Strong Localized Perturbations: Theory and Applications

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1 Introduction

The method of matched asymptotic expansions is a powerful systematic analytical method for asymptotically calculating solutions to singularly perturbed PDE problems. It has been successfully used in a wide variety of applications (cf. [**5**], [**6**]).

In the first week of the class we consider various classes of perturbation problems with localized imperfections in multi-dimensional domains. A perturbation of large magnitude but small extent will be called a strong localized perturbation. It may be contrasted with a weak perturbation, which is of small magnitude but may have large extent. We shall show how to calculate the effects of strong localized perturbations on the solutions of elliptic PDE problems and reaction-diffusion systems.

The examples of strong localized perturbations that we will consider are the removal of a small subdomain from the domain of a problem with the imposition of a boundary condition on the boundary of the resulting hole, a large alteration of the boundary condition on a small region of the boundary of the domain, a large but localized change in the coefficients of the differential operator, and nonlinear reaction diffusion problems where the nonlinearity is effectively localized in the domain.

Strong localized perturbations are singular perturbations in the sense that they produce large changes in the solutions of the problems in which they occur. However, these large changes are themselves localized. Consequently, the perturbed solutions can be constructed by the method of matched asymptotic expansions. An inner expansion can describe the large changes in the solution in a neighborhood of the strong perturbation. An outer expansion, valid in the region away from the strong perturbation can account for the relatively small effects that the perturbation produces there. These two expansions can be matched to determine the undetermined coefficients in both of them.

For strong localized perturbations in a 2-D domain, the asymptotic expansion of the solution often leads to infinite logarithmic series in powers of $\nu = -1/\log \varepsilon$, where ε is a small positive parameter, it is well-known that this method may be of only limited practical use in approximating the exact solution accurately. This difficulty stems from the fact that $\nu \to 0$ very slowly as ε decreases. Therefore, unless many coefficients in the infinite logarithmic series can be obtained analytically, the resulting low order truncation of this series will typically not be very accurate unless ε is very small. Singular perturbation problems involving infinite logarithmic expansions arise in many areas of application in two-dimensional spatial domains, including; low Reynolds number fluid flow past cylindrical bodies, eigenvalue problems in perforated domains, the calculation of the mean first passage time for Brownian motion in a domain with small traps, localized spot patterns for reaction-diffusion systems, etc.

One primary goal of the notes in this first week of class is to highlight how a common analytical framework can be used to treat a wide range of problems with strong localized perturbations arising from different areas of application.

2 Strong Localized Perturbations in 3-D

We first recall a few basic results from potential theory. Suppose that $\Delta u - k^2 u = \delta(x - x_0)$ with $x \in \Omega \in \mathbb{R}^3$ and that $k \geq 0$ a constant. Here Δ is the 3-D Laplacian. Then,

$$u \sim -\frac{1}{4\pi|x - x_0|} \quad \text{as} \quad x \to x_0 \,. \tag{2.1}$$

To derive this simple result, we introduce a small sphere of radius σ about x_0 so that $\Omega_\sigma = \{x| \, |x - x_0| \leq \sigma\}$. Then we define $r = |x - x_0|$, and we look for a local radially symmetric solution to

$$\Delta u - k^2 u = u_{rr} + \frac{2}{r} u_r - k^2 u = 0 \,,$$

for $r > 0$, that has a singularity at $r = 0$. We get $u = Ae^{-kr}/r$ for some constant A. Upon integrating the PDE over Ω_σ, and then applying the divergence theorem, we obtain

$$\int_{\Omega_\sigma} \Delta u \, dx - k^2 \int_{\Omega_\sigma} u \, dx = \int_{\Omega_\sigma} \delta(x - x_0) \, dx = 1 \,,$$
$$\int_{\partial\Omega_\sigma} \nabla u \cdot n \, dS - k^2 \int_{\Omega_\sigma} u \, dx = 4\pi \left(r^2 \frac{\partial u}{\partial r}\bigg|_{r=\sigma} \right) - k^2 \int_{\Omega_\sigma} u \, dx = 1 \,.$$

Upon substituting $u = Ar^{-1}e^{-kr}$ into the formula above, and taking the limit as $\sigma \to 0$, we obtain $A = -1/4\pi$, which yields (2.1).

