Competition and Oscillatory Instabilities of Spike Patterns in the Gierer-Meinhardt and Gray-Scott Models

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Outline of the Talk

1. The Stability of Multi-Spike Patterns for the Equilibrium Problem
   - The GM and GS models (bounded domain)
   - Stability: The NLEP (nonlocal eigenvalue problem)
   - Competition and synchronous oscillatory instabilities: theory
   - Numerical experiments to confirm the theory
   - Special limiting cases
   - Remarks on small eigenvalues
2. The Dynamics of Two-Spike Patterns for the GM and GS Models
   - The existence of quasi-equilibrium two-spike patterns
   - The slow dynamics of quasi-equilibria
   - A non-gradient flow "coarsening" process: Dynamic bifurcations and the triggering of competition and oscillatory instabilities of quasi-equilibria.

The GM Model (Gierer, Meinhardt 1972)

On $-1 < x < 1$, the activator a and the inhibitor h satisfy the dimensionless system (for $p > 1$, $q > 0$, $m > 1$, $s \geq 0$)

$$a_t = \varepsilon^2 a_{xx} - a + \frac{a^p}{h^q}\,, \qquad a_x = 0\,,\ x = \pm 1$$

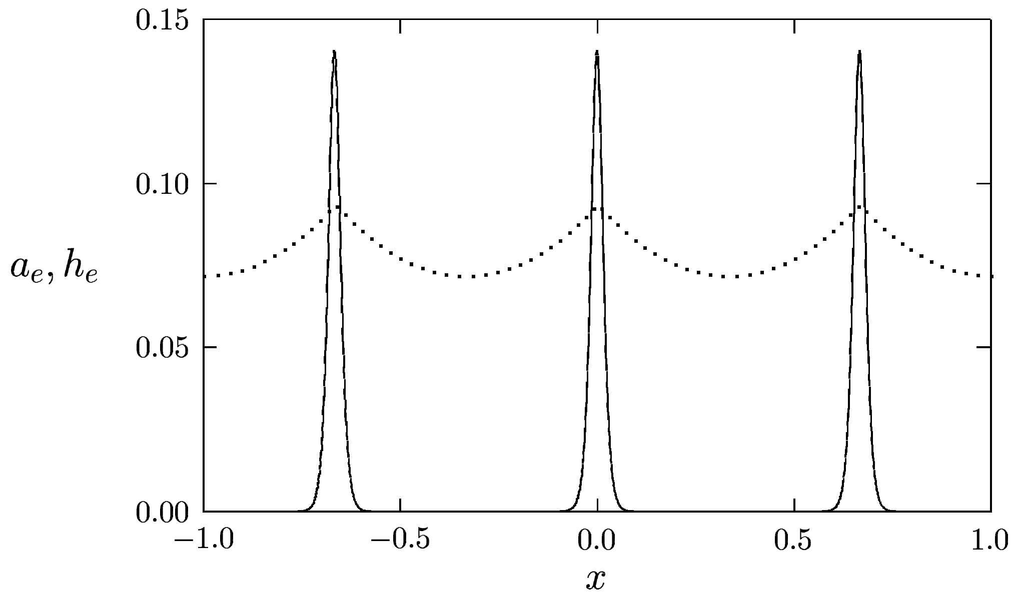
$$\tau h_t = D h_{xx} - h + \varepsilon^{-1} \frac{a^m}{h^s}\,, \qquad h_x = 0\ \ x = \pm 1\,.$$

A model for biological morophogenesis, sea-shells patterns etc.. (Meinhardt 1982, 1995). Classical GM model exponent set is $(p, q, m, s) = (2, 1, 2, 0)$. It is assumed that $\zeta \equiv \frac{qm}{(p-1)} - (s+1) > 0$.

- **Semi-Strong Interaction Regime:** $\varepsilon \ll 1$ and $D > 0$ with $D = O(1)$. Here a is localized near a spike, and h varies globally across the domain. In this regime, the stability properties are intricate.
- **Weak-Interaction Regime:** $\varepsilon \ll 1$ and $D = O(\varepsilon^2)$. Both a and h are coupled near the core of the spike. Self-replication behavior occurs in this regime (Doelman, Van der Ploeg; Nishiura; Kolokolnikov, Ward, Wei).

The Equilibrium Problem: Semi-Strong Regime

Symmetric Three-Spike Equilibrium Solution in Semi-Strong Regime



Construction of Equilibrium Solution:

- Matched asymptotic analysis (Iron, Wei, Ward, Physica D)
- Lyapunov-Schmidt reduction (Wei, Winter)
- Infinite line problem: geometric singular perturbation theory (Doelman et al, Indiana J.)

Symmetric k-Spike Equilibria

Principal Result : Let $x_j = -1 + (2j-1)/k$. Then, for $\varepsilon \to 0$, a symmetric k-spike equilibrium solution satisfies

$$a_e(x) \sim H^{\gamma} \sum_{j=1}^{k} w\left[\varepsilon^{-1}(x - x_j)\right] , \quad h_e(x) \sim \frac{H}{a_g} \sum_{j=1}^{k} G_0(x; x_j) \, .$$

Here $\gamma = q/(p-1)$ and, on $-\infty < y < \infty$, $w(y)$ is the homoclinic of

$$w^{''} - w + w^p = 0 \, ; \quad w(\pm\infty) = 0 \, , \quad w(0) > 0 \, , \quad w^{'}(0) = 0 \, .$$

The Green's function $G_0(x; x_j)$ satisfies

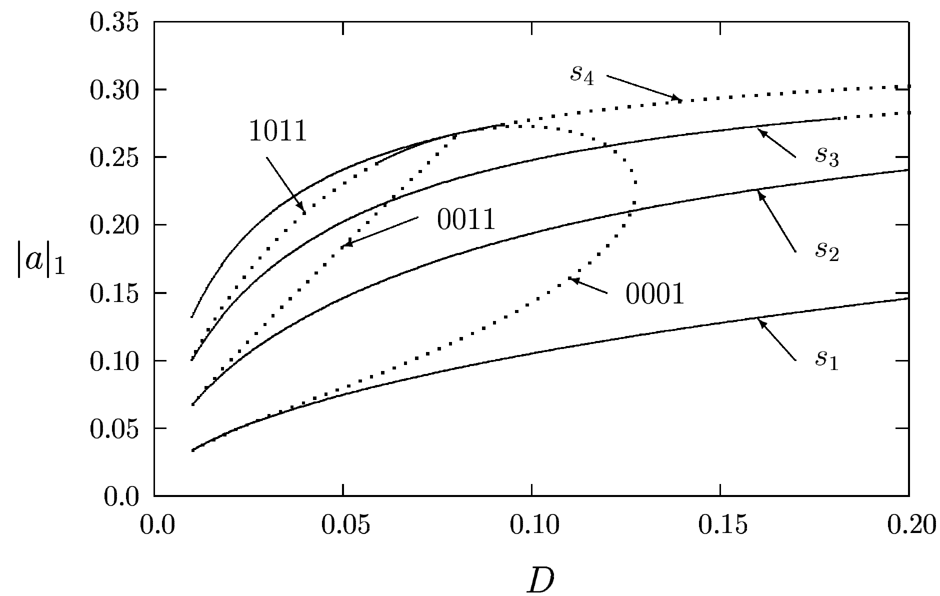
$$DG_{0xx} - G_0 = -\delta(x - x_j) \, , \quad -1 < x < 1 \, ; \quad G_{0x}(\pm 1; x_j) = 0 \, .$$

The constants are $\theta_0 \equiv D^{-1/2}$, $b_m \equiv \int_{-\infty}^{\infty} [w(y)]^m \, dy$, and

$$H^{\gamma m - (s+1)} \equiv \frac{1}{b_m a_g} \, , \quad a_g \equiv \sum_{j=1}^{k} G_0(x_j; x_k) = \left[\frac{2}{\theta_0} \tanh\left(\frac{\theta_0}{k} \right) \right]^{-1} \, .$$

Bifurcation Diagram: k-Spike Equilibria

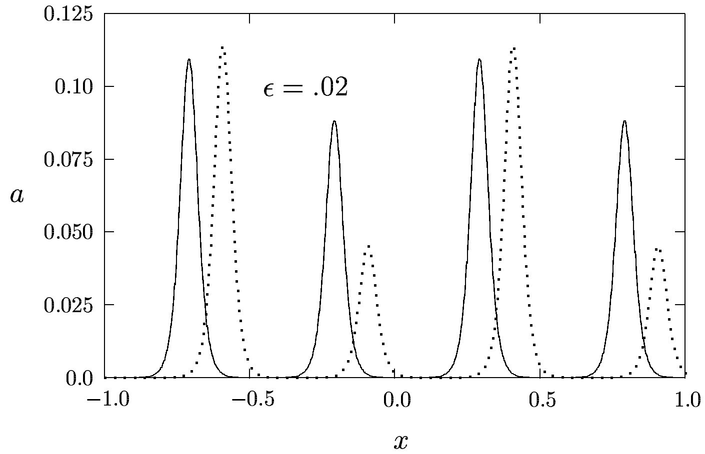
Bifurcation Diagram of k-Spike Equilibria for $(2,1,2,0)$ exponent set with $\varepsilon = 0.02$. Solid (dashed) lines indicate stability (instability) with respect to the large eigenvalues when $\tau = 0$. Note: $|a|_1 = kH^\gamma$.



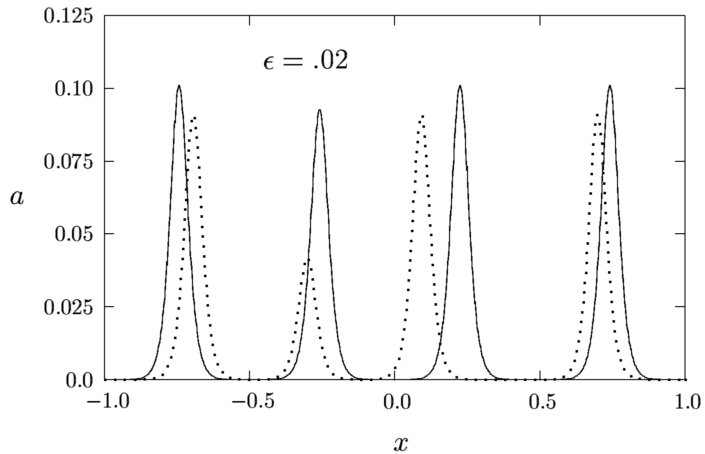
- Asymmetric spike equilibria in the form $SSBS\ldots$ can also exist (Ward, Wei, EJAM; Doelman et al. MAA). The ones bifurcating from the symmetric s_4 branch are shown. Note $0001 \to SSSB$.
- $k-1$ asymmetric equilibrium branches bifurcate at the point where $k-1$ small eigenvalues for the symmetric branch s_k simultaneously cross through zero.

