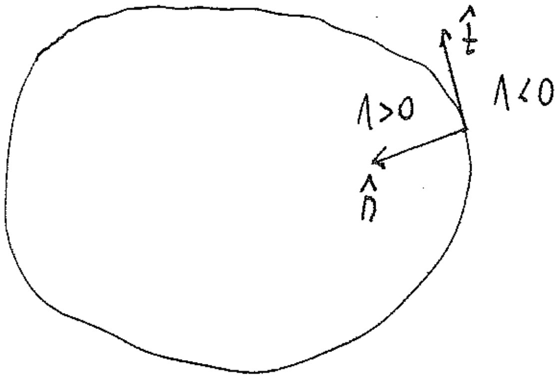
BOUNDARY FITTED COORDINATES

LET Λ BE DISTANCE FROM $\partial\Omega$ TO A POINT INSIDE Ω. LET s BE ARCLENGTH WHEN ON $\partial\Omega$. AS SIGN CONVENTION WE TAKE



$\Lambda > 0$ INSIDE Ω

$\Lambda < 0$ OUTSIDE Ω

SUPPOSE $\partial\Omega$ IS SMOOTH SO THAT $\exists$ A TANGENT (UNIT) $\hat{t}$ AND A NORMAL $\hat{n}$ AT EACH POINT ON $\partial\Omega$. LET $(\gamma_1(s), \gamma_2(s))$ PARAMETRIZE $\partial\Omega$, WHERE s IS ARCLENGTH ON $\partial\Omega$.

THEN $\dfrac{d\hat{t}}{ds} = \kappa\,\hat{n}$ WHERE κ IS CURVATURE OF $\partial\Omega$ AT s (WITH SIGN $\kappa > 0$ IF Ω IS CONVEX AS SEEN FROM INSIDE)

NOW LET (X_1, X_2) BE ANY POINT NEAR $\partial\Omega$ IN CARTESIAN COORDINATES. WE SEEK TO CHANGE VARIABLES AS

$$(1)\qquad \begin{aligned} X_1 &= \gamma_1(s) + \Lambda\, n_1(s) \\ X_2 &= \gamma_2(s) + \Lambda\, n_2(s) \end{aligned} \qquad \text{WHERE} \quad \begin{aligned} \hat{\underline{n}} &= (n_1, n_2) \\ \hat{\underline{t}} &= (\gamma_1', \gamma_2') \end{aligned}$$

OR IN VECTOR FORM $\underline{X} = \underline{\gamma} + \Lambda\,\hat{n}$.

NOW DIFFERENTIATE WRT X_1 AND X_2 AND DEFINE $\hat{\underline{e}}_{X_1} = (1,0)$, $\hat{\underline{e}}_{X_2} = (0,1)$.

THEN

$$(2)\qquad \left\{ \begin{aligned} \hat{\underline{e}}_{X_1} &= \underline{\gamma}'(s)\frac{\partial s}{\partial X_1} + \frac{\partial \Lambda}{\partial X_1}\,\hat{\underline{n}} + \Lambda\,\hat{\underline{n}}'(s)\frac{\partial s}{\partial X_1} \\ \hat{\underline{e}}_{X_2} &= \underline{\gamma}'(s)\frac{\partial s}{\partial X_2} + \frac{\partial \Lambda}{\partial X_2}\,\hat{\underline{n}} + \Lambda\,\hat{\underline{n}}'(s)\frac{\partial s}{\partial X_2} \end{aligned} \right.$$

NOW SINCE $\hat{\underline{n}} \cdot \hat{\underline{n}} = 1$ WE HAVE $\dfrac{d\hat{\underline{n}}}{ds} \cdot \hat{\underline{n}} = 0$ AND SO $\dfrac{d\hat{\underline{n}}}{ds}$ IS $\perp$ TO $\hat{\underline{n}}$.

THUS $\dfrac{d\hat{\underline{n}}}{ds} = \alpha \hat{\underline{t}}$ FOR SOME α.

NOW TO FIND α, $\dfrac{d^2\hat{\underline{t}}}{ds^2} = \kappa'(s)\,\hat{\underline{n}} + \kappa \dfrac{d\hat{\underline{n}}}{ds}$

TAKE DOT PRODUCT WITH $\hat{\underline{t}}$: $\dfrac{d^2\hat{\underline{t}}}{ds^2} \cdot \hat{\underline{t}} = 0 + \kappa \dfrac{d\hat{\underline{n}}}{ds} \cdot \hat{\underline{t}} = \kappa\alpha$.

BUT $0 = \dfrac{d}{ds}\left[\dfrac{d\hat{\underline{t}}}{ds} \cdot \hat{\underline{t}}\right] = \dfrac{d^2\hat{\underline{t}}}{ds^2} \cdot \hat{\underline{t}} + \dfrac{d\hat{\underline{t}}}{ds} \cdot \dfrac{d\hat{\underline{t}}}{ds} = \kappa\alpha + \kappa^2\,\hat{\underline{n}} \cdot \hat{\underline{n}} = \kappa\alpha + \kappa^2 = 0$.

THUS $\alpha = -\kappa$.

WE HAVE (3) $\left\{ \dfrac{d\hat{\underline{t}}}{ds} = \kappa\,\hat{\underline{n}} \right.$ AND $\dfrac{d\hat{n}}{ds} = -\kappa\,\hat{\underline{t}}$

NOW IN (2), WE USE (3) AND $\hat{\underline{t}} = \underline{\gamma}'(s)$.

THIS GIVES

$$\hat{\underline{e}}_{x_1} = \hat{\underline{t}}\,\frac{\partial s}{\partial x_1} - \kappa\,\hat{\underline{t}}\,n\,\frac{\partial s}{\partial x_1} + \frac{\partial n}{\partial x_1}\,\hat{\underline{n}},$$ WITH SIMILAR EXPRESSION FOR $\hat{e}_{x_2}$

WE GET

$$(4)\quad \left\{ \begin{aligned} \hat{\underline{e}}_{x_1} &= \hat{\underline{t}}(s)\,(1-\kappa n)\,\frac{\partial s}{\partial x_1} + \hat{\underline{n}}(s)\,\frac{\partial n}{\partial x_1} \\ \hat{\underline{e}}_{x_2} &= \hat{\underline{t}}(s)\,(1-\kappa n)\,\frac{\partial s}{\partial x_2} + \hat{\underline{n}}(s)\,\frac{\partial n}{\partial x_2} \end{aligned} \right.$$

• NOW TAKE DOT PRODUCT WITH $\hat{\underline{t}} = (t_1, t_2)$. WE GET

$$t_1 = (1-\kappa n)\,\partial s/\partial x_1 \quad \text{AND} \quad t_2 = (1-\kappa n)\,\partial s/\partial x_2$$

WRITING THIS AS A VECTOR WE OBTAIN

$$\nabla S = \left(\frac{\partial S}{\partial x_1}, \frac{\partial S}{\partial x_2} \right) = \frac{\hat{t}}{1-\kappa\eta}.$$

NOW TAKE DOT PRODUCT WITH $\hat{\underline{n}}$. WE GET $\nabla\eta = \left(\frac{\partial\eta}{\partial x_1}, \frac{\partial\eta}{\partial x_2} \right) = \hat{n}$.

WE WRITE THESE TWO RELATIONS AS

$$(5) \qquad \nabla\eta = \hat{n}, \qquad \nabla S = \frac{\hat{t}}{1-\kappa\eta}. \qquad \text{WE NEED } \eta < 1/\kappa = R$$

R = Radius of curvature.