The key point here is that the leading-order behavior of the singularity is independent of the lower-order term in the operator. In particular, for a problem in 3-D of the form

$$\Delta u + p(x)u = \delta(x - x_0) \,,$$

where $p(x)$ is a smooth function, we will always have to leading order as $x \to x_0$ that $u \sim 1/(4\pi|x - x_0|)$. However, the higher order behavior of u as $x \to x_0$ does depend on $p(x)$.

Therefore, if we want to solve in $\mathbb{R}^3$ the problem

$$\Delta u = 0 \,, \qquad x \in \Omega \backslash \{x_0\} \,; \qquad u = 0 \,, \qquad x \in \partial\Omega \,,$$
$$u \sim \frac{A}{|x - x_0|} \qquad x \to x_0 \,,$$

we use the formal correspondence

$$-\frac{1}{4\pi|x - x_0|} \to \delta(x - x_0) \,,$$

to get $\frac{A}{|x - x_0|} \to -4\pi A \delta(x - x_0)$. Thus, the problem above can be written in the domain Ω in terms of a singular "delta forcing" as

$$\Delta u = -4\pi A \delta(x - x_0) \,, \qquad x \in \Omega \,; \qquad u = 0 \,, \qquad x \in \partial\Omega \,.$$

2.1 Eigenvalue Asymptotics in $\mathbb{R}^3$

Our first example in the asymptotic theory of strong localized perturbations concerns finding the effect on an eigenvalue of the Laplacian when a small hole is cut out from a bounded 3-D domain. Let Ω be a 3-D bounded domain with a hole, or "trap", of "radius" $\mathcal{O}(\epsilon)$, that is removed from Ω. We consider the perturbed problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{for } x \in \Omega\backslash\Omega_\epsilon \\ u = 0 & \text{for } x \in \partial\Omega \\ u = 0 & \text{for } x \in \partial\Omega_\epsilon \\ \int_{\Omega\backslash\Omega_\epsilon} u^2\, dx = 1\,. \end{cases} \tag{2.2}$$

We assume that Ω_ϵ shrinks to a point x_0 as $\epsilon \to 0$. For instance, Ω_ϵ could be the sphere $|x - x_0| \leq \epsilon$, but more generally we will allow for holes of arbitrary shape. However, our assumption that Ω_ϵ tends to a point x_0 as $\epsilon \to 0$, precludes the more challenging case where Ω_ϵ tends to a finite length curve or surface, each having zero measure in 3-D, as $\epsilon \to 0$.

The unperturbed problem corresponding to (2.2), where the hole is absent, is

$$\begin{cases} \Delta \phi + \mu \phi = 0 & \text{for } x \in \Omega \\ \phi = 0 & \text{for } x \in \partial\Omega \\ \int_{\Omega} \phi^2\, dx = 1\,. \end{cases} \tag{2.3}$$

This is a classical problem. It is well-known that there are an infinite set of eigenpairs μ_j, $\phi_j(x)$, for $j = 0, 1, ...,$ with the property that $0 < \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \ldots$. In addition, the eigenfunctions have the orthogonality property $\int_\Omega \phi_j \phi_k \text{ d}x = 0$ for $j \neq k$, with $\phi_0(x) > 0$ for $x \in \Omega$. The eigenfunctions form a complete set in the sense that any function in L^2 can be expanded in terms of them. The first eigennpair μ_0 and ϕ_0 are non-degenerate, but in general when the domain is symmetric other eigenstates for $j = 1, \ldots$ can be degenerate, i.e. more than one independent eigenfunction for an eigenvalue μ_j with $j \geq 1$. In this case these independent eigenfunctions, corresponding to the same eigenvalue, can be orthogonalized via a Gram-Schmidt process.

Our goal is to show formally that there is an eigenvalue of (2.2) for which $\lambda(\epsilon) \to \mu_0$ as $\epsilon \to 0$, and to calculate the precise rate of convergence as $\epsilon \to 0$. This is done using the method of matched asymptotic expansions.