Asymmetric Equilibria: $(2, 1, 2, 0)$ and $\varepsilon = 0.02$

BSBS pattern: $D = .079$ (solid curve) and $D = 0.06$ (dotted curve).



BSBB pattern: $D = .075$ (solid curve) and $D = 0.04$ (dotted curve).



Stability of k-Spike Symmetric Equilibria

1. **Large Eigenvalues:** Instabilities on a fast $O(1)$ time-scale governing the stability of the spike profiles. Leads to a nonlocal eigenvalue problem (NLEP).
   - We let $a = a_e + e^{\lambda t}\phi(x)$ and $h = h_e + e^{\lambda t}\eta(x)$ and linearize.
   - The eigenfunction is localized so that $\phi(x) = \sum_{j=1}^{k} c_j \Phi\left[\varepsilon^{-1}(x - x_j)\right]$, where $\int_{-\infty}^{\infty} w\Phi\, dy \neq 0$.
   - The perturbation η is solved on each subinterval $[x_{j-1}, x_j]$ and is patched together for C^1 continuity.
   - There are k vectors for the vector $(c_1, \ldots, c_k)^t$, which arise as the eigenvectors of a certain Green's matrix. This gives k possible modes of instability.
   - The result is a nonlocal eigenvalue problem for $\Phi(y)$.
2. **Small Eigenvalues** of $O(\varepsilon^2)$: Instabilities on a long $O(\varepsilon^{-2})$ time-scale. Translational instabilities. Associated with the bifurcation of asymmetric spike patterns. Analysis by a SLEP (singular limit eigenvalue problem) method for spikes. The SLEP method for front-back structures developed by Nishiura.

The NLEP: GM Model

Principal Result (NLEP): Assume that $0 < \varepsilon \ll 1$ and $\tau \geq 0$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues satisfy the NLEP on $-\infty < y < \infty$:

$$L_0 \Phi - \chi_j w^p \left(\frac{\int_{-\infty}^{\infty} w^{m-1} \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} \right) = \lambda \Phi \,, \quad \Phi \to 0 \,, \quad |y| \to \infty \,.$$

The local operator L_0 *is* $L_0 \Phi \equiv \Phi^{''} - \Phi + p w^{p-1} \Phi$. *There are* k *choices for the multiplier* $\chi_j = \chi_j(z; j)$ *given by*

$$C_j(\lambda) \equiv \frac{1}{\chi_j} \equiv \frac{s}{qm} + \frac{\sqrt{1+z}}{qm \tanh\left(\theta_0/k\right)} \left[\tanh\left(\theta_\lambda/k\right) + \frac{\left(1 - \cos\left[\pi(j-1)/k\right]\right)}{\sinh\left(2\theta_\lambda/k\right)} \right] \,,$$

where $z \equiv \tau\lambda$, $\theta_\lambda \equiv \theta_0 \sqrt{1+z}$, and $\theta_0 \equiv D^{-1/2}$.

The NLEP: GM Model: Remarks

The goal is to determine the range of values of D and τ where Re$(\lambda) < 0$, which yields stability on an $O(1)$ time-scale.

- The NLEP can be derived rigorously as in Wei, Winter (2004) using a Lyapunov-Schmidt reduction.
- When $O(\varepsilon^2) \ll D \ll 1$, the k multipliers collapse onto one limiting multiplier

$$\chi_j \to \chi_\infty \equiv \frac{qm}{s + \sqrt{1 + \tau\lambda}} .$$

  The NLEP with this limiting multiplier, corresponding to the infinite-line problem, was studied in Doelman et al.(Indiana J.)
- For the case of one spike where $k = 1$, then

$$\chi_1 = qm \left[s + \sqrt{1 + \tau\lambda} \left(\frac{\tanh \theta_0}{\tanh \theta_\lambda} \right) \right]^{-1} .$$

Reformulation of the NLEP

The large eigenvalues are the union of the roots of $g_j(\lambda) = 0$, where

$$g_j(\lambda) \equiv C_j(\lambda) - f(\lambda)\,, \quad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w^{m-1}\psi\, dy}{\int_{-\infty}^{\infty} w^m\, dy}\,, \quad C_j(\lambda) \equiv \frac{1}{\chi_j(\tau\lambda; j)}\,,$$

$$L_0\psi \equiv \psi^{''} - \psi + pw^{p-1}\psi = \lambda\psi + w^p\,; \quad \psi \to 0 \;\; |y| \to \infty\,.$$

Lemma: (Lin, Ni, Takagi) There is a unique positive eigenvalue ν_0 of the local eigenvalue problem $L_0\Phi = \nu\Phi$. Thus, since $\psi = (L_0 - \lambda)^{-1} w^p$, each $g_j(\lambda)$ is analytic in Re$(\lambda) > 0$ except at the simple pole $\lambda = \nu_0 > 0$.

Lemma: Let $\tau > 0$, and assume that $g_j(i\lambda_I) \neq 0$. Then, the number of eigenvalues M of NLEP in Re$(\lambda) > 0$ is

$$M = \frac{5k}{4} + \frac{1}{\pi}\sum_{j=1}^{k} \left[\text{arg}\, g_j\right]_{\Gamma_I}\,, \quad \Gamma_I = i\lambda_I\,.$$

Here $\left[\text{arg}\, g_j\right]_{\Gamma_I}$ denotes the change in the argument of g_j along the semi-infinite imaginary axis traversed in the downwards direction.

Stability of Multi-Spike Solutions

1. Qualitative Results:
   - For a fixed τ, eigenvalues can cross through the origin onto the positive real axis as D is increased. As D is increased the system tends to the shadow system, for which a k-spike solution for the classical GM model has $k-1$ real positive eigenvalues. This instability is an over-crowding instability.
   - For a fixed D, eigenvalues can only enter the right half-plane via a Hopf bifurcation as τ is increased. As τ is increased, the inhibitor responds more sluggishly to changes in the activator concentration. This delay effect leads to an oscillatory feedback loop.
2. Ingredients of Rigorous Mathematical Analysis
   - Behavior of $f(\lambda)$ on the imaginary axis and on the real axis in terms of the homoclinic solution (Ward, Wei, JNS (2003)) to calculate the winding number criterion. Here is where restrictions on the exponent set arises.
   - Behavior of $C_j(\lambda)$ on the imaginary and real axis. This is determined by the spike interaction, D, and the Green's function. (Ward, Wei, JNS (2003)).

Real Eigenvalues: A Key Proposition

Proposition: Let $\lambda = \lambda_R > 0$ be real. Then, for $\tau > 0$ and $D > 0$

$$C_k(\lambda_R) > C_{k-1}(\lambda_R) > \ldots > C_1(\lambda_R) > 0\,,$$

$$0 < C_k^{'}(\lambda_R) < C_{k-1}^{'}(\lambda_R) < \ldots < C_1^{'}(\lambda_R)\,.$$

$$C_j^{''}(\lambda_R) < 0\,, \;\; C_j^{'}(\lambda_R) = O(\tau^{1/2})\,, \;\; \tau \gg 1\,, \quad j = 1, \ldots, k\,.$$

Let $B_j(D) \equiv C_j(0)$ for $j = 1, .., k$. Then, $B_j(D)$ is independent of τ, and

$$B_j^{'}(D) > 0\,, \quad j = 2, .., k\,; \quad B_1(D) = \frac{s+1}{qm}\,.$$

Finally, for $j = 2, .., k$, we have $B_j(D) = 1/(p-1)$ when $D = D_j$, where

$$D_j \equiv \frac{4}{k^2 \left[\log\left(a_j + \sqrt{a_j^2 - 1}\right)\right]^2}\,, \quad a_j \equiv 1 + \left[1 - \cos\left(\frac{\pi(j-1)}{k}\right)\right] \zeta^{-1}\,.$$

Note that $D_{j-1} > D_j$ for $j = 3, .., k$. We label $D_1 = \infty$.

Real Eigenvalues

Proposition (Real Eigenvalues): Let $k \geq 2$, and suppose that either $m = p + 1$ and $p > 1$, or $m = p = 2$. Suppose that j^* satisfies $2 \leq j^* \leq k$ such that $D_{j^*} < D < D_{j^*-1}$. Then, for any $\tau > 0$, the number of eigenvalues M of NLEP in the right half-plane satisfies

$$1 + k - j^* \leq M \leq k - 1 + j^* \,.$$

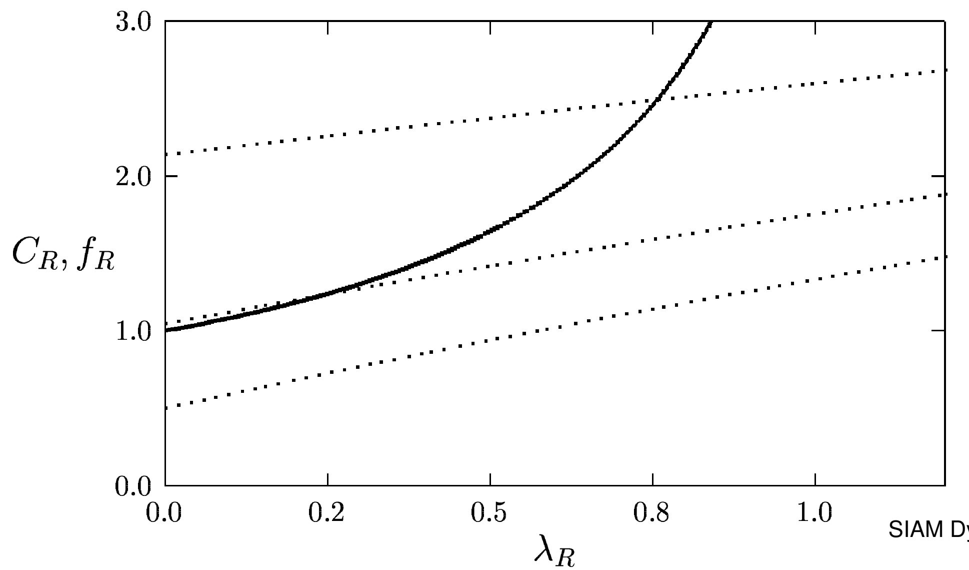
Moreover, there are at least $M_R = 1 + k - j^*$ positive real eigenvalues on $0 < \lambda_R < \nu_0$ for any $\tau > 0$.

