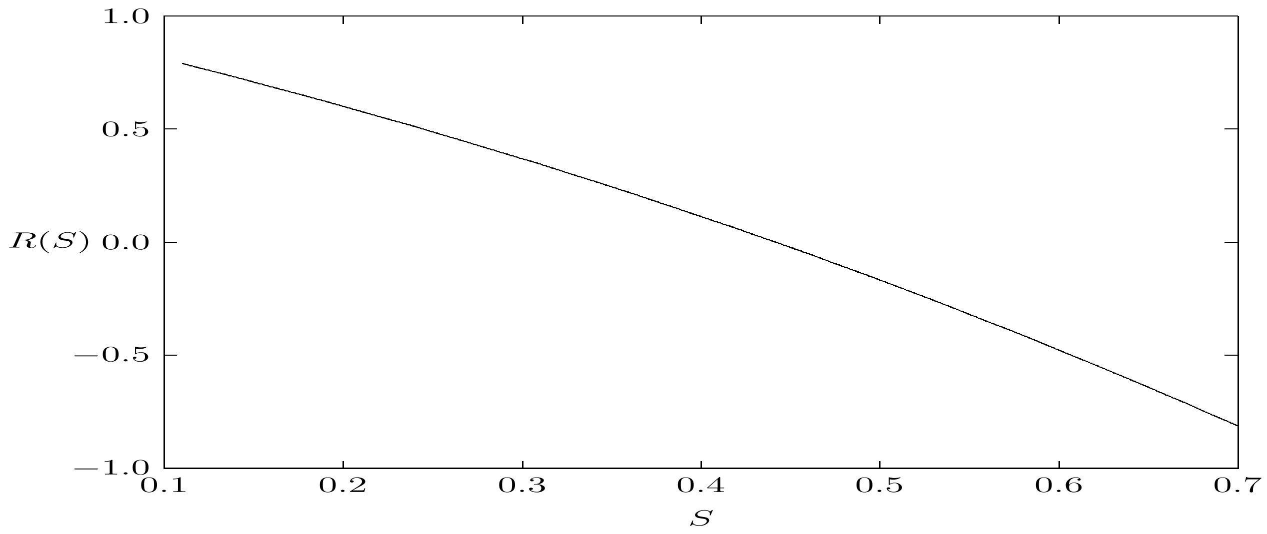
$$\Psi_H \sim r\sin\theta\,, \quad \text{as} \quad r\to\infty\,, \tag{2.13c}$$

$$\Psi_H \sim Sr\log r\sin\theta\,, \quad \text{as} \quad r\to 0\,. \tag{2.13d}$$

We solve this problem for a range of values of S, and in terms of this solution we identify the regular part $R = R(S)$ of this singularity structure by the following limiting process

$$\Psi_H - Sr\log r\sin\theta = R(S)r\sin\theta + o(r)\,, \quad \text{as} \quad r\to 0\,. \tag{2.14}$$

Slow Viscous Flow: Hybrid III



By matching the singularity structure for Ψ_H with the required behavior in terms of $A^\star$, we conclude that $A^\star(z)$ and $\nu(z)$, with $z \equiv \varepsilon d_f$, are given parametrically in terms of the singularity strength S and its regular part $R(S)$ by

$$\nu(z) = -\frac{1}{\log\left[ze^{1/2}\right]} = \frac{S}{R(S)}\,, \qquad A^\star(z) = R(S)\,. \tag{2.15}$$

Remark: Can $R = R(S)$ be found analytically? (one possibility is to look at the special exact solutions to the full 2-D incompressible NS equations found in K.Ranger, Stud. Appl. Math., **94**(2), (1995), p. 169–181.)