We now look for an eigenpair of (2.2) near the unique principal eigenpair $\phi_0(x)$, μ_0. We proceed by the method of matched asymptotic expansions. We first expand the eigenvalue for (2.2) as

$$\lambda \sim \mu_0 + \nu(\epsilon)\lambda_1 + \cdots\,,$$

where $\nu(\epsilon) \to 0$ as $\varepsilon \to 0$ is some gauge function to be determined.

In the outer (or global) region away from the hole, we expect that the perturbation has little influence on the eigenfunction, and so we expand

$$u = \phi_0(x) + \nu(\epsilon) u_1 + \cdots\,.$$

Now since $\Omega_\epsilon \to \{x_0\}$ as $\epsilon \to 0$, it follows that u_1 satisfies

$$\begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 & \text{for } x \in \Omega \backslash \{x_0\} \\ u_1 = 0 & \text{for } x \in \partial\Omega \\ \int_\Omega u_1 \phi_0 \, \mathrm{d}x = 0 \,. \end{cases} \tag{2.4}$$

Next, we construct an inner (or local) expansion near the hole. We let $y = \varepsilon^{-1}(x - x_0)$ and we define $v(y;\epsilon) = u(x_0 + \epsilon y)$. Then, $v(y)$ satisfies

$$\begin{cases} \Delta_y v + \lambda \epsilon^2 v = 0 & \text{for } x \notin \Omega_0 \\ v = 0 & \text{for } x \in \partial\Omega_0 \,. \end{cases} \tag{2.5}$$

Here $\Omega_0 = \varepsilon^{-1}\Omega_\epsilon$ is the magnified hole as expressed in the stretched y variable. Then, we expand $v = v_0 + \nu(\epsilon) v_1 + \cdots$, to obtain that v_0 satisfies

$$\begin{cases} \Delta_y v_0 = 0 & \text{for } y \notin \Omega \\ v_0 = 0 & \text{for } y \in \partial\Omega \\ v_0 \to \phi_0(x_0) & \text{as } |y| \to \infty \,. \end{cases} \tag{2.6}$$

The matching condition between the outer and inner solutions is that as $x \to x_0$ the outer expansion must agree with the far-field behavior as $|y| \to \infty$ of the inner expansion. We write this formally as

$$\phi_0(x) + \nu(\epsilon) u_1 + \cdots \sim v_0 + \nu(\epsilon) v_1 + \cdots \,, \quad \text{as } x \to x_0 \text{ and } |y| \to \infty \,. \tag{2.7}$$

Now we write the solution to (2.6) as

$$v_0 = \phi_0(x_0) \left(1 - v_c(y)\right) \,, \tag{2.8}$$

where $v_c(y)$ satisfies the classic electrostatic capacitance problem ([**4**]), given by

$$\begin{cases} \Delta_y v_c = 0 & \text{for } y \notin \Omega_0 \\ v_c = 1 & \text{for } y \in \partial\Omega_0 \\ v_c \to 0 & \text{as } |y| \to \infty \,. \end{cases} \tag{2.9 a}$$

With the exception of a few simple shapes Ω_0, the solution for v_c cannot be found in closed form. However, it does have the well-known far-field asymptotic behavior

$$v_c \sim \frac{C}{|y|} + \frac{\mathbf{p} \cdot y}{|y|^3} + \mathcal{O}(|y|^{-3}) \,, \quad \text{as } |y| \to \infty \,, \tag{2.9 b}$$

where $C > 0$ is called the electrostatic capacitance of Ω_0, and the vector $\mathbf{p}$, is called the dipole moment of Ω_0 (cf. [**4**]). Further terms in the far-field asymptotics are the quadrapole terms etc.., but these are not needed.

As a remark, for the special case of a spherical trap of radius ε, then $\Omega_\epsilon = \{x \,|\, |x - x_0| \le \epsilon\}$ and $\Omega_0 = \{y \,|\, |y| \le 1\}$. We let $r = |y|$ so that in $\mathbb{R}^3$, $v_c = v_c(r)$ satisfies

$$\begin{cases} v_c'' + \frac{2}{r} v_c' = 0 & \text{for } r \ge 1 \\ v_c = 1 & \text{for } r = 1 \\ v_c \to 0 & \text{as } r \to \infty \,. \end{cases}$$

The explicit solution for v_c is readily found as $v_c = 1/r$ for $r \ge 1$, so that $C = 1$ and $\mathbf{p} = \mathbf{0}$.