1. **Near-Shadow Limit:** If $j^* = 2$, then for $D > D_2$ and any $\tau > 0$ the number of positive real eigenvalues satisfies $k - 1 \leq M \leq k + 1$, and are on the range $0 < \lambda_R < \nu_0$.
2. **Competition Instability:** If $j^* = k$, then $1 \leq M \leq 2k - 1$. For $\tau \ll 1$, then $M = 1$, and the unstable eigenvalue is on the positive real axis.
3. **Large τ:** Suppose that $0 < D < D_k$ and $\tau \gg 1$. Then, $M = 2k$, and they are all real.
4. **Zero τ:** For $\tau = 0$, then $M = 0$ when $D < D_k$.

Graphical Illustration of the Proof

Sketch of Proof of (1): When $D > D_2$ then $C_j(0) > f(0)$ for $j = 2, .., k$. One can prove that $C_j(\lambda_R)$ is concave and $f = f_R(\lambda_R)$ is convex for either $m = p + 1$ and $p > 1$, or $m = p = 2$. Therefore, we have at least $k - 1$ real eigenvalues. If τ is large enough, then C_1 intersects f_R twice, which gives two more real eigenvalues. By examining the winding number criterion and the properties of g_j on the imaginary axis, then these are the only eigenvalues in Re$(\lambda) > 0$. Hence, $k - 1 \leq M \leq k + 1$.

Figure: Plots of $f_R(\lambda_R)$ (heavy solid curve) and $C_j(\lambda_R)$ (dashed curves) for $j = 1$ (bottom curve), $j = 2$ (middle curve), and $j = 3$ (top curve), for a three-spike solution for the exponent set $(2, 1, 2, 0)$. For $\tau = 2.0$ and $D = 0.52 > D_2$, then $2 \leq M \leq 4$.



Hopf Bifurcations

Proposition: Let $k \geq 2$ and suppose that either $m = p + 1$ and $p > 1$, or $m = 2$ and $1 < p \leq 5$. Then, a multi-spike solution is unstable for any $\tau \geq 0$ when $D > D_k$. Next, suppose that $0 < D < D_k$, with $m = 2$ and $1 < p \leq 5$. Then, there exists a value $\tau_0(D; k)$ such that a multi-spike solution is stable on an $O(1)$ time-scale for $0 \leq \tau < \tau_0(D; k)$.

Sketch of Proof:

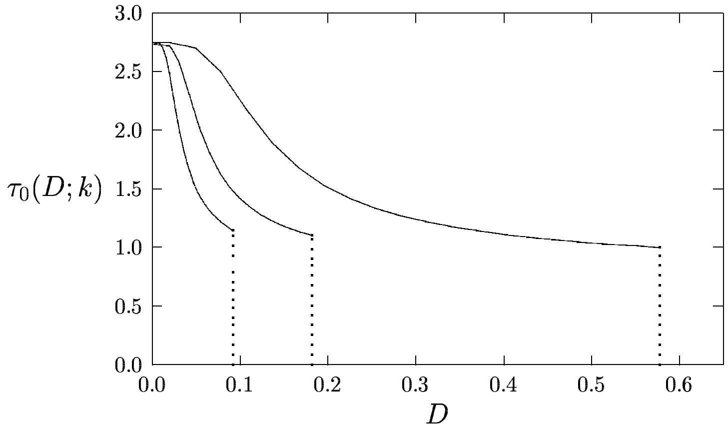
- When $D > D_k$ there is always an eigenvalue on the positive real axis for any $\tau > 0$ because $C_k(0) > f(0)$.
- For $D < D_k$ there are no eigenvalues on the positive real axis. They can only cross into the right half-plane through a Hopf bifurcation at some critical value of τ. The critical value is

$$0 \leq \tau < \tau_0(D; k) \equiv \text{Min}\left(\tau_{0j}(D)\,;\, j = 1, .., k\right) \,.$$

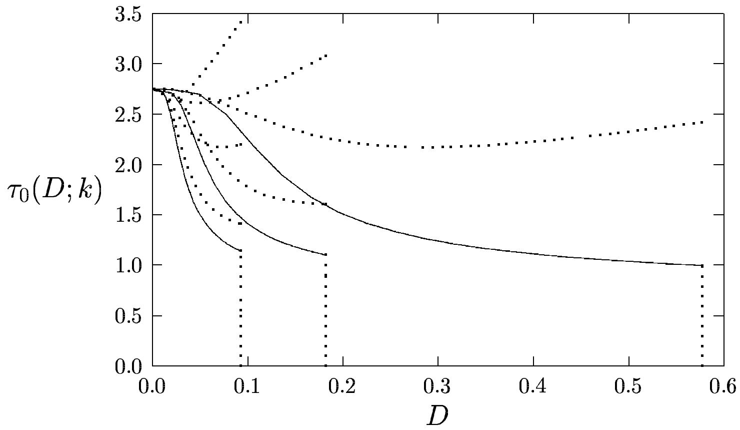
  An ordering principle determines which mode sets τ_0. Most typically $j = 1$. This gives a synchronous oscillatory instability.
- **Open Theoretical Problem:** Is τ_0 unique? i.e. is there a strict transversal crossing condition?

The Instability Diagram

Plot of $\tau_0(D;k)$ versus D for $k = 2, 3, 4$ for exponent set $(2, 1, 2, 0)$. Vertical lines at $D_2 = 0.5767$, $D_3 = 0.1810$, and $D_4 = 0.0915$.



Plot of τ_{0j} for $k = 2, 3, 4$. These dashed curves are τ values where additional pair of complex conjugate eigenvalues first enter Re$(\lambda) > 0$.



The Initial Instability

(Competition Instability) Let $k \geq 2$ and suppose that $D_k < D < D_{k-1}$ and τ is sufficiently small. Then, there is exactly one eigenvalue λ_R on the positive real axis. The unstable eigenfunction has the form

$$a = a_e + \delta e^{\lambda_R t} \phi \,, \quad \phi(x) = \sum_{j=1}^{k} c_j \psi \left[\varepsilon^{-1}(x - x_j) \right] \,,$$

$$c_j = \cos\left(\frac{\pi(k-1)}{k} (j - 1/2) \right) \quad j = 1, .., k \,.$$

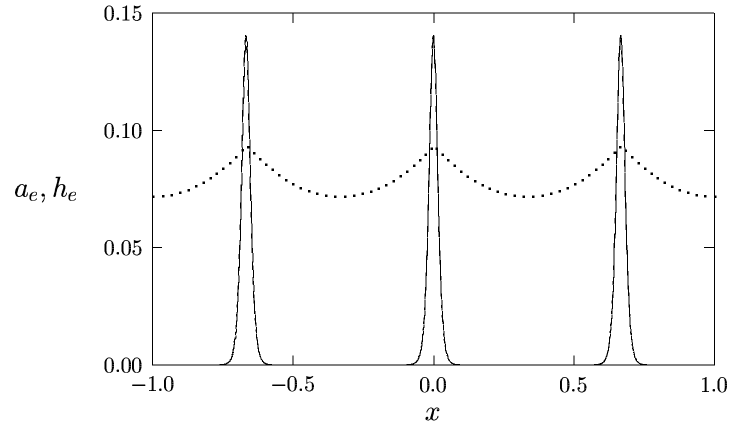
(Synchronous Oscillatory Instability) Suppose $0 < D < D_k$ and $\tau = \tau_0(D; k)$. Then, there is one pair of complex conjugate eigenvalues on the imaginary axis, and

$$a = a_e + \delta e^{i\lambda_I^0 t} \phi + \text{c.c} \,, \;\; \phi(x) = \sum_{j=1}^{k} c_j \psi \left[\varepsilon^{-1}(x - x_j) \right] \,, \quad c_j = 1$$

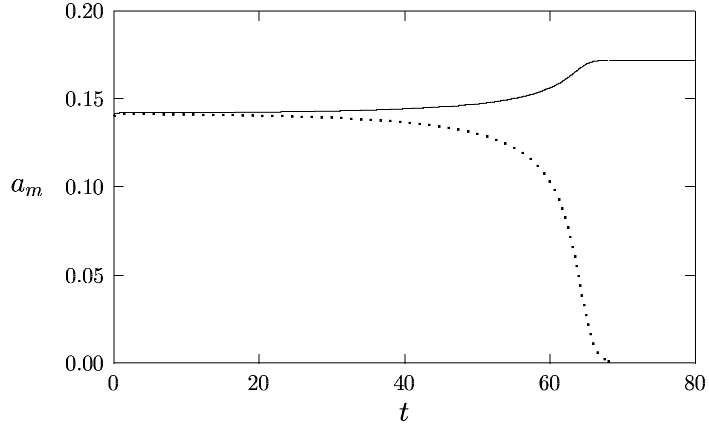
Qualitative analysis of competition instabilities given in Kerner and Osipov (Autosolitons, Kluwer) as “activator-repumping sign-altering fluctuations”.

Numerical Experiment: I

Equilibrium solution for $k = 3$, $(2, 1, 2, 0)$, and $D = 0.19$.

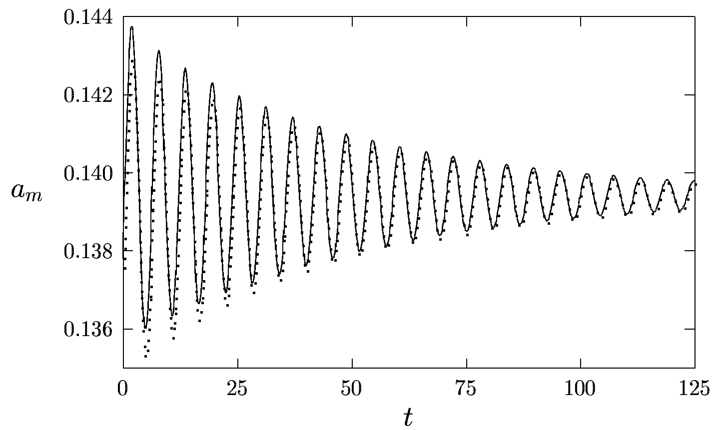


Competition instability: Spike amplitudes versus t for $\tau = .02$ and $D = 0.19$. Dashed curve is a_{m2}, solid curves a_{m1}, a_{m3}. Second spike is killed.

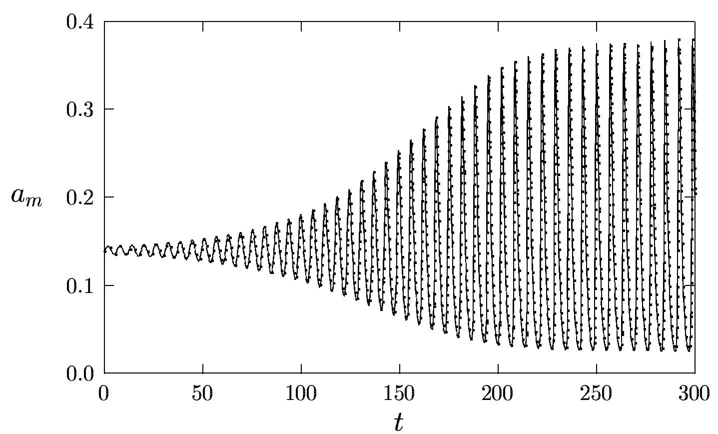


Numerical Experiment: II

Decaying Synchronous Oscillation: $D = 0.1695$ and $\tau = 1.05$. Solid curves a_{m1} and a_{m3}, dashed curve is a_{m2}.