STEP 2 WE WANT TO TAKE AN ARBITRARY FUNCTION $u(x_1, x_2)$ AND CALCULATE $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right)$ IN TERMS OF NEW COORDINATE SYSTEM.

NOW $\nabla u = u_{x_1} \hat{\underline{e}}_{x_1} + u_{x_2} \hat{e}_{x_2}$. NOW USE (4) TO GET

$$\nabla u = (u_S S_{x_1} + u_\eta \eta_{x_1}) \left[(1-\kappa\eta) S_{x_1} \hat{\underline{t}} + \eta_{x_1} \hat{\underline{n}} \right]$$
$$+ (u_S S_{x_2} + u_\eta \eta_{x_2}) \left[(1-\kappa\eta) S_{x_2} \hat{t} + \eta_{x_2} \hat{\underline{n}} \right]$$

SO

$$\nabla u = u_S (1-\kappa\eta) S_{x_1}^2 \hat{\underline{t}} + u_S (1-\kappa\eta) S_{x_2}^2 \hat{\underline{t}} + u_\eta \hat{\underline{n}} (\eta_{x_1}^2 + \eta_{x_2}^2)$$
$$+ u_S (S_{x_1}\eta_{x_1} + S_{x_2}\eta_{x_2}) \hat{\underline{n}} + u_\eta (1-\kappa\eta) [\eta_{x_1} S_{x_1} + \eta_{x_2} S_{x_2}] \hat{t}.$$

BUT $S_{x_1}\eta_{x_1} + S_{x_2}\eta_{x_2} = \nabla S \cdot \nabla\eta = 0$ BY (5).

THUS $\nabla u = u_S (1-\kappa\eta) |\nabla S|^2 \hat{\underline{t}} + u_\eta \hat{\underline{n}} |\nabla\eta|^2$.

NOW $|\nabla S|^2 = \nabla S \cdot \nabla S = \frac{1}{(1-\kappa\eta)^2}$ AND $\nabla\eta \cdot \nabla\eta = |\nabla\eta|^2 = 1$.

WE CONCLUDE THAT

$$\boxed{\nabla u = \frac{u_S}{1-\kappa\eta} \hat{t} + u_\eta \hat{\underline{n}}}. \qquad (6)$$

NEXT WE CALCULATE $\Delta\Lambda$. SINCE $\hat{\underline{n}} = \hat{\underline{n}}(s)$ FROM (5)

WE HAVE $$\nabla\cdot[\nabla\Lambda] = \frac{d}{ds}\hat{\underline{n}}\cdot\nabla s = -\kappa\hat{\underline{t}}\cdot\frac{\hat{\underline{t}}}{1-\kappa\Lambda} = -\frac{\kappa}{1-\kappa\Lambda}.$$

WHERE WE HAVE USED $\frac{d\hat{\underline{n}}}{ds} = -\kappa\hat{\underline{t}}$ AND $\nabla s = \frac{\hat{\underline{t}}}{1-\kappa\Lambda}$ FROM (4) AND (5).

THIS GIVES $$\Delta\Lambda = -\frac{\kappa}{1-\kappa\Lambda} \qquad (6).$$

WE FINALLY CALCULATE Δs.

$$\nabla\cdot(\nabla s) = \frac{\partial}{\partial x_1}\left(\frac{\hat{\underline{t}}}{1-\kappa\Lambda}\right) + \frac{\partial}{\partial x_2}\left(\frac{\hat{\underline{t}}}{1-\kappa\Lambda}\right)$$

$$= \frac{\hat{\underline{t}}'(s)\cdot\nabla s}{1-\kappa\Lambda} + \hat{\underline{t}}\left[\frac{\partial}{\partial x_1}(1-\kappa\Lambda)^{-1} + \frac{\partial}{\partial x_2}(1-\kappa\Lambda)^{-2}\right]$$

BUT $\nabla s = \frac{\hat{\underline{t}}}{1-\kappa\Lambda}$ AND SO $\hat{\underline{t}}'(s)\cdot\hat{\underline{t}}(s) = 0$ SINCE $\hat{\underline{t}}\cdot\hat{\underline{t}} = 1$.

THUS $$\nabla\cdot(\nabla s) = \hat{\underline{t}}\cdot\left[-(1-\kappa\Lambda)^{-2}\left[-\kappa'(s)\Lambda\nabla s - \kappa \quad \nabla\Lambda\right]\right]$$

BUT $\nabla\Lambda = \hat{n}$ AND $\hat{n}\cdot\hat{t} = 0$. THUS THE SECOND TERM VANISHES.

NOW $\nabla s = \frac{\hat{t}}{1-\kappa\Lambda}$ SO $$\nabla\cdot(\nabla s) = \hat{t}\cdot\left[\frac{\kappa'(s)\Lambda}{(1-\kappa\Lambda)^3}\hat{t}\right] = \frac{\kappa'(s)\Lambda}{(1-\kappa\Lambda)^3}.$$

WE CONCLUDE THAT

$$\Delta s = \frac{\kappa'(s)\Lambda}{(1-\kappa\Lambda)^3}. \qquad (7)$$

FINALLY, UNDER THE VARIABLE CHANGE $\underline{X} \rightarrow (\eta, s)$ WE HAVE BY TAKING PARTIAL DERIVATIVES THAT

$$\Delta u = u_{\eta\eta} |\nabla \eta|^2 + u_{ss} |\nabla s|^2 + 2 u_{s\eta} \nabla s \cdot \nabla \eta + u_\eta \Delta \eta + u_s \Delta s .$$

WHERE $\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2}$, $\nabla = \left(\partial/\partial x_1 , \partial/\partial x_2 \right)$.

NOW WE USE OUR RESULTS THAT

$$\Delta s = \frac{\kappa'(s)\eta}{(1-\kappa\eta)^3} , \qquad \Delta \eta = \frac{-\kappa}{1-\kappa\eta} , \qquad |\nabla s|^2 = \frac{1}{(1-\kappa\eta)^2} , \qquad |\nabla \eta|^2 = 1 .$$

THIS GIVES

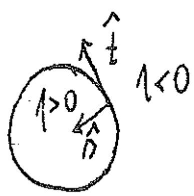
$$\Delta u \equiv u_{\eta\eta} - \frac{\kappa}{1-\kappa\eta} u_\eta + \frac{1}{(1-\kappa\eta)^2} u_{ss} + \frac{\kappa'(s)\eta}{(1-\kappa\eta)^3} u_s .$$

WE CAN WRITE THIS MORE ELEGANTLY AS

$$(8) \quad \boxed{\Delta u \equiv u_{\eta\eta} - \frac{\kappa}{1-\kappa\eta} u_\eta + \frac{1}{(1-\kappa\eta)} \frac{\partial}{\partial s} \left(\frac{u_s}{(1-\kappa\eta)} \right)}$$

THIS IS OUR KEY RESULT

REMARK (i) RECALL THE PICTURE



$\eta > 0$ INSIDE

$\kappa > 0$ IF Ω IS CONVEX AS SEEN FROM INSIDE

IT IS DEFINED ONLY FOR

$$0 < \eta < 1/\kappa_{MAX} \qquad \kappa_{MAX} = \max_{\kappa>0} (\kappa) .$$

(ii) THE FORMULA (6) ON BOTTOM OF PAGE B3 CAN BE USED TO CALCULATE AND LOWER ORDER DERIVATIVE TERM $\partial u/\partial x_1$ AND $\partial u/\partial x_2$.