The capacitance C, defined by (2.9), has two key properties. Firstly, it is invariant under rotations of the trap shape. Secondly, with respect to all trap shapes Ω_0 of the same volume, C is minimized for a spherical-shaped trap.

Trap Shape $\Omega_0 = \varepsilon^{-1}\Omega_\varepsilon$	Capacitance C
sphere of radius a	$C = a$
hemisphere of radius a	$C = 2a\left(1 - \frac{1}{\sqrt{3}}\right)$
flat disk of radius a	$C = \frac{2a}{\pi}$
prolate spheroid with semi-major and minor axes a, b	$C = \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}$
oblate spheroid with semi-major and minor axes a, b	$C = \frac{\sqrt{a^2-b^2}}{\cos^{-1}(b/a)}$
ellipsoid with axes a, b, and c	$C = 2\left(\int_0^\infty (a^2+\eta)^{-1/2}(b^2+\eta)^{-1/2}(c^2+\eta)^{-1/2}\, d\eta\right)^{-1}$

Table 1. Capacitance C of some simple trap shapes, defined from the solution to (2.9).

This second statement is a famous isoperimetric inequality due to G. Szegö [**9**]. Although C must in general be calculated numerically from (2.9), typically using a boundary integral method, when Ω_0 has an arbitrary shape, it is known analytically for some simple shapes. Many of these tabulated results, which are typically found by solving the exterior Laplace's equation in special separable coordinate systems, are summarized in Table 1. The capacitance C is also known in a few other situations. For instance, for the case of two overlapping identical spheres of radius εa, that intersect at exterior angle ψ, with $0 < \psi < \pi$, then C is given by (cf. [**2**])

$$C = 2\,a\,\sin\left(\frac{\psi}{2}\right)\int_0^\infty \left[1 - \tanh(\pi\tau)\tanh\left(\frac{\psi\tau}{2}\right)\right] d\tau\,. \tag{2.10}$$

For $\psi \to 0$, (2.10) reduces to the well-known result $C = 2a\log 2$ for the capacitance of two touching spheres. In the analysis below, we will treat C as either a known or easily-computed quantity.

Now we return to v_0 and write its far-field behavior as

$$v_0 \sim \phi(x_0)\left(1 - \frac{C}{|y|} + \cdots\right), \quad \text{as} \quad |y| \to \infty\,.$$

We let $y = \varepsilon^{-1}(x - x_0)$ and use the matching condition of (2.7) to obtain

$$\phi_0(x_0) + \nu(\epsilon)u_1 \sim \phi_0(x_0) - \phi_0(x_0)\frac{\epsilon C}{|x - x_0|} + \cdots, \quad \text{as } x \to x_0\,.$$

This determines both the gauge function $\nu(\varepsilon)$ and the singularity behavior of u_1 as $x \to x_0$ as

$$\nu(\epsilon) = \epsilon\,, \qquad u_1 \to -\phi_0(x_0)\frac{C}{|x - x_0|}\,, \quad \text{as} \quad x \to x_0\,.$$

With the singularity behavior of u_1 as $x \to x_0$ now provided by the matching condition, we return to (2.4) and write the problem for u_1 as

$$\begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1\phi_0 & \text{for } x \in \Omega\backslash\{x_0\} \\ u_1 = 0 & \text{for } x \in \partial\Omega \\ u_1 \to -\phi_0(x_0)\frac{C}{|x-x_0|} & \text{as } x \to x_0 \\ \int_\Omega u_1\phi_0\, dx = 0\,. & \end{cases}$$

Upon using the correspondence $\frac{-1}{4\pi|x-x_0|} \to \delta(x - x_0)$, this problem is equivalent to

$$\begin{cases} Lu := \Delta u_1 + \mu_0 u_1 = -\lambda_1\phi_0 + 4\pi C\phi_0(x_0)\delta(x - x_0) & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

To determine λ_1, we use a solvability condition. This condition arises since the homogeneous problem $L\phi = 0$ in Ω, with $\phi = 0$ on $\partial\Omega$, has a one-dimensional nullspace of L, i.e. that $\phi = \phi_0$ is the unique element in the nullspace (up to a multiplicative constant).