Synchronous Oscillatory Instability: $D = 0.1695$ and $\tau = 1.15$. Solid curves a_{m1} and a_{m3}, dashed curve is a_{m2}. Hopf bifurcation is at $\tau_0 = 1.128$ and is apparently supercritical



The Gray-Scott Model

$$V_T = D_V V_{XX} - (F+k)V + UV^2\,, \quad V_X = 0\,,\ X = 0, L$$

$$U_T = D_U U_{XX} + F(1-U) - UV^2\,, \quad U_X = 0\,,\ X = 0, L$$

This can be very conveniently transformed on $-1 < x < 1$ to (Muratov)

$$v_t = \varepsilon^2 v_{xx} - v + Auv^2\,, \quad v_x = 0\,,\ x = \pm 1$$

$$\tau u_t = D u_{xx} + (1-u) - uv^2\,, \quad u_x = 0\,,\ x = \pm 1$$

where $D = \frac{4D_U}{FL^2}$, $\varepsilon^2 = \frac{4D_v}{L^2(F+k)} \ll 1$, $\tau = \frac{F+k}{F} > 1$, and $A = \frac{\sqrt{F}}{F+k}$.

Most work in this field has focused on pattern formation from a spatially uniform state that is near the transition from linear stability to linear instability. With this restriction, standard bifurcation-theoretic tools such as amplitude equations have been used with considerable success (ref: Cross and Hohenburg (Rev. Mod. Physics 1993)). It is unclear whether the patterns presented here will yield to these standard technologies.

John Pearson: Complex Patterns in a Simple System, Science 1993

Different Asymptotic Regimes

1. **Semi-Strong Interaction Regime:** $\varepsilon \ll 1$ and $D = O(1)$
   - **Low Feed-Rate Regime** $A = O(\varepsilon^{1/2})$**:** Saddle-node bifurcation of equilibria. Similar NLEP to GM model. Competition and oscillatory instabilities of spike profile (KWW, Studies App Math; Muratov and Osipov, NLEP problem for one spike on infinite line).
   - **Intermediate Regime** $O(\varepsilon^{1/2}) \ll A \ll O(1)$**:** Scaling laws for Hopf bifurcation (NLEP) and traveling wave instability of small eigenvalues. (Doelman et al. Physica D; Muratov and Osipov, SIAM, Physica D)
   - **High Feed-Rate Regime** $A = O(1)$**:** Saddle-node bifurcations, simultaneous pulse-splitting behavior, traveling wave instabilities (Doelman et al.; Muratov Osipov; KWW).
2. **Weak-Interaction Regime:** $\varepsilon \ll 1$ and $D = O(\varepsilon^2)$. Hierarchy of saddle-node bifurcations, pulse-splitting of edge-splitting type, traveling waves (Nishiura et al.).

Low-Feed: Symmetric Equilibria: $A = \varepsilon^{1/2}\mathcal{A}$

Principal Result : Assume that $0 < \varepsilon \ll 1$, $\mathcal{A} = O(1)$, and $D = O(1)$. Then, when $\mathcal{A} > \mathcal{A}_{ke}$, there are two symmetric k-spike equilibria given by

$$\nu_{\pm}(x) \sim \frac{1}{\mathcal{A}U_{\pm}} \sum_{j=1}^{k} w\left[\varepsilon^{-1}(x - x_j)\right], \quad u_{\pm}(x) \sim 1 - \frac{(1 - U_{\pm})}{a_g} \sum_{j=1}^{k} G(x; x_j),$$

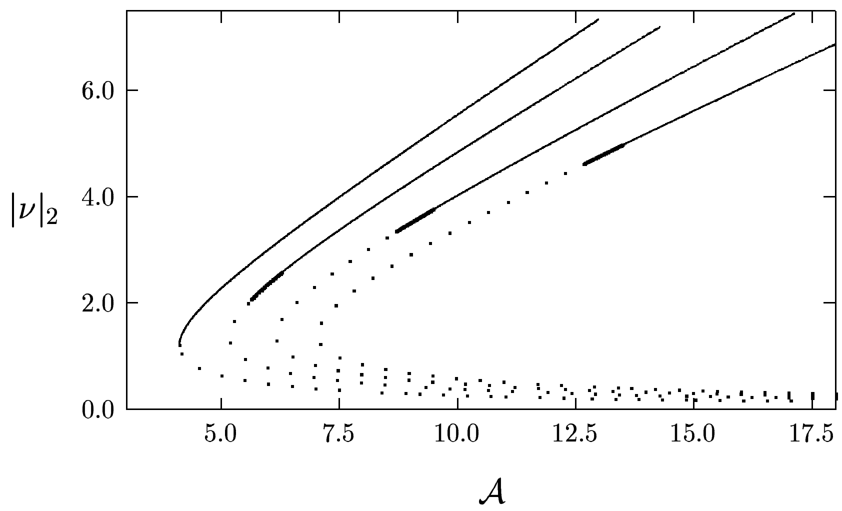
Here $x_j = -1 + \frac{(2j-1)}{k}$, $a_g \equiv \sum_{i=1}^{k} G(x_j; x_i)$, $w(y) = \frac{3}{2}\text{sech}^2(y/2)$, and $G(x; x_j)$ is the Green's function satisfying $DG_{xx} - G = -\delta(x - x_j)$ with $G_x(\pm 1; x_j) = 0$.
The solution branches are parameterized by s_g, with $0 < s_g < \infty$, defined by $s_g \equiv (1 - U_{\pm})/U_{\pm}$. With $\theta_0 \equiv D^{-1/2}$, we have

$$\mathcal{A} = \mathcal{A}_{ke} \frac{(1 + s_g)}{2\sqrt{s_g}}, \quad s_g \equiv \frac{1 - U_{\pm}}{U_{\pm}}, \quad \mathcal{A}_{ke} \equiv \sqrt{\frac{12\theta_0}{\tanh(\theta_0/k)}}.$$

The large solution u_-, ν_- is for $1 < s_g < \infty$, while the small solution u_+, ν_+ corresponds to $0 < s_g < 1$. The existence threshold $\mathcal{A}_{ke}$ corresponds to $s_g = 1$.

Bifurcation Diagram: k-Spike Equilibria

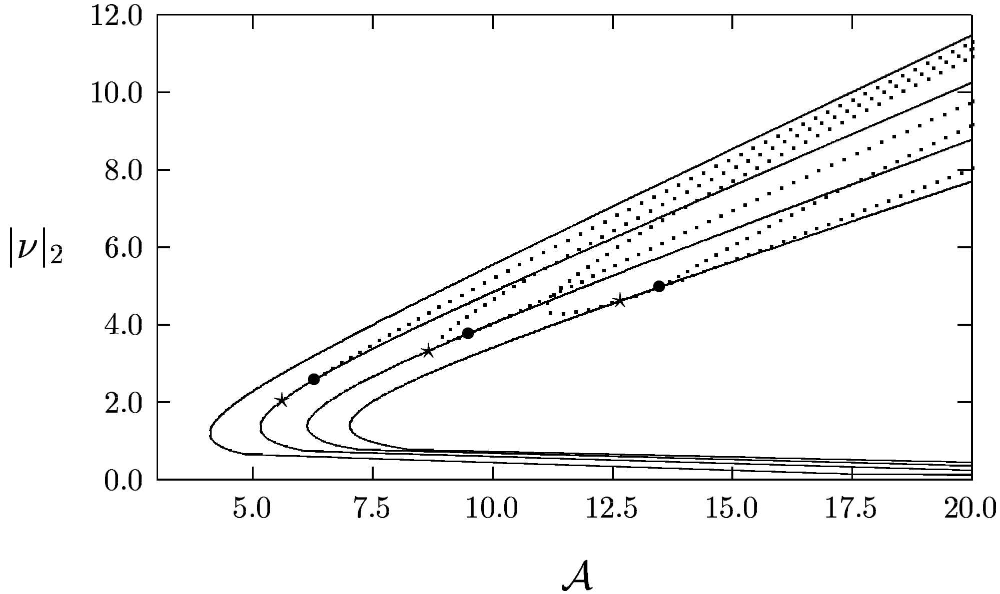
Bifurcation diagram of $|v|$ versus $\mathcal{A}$, where $A = \varepsilon^{1/2}\mathcal{A}$ for symmetric k-spike equilibria when $D = 0.75$ and $k = 1, \ldots, 4$,



- The saddle-node values $\mathcal{A}_{ke}$ increase with k.
- The heavy solid lines are stable only with respect to the large eigenvalues when $\tau < \tau_{hL}$.
- The solid lines are stable with respect to both the large and small eigenvalues when $\tau < \tau_{hL}$. The critical values $\mathcal{A}_{kL}$ correspond to where the dashed and heavy solid lines intersect (for $k = 1$, $\mathcal{A}_{kL} = \mathcal{A}_{ke}$).

Bifurcation Diagram: Asymmetric Equilibria

Bifurcation diagram of $|v|$ versus $\mathcal{A}$, where $A = \varepsilon^{1/2}\mathcal{A}$ for symmetric and asymmetric k-spike equilibria when $D = 0.75$ and $k = 1, \ldots, 4$,



<u>**Principal Result**</u>: Let $\tau > 0$, $k > 1$, and $\tau = O(1)$. For the large solution u_-, ν_-, there is stability with respect to the small eigenvalues λ_j for $j = 1, .., k-1$ when $\mathcal{A} > \mathcal{A}_{kS} \equiv \mathcal{A}_{ke}\left[\tanh\left(\frac{2\theta_0}{k}\right)\right]^{-1}$. Note $\lambda_k < 0$. As $\mathcal{A}$ crosses below $\mathcal{A}_{kS}$, $\lambda_j = 0$ for $j = 1, \ldots, k-1$, leading to the existence of $k-1$ asymmetric equilibrium solution branches.