(iii) CONSIDER A CIRCLE OF RADIUS R. THEN

$$s = R\varphi, \quad \kappa = 1/R, \quad \kappa' = 0, \quad n = R - r \quad \text{FOR} \quad 0 < r < R.$$

OUR RESULT (8) BECOMES WITH $u_s = u_\varphi R$, $u_{nn} = u_{rr}$, $u_n = -u_r$.

$u_{ss} = R^2 u_{\varphi\varphi}$

$$\Delta u = u_{rr} + \frac{(1/R)\, u_r}{\left(1 - \frac{1}{R}(R-r)\right)} + \frac{R^2}{\left(1 - \frac{1}{R}(R-r)\right)^2} u_{\varphi\varphi}$$

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}$$ $\rightarrow$ POLAR COORDINATES AS EXPECTED

<u>EXAMPLE</u> SUPPOSE $u_t = D\Delta u$ IN $\Omega \subset \mathbb{R}^2$, Ω closed, $D>0$ CONSTANT.

WHERE $\partial\Omega$ IS SMOOTH. SUPPOSE THAT

$$u = u_b \quad \text{ON} \quad \partial\Omega \quad \forall\, t \geq 0$$
$$u(\underset{\sim}{x}, 0) = 0 \quad \text{IN} \quad \Omega \quad (t=0).$$

FIND THE ASYMPTOTIC BEHAVIOR AS $t \to 0$ OF THE SOLUTION.

<u>ANALYSIS</u> WE TAKE LAPLACE TRANSFORMS $\bar{U}(\underset{\sim}{x}, \sigma) = \int_0^\infty e^{-\sigma t}\, u(\underset{\sim}{x}, t)\, d\underset{\sim}{x}$.

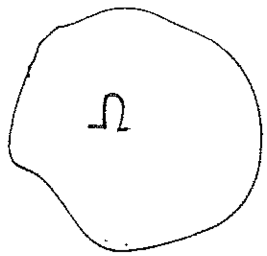
THEN $$\sigma \bar{U} = D\Delta \bar{U}.$$

SHORT TIME $\Rightarrow$ $\sigma \to \infty$. DEFINE $\varepsilon = D/\sigma \ll 1$.

THEN
$$\varepsilon \Delta \bar{U} - \bar{U} = 0 \quad \text{IN} \quad \Omega$$
$$\bar{U} = u_b/\sigma \quad \text{ON} \quad \partial\Omega.$$

LET $\bar{U} = \frac{1}{\sigma}\bar{V}$ SO THAT

$$\left\{ \begin{array}{ll} \varepsilon\Delta\bar{V} - \bar{V} = 0 & \text{IN } \Omega \\ \bar{V} = u_b & \text{ON } \partial\Omega. \end{array} \right.$$



THE OUTER SOLUTION IS $\bar{V} = 0$. NEAR $\partial\Omega$ WE INTRODUCE THE BODY FITTED COORDINATES (η, s). THIS GIVES

$$\varepsilon\left(\bar{V}_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}\bar{V}_\eta + \frac{1}{(1-\kappa\eta)}\frac{\partial}{\partial s}\left(\frac{\bar{V}_s}{1-\kappa\eta}\right)\right) - \bar{V} = 0 \qquad \underline{\text{NEAR}}\ \partial\Omega$$

$$\bar{V} = u_b \quad \text{ON} \quad \partial\Omega$$

NOW INTRODUCE A STRETCHED NORMAL DISTANCE $\hat{\eta} = \eta/\varepsilon^{1/2}$.

$$\bar{V}_{\hat{\lambda}\hat{\lambda}} - K\varepsilon^{1/2}\bar{V}_{\lambda} + O(\varepsilon) - \bar{V} = 0.$$

WE EXPAND $\bar{V} = \bar{V}_0 + \varepsilon^{1/2}\bar{V}_1 + \dots$

WE OBTAIN
$$\begin{cases} \bar{V}_{0\hat{\lambda}\hat{\lambda}} - \bar{V}_0 = 0 & \text{IN } \hat{\lambda} \geq 0 \\ \bar{V}_0 = Ub \text{ ON } \hat{\lambda} = 0\,; & \bar{V}_0 \to 0 \text{ AS } \hat{\lambda} \to \infty \end{cases}$$

$$\begin{cases} \bar{V}_{1\hat{\lambda}\hat{\lambda}} - \bar{V}_1 = K\,\bar{V}_{0\lambda} & \text{IN } \hat{\lambda} \geq 0 \qquad K = K(s) \\ \bar{V}_1 = 0 \text{ ON } \hat{\lambda} = 0\,; & \bar{V}_1 \to 0 \text{ AS } \hat{\lambda} \to +\infty. \end{cases}$$

THE SOLUTION IS $\bar{V}_0 = Ub\, e^{-\hat{\lambda}}$ TO THE FIRST PROBLEM.

AND AT NEXT ORDER $\bar{V}_{1\hat{\lambda}\hat{\lambda}} - \bar{V}_1 = -Ub\,K\,e^{-\hat{\lambda}}$

THE PARTICULAR SOLUTION IS $\bar{V}_1 = Ub\,K/_2\, e^{-\hat{\lambda}}\,\hat{\lambda}$, WHICH ALSO SATISFIES $\bar{V}_1 = 0$ ON $\hat{\lambda} = 0$ AND $\bar{V}_1 \to 0$ AS $\hat{\lambda} \to +\infty$.

THUS WE HAVE THE 2-TERM ASYMPTOTICS

$$\bar{U} = \frac{1}{\sigma}\left[Ub\, e^{-\lambda/\varepsilon^{1/2}} + \frac{Ub\,K}{2}\,\varepsilon^{1/2}\,\hat{\lambda}\, e^{-\lambda/\varepsilon^{1/2}} \right] = \frac{Ub}{\sigma}\left[e^{-\lambda/\varepsilon^{1/2}} + \frac{K\lambda}{2}\, e^{-\lambda/\varepsilon^{1/2}} \right]$$

NOW $\varepsilon = D/\sigma$. SO

$$\bar{U} = Ub\left[\frac{1}{\sigma}\, e^{-\lambda/\sqrt{D}\sqrt{\sigma}} + \frac{K\lambda}{2}\, e^{-\lambda/\sqrt{D}\sqrt{\sigma}} \right]$$

NOW INVERT: $\mathcal{L}^{-1}[\bar{U}] = U(\underline{x},t)$

NOW RECALL $\mathcal{L}^{-1}\left(\frac{e^{-\lambda\sqrt{\sigma}}}{\sigma}\right) = \mathrm{erfc}\left(\frac{\lambda}{2\sqrt{t}}\right)$ AND $\mathcal{L}^{-1}\left(e^{-\lambda\sqrt{\sigma}}\right) = \frac{\lambda}{2\sqrt{\pi}\,t^{3/2}}\, e^{-\lambda^2/4t}$

THUS $$U(\underline{x},t) \sim Ub\,\mathrm{erfc}\left(\frac{\lambda}{2\sqrt{D t}}\right) + \frac{K\lambda\, Ub}{2\sqrt{\pi D}\, t^{3/2}}\exp\left(-\lambda^2/4Dt\right) \qquad t \ll 1.$$

THEN $\bar{V}_{\hat{\eta}\hat{\eta}} - \kappa\varepsilon^{1/2}\bar{V}_{\hat{\eta}} + O(\varepsilon) - \bar{V} = 0$