To derive this solvability condition, we simply integrate Lu over Ω and use the Green's second identity to get

$$\int_\Omega (\phi_0 L u_1 - u_1 L\phi_0)\, \mathrm{d}x = \int_{\partial\Omega} (\phi_0 \partial_n u_1 - u_1 \partial_n \phi_0)\, dS\,.$$

Since both $\phi_0 = u_1 = 0$ on $\partial\Omega$, and $L\phi_0 = 0$, we obtain that the equation above reduces to

$$0 = \int_\Omega \phi_0 L u_1\, dx = \int_\Omega \phi_0 (-\lambda_1 \phi_0 + 4\pi C \phi_0(x_0)\delta(x - x_0))\, dx\,,$$

which determines λ_1 as

$$\lambda_1 = \frac{4\pi C [\phi_0(x_0)]^2}{\int_\Omega \phi_0^2 dx}\,.$$

In summary with $\nu(\epsilon) = \epsilon$, we obtain the following two-term result for the expansion of the principal eigenvalue:

$$\lambda \sim \mu_0 + \epsilon\lambda_1 + \cdots\,, \qquad \lambda_1 = \frac{4\pi C[\phi_0(x_0)]^2}{\int_\Omega \phi_0^2\, dx}\,. \tag{2.11}$$

This is the main result of this section. A rigorous derivation of this leading-order result, and similar leading-order results in different dimensions, is due to Ozawa [**7**], [**8**]. A higher order analysis in different dimensions, and various related problems, as obtained by the method of matched asymptotic expansions, is given in [**10**].

We now consider a specific geometry where an exact solution to (2.2) can be found, which serves as a partial check on our asymptotic result (2.11).

Example: (Concentric Spheres)

Consider the special case of two concentric spheres with an inner sphere of small radius. The radially symmetric eigenfunctions, under Dirichlet conditions, satisfy

$$\begin{cases} u_{rr} + \frac{2}{r} u_r + \lambda u = 0 & \text{for } \epsilon < r < 1\,, \\ u(1) = 0\,, \quad u(\epsilon) = 0\,. \end{cases}$$

By making the substitution $u = f(r)/r$, we readily find that $f'' + \lambda r = 0$. As such, by expressing the solution for f in terms of a convenient phase shift, we obtain that the exact eigenfunction is $u = r^{-1} \sin\left(\sqrt{\lambda}(r-\epsilon)\right)$. By satisfying $u(1) = 0$, we get $\sqrt{\lambda}(1-\epsilon) = n\pi$. The lowest eigenvalue, for which $n = 1$, is $\lambda = \frac{\pi^2}{(1-\epsilon)^2} \sim \pi^2(1 + 2\epsilon + \cdots)$. Using the binomial series, we get $\lambda \sim \pi^2 + 2\epsilon\pi^2 + \cdots$.

Now use the asymptotic formula given in (2.11). In (2.11), we set $\mu_0 = \pi^2$, $\phi_0 = r^{-1}\sin(\pi r)$, so that $\phi_0(0) = \lim_{r\to 0} \frac{\sin(\pi r)}{r} = \pi$. In addition, $\int_\Omega \phi_0^2\, dx = 4\pi \int_0^1 \left(r^{-2}\sin^2(\pi r)\right) r^2\, dr = 2\pi$. Then (2.11) with $C = 1$ yields $\lambda \sim \pi^2 + 2\epsilon\pi^2 + \cdots$, in agreement with the expansion of the exact eigenvalue relation as shown above. ■

We now make some remarks on possible extensions of our main result, and the limitations on our analysis.