The NLEP: GS Model, Low Feed-Rate

Principal Result (NLEP): Assume that $0 < \varepsilon \ll 1$ and $\tau \geq 0$. Then, with $\Phi = \Phi(y)$, the $O(1)$ eigenvalues satisfy the NLEP on $-\infty < y < \infty$:

$$L_0\Phi - \chi_{gsj} w^2 \left(\frac{\int_{-\infty}^{\infty} w\Phi \, dy}{\int_{-\infty}^{\infty} w^2 \, dy} \right) = \lambda\Phi \,, \quad \Phi \to 0 \,, \ \ |y| \to \infty \,.$$

The local operator L_0 is $L_0\Phi \equiv \Phi^{''} - \Phi + 2w\Phi$.

Spectral Equivalence Principle: The k-choices for the multipliers χ_{gsj} are precisely the ones for a GM model with exponent set (p, q, m, s) where

$$p = 2 \,, \quad q = s_g \,, \quad m = 2 \,, \quad s = s_g \,.$$

Recall that s_g parameterizes the k-spike symmetric equilibria as

$$s_g \equiv \frac{1 - U_{\pm}}{U_{\pm}} \,.$$

Large Eigenvalues: Main Result

<u>**Proposition**</u>: Let $\tau > 0$, $k > 1$, $D = O(1)$, and consider the large solution. For $\mathcal{A} > \mathcal{A}_{kL}$, the solution will be stable with respect to the large eigenvalues when $0 < \tau < \tau_{hL}$. An oscillation (usually synchronous) occurs when τ increases past some (possibly non-unique) τ_{hL}. Alternatively, suppose that $\mathcal{A}_{ke} < \mathcal{A} < \mathcal{A}_{kL}$, then the solution is unstable for any $\tau > 0$ due to a competition instability. The threshold value is

$$\mathcal{A}_{kL} = \mathcal{A}_{ke} \frac{\left((\gamma_k/2) + 2\sinh^2(\theta_0/k)\right)}{\left(\left[(\gamma_k/2) + 2\sinh^2(\theta_0/k)\right]^2 - (\gamma_k/2)^2\right)^{1/2}},$$

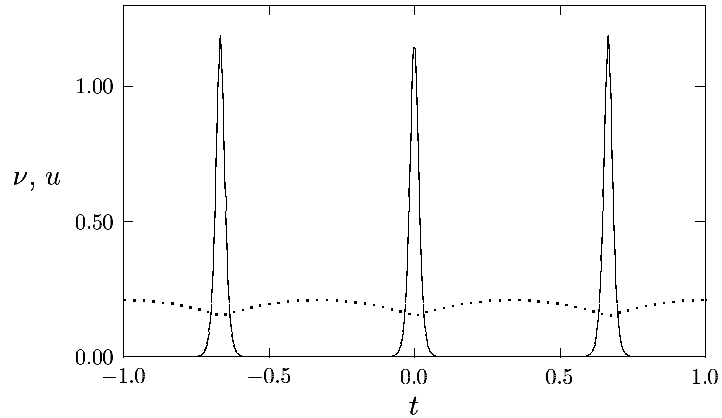
with $\gamma_k = 1 + \cos(\pi/k)$. The small solution is always unstable.

Remarks on Large Eigenvalues

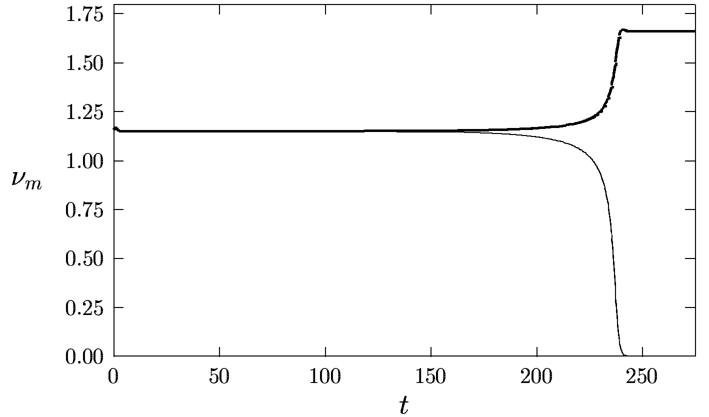
- For a GM model with set (p,q,m,s), it is assumed that $\zeta = \frac{qm}{p-1} - (s+1) > 0$. For the GS model, $\zeta = s_g - 1$. Hence, for the small solution where $0 < s_g < 1$, then $\zeta < 0$, and we get instability as a result of k positive eigenvalues.
- Large solution has oscillatory or competition instabilities depending on D, τ, and $\mathcal{A}$.
- A long domain is equivalent to setting $D \ll 1$ (Muratov, Osipov, SIAM), to get $\frac{1}{\chi_{gsj}} \to \frac{1}{2} + \frac{\sqrt{1+\tau\lambda}}{2s_g}$. With this multiplier, there are no competition instabilities, and only Hopf bifurcations are possible. The overcrowding effect associated with a finite domain lead to competition instabilities.
- The inequality $\mathcal{A}_{ke} < \mathcal{A}_{kL} < \mathcal{A}_{ks}$ holds. Hence, interesting dynamics should occur for $\mathcal{A}$ between $\mathcal{A}_{kL}$ and $\mathcal{A}_{kS}$. Note, $\mathcal{A}_{kS}$ and $\mathcal{A}_{KL}$ tend to $\mathcal{A}_{ke}$ as $D \to 0$.

Numerical Experiment: I

Equilibrium solution for $k = 3$, $D = 0.75$, $\varepsilon = 0.01$, and $\mathcal{A} = 8.6$.

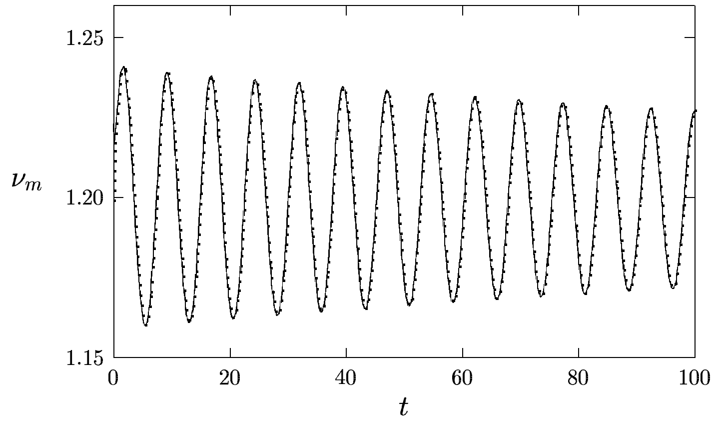


Competition instability: Let $\tau = 2.0 < \tau_{hL}$. Since $\mathcal{A} < \mathcal{A}_{3L} \approx 8.686$, there is a competition instability and the second spike is killed.

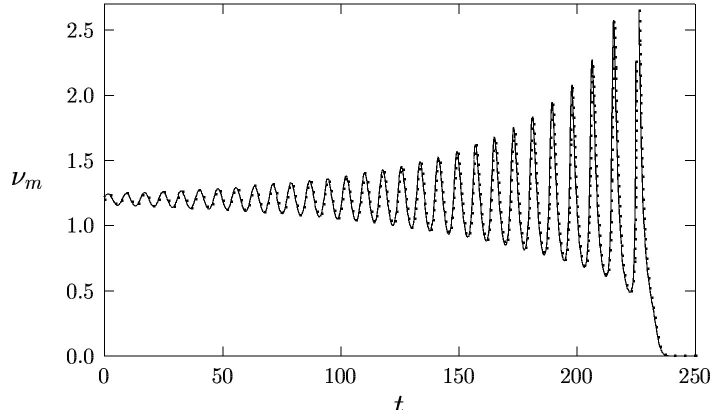


Numerical Experiment: II

Decaying Synchronous Oscillation: This occurs for $\tau = 7.25 < \tau_{hL} \approx 7.5$ and $\mathcal{A} = 8.86 > \mathcal{A}_{3L}$.



Sychnronous Oscillatory Instability: This occurs for $\tau = 7.6 > \tau_{hL}$ and $\mathcal{A} = 8.86 > \mathcal{A}_{3L}$. The synchronous oscillatory instability is apparently subcritical.



The Intermediate Regime: $s_g \to \infty$.

Intermediate Limit: Let $\mathcal{A}/\mathcal{A}_{ke} \gg 1$ and $\mathcal{A} \ll O(\varepsilon^{-1/2})$, and assume that $D = O(1)$. Then, we have $s_g \gg 1$, and

$$s_g = \frac{1-U_-}{U_-} \sim \frac{4\mathcal{A}^2}{\mathcal{A}_{ke}^2} - 2 + o(1)\,.$$

In this limit, the k-independent choices for the multiplier collapse onto one limiting function

$$\frac{1}{\chi_{gsj}} \to \frac{1}{\chi_{gs}^{\infty}} \equiv \frac{1}{2}\left[1+\sqrt{\tau_0\lambda}\right]\,, \qquad \tau \sim \tau_0 \tanh^2\left(\theta_0/k\right) s_g^2\,.$$

- There are no competition instabilities with this limiting NLEP. It is equivalent to the NLEP problem studied for periodic-spike patterns in Doelman et al. (Physica D), Doelman et al. (Memoirs of AMS).
- Since the thresholds wrt τ for the k modes of instability are asymptotically the same, there is no theoretical guarantee of synchronous oscillatory instabilities.

Intermediate: Large Eigenvalue Results

Proposition: Let $\varepsilon \ll 1$, and suppose that $O(1) \ll \mathcal{A} \ll O(\varepsilon^{-1/2})$. Then, for $D = O(1)$, the symmetric k-spike large solution u_-, ν_- is stable with respect to the large eigenvalues when $\tau < \tau_h^\infty$, where

$$\tau_h^\infty \sim \frac{\mathcal{A}^4 D}{9} \tanh^4\left(\theta_0/k\right) \tau_{0h} \left(1 - \frac{6\theta_0}{\mathcal{A}^2 \tanh(\theta_0/k)}\right)^2 + o(1)\,.$$

Here $\tau_{0h} \approx 1.748$. Hence, $\tau_h^\infty \gg 1$ in this regime.

This is asymptotically equivalent to the main stability result of Doelman et al. (Physica D) for periodic patterns.

As for competition instabilities when $\tau = O(1)$, they only occur for closely spaced spikes when the interspike distance L falls below some threshold L_m. There are no such instabilities for $O(\varepsilon^{1/2}) \ll A \ll O(1)$ when

$$L > L_m \sim \left(\frac{12\gamma_k D\varepsilon}{A^2}\right)^{1/3}\,, \quad \gamma_k \equiv 1 + \cos\left(\pi/k\right)\,, \quad A = \varepsilon^{1/2}\mathcal{A}\,.$$

A related scaling law for the Brusselator was found by Kerner, Osipov.