WE EXPAND $\bar{V} = \bar{V}_0 + \varepsilon^{1/2}\bar{V}_1 + \dots$

THIS GIVES THE TWO PROBLEMS:

$$\left\{\begin{array}{l} \bar{V}_{0\hat{\eta}\hat{\eta}} - \bar{V}_0 = 0 \quad \text{IN} \quad \hat{\eta} \geq 0 \\ \bar{V}_0 = U_b \;\; \text{ON} \;\; \hat{\eta} = 0, \quad \bar{V}_0 \to 0 \;\; \text{AS} \;\; \hat{\eta} \to +\infty \end{array}\right.$$

$$\left\{\begin{array}{l} \bar{V}_{1\hat{\eta}\hat{\eta}} - \bar{V}_1 = \kappa\bar{V}_{0\hat{\eta}} \quad \text{IN} \quad \hat{\eta} \geq 0 \\ \bar{V}_1 = 0 \;\; \text{ON} \;\; \hat{\eta} = 0; \quad \bar{V}_1 \to 0 \;\; \text{AS} \;\; \hat{\eta} \to +\infty. \end{array}\right.$$

THE LEADING ORDER SOLUTION IS $\bar{V}_0 = U_b e^{-\hat{\eta}}$. AT NEXT ORDER WE GET

$$\bar{V}_{1\hat{\eta}\hat{\eta}} - \bar{V}_1 = -\kappa U_b e^{-\hat{\eta}} \longrightarrow \bar{V}_1 = -\kappa U_b W \quad \text{SO} \quad W'' - W = e^{-\hat{\eta}}.$$

WE GET $W = -\frac{1}{2}\hat{\eta}e^{-\hat{\eta}}$. THIS SATISFIES $\bar{V}_1 = 0$ ON $\hat{\eta} = 0$ AND $\bar{V}_1 \to 0$ AS $\hat{\eta} \to \infty$.

WE HAVE $\bar{V}_1 = \dfrac{\kappa U_b}{2}\hat{\eta}e^{-\hat{\eta}}$.

NOW WE WRITE THE TWO-TERM EXPANSION AS (WITH $\hat{\eta} = \eta/\varepsilon^{1/2}$)

$$\bar{U} \sim \frac{1}{\sigma}\left[U_b e^{-\eta/\varepsilon^{1/2}} + \varepsilon^{1/2}\left(\frac{U_b\kappa}{2}\hat{\eta}e^{-\eta/\varepsilon^{1/2}}\right)\right]$$

WE HAVE $\hat{\eta}\varepsilon^{1/2} = \eta$ AND SO $\bar{U} \sim \dfrac{U_b}{\sigma}\left[\left(1 + \dfrac{\kappa}{2}\eta\right)e^{-\eta/\varepsilon^{1/2}}\right]$

FINALLY RECALL $\varepsilon = D/\sigma \ll 1$ SO THAT $\bar{U} = \dfrac{U_b}{\sigma}\left[\left(1 + \dfrac{\kappa}{2}\eta\right)e^{-\eta/\sqrt{D}\,\sqrt{\sigma}}\right]$

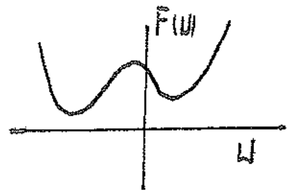
RECALL THAT WE MUST INVERT THE TRANSFORM: $\mathcal{L}^{-1}[\bar{U}] = U$ FOR $t \ll 1$.

THUS RECALLING $\mathcal{L}^{-1}\left(e^{-\eta\sqrt{\sigma}}/\sigma\right) = \text{ERFC}\left(\eta/2\sqrt{t}\right)$ WITH $\text{ERFC}(z) \equiv \dfrac{2}{\sqrt{\pi}}\displaystyle\int_z^\infty e^{-y^2}dy$

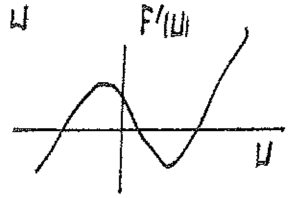
WE GET $\quad U(x,t) \sim U_b\left(1 + \dfrac{\kappa}{2}\eta\right)\text{ERFC}\left(\eta/2\sqrt{Dt}\right) \quad$ FOR $t \ll 1$.

WHERE κ IS CURVATURE OF BOUNDARY.

CAHN-HILLIARD → MULLINS-SEKERKA



F double well potential



WE WRITE THE CH MODEL AS

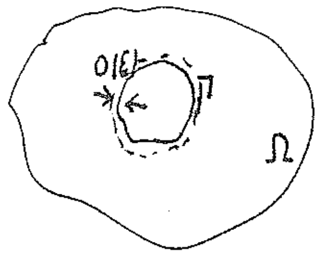
$$u_t = \Delta\left[F'(u) - \varepsilon^2 \Delta u \right] \quad \text{in } \Omega$$

WITH $\partial_n u = \partial_n \left[F'(u) - \varepsilon^2 \Delta u \right] = 0$ ON $\partial\Omega$.

WE WILL INTRODUCE SLOW TIME $\tau = \varepsilon t$ AND W AS

$$(1) \quad \begin{cases} \varepsilon u_\tau = \Delta W & W_n = 0 \text{ on } \partial\Omega \\ \varepsilon^2 \Delta u - F'(u) = -W & u_n = 0 \text{ on } \partial\Omega \end{cases}$$

NOW WE CONSIDER LIMIT $\varepsilon \to 0$ WHERE u IS A LAYERED SOLUTION SHOWN AS

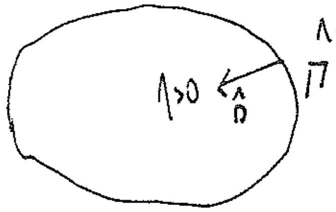


IN LIMIT $\varepsilon \to 0$ THE "CENTER" OF THE LAYER IS A CURVE Γ.

THE QUESTION IS HOW DOES Γ EVOLVE IN TIME.

WE LET Λ BE DISTANCE FROM Γ WITH $\Lambda > 0$ IF $\underline{x}$ IS INSIDE Γ

AND LET s BE ORTHOGONAL COORDINATE REPRESENTING ARC LENGTH ON Γ.



RECALL FOR A FUNCTION ϕ HOW TO CONVERT TO (Λ, s) COORDINATES.

WE OBTAIN FROM (1)

$$(2) \quad \begin{cases} W_{\Lambda\Lambda} - \dfrac{\kappa}{1-\kappa\Lambda} W_\Lambda + \dfrac{1}{(1-\kappa\Lambda)} \partial_s \left(\dfrac{W}{1-\kappa\Lambda} \right) = \varepsilon u_\tau = \varepsilon \left(u_\Lambda \Lambda' + u_s s' \right) \\[2ex] \varepsilon^2 \left(u_{\Lambda\Lambda} - \dfrac{\kappa}{1-\kappa\Lambda} u_\Lambda + \dfrac{1}{(1-\kappa\Lambda)} \partial_s \left(\dfrac{u}{1-\kappa\Lambda} \right) \right) - F'(u) = -W. \end{cases}$$

$\Lambda' = d\Lambda / d\tau$.

REMARK (i) $\Lambda' = d\Lambda/d\tau = V$ NORMAL VELOCITY TO Γ.

(ii) $\kappa > 0$ IF Γ IS CONVEX AS SEEN FROM THE INSIDE.