Slow Viscous Flow: Hybrid IV

- In our hybrid formulation the cylinder is replaced by the singularity structure (2.13) that was derived by exploiting the far-field form of the infinite-order logarithmic expansion in the Stokes region.
- Instead of computing solutions to an infinite sequence of problems, the hybrid method requires the solution to a parameter-dependent problem, with singular behavior in terms of the parameter S.
- The (nonlinear) regular part $R = R(S)$ of this singularity behavior is calculated in terms of the solution by a limiting process.
- Then $A^{\star}(z)$, in terms of $z = \varepsilon d_f$, is obtained parametrically.
- In terms of $A^{\star}(z)$ and d_f, the drag coefficient is

$$C_D = \frac{4\pi}{\varepsilon}\left[\nu(z)A^{\star}(z) + \cdots\right], \qquad \nu(z) = \frac{-1}{\log\left[z\,e^{1/2}\right]}, \qquad z = \varepsilon d_f\,.$$

- The curve $A^{\star}(z)$ vs. z can be used for a cylinder of arbitrary cross-section, by calculating only a single constant d_f from the numerical solution to the canonical Stokes problem This feature provides a significant advantage over a direct numerical approach on the full problem.

Slow Viscous Flow: Hybrid V

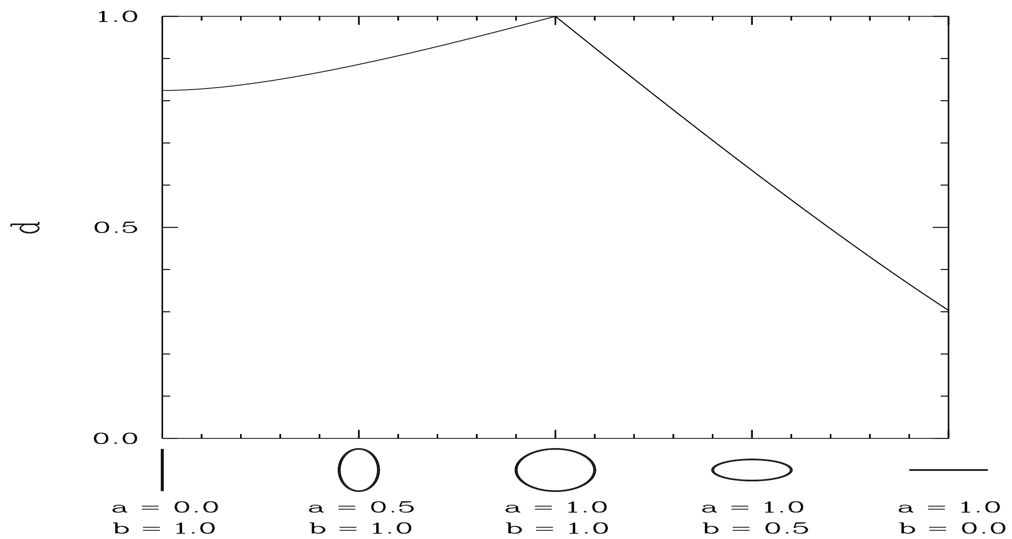
The constant d_f can be determined analytically in only a few cases:

- For a circular cross-section, $d_f = 1$ since ψ_c is given by

$$\psi_c = \left(\rho \log \rho - \frac{\rho}{2} + \frac{1}{2\rho}\right) \sin\theta \,. \tag{2.16}$$

- For the elliptical cross-section $(x/a)^2 + (y/b)^2 = 1$ where $\max(a, b) = 1$, then

$$d_f = \left(\frac{a+b}{2}\right) \exp\left[\frac{b-a}{2(b+a)}\right] \,. \tag{2.17}$$



Slow Viscous Flow: Hybrid VI

The constant d_f can be computed numerically for the family of symmetric Karman-Trefftz airfoils.. The mapping function, $z = z(\sigma)$, for the boundary of these profiles is

$$z(\sigma) = \beta_0 kc \left[\frac{(\xi + c)^k + (\xi - c)^k}{(\xi + c)^k - (\xi - c)^k} \right] , \qquad \xi \equiv \sigma^{-1} + c - 1 , \tag{2.18a}$$

where $\sigma = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. By fixing the length of the airfoil to be 2, we find that the mapping constant β_0 is given in terms of k and c by

$$\beta_0 = \frac{\left[1 - (1 - c)^k\right]}{kc} . \tag{2.18b}$$

The boundary of the airfoil is obtained by setting $\sigma = e^{i\theta}$.