Equivalence Principles

We have shown that there is an NLEP equivalence principle for k-spike solutions between the GM model and the GS model in the low feed-rate regime. At each point on the GS equilibrium bifurcation diagram, the k-independent choices for the GS multipliers are the same as those for a generalized GM model with a specific exponent set. The k different multipliers, give multiple modes of instability. The onset of such instabilities are the synchronous oscillatory and competition instabilities. A subclass of this equivalence principle is the one which yields NLEP problems of the form (Doelman et al.)

$$\Phi'' - \Phi + pw^{p-1}\Phi - \chi w^p \left(\frac{\int_{-\infty}^{\infty} w^m \Phi \, dy}{\int_{-\infty}^{\infty} w^m \, dy} \right) = \lambda \Phi \,, \qquad \chi \equiv \frac{a}{b + \sqrt{c + d\lambda}} \,.$$

NLEP's of this form occur for a k-spike equilibrium for the GM model when $D \ll 1$ (equivalent to infinite-line problem), and for a k-spike solution for the GS model in the intermediate regime $s_g \to \infty$ where there is a universal NLEP problem, that determines a scaling law for the stability threshold. Such a universal NLEP is not associated with competition instabilities, and there is no theoretical prediction of synchronicity in the spike oscillations.

Comparison of Two Slow Dynamics Processes

1. **Dynamics of Quasi-Equilibria: Cahn-Hilliard, Allen-Cahn:**
   - **Metastable dynamics** for widely-spaced heteroclinic layers.
   - **Collapse Events** on $O(1)$ time-scale as k-layer solutions cascade to $k-2$ layer solutions from pairwise collapse of nearest neighbours.
   - **Minimum Energy** yields the final equilibrium state of no interfaces (Allen-Cahn), or one-layer from mass conservation (Cahn-Hill).
2. **Dynamics of Quasi-Equilibria: GM and GS Models**
   - **No Variational Structure**: Below certain thresholds on D and τ that depend on k, all equilibrium solutions with $\leq k$ spikes are stable.
   - **Slow Algebraic Motion**: Slow dynamics with speed $O(\varepsilon^2)$ determined by the global variable. Slow dynamics only when a profile stability condition wrt the large eigenvalues is satisfied. Stability thresholds depend on instantaneous spike locations.
   - **Transitions: Dynamic Bifurcations** can occur if stability boundaries are crossed before ODE reaches its equilibrium. **Yields dynamic oscillatory or dynamic competition instabilities.** We label as **static competition and oscillatory instabilities** those that arise immediately due to parameters being in the unstable zone at time $t = 0$.

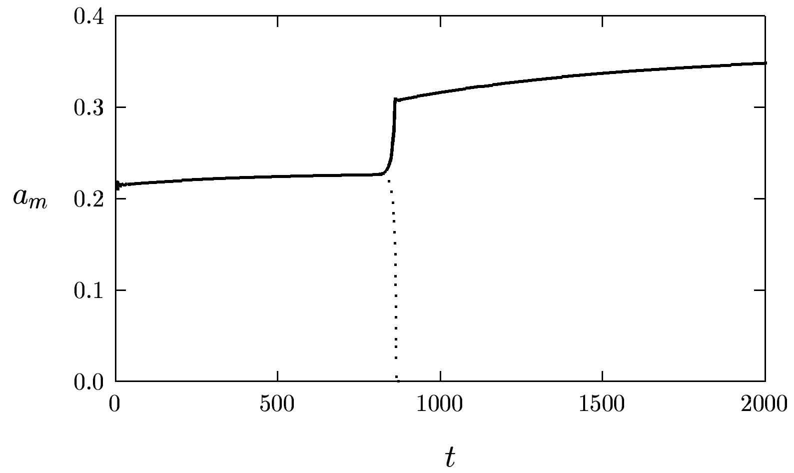
Finite Versus Infinite Domain: GS and GM

1. **Finite Domain** $-1 < x < 1$: Symmetric Pattern With $x_0 = -x_1 = \alpha$
   - Derive ODE for spike locations (formal asymptotics). Obtain ODE for α, which has fixed point at $\alpha = 1/2$. Equivalence principle in dynamics between GM and low feed-rate GS.
   - Rigorous analysis of NLEP governing profile instabilities. There are two distinct multipliers. One governs competition and the other oscillatory instabilities.
   - In addition, to **static** competition and oscillatory instabilities that occur at $t = 0$ there can also **dynamic** instabilities due to slow eventual passage into unstable regimes.
2. **Infinite Domain** $-\infty < x < \infty$
   - ODE's for slow dynamics for GS by geometric singular perturbations. **Static** oscillatory instability by NLEP for GS in intermediate regime (Doelman et al. (SIAM))
   - **Static** competition instabilities do occur for closely spaced spikes. However, **Dynamic** instabilities do not occur for GS and GM models due to repulsive interaction (SWR, (SIADS)).
   - ODE's for more general RD models, and other bifurcations. (Doelman, Kaper, (SIADS)).

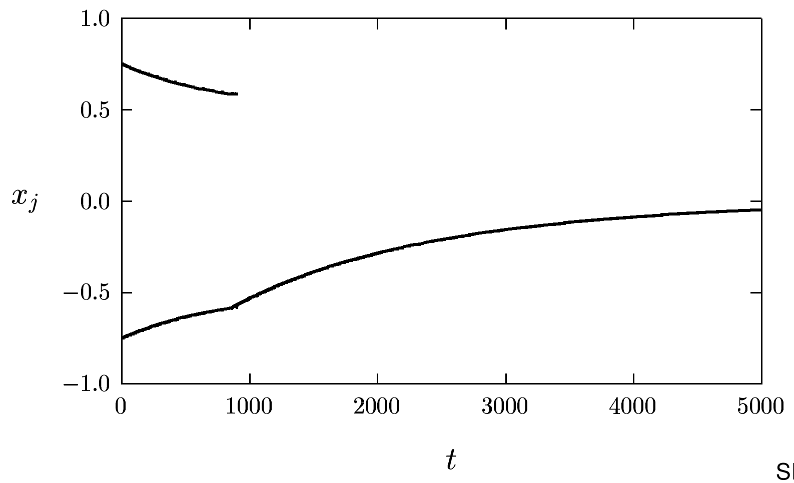
Two-Spike GM: Dynamic Competition

$\varepsilon = 0.025$, $D = 0.75$, $\tau = 0.9$, $x_1(0) = -x_2(0) = 0.75$.

Spike amplitudes versus t



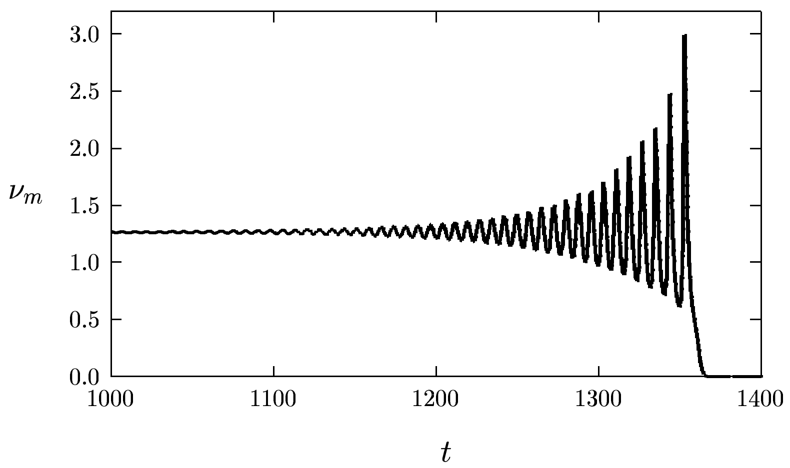
Spike locations versus t.



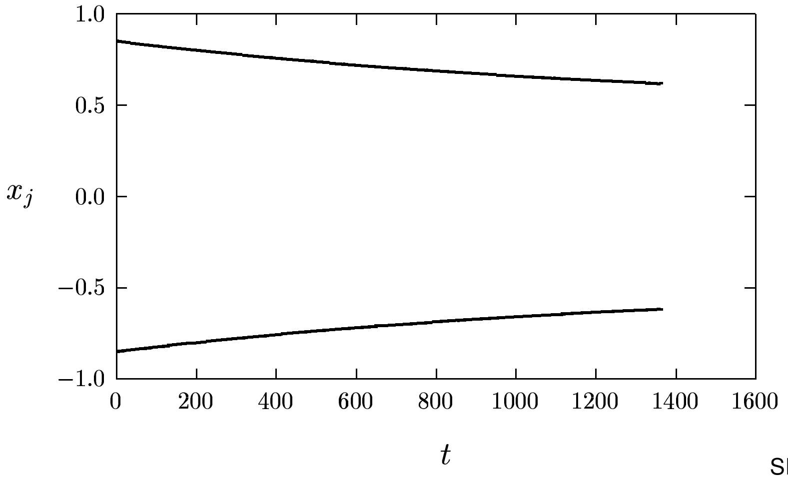
Two-Spike GS: Dynamic Oscillatory

$\varepsilon = 0.015$, $D = 2.25$, $\mathcal{A} = 6.5$, $\tau = 5.3$, $x_1(0) = -x_2(0) = 0.85$.

Synchronous oscillation in the amplitudes (subcritical?)



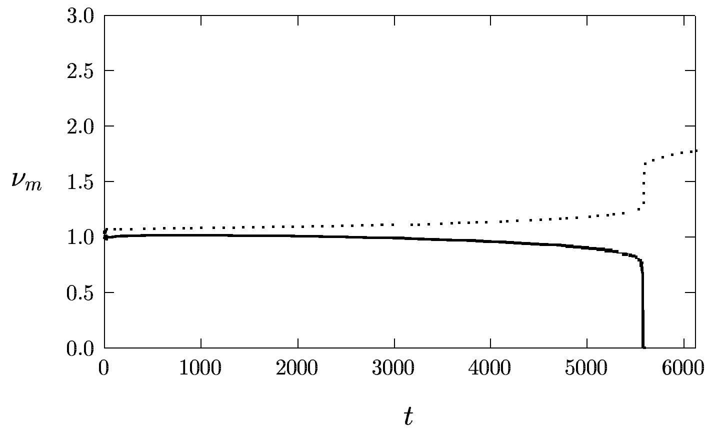
Spike locations versus t.