OUTER EXPANSION

IN THE OUTER REGION AWAY FROM $O(\varepsilon)$ REGION NEAR Γ WE EXPAND $\quad W = W_0 + \varepsilon W_1 + \cdots \qquad U = U_0 + \varepsilon U_1 + \cdots$

FROM (1) WE OBTAIN AWAY FROM Γ THAT

$$(3)\quad \begin{cases} O(1): & \Delta W_0 = 0, \quad F'(U_0) = -W_0, \quad \partial_n W_0 = 0 \text{ ON } \partial\Omega \\ O(\varepsilon): & \Delta W_1 = U_{0\tau}, \quad F''(U_0)U_1 = W_1, \quad \partial_n W_1 = 0 \text{ ON } \partial\Omega. \end{cases}$$

INNER EXPANSION WE INTRODUCE LOCAL NORMAL COORDINATE $\hat{\Lambda} = \Lambda/\varepsilon$ AND REPLACE $U \to \bar{U}$ AND $W \to \bar{W}$. THEN FROM (2),

$$\frac{1}{\varepsilon^2}\bar{W}_{\hat\Lambda\hat\Lambda} - \frac{\kappa}{\varepsilon(1-\kappa\varepsilon\hat\Lambda)}\bar{W}_{\hat\Lambda} + \frac{1}{(1-\kappa\varepsilon\hat\Lambda)}\partial_s\left(\frac{\bar{W}}{(1-\kappa\varepsilon\hat\Lambda)}\right) = \varepsilon\left(\frac{1}{\varepsilon}\bar{U}_{\hat\Lambda}V + \bar{U}_s\right)$$

$$\varepsilon^2\left(\frac{1}{\varepsilon^2}\bar{U}_{\hat\Lambda\hat\Lambda} - \frac{\kappa}{\varepsilon(1-\kappa\varepsilon\hat\Lambda)}\bar{U}_{\hat\Lambda} + \frac{1}{(1-\kappa\varepsilon\hat\Lambda)}\partial_s\left(\frac{\bar{U}}{(1-\kappa\varepsilon\hat\Lambda)}\right)\right) - F'(\bar{U}) = -\bar{W}.$$

WE THEN EXPAND

$$\bar{W} = \bar{W}_0 + \varepsilon\bar{W}_1 + \varepsilon^2\bar{W}_2 + \cdots \qquad \bar{U} = \bar{U}_0 + \varepsilon\bar{U}_1$$

WE GET (4a) $\bar{W}_{0\hat\Lambda\hat\Lambda} = 0$

(4b) $\bar{W}_{1\hat\Lambda\hat\Lambda} - \kappa\bar{W}_{0\hat\Lambda} = 0$

(4c) $\bar{W}_{2\hat\Lambda\hat\Lambda} - \kappa\bar{W}_{1\hat\Lambda} - \underline{\kappa^2\hat\Lambda\bar{W}_{0\hat\Lambda}} + \bar{W}_{0ss} = \bar{U}_{0\hat\Lambda}V$

REMARK (i) IN OBTAINING UNDERLINED TERM WE USED BINOMIAL SERIES ON $-\kappa/\varepsilon(1-\kappa\varepsilon\hat\Lambda)\,\bar{W}_{\hat\Lambda} \sim -\frac{\kappa}{\varepsilon}(1+\kappa\varepsilon\hat\Lambda+\ldots)\bar{W}_{\hat\Lambda}$

NOW FROM $\bar{U}$ EQUATION

(5a) $\bar{U}_{0\hat{\Lambda}\hat{\Lambda}} - F'(\bar{U}_0) = -\bar{W}_0.$

AND AT ONE HIGHER ORDER

(5b) $\bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - F''(\bar{U}_0)\bar{U}_1 = \kappa \bar{U}_{0\hat{\Lambda}} - \bar{W}_1$

NOW THE SOLUTION TO (4a) IS $\bar{W}_0$ = linear IN Λ. WE MUST HAVE TO MATCH TO OUTER THAT THERE IS NO LINEAR GROWTH AT ∞. THUS $\bar{W}_0$ = CONSTANT. THIS IMPLIES THAT OUTER SOLUTION W_0 IS A GLOBAL CONSTANT AS WELL.

THEN (5a) BECOMES

(6) $\bar{U}_{0\hat{\Lambda}\hat{\Lambda}} - F'(\bar{U}_0) = -\bar{W}_0 \qquad -\infty < \Lambda < \infty$

TO MATCH TO OUTER SOLUTION WHERE $F'(U_0) = -W_0$ IS NEEDED WE MUST FIND CONSTANTS $\bar{U}_{0-}$, $\bar{U}_{0+}$ AND $\bar{W}_{0m}$ THAT SATISFY

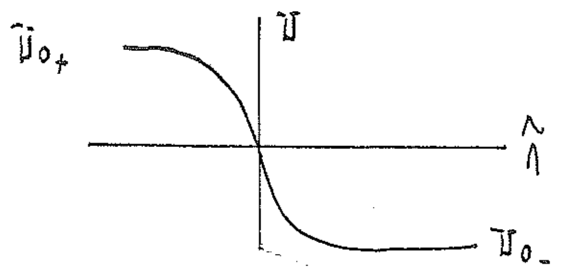
$$(7a)\quad \left\{ \begin{array}{l} F'(\bar{U}_{0+}) = -\bar{W}_{0m} \\ F'(\bar{U}_{0-}) = -\bar{W}_{0m} \end{array} \right.$$

AND TO HAVE A HETEROCLINIC $\displaystyle\int_{\bar{U}_{0-}}^{\bar{U}_{0+}} [-F'(\bar{U}_0) + \bar{W}_{0m}]\, d\bar{U} = 0.$ (7b)

TO DECAY AT ∞ WE REQUIRE

$$F''(\bar{U}_{0-}) > 0, \quad F''(\bar{U}_{0+}) > 0.$$

THEN $\exists$ A HETEROCLINIC TO (6) WITH $\bar{U}_0 \to \bar{U}_{0-}$ AS $\hat{\Lambda} \to +\infty$ (INside Γ) AND $\bar{U}_0 \to \bar{U}_{0+}$ AS $\hat{\Lambda} \to -\infty$ (OUTSIDE Γ).



WE DEFINE $\bar{U}_0$ UNIQUELY BY CONVENIENTLY SPECIFYING $\bar{U}_0 = \dfrac{(\bar{U}_{0+} + \bar{U}_{0-})}{2}$ ON $\hat{\Lambda} = 0$.

REMARKS (i) THUS Γ IS THE LEVEL CURVE WHERE $\bar{U}_0 = (\bar{U}_{0+} + \bar{U}_{0-})/2$

(ii) RECALL FROM BASIC HETEROCLINIC THEORY THAT

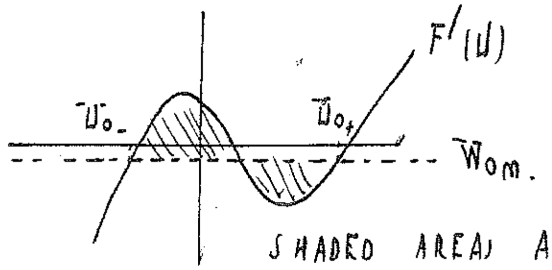
$$\phi'' + Q(\phi) = 0$$

HAS A HETEROCLINIC IF $\phi \to \begin{cases} \phi_+ & \text{AS } \Lambda \to \infty \\ \phi_- & \text{AS } \Lambda \to -\infty \end{cases}$

WHERE $Q(\phi_\pm) = 0$, $Q'(\phi_\pm) < 0$, $\int_{\phi_-}^{\phi_+} Q(\phi)\, d\phi = 0$.

THIS IS PRECISELY THE CONDITIONS (7a), (7b).

GRAPHICALLY WE HAVE



SHADED AREAS ARE SAME.