Slow Viscous Flow: Hybrid VII

δ	θ_T	k	c	d_f	b
.050	0°	2.000	0.961	0.328	0.040
.080	5°	1.972	0.952	0.344	0.066
.100	13°	1.928	0.960	0.354	0.082
.120	16°	1.910	0.954	0.364	0.098
.120	20°	1.889	0.968	0.363	0.096
.200	25°	1.861	0.915	0.410	0.170

Table 1: Numerical values for d_f corresponding to the Karman Trefftz airfoils (2.18). The tail angle (in degrees) is θ_T, and the thickness ratio is δ. The last column gives the value of b for an ellipse, with $a = 1$, that has the same value of d_f as the corresponding airfoil.

Slow Viscous Flow: Hybrid VIII

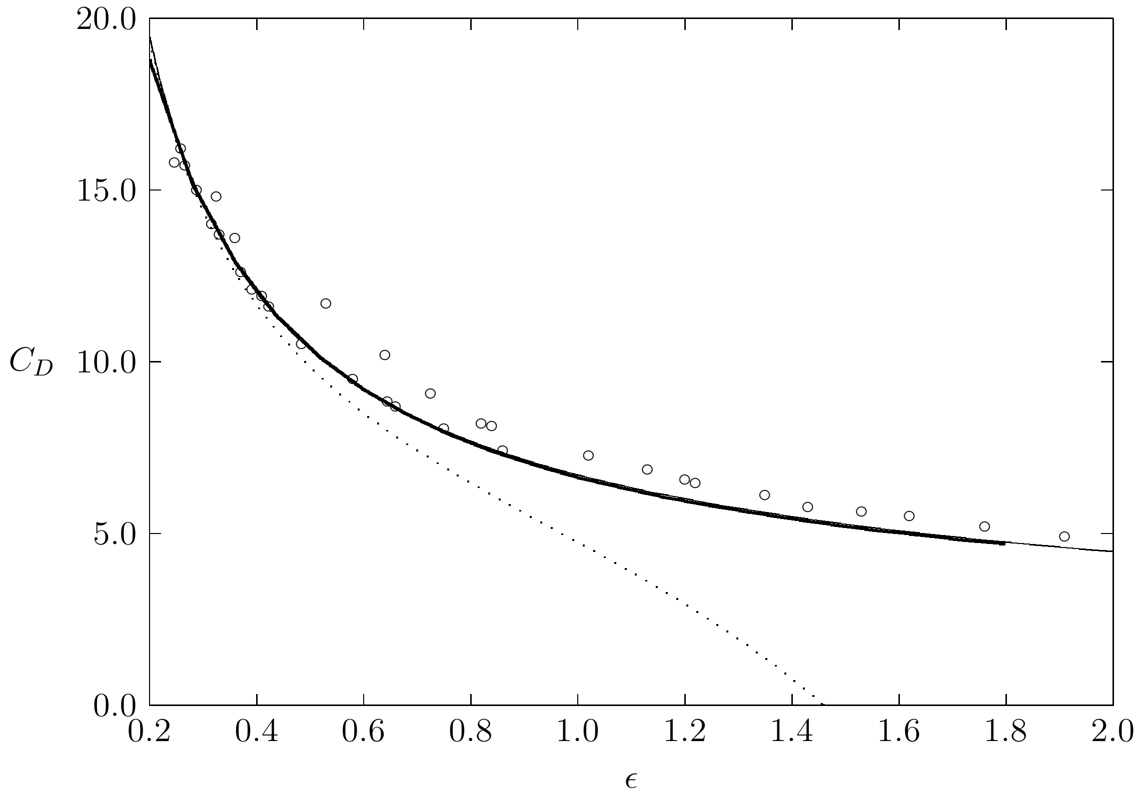


Figure 1: The drag coefficient C_D versus the Reynolds number ε. for a circular cylinder; the hybrid result (solid curve), the full numerical results (heavy solid curve), the three-term Kaplun result (dotted curve), and the experimental results of Tritton.