Remarks:

(1) **Multiple Holes:** If there are N small holes that are separated by $\mathcal{O}(1)$ distances, we can proceed in a similar

way so as to obtain the leading-order asymptotics

$$\lambda \sim \mu_0 + 4\pi\epsilon \sum_{j=1}^{N} C_j \frac{[\phi_0(x_j)]^2}{\int_\Omega \phi_0^2 \, dx} + \cdots ,$$

where C_j is the capacitance of the j^{th} hole centered at $x_j \in \Omega$. Therefore, to leading order the effect on the eigenvalue is simply a superposition of the individual effects of the N traps.

(2) **Degenerate Eigenfunctions:** We assumed that the unperturbed eigenfunction is non-degenerate. In the degenerate case, where there are several independent eigenfunctions for the same eigenvalue μ_j, we can proceed in a similar way, replacing $\phi_j(x)$ by $\phi_j(x) = \sum_i^N a_i \phi_{ji}(x)$ in the theory, where ϕ_{ji} for $i = 1, \ldots, m$ are the m-eigenstates, made into an orthonormal set, for the eigenvalue μ_j. There will then be m-solvability conditions to derive an $m \times m$ linear algebraic system for the vector $\mathbf{a} = (a_1, \ldots, a_m)^T$ and the eigenvalue λ_1. The key issue is whether the effect of the hole breaks the degeneracy in the leading-order eigenstate μ_j.

(3) **High Frequency Asymptotics:** Our analysis is effectively limited to the case of "small eigenvalues". In particular, since $\mu_j \to +\infty$ as $j \to \infty$, the eigenfunctions corresponding to these high eigenstates will exhbit rapid spatial variation across the domain. As such, the effect of a small hole on these eigenstates will be rather pronounced, and in fact we cannot make the approximation in (2.5) that $\epsilon^2 \mu_j \ll 1$. In other words, our asymptotic analysis for $\epsilon \to 0$ is not *uniformly valid* as $n \to \infty$ for μ_n.

(4) **Viewpoint of Numerics:** To solve (2.2) numerically using numerical methods, one needs to have a refined mesh near the hole to resolve any large gradients in the solution. Then, inverse iteration or a similar numerical procedure, needs to be used to extract the numerical approximation of the principal eigenvalue. A key issue is that, since the domain is changing as ϵ is decreased, one must re-mesh and re-compute at each different ϵ. In contrast, the leading order asymptotics (2.11) gives the eigenvalue correction for all ϵ small. Moreover, by changing the shape of the trap, one needs to only compute a single quantity, namely the capacitance C, to determine the eigenvalue.

2.2 Reflecting Boundary Conditions: The Neumann Problem

An important problem with regards to applications, and which we discuss below, is where the condition $u = 0$ on $\partial\Omega$ in (2.2) is replaced by the no-flux or Neumann condition $\partial_n u = 0$ on $\partial\Omega$. In this case, it is readily seen that the unperturbed problem

$$\begin{cases} \Delta\phi + \mu\phi = 0 & \text{for } x \in \Omega \\ \partial_n \phi = 0 & \text{for } x \in \partial\Omega \\ \int_\Omega \phi^2 \, dx = 1 \,, \end{cases}$$

has the principal eigenpair

$$\mu_0 = 0 \,, \qquad \phi_0 = \frac{1}{|\Omega|^{1/2}} \,, \tag{2.12}$$

where $|\Omega|$ is the volume of Ω. In this case, we can readily calculate from (2.11) that

$$\lambda \sim \frac{4\pi\epsilon C}{|\Omega|} \,. \tag{2.13}$$

We consider a simple application of the result (2.13) to a Brownian motion problem with a constant diffusivity D. Consider Brownian motion in a 3-D domain, with a reflecting outer wall, that has a small trap of "radius" $\mathcal{O}(\epsilon)$. Let $p(x,t)$ denote the probability that the diffusing particle is at point $x \in \Omega\backslash\Omega_\epsilon$ at time t, and has not yet been absorbed by the trap. A schematic plot of the domain is shown in Fig. 1.