IN SUMMARY, TO LEADING ORDER WE HAVE THE FOLLOWING

$$(8) \begin{cases} W \sim \bar{W}_{0m} \quad \text{EVERYWHERE (IN OUTER REGION AND NEAR } \Gamma). \\ U \sim \bar{U}_{0-} \quad \text{INSIDE } \Gamma \ ; \quad U \sim \bar{U}_{0+} \quad \text{OUTSIDE } \Gamma \\ U \sim \bar{U}_0(\hat{\Lambda}) \quad \text{IN } O(\varepsilon) \text{ REGION NEAR } \Gamma. \end{cases}$$

SINCE $\bar{U}_{0\pm}$ ARE INDEPENDENT OF $\hat{\tau}$, THEN FROM THE $O(\varepsilon)$ EQUATION IN (3), WE OBTAIN THAT THE OUTER CORRECTION FOR W_1 SATISFIES

$$(9) \quad \Delta W_1 = 0 \quad \text{AWAY FROM } \Gamma.$$

OUR GOAL IS TO DERIVE THE TRANSMISSION (OR EQUIVALENTLY JUMP CONDITIONS) FOR W_1 ACROSS Γ.

NOW FROM (4b) WE OBTAIN SINCE $\bar{W}_0$ IS A CONSTANT THAT

$$\bar{W}_{1\hat{\Lambda}\hat{\Lambda}} = 0 \longrightarrow \bar{W}_1 = A\hat{\Lambda} + B.$$

THE MATCHING CONDITION IS THAT

$$\bar W_{0m} + \varepsilon W_1 + \cdots \sim \bar W_{0m} + \varepsilon \bar W_1 + \cdots$$

$$\text{AS } x \to \Gamma \qquad \hat\lambda \to \pm\infty .$$

THUS WE CANNOT ALLOW $\bar W_1$ TO GROW AS $\hat\lambda \to \pm\infty$. CONSEQUENTLY,

(10) $\bar W_1 = B$ WHERE B IS INDEPENDENT OF $\hat\lambda$.

THEN FROM (5b) WE HAVE

$$L\bar U_1 \equiv \bar U_{1\hat\lambda\hat\lambda} - F''(\bar U_0)\bar U_1 = \kappa \bar U_{0\hat\lambda} - B .$$

BUT $L\bar U_0' = 0$ IS SOLVABILITY CONDITION, SO THAT

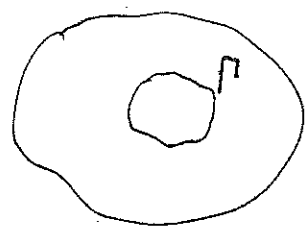
$$\kappa \int_{-\infty}^{\infty} \bar U_{0\hat\lambda}^2 \, d\tilde\lambda = B \int_{-\infty}^{\infty} \bar U_{0\hat\lambda} \, d\hat\lambda = B\left(\bar U_{0+} - \bar U_{0-} \right)$$

WE CONCLUDE THAT

(11) $$\bar W_1 = B = \frac{\kappa \int_{-\infty}^{\infty} \bar U_{0\hat\lambda}^2 \, d\hat\lambda}{\bar U_{0+} - \bar U_{0-}}$$

THE MATCHING CONDITION THEN YIELDS FOR OUTER PROBLEM THAT $W_1 \to B$ AS WE APPROACH Γ.

THIS GIVES MULLINS-SEKERKA FLOW



(12) $$\begin{cases} \Delta W_1 = 0 & \text{INSIDE AND OUTSIDE } \Gamma. \\ \partial_n W_1 = 0 & \text{ON } \partial\Omega \\ W_1 = \gamma\kappa & \text{ON } \Gamma \end{cases} \qquad \gamma \equiv \frac{\int_{-\infty}^{\infty} \bar U_{0\hat\lambda}^2 \, d\lambda}{\bar U_{0+} - \bar U_{0-}}$$

WHERE $\kappa = \kappa(s)$ IS THE CURVATURE OF Γ.

THE <u>FINAL STEP</u> IN THE CALCULATION IS TO DETERMINE THE NORMAL VELOCITY V.

NOW SINCE $\overline{W}_{1\hat{\Lambda}} = 0$, $\overline{W}_0 = $ CONSTANT, THEN EQUATION (4C) YIELDS THAT

$$\overline{W}_{2\hat{\Lambda}\hat{\Lambda}} = \overline{U}_{0\hat{\Lambda}} V.$$

SINCE $\overline{U}_{0\hat{\Lambda}}$ IS TENDING TO ZERO EXPONENTIALLY AT ∞ WE HAVE $W_{2\hat{\Lambda}\hat{\Lambda}} = o(1)$ FOR $\hat{\Lambda} \to \pm\infty$.

WE INTEGRATE FROM $-\infty < \hat{\Lambda} < \infty$ TO OBTAIN

$$(13) \qquad \overline{W}_{2\hat{\Lambda}}(\infty) - \overline{W}_{2\hat{\Lambda}}(-\infty) = \overline{V} \int_{-\infty}^{\infty} \overline{U}_{0\hat{\Lambda}}\, d\hat{\Lambda} = \overline{V} \left[\overline{U}_{0+} - \overline{U}_{0-} \right]$$

NOW (13) GIVES A JUMP CONDITION FOR DERIVATIVE OF $\overline{W}_2$.

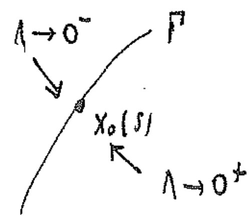
WE MUST CONSIDER MATCHING CONDITION FOR W.

$$(14) \qquad \overline{W}_{0m} + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots \sim \overline{W}_{0m} + \varepsilon \overline{W}_1 + \varepsilon^2 \overline{W}_2 + \cdots$$

$\Lambda \to 0$ (outer limit) AS $\hat{\Lambda} \to \pm\infty$.

<u>REMARK</u> (i) THE JUMP IN DERIVATIVE OF W_1 ACROSS Γ IS RELATED TO JUMP IN (13) FOR $\overline{W}_{2\hat{\Lambda}}$.

OUR FINAL STEP IS TO DO THIS MATCHING PROPERLY. WE EXPAND LHS OF (14) AS $\underline{X} \to \underline{X}_0(S)$ WHERE $\underline{X}_0(S)$ IS A POINT ON Γ.



NOW AS WE APPROACH FROM $\pm$ DIRECTION WE GET BY TAYLOR

$$W_1 \sim W_1\Big|_{\Lambda=0} + \nabla W_1 \cdot (\underline{X} - \underline{X}_0) + \cdots$$

NOW USE FORMULA (6) ON PAGE (B3) THAT $\nabla W_1 = \dfrac{W_S}{1-K\Lambda}\hat{t} + W_{1\Lambda}\hat{n}$

AND FROM (1) ON (B1) THAT

$$\underline{X} = \underline{X}_0(S) + \Lambda \hat{n}(S)$$

THIS GIVES $\nabla W_1 \cdot (\underline{X} - \underline{X}_0) = \left(\dfrac{W_{1S}}{1-K\Lambda}\hat{t} + W_{1\Lambda}\hat{n} \right) \cdot \left(\Lambda \hat{n} \right) = W_{1\Lambda} \Lambda.$ (15)

WE CONCLUDE THAT THE LEFT SIDE OF (14) AS WE APPROACH THE INTERFACE Γ IS

$$\bar{W}_{0m} + \varepsilon \left[W_1\big|_{\Lambda=0} + W_{1\Lambda}\big|_{\Lambda=0^+} \Lambda \right] + \varepsilon^2 W_2\big|_{\Lambda=0} + \cdots \qquad \text{AS } \Lambda \to 0^+ \text{ (FROM INSIDE)}$$