Slow Viscous Flow: Hybrid IX

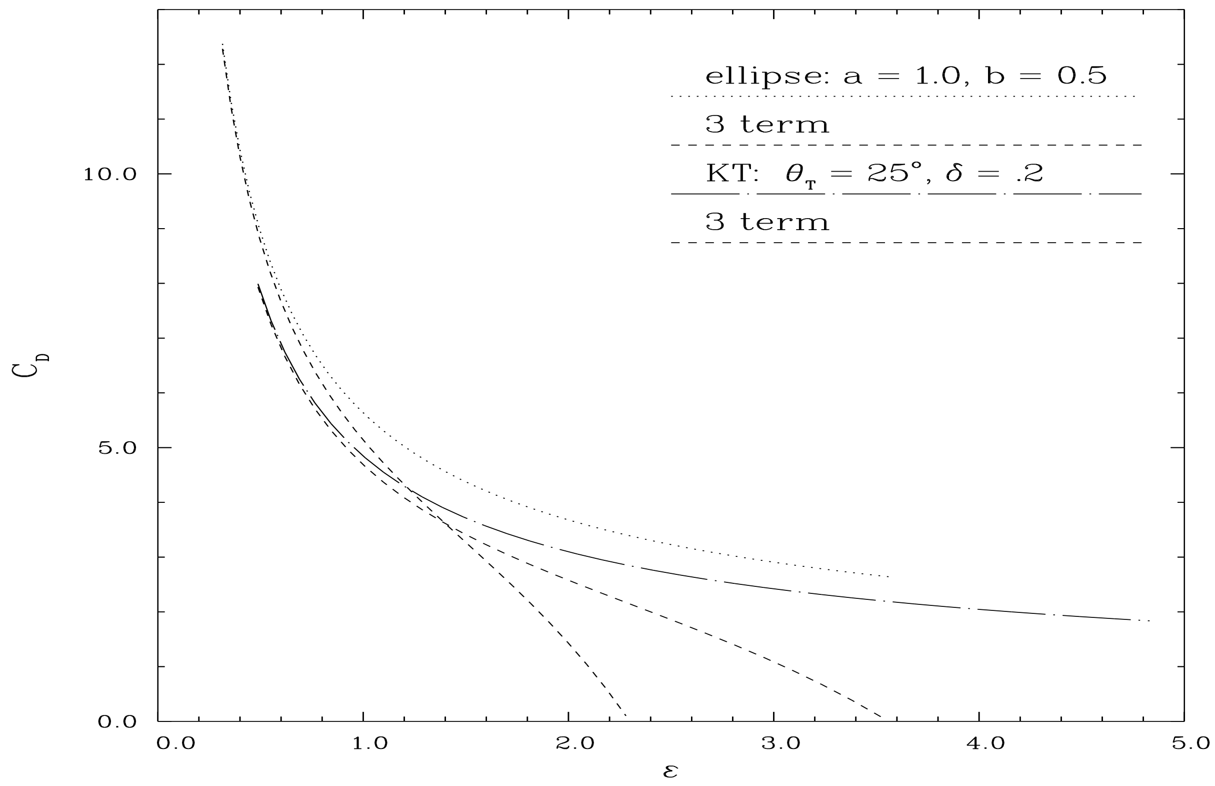


Figure 2: The drag coefficient C_D versus the Reynolds number ε. the hybrid result (solid curves) is compared with the three-term Kaplun result for a cylindrical body of either an elliptical or a Karman-Trefftz airfoil cross-section.