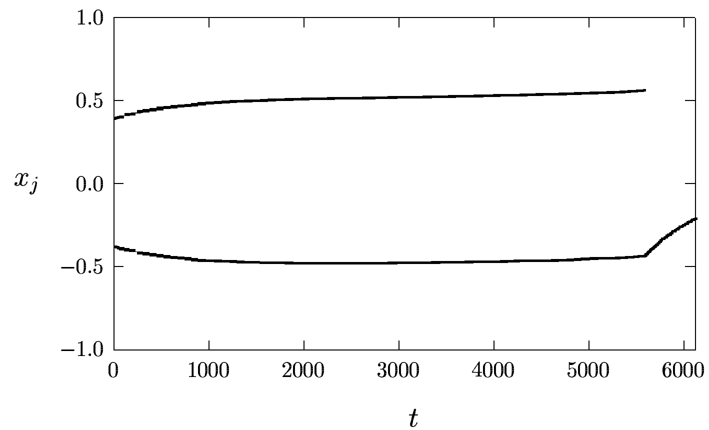
2-Spike GS: Unstable Small Eigenvalues

$\varepsilon = 0.015$, $D = 0.75$, $\mathcal{A} = 6.1$, $\tau = 4.1$, $x_1(0) = 0.39$, $x_2(0) = -0.38$. Theory yields stability wrt NLEP when $x_1(0) = -x_2(0) < 0.39$. Equilibrium unstable wrt small eigenvalues.

Spike amplitudes versus t



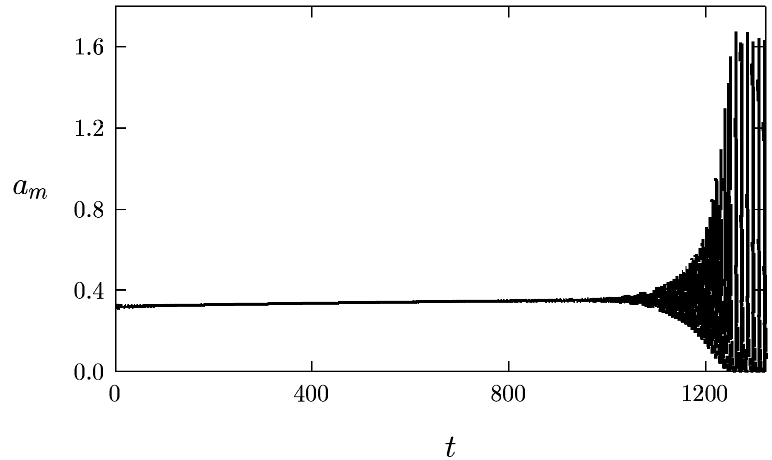
Spike locations versus t.



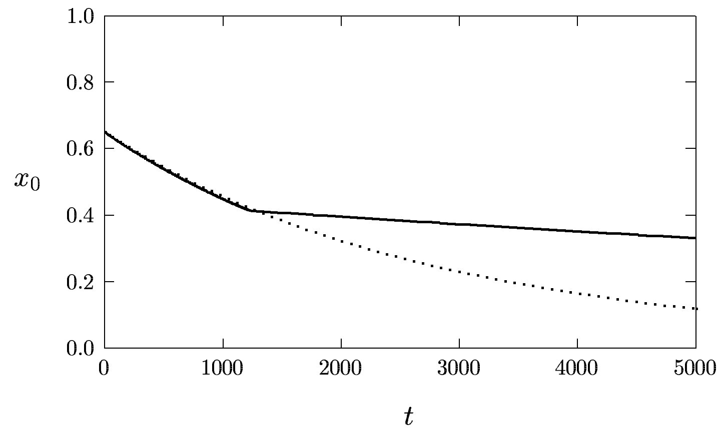
1-Spike GM: Dynamic Oscillatory

$\varepsilon = 0.02$, $D = 1.0$, $\tau = 1.4$, $x_1(0) = 0.65$.

Spike amplitude versus t (supercritical?)



Slow drift with pulse oscillations. Full numerical result (heavy solid) and incorrectly extrapolated asymptotics (dashed).



GM Competition Instability

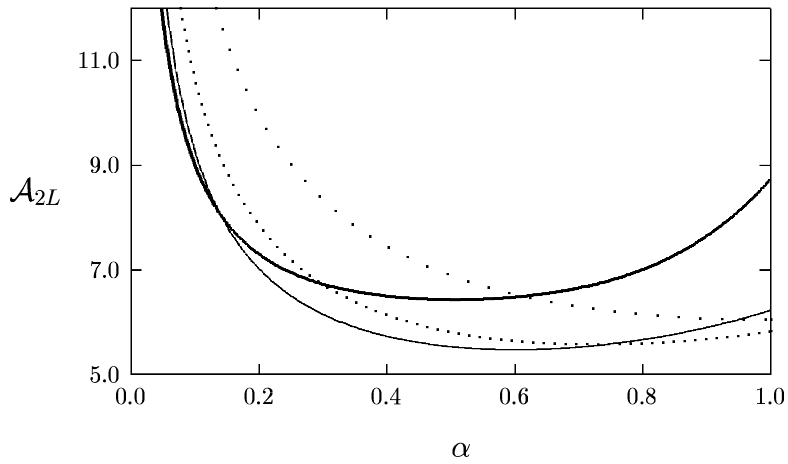
Proposition: Suppose that $0 < D < D_* \equiv \left[\log(\sqrt{2}+1)\right]^{-2} \approx 1.287$ and $0 \le \tau < \tau_H$ for some $\tau_H > 0$. Define α_c in $0 < \alpha_c < 1$ by

$$\alpha_c \equiv \frac{1}{2\theta_0} \log\left[\frac{2+\coth\theta_0}{2-\coth\theta_0}\right], \qquad \theta_0 = D^{-1/2}.$$

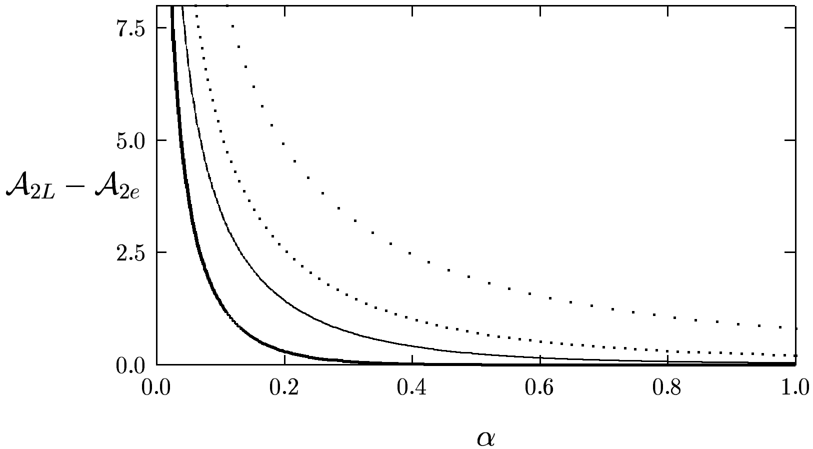
- When $0 < \alpha < \alpha_c$, the quasi-equilibrium solution is unstable as a result of a unique real positive eigenvalue.
- Alternatively, for $\alpha_c < \alpha < 1$, and for $0 \le \tau < \tau_H$, the two-spike solution is stable on an $O(1)$ time-scale.
- For $D > D_* \approx 1.287$, the two-spike quasi-equilibrium solution is unstable for any α with $0 < \alpha < 1$, and for any $\tau > 0$.
- When the initial spike location $\alpha(0)$ satisfies $\alpha_c < \alpha(0) < 1$, with $\alpha_c > 1/2$, there is a **dynamic competition instability** as α crosses below α_c as it tends to its equilibrium at $\alpha = 1/2$.

GS Existence and Competition Instability Thresh

GS competition instability threshold $\mathcal{A}_{2L}$ for $D = 0.1$ (heavy solid), $D = 0.5$ (solid), $D = 1.0$ (dotted), $D = 2.306$ (widely spaced dots).

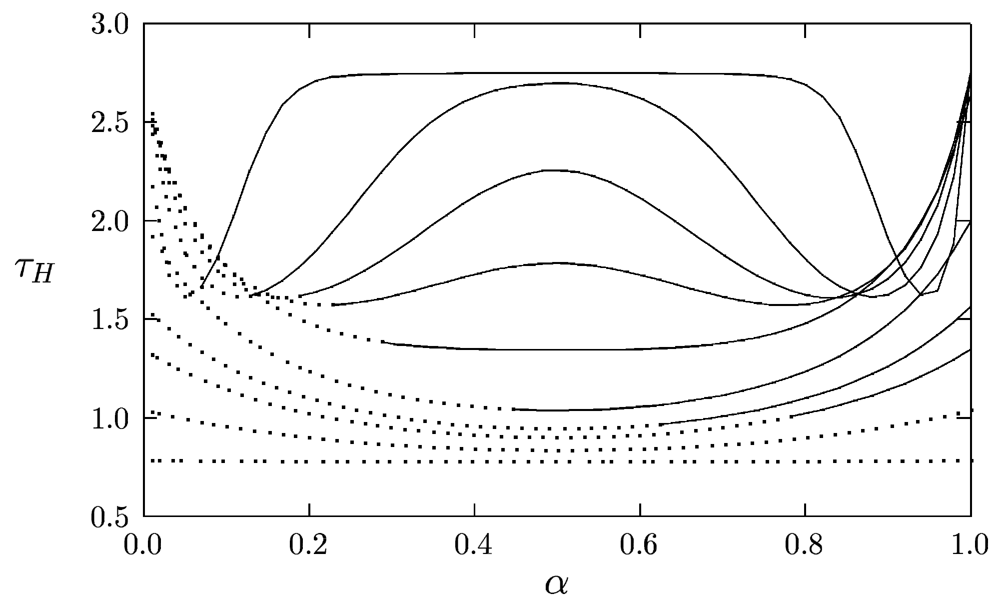


The difference $\mathcal{A}_{2L} - \mathcal{A}_{2e}$ (same labels for D)



GM Dynamic Oscillatory Instabilities: 2-Spikes

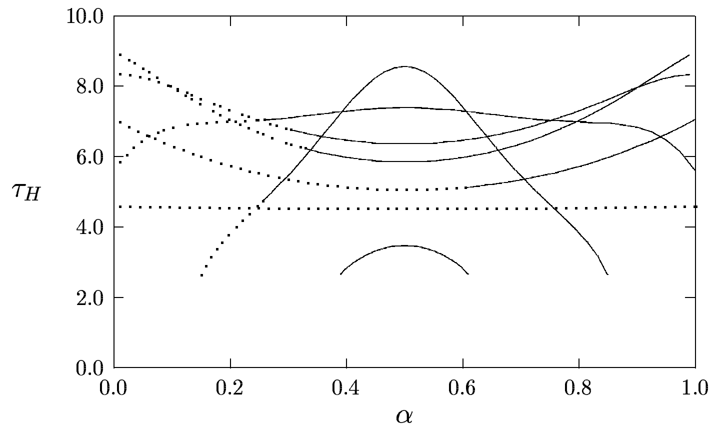
GM Model: $\tau_H(\alpha)$ for $D = (50, 2, 1, 0.75, 0.5, 0.25, 0.15, 0.10, 0.05, .01)$. Lower values of D yield higher values for $\tau_H(1/2)$. For the dotted segments there is a competition instability since $\alpha < \alpha_c$.