AND
$$\bar{W}_{0m} + \varepsilon \left[W_1\big|_{\Lambda=0^+} W_{1\Lambda}\big|_{\Lambda=0^-} \Lambda \right] + \varepsilon^2 W_2\big|_{\Lambda=0} + \cdots \qquad \text{AS } \Lambda \to 0^- \text{ (FROM OUTSIDE)}$$

NOW LET $\Lambda = \varepsilon \hat{\Lambda}$ SO WE WRITE IN TERMS OF INNER VARIABLE

$$(16) \qquad \bar{W}_{0m} + \varepsilon W_1\big|_{\Lambda=0} + \varepsilon^2 W_{1\Lambda}\big|_{\Lambda=0^{\pm}} \hat{\Lambda} + \varepsilon^2 W_2\big|_{\Lambda=0} \qquad \text{AS } \Lambda \to 0^{\pm} \qquad \begin{array}{l} + \to \text{INSIDE} \\ - \to \text{OUTSIDE.} \end{array}$$

THIS MUST MATCH WITH RHS OF (14). RECALL THAT $\bar{W}_1 =$ CONSTANT AND SO RHS OF (14) IS

$$(17) \qquad \bar{W}_{0m} + \varepsilon \bar{W}_1 + \varepsilon^2 \bar{W}_2 + \cdots \qquad \text{AS } \hat{\Lambda} \to \pm\infty.$$

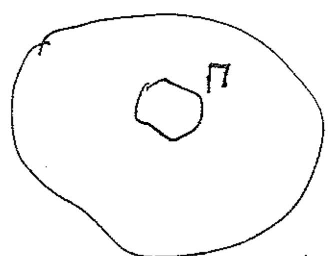
WE HAVE FROM (13) THAT $\bar{W}_{2\hat{\Lambda}}(\infty) - \bar{W}_{2\hat{\Lambda}}(-\infty) = V\left[\bar{U}_{0+} - \bar{U}_{0-}\right]$.

THIS MEANS THAT
$$\bar{W}_2 \sim W_{1\Lambda}\big|_{\Lambda=0^+} \hat{\Lambda} \qquad \text{AS} \qquad \hat{\Lambda} \to +\infty$$
$$\bar{W}_2 \sim W_{1\Lambda}\big|_{\Lambda=0^-} \hat{\Lambda} \qquad \text{AS} \qquad \hat{\Lambda} \to -\infty,$$

IS THE REQUIRED MATCHING CONDITION.

WE CONCLUDE THAT $\left[\bar{W}_{2\hat{\Lambda}}\right]\Big|_{-\infty}^{\infty} = \left[W_{1\Lambda}\right]\Big|_{0^-}^{0^+} = V\left[\bar{U}_{0+} - \bar{U}_{0-}\right].$

FINALLY, WE HAVE <u>MULLINS-SEKERKA</u> FLOW:



$$(MS) \left\{ \begin{array}{ll} \Delta W_1 = 0 & \text{INSIDE AND OUTSIDE } \Gamma \\ \partial_n W_1 = 0 & \text{ON } \partial\Omega \\ W_1 = \gamma \kappa & \text{ON } \Gamma \\ \left[W_{1\Lambda}\right] = V\left[\bar{U}_{0+} - \bar{U}_{0-}\right] & \end{array} \right.$$

$\gamma = \left(\int_{-\infty}^{\infty} \bar{U}_{0\hat{\Lambda}}^2 \, d\Lambda\right) \Big/ \left(\bar{U}_{0+} - \bar{U}_{0-}\right)$ $\qquad [W_{1\Lambda}] = W_{1\Lambda}\big|_{IN} - W_{1\Lambda}\big|$

V = NORMAL VELOCITY TO Γ. $V > 0$ SHRINKING SPHERE

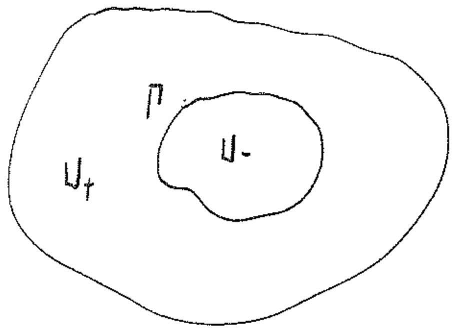
MOTION BY CURVATURE (INTERFACE)

$$u_t = \varepsilon^2 \Delta u + Q(u) \quad \text{IN } \Omega$$

$$\partial_n u = 0 \quad \text{ON } \partial\Omega.$$

ASSUME $\quad Q(u_\pm) = 0, \quad Q'(u_\pm) < 0, \quad \int_{u_-}^{u_+} Q(u)\,du = 0.$

ON AN $O(1)$ TIME-SCALE $u \to u_+$ OR $u \to u_-$ SEPARATED BY AN INTERFACE Γ.



Γ IS CENTERLINE OF INTERFACE OF WIDTH $O(\varepsilon)$, AND IS LEVEL CURVE

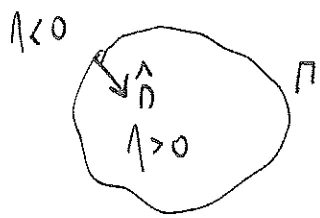
$$u\big|_\Gamma = (u_+ + u_-)/2$$

WHAT IS EVOLUTION OF Γ?

- IN OUTER REGION $u \sim u_-$ OR $u \sim u_+$.
- NEAR Γ WE INTRODUCE (η, s) COORDINATES

$$u_t = \varepsilon^2\left(u_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}\, u_\eta + \frac{1}{1-\kappa\eta}\, \partial_s\left(\frac{1}{1-\kappa\eta}\, \partial_s u \right)\right) + Q(u)$$

NOW



- WITH κ CURVATURE OF Γ, $\kappa = \kappa(s)$
- s IS ARClength ON Γ
- η IS distance FROM Γ.

NOW WE LET $\hat{\eta} = \eta/\varepsilon$. THEN $u \to \bar{u}$ WITH $\eta = \eta(\tau)$ $\quad \tau = \varepsilon^p t$

$$\varepsilon^{p-1}\, \bar{u}_{\hat\eta}\, \dot{\eta} + \bar{u}_s\, \varepsilon^p\, \dot{s} = \bar{u}_{\hat\eta\hat\eta} - \kappa\varepsilon\, \bar{u}_{\hat\eta} + \varepsilon^2\, \bar{u}_{ss} + Q(\bar{u})$$

$$\text{where} \quad \dot{\eta} = d\eta/d\tau$$

MC2

WE CHOOSE $p = 2$ AND EXPAND

$$\bar{U} = \bar{U}_0 + \varepsilon \bar{U}_1 + \cdots$$

$$(1)\quad \begin{cases} \bar{U}_{0\hat{\Lambda}\hat{\Lambda}} + Q(\bar{U}_0) = 0, & -\infty < \hat{\Lambda} < \infty \\ \bar{U}_0 \to U_- & \text{AS} \quad \hat{\Lambda} \to +\infty \\ \bar{U}_0 \to U_+ & \text{AS} \quad \hat{\Lambda} \to -\infty \end{cases}$$

$\exists$ A SOLUTION TO (1), AND IT IS UNIQUE BY SPECIFYING $\bar{U}_0(0) = \dfrac{U_+ + U_-}{2}$.

NOW AT NEXT ORDER,

$$(2)\quad \begin{cases} \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} + Q'(\bar{U}_0)\bar{U}_1 = \bar{U}_{0\hat{\Lambda}}\,\dot{\Lambda} + \kappa\,\bar{U}_{0\hat{\Lambda}}. \\ \bar{U}_1 \to 0 \quad \text{AS} \quad \hat{\Lambda} \to \pm\infty. \end{cases}$$

NOTICE $\dot{\Lambda} = \bar{V}$ NORMAL VELOCITY TO Γ. NOTICE $\bar{V} > 0$ IF EXPANDING.