Problem 5 From Workshop Notes

Problem 5: Consider the Biharmonic equation in the two-dimensional concentric annulus, formulated as

$$\triangle^2 u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon\,, \tag{5.1a}$$

$$u = f\,, \quad u_r = 0\,, \quad \text{on} \quad r = 1\,, \tag{5.1b}$$

$$u = u_r = 0\,, \quad r = \varepsilon\,. \tag{5.1c}$$

Here Ω is the unit disk centered at the origin, containing a small hole of radius ε centered at $\mathbf{x} = 0$, i.e. $\Omega_\varepsilon = \{\mathbf{x} \mid |\mathbf{x}| \leq \varepsilon\}$. Consider the following two choices for f: **Case I:** $f = 1$. **Case II:** $f = \sin\theta$. For each of these two cases calculate the exact solution, and from it determine an approximation to the solution in the outer region $|\mathbf{x}| \gg \mathcal{O}(\varepsilon)$. Can you re-derive these results from singular perturbation theory in the limit $\varepsilon \to 0$?.

Remark 1: The leading-order outer problem for Case I is different from what you might expect.

Remark 2: For Case 2 one can sum an infinite logarithmic expansion in a similar way as for slow viscous flow. The result can then be verified from the exact solution.

Problem 6 From Workshop Notes: I

Problem 6: *Consider the following convection-diffusion equation for $T(\mathbf{X})$, with $\mathbf{X} = (X_1, X_2)$ posed outside two circular disks Ω_j for $j = 1, 2$ of a common radius a, and with a center-to-center separation $2L$ between the two disks:*

$$\kappa \triangle T = \mathbf{U} \cdot \nabla T \,, \quad \mathbf{X} \in \mathbb{R}^2 \backslash \cup_{j=1}^{2} \Omega_j \,, \tag{6.1a}$$

$$T = T_j \,, \quad \mathbf{X} \in \partial \Omega_j \,, \quad j = 1, 2 \,, \tag{6.1b}$$

$$T \sim T_\infty \,, \quad |\mathbf{X}| \to \infty \,. \tag{6.1c}$$

Here $\kappa > 0$ is constant, T_j for $j = 1, 2$ and T_∞ are constants, and $\mathbf{U} = \mathbf{U}(\mathbf{X})$ is a given bounded flow field with $\mathbf{U}(\mathbf{X}) \to (\mathbf{U}_\infty, \mathbf{0})$ as $|\mathbf{X}| \to \infty$, where U_∞ is constant.

- Non-dimensionalize (6.1) in terms of U_∞ and the length-scale $\gamma = \kappa / U_\infty$ to derive a convection-diffusion equation outside of two circular disks of radii $\varepsilon \equiv U_\infty a / \kappa$, with inter-disk separation $2L\varepsilon / a$. Here ε is the Peclet number.

Problem 6 From Workshop Notes: II

- In the low Peclet number limit $\varepsilon \to 0$ show how a hybrid asymptotic-numerical solution can be implemented to sum the infinite logarithmic expansions for two different distinguished limits: **Case 1:** $L/a = \mathcal{O}(1)$. **Case 2:** $L/a = \mathcal{O}(\varepsilon^{-1})$.
- For a uniform flow with $\mathbf{U} = (U_\infty, 0)$ for $\mathbf{X} \in \mathbb{R}^2$, determine the required Green's function and its regular part.

Remark: For Case 1, we require an explicit formula for the logarithmic capacitance, d, of two disks of a common radius, a, and with a center-to-center separation of $2l$. The result is

$$\log d = \log(2\beta) - \frac{\xi_c}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\xi_c}}{m\cosh(m\xi_c)}, \tag{6.2}$$

where β and ξ_c are determined in terms of a and l by

$$\beta = \sqrt{l^2 - a^2}; \qquad \xi_c = \log\left[\frac{l}{a} + \sqrt{\left(\frac{l}{a}\right)^2 - 1}\right]. \tag{6.3}$$

References

Our paper available at: http://www.math.ubc.ca/ ward/prepr.html

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