Conjecture: For a symmetric two-spike solution to the GM model, we conjecture that $0.771 < \tau_H < 2.75$ for any $\alpha > 0$. The lower bound corresponds to the shadow limit $D = \infty$, while the upper bound corresponds to the infinite-line problem obtained by taking the limit $D \to 0$ (with $D \gg O(\varepsilon^2)$).

GS Dynamic Oscillatory Instabilities: 2-Spikes

GS Model: $\tau_H(\alpha)$ when $\mathcal{A} = 6.5$ labeled by $(\tau_H(1/2), D)$: $(3.47, 0.1)$, $(8.56, 0.2)$, $(7.39, 0.5)$, $(6.36, 0.75)$, $(5.84, 1.0)$, $(5.04, 2.25)$, and $(4.5, 50)$. For the dotted segments there is a competition instability



In the intermediate regime, $O(\varepsilon)^{1/2} \ll A \ll O(1)$, then $\tau_H = \tau_H(\alpha)$ and

$$\tau_H \sim \frac{1.75\varepsilon^{-2}A^4 D}{9\omega^4}\left(1 - \frac{6\varepsilon\omega}{A^2\sqrt{D}}\right)^2, \quad \omega = 2\left[\tanh(\theta_0\alpha) + \tanh(\theta_0(1-\alpha))\right]^{-1}.$$

Note that $\tau_H(\alpha)$ has a maximum at $\alpha = 1/2$, and τ_H is convex. Thus, there are **no dynamic oscillatory instabilities in the intermediate regime.** Also, no dynamic competition instabilities in this regime. A static such instability can occur only when $\alpha = O(\varepsilon/A^2)$.

GS Model: Two-Spike Evolution

Principal Result: Consider a symmetric quasi-equilibrium two-spike solution for the GS model where the spikes are located at $\alpha \equiv x_1 = -x_0 > 0$, with $\alpha \gg O(\varepsilon)$ and $1-\alpha \gg O(\varepsilon)$. Suppose that $\mathcal{A} > \mathcal{A}_{2e}$, where $\mathcal{A}_{2e} = \mathcal{A}_{2e}(\alpha)$ is the existence threshold given by

$$\mathcal{A}_{2e} = \sqrt{\frac{12\theta_0}{\sinh\theta_0}} \left(\cosh\theta_0 + \cosh\left[2\theta_0\left(\alpha - 1/2\right)\right]\right)^{1/2}, \quad \theta_0 \equiv D^{-1/2}.$$

Then, for $0 < \varepsilon \ll 1$ and $\tau = 0(1)$, and assuming that the quasi-equilibrium solution is stable on an $O(1)$ time scale, the spike locations $\alpha \equiv x_1 = -x_0$ satisfy the ODE

$$\frac{d\alpha}{dt} \sim \varepsilon^2 \theta_0 s_g \left[\tanh(\theta_0(1-\alpha)) - \tanh(\theta_0\alpha)\right], \qquad \theta_0 = D^{-1/2}.$$

The equilibrium is $\alpha = 1/2$. Here $s_g = s_g(\alpha)$ is defined by

$$s_g = 2\left[1 \pm \sqrt{1 - \left(\frac{\mathcal{A}_{2e}}{\mathcal{A}}\right)^2}\right]^{-1} - 1.$$

GM Model: Two-Spike Evolution

Principal Result: Consider a symmetric quasi-equilibrium two-spike solution for the GM model with spikes at $\alpha \equiv x_1 = -x_0 > 0$, where $\alpha \gg O(\varepsilon)$ and $1 - \alpha \gg O(\varepsilon)$. Then, for $0 < \varepsilon \ll 1$ and $\tau = O(1)$, and assuming that the quasi-equilibrium solution is stable on an $O(1)$ time scale, the dynamics of such a solution with $\sigma = \varepsilon^2 t$ is given by

$$a(x, \sigma) \sim a_e \equiv H^\gamma \left[w\left(\varepsilon^{-1}[x - x_0(\sigma)]\right) + w\left(\varepsilon^{-1}[x - x_1(\sigma)]\right) \right] .$$

Here $\gamma \equiv q/(p-1)$. The spike locations $\alpha \equiv x_1 = -x_0$, with an equilibrium at $\alpha = 1/2$, satisfy the ODE

$$\frac{d\alpha}{dt} \sim \frac{\varepsilon^2 q \theta_0}{(p-1)} \left[\tanh(\theta_0(1-\alpha)) - \tanh(\theta_0 \alpha) \right] , \qquad \theta_0 = D^{-1/2} .$$

Notice that the GM and GS dynamics are also related by the equivalence principle GM:(p, q, m, s) $\rightarrow$ GS:$(2, s_g, 2, s_g)$.

Stability of the Profile: The NLEP

Principal Result: Let α be fixed. Then, the stability of the two-spike profile is determined by the union of the zeroes of $g_{\pm}(\lambda) = 0$, where

$$g_{\pm}(\lambda) \equiv C_{\pm}(\lambda) - f(\lambda)\,, \quad C_{\pm}(\lambda) \equiv \frac{1}{\chi_{\pm}}\,, \quad f(\lambda) \equiv \frac{\int_{-\infty}^{\infty} w\,(L_0 - \lambda)^{-1}\, w^2\, dy}{\int_{-\infty}^{\infty} w^2\, dy}$$

For the GS and the GM models, the functions $C_{\pm}$ take the form

$$C_{gs\pm} \equiv \frac{1}{2} + \frac{\sqrt{1+\tau\lambda}}{2s_g}\kappa_{\pm}(\tau\lambda)\,, \qquad C_{gm\pm} \equiv \frac{\sqrt{1+\tau\lambda}}{2}\kappa_{\pm}(\tau\lambda)\,.$$

One multiplier corresponds to competition instabilities, the other to synchronous oscillations. Here $\kappa_{\pm} = \kappa_{\pm}(\tau\lambda)$ are defined by

$$\kappa_{+} = \frac{\tanh(\theta_{\lambda}\alpha) + \tanh(\theta_{\lambda}(1-\alpha))}{\tanh(\theta_0\alpha) + \tanh(\theta_0(1-\alpha))}\,, \;\; \kappa_{-} = \frac{\coth(\theta_{\lambda}\alpha) + \tanh(\theta_{\lambda}(1-\alpha))}{\tanh(\theta_0\alpha) + \tanh(\theta_0(1-\alpha))}\,.$$

where $\alpha \equiv x_1 = -x_0$.

GS Competition Instability

Proposition: Suppose that $0 \leq \tau < \tau_H$ and that $\mathcal{A}$ satisfies $\mathcal{A}_{2e} < \mathcal{A} < \mathcal{A}_{2L}$, where $\mathcal{A}_{2e}$ is the existence threshold. Then, the quasi-equilibrium solution is unstable as a result of a unique eigenvalue in Re$(\lambda) > 0$ located on the positive real axis. The threshold $\mathcal{A}_{2L}$ is given by

$$\mathcal{A}_{2L} \equiv \mathcal{A}_{2e} \frac{[1 + \coth(\theta_0)\coth(\theta_0 \alpha)]}{2\sqrt{\coth(\theta_0)\coth(\theta_0 \alpha)}} \,.$$

- Alternatively, for $0 < \tau < \tau_H$, the solution is stable on an $O(1)$ time-scale when $\mathcal{A} > \mathcal{A}_{2L}$.
- Suppose that the initial spike location $\alpha(0)$ satisfies $1/2 < \alpha(0) < 1$ and that $D > D_{2gs} \approx 2.3063$. Suppose that $\mathcal{A}$ satisfies $\mathcal{A}_{2L}(\alpha(0)) < \mathcal{A} < \mathcal{A}_{2L}(1/2)$. Then, there is a **dynamic competition instability** before the spikes reach their stable equilibria at $\alpha = 1/2$.

Open Problems

- **Uniqueness:** Is the Hopf bifurcation threshold uniquely determined for a one-spike solution to the GS and GM model? (assume exponent set (p, q, m, s) for GM model gives stability when $\tau = 0$.)
- **Asynchronous:** Find parameter regimes where asynchronous spike oscillations occur robustly via an NLEP formulation.
- **Weakly Nonlinear:** Are the oscillatory instabilities in the spike profiles supercritical for GM and subcritical for GS? (spectral equivalence principle does not extend into weakly nonlinear regime).
- **Transition Behavior:** Examine transition behavior from semi-strong to weak-interaction regime of Nishiura et al.
- **Coarsening Process:** Provide a description of "coarsening" process for k-spike quasi-equilibria. Can we get chaotic spike motion for GS in the intermediate regime?
- **Multi-Dimensions:** Since our PDE based-analysis is free of hypergeometric functions, Evans functions, and Geometric Singular Perturbation Theory, it can readily be extended to study the stability and dynamic bifurcations of spikes in multi-dimensions (Wei and Winter).

References

1. **Equilibrium Theory:**
   - D. Iron, M. J. Ward, J. Wei, *The Stability of Spike Solutions to the One-Dimensional Gierer-Meinhardt Model*, Physica D. Vol. 150, (2001) pp. 25-62.
   - M. J. Ward, J. Wei, *Hopf Bifurcation and Oscillatory Instabilities of Spike Solutions for the One-Dimensional Gierer-Meinhardt Model*, Journal of Nonlinear Science, Vol. 13, No. 2, (2003), pp. 209-264.
   - T. Kolokolnikov, M. J. Ward, J. Wei, *The Existence and Stability of Spike Equilibria in the One-Dimensional Gray-Scott Model: The Low Feed Rate Regime*, to appear, Studies in Applied Math.
   - T. Kolokolnikov, M. J. Ward, J. Wei, *Slow Translational Instabilities of Spike Patterns in the One-Dimensional Gray-Scott Model*, submitted, Interfaces and Free Boundaries, Jan. 2005.

2. **Dynamics of Quasi-Equilibria:**
   - D. Iron, M. J. Ward, *The Dynamics of Multi-Spike Solutions for the One-Dimensional Gierer-Meinhardt Model*, SIAM J. Appl. Math., Vol. 62, No. 6, (2002), pp. 1924-1951.
   - W. Sun, M. J. Ward, R. Russell, *The Slow Dynamics of Two-Spike Solutions for the Gray-Scott and Gierer-Meinhardt Systems: Competition and Oscillatory Instabilities*, accepted, SIADS.