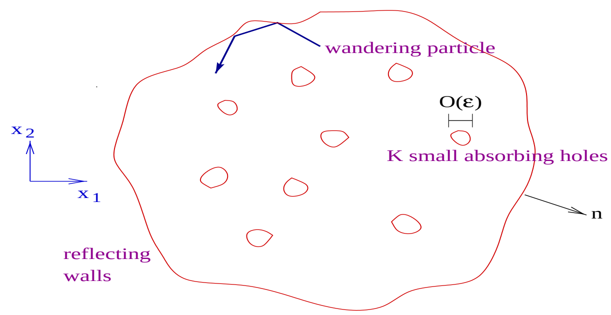

Figure 1. A schematic plot of Bronwian motion in a domain with K small traps.

Assuming that the Brownian motion starts at some given $\xi \in \Omega\backslash\Omega_\epsilon$, it is well-known that $p(x,t)$ satisfies the diffusion equation

$$\begin{cases} p_t = D\Delta p & \text{for } x \in \Omega\backslash\Omega_\epsilon \\ \partial_n p = 0 & \text{for } x \in \partial\Omega \\ p = 0 & \text{for } x \in \partial\Omega_\epsilon \\ p(x,0) = \delta(x-\xi) & \end{cases} \tag{2.14}$$

We assume, as before, that Ω_ϵ is a small trap centered at x_0, for which $\Omega_\epsilon \to x_0$ as $\epsilon \to 0$.

We now relate the long-time behavior of $p(x,t)$ for $t \to \infty$ to our singularly perturbed eigenvalue problem. We can readily separate variables and represent $p(x,t)$ as an eigenfunction expansion of the form

$$p(x,t) = \sum_{n=0}^{\infty} b_n u_n(x) e^{-\lambda_n D t} \,,$$

where b_n are constants to be determined. Here u_n, λ_n are the eigenpairs for $n \geq 0$ of the Neumann problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{for } x \in \Omega\backslash\Omega_\epsilon \\ \partial_n u = 0 & \text{for } x \in \partial\Omega \\ u = 0 & \text{for } x \in \partial\Omega_\epsilon \\ \int_{\Omega\backslash\Omega_\epsilon} u^2 \, dx = 1 \,. & \end{cases} \tag{2.15}$$

By satisfying the initial condition for p, and then using the orthogonality of the eigenstates, we have that

$$\delta(x-\xi) = \sum_{n=0}^{\infty} b_n u_n(x) \,,$$

yields that $b_n = u_n(\xi)$. Thus, the solution to (2.14) can be written as the eigenfunction expansion

$$p(x,t) = \sum_{n=0}^{\infty} u_n(\xi) u_n(x) e^{-\lambda_n D t} \,. \tag{2.16}$$

Next, for $t \gg 1$, we observe that the first term in the eigenfunction series of (2.16) dominates, and so we have

$$p(x,t) \sim u_0(\xi) u_0(x) e^{-\lambda_0 D t}\,, \qquad \text{for} \quad t \gg 1\,.$$

However, we have previously worked out the asymptotics for $\epsilon \to 0$ of the principal eigenpair of (2.15). In particular, from (2.12) and (2.13) (in a slightly different notation), we have $u_0 \sim |\Omega|^{-1/2}$ and $\lambda_0 \sim 4\pi\varepsilon/|\Omega|$. This yields the long-time behavior

$$p(x,t) \sim \frac{1}{|\Omega|} e^{-t/T}\,, \qquad T \equiv \frac{|\Omega|}{4\pi\epsilon C}\,. \tag{2.17}$$

The time-scale T for the capture of the wandering particle is inversely proportional to the capacitance of the trap. For a trap that has many appendages so that it is far from a simple spherical shape, the capacitance can be rather large. For such a trap, the estimate (2.17) shows that the Brownian particle is captured quicker than for a simple spherical shape (as expected intuitively).

The key observation here, and a limitation of our analysis so far, is that the leading-order estimate of T and λ_0 is independent of the location x_0 of the trap. A key question is the following:

Q1: *Calculate the next term in the asymptotic expansion of the principal eigenvalue for this Neumann problem (2.15) , so as to determine the effect on the eigenvalue of the location of the trap inside the domain. Extend to the case of multiple traps. This problem has been worked out recently in [**1**].*

A second interesting question is to study the asymptotics for $p(x,t)$ for $\mathcal{O}(1)$ times for a small trap, and not simply for $t \gg 1$. This has been recently considered in [**3**].

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