NOW DEFINE $L\bar{U}_1 \equiv \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} + Q'(\bar{U}_0)\bar{U}_1$.

SINCE $L\bar{U}_{0\hat{\Lambda}} = (\bar{U}_{0\hat{\Lambda}\hat{\Lambda}})_{\hat{\Lambda}} + Q'(\bar{U}_0)\bar{U}_{0\hat{\Lambda}} = 0$

WE MUST HAVE FOR SOLVABILITY OF (2) THAT

$$\dot{\Lambda}\int_{-\infty}^{\infty} (\bar{U}_{0\hat{\Lambda}})^2\, d\hat{\Lambda} = -\kappa \int_{-\infty}^{\infty} (\bar{U}_{0\hat{\Lambda}})^2\, d\hat{\Lambda}.$$

THUS $\bar{V} = -\kappa$. $\qquad \bar{V} = \dfrac{d\Lambda}{d\tau}$.

IN TERMS OF ORIGINAL TIME-SCALE $\tau = \varepsilon^2 t$, THEN

$$V = -\varepsilon^2 \kappa.$$

MC3

THUS IF Γ IS A SPHERE OF RADIUS ρ, WE HAVE $\kappa = 1/\rho$

AND $V = d\rho/dt$.

THUS $d\rho/dt = -1/\rho$

IF $\rho(0) = \rho_0$, THEN ρ VANISHES IN FINITE TIME,

$$\rho^2 - \rho_0^2 = -t$$

OR $\rho = \sqrt{\rho_0^2 - t}$ FOR $0 < t < (\rho_0^2)$.

MOTION OF STRIPES FORMED BY HOMOCLINIC PULSES

WE CONSIDER FIRST THE SEMI-STRONG REGIME

$$
(1)\quad
\begin{aligned}
V_t &= \varepsilon^2 \Delta V - V + g(U,V) && \text{IN } \Omega \\
\tau U_t &= D \Delta U - U + \frac{1}{\varepsilon} f(U,V) && \text{IN } \Omega.
\end{aligned}
$$

$$\partial_n U = \partial_n V = 0, \quad \text{ON } \partial\Omega$$

ASSUMPTIONS (i) ASSUME THAT FOR U_0 CONSTANT, THAT $\exists$ A HOMOCLINIC PULSE SOLUTION TO

$$\bar{V}_{yy} - \bar{V} + g(U_0, \bar{V}) = 0 \quad \text{IN} \quad -\infty < y < \infty$$

WITH $\bar{V} \to 0$ AS $|y| \to \pm\infty$

WE NEED $g(U_0, 0) = 0$

AND $f(U, 0) = 0.$

(ii) ASSUME ALSO THAT THIS PULSE SOLUTION IS <u>STABLE</u> WRT NONLOCAL EIGENVALUE ON AN $O(1)$ TIME SCALE. I.e. THAT A STRIPE WILL <u>NOT</u> BREAK UP INTO SPOTS.

THIS IS THE KEY ASSUMPTION.

IT IS SHOWN IN SECTION 2.3 OF KOLOKOLNIKOV ET AL. "STABILITY OF A STRIPE FOR THE GM MODEL..." (2005) THAT THIS IS SATISFIED WHEN

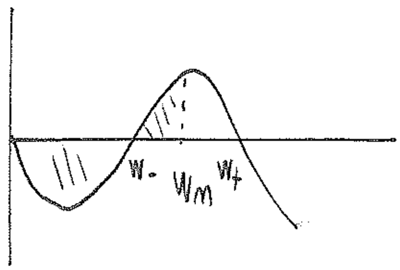
$$g(U,V) = \frac{V^2}{U(1+\kappa V^2)} \qquad f(U,V) = V^2$$

AND κ large enough.

IN THIS CASE THE HOMOCLINIC PROFILE IS

$$W'' + Q(W) = 0 \qquad Q(W) = -W + \frac{W^2}{1 + bW^2} \qquad b = \bar{U}^2 K.$$

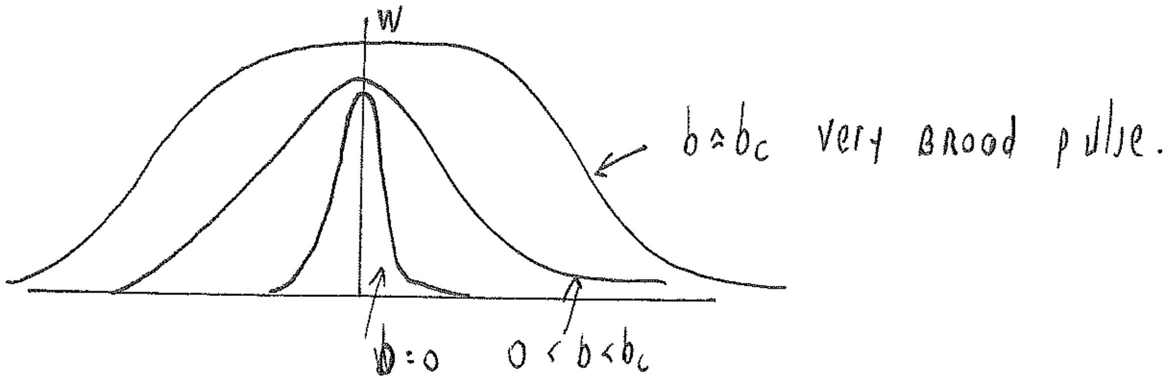
FOR $0 \leq b < 1/4 \rightarrow W = 0$ AND $W = W_+ = \frac{1}{2b}\left[1 + \sqrt{1-4b}\right]$ ARE SADDLE POINTS, WITH W_- IN $0 < W_- < W_+$ A CENTER, $W_- = \frac{1}{2b}\left[1 - \sqrt{1-4b}\right]$



NOW $\exists$ A PULSE WHEN $\exists$ W_m IN $W_- < W_m < W_+$ SUCH THAT $\int_0^{W_m} Q\, dW = 0$

HOWEVER AS $b \rightarrow 1/4$, $W_- \rightarrow W_+$ AND EQUAL AREA CONDITION IS IMPOSSIBLE. SETTING $\int_0^{W_+} Q\, dW = 0$ GIVES $b = b_c = .211376 < 1/4$.

SO A <u>PULSE</u> <u>EXISTS</u> IF $b < b_c = .211376 < \bar{U}^2 K$.



THUS NEED $\bar{U}^2 K < .211376$ FOR existence of pulse.

NOW LET'S ASSUME WE HAVE STABILITY WRT BREAKUP.

WHAT IS EVOLUTION OF THE INTERFACE FOR (1)? THE INTERFACE IS DEFINED BY CURVE Γ WHERE V HAS MAXIMUM

WE WILL INTRODUCE LAYER COORDINATES:

$$V_n n' + V_s s' = \varepsilon^2 \left(V_{nn} - \frac{\kappa}{1-\kappa n} V_n + \frac{1}{(1-\kappa n)} \partial_s \left(\frac{1}{1-\kappa n} V_s \right) \right) + g(U,V) - V$$

$$\tau \left(U_n n' + U_s s' \right) = D \left(U_{nn} - \frac{\kappa}{1-\kappa n} U_n + \frac{1}{1-\kappa n} \partial_s \left(\frac{1}{1-\kappa n} U_s \right) \right) + \frac{1}{\varepsilon} f(U,V) - U.$$

NOW INTRODUCE $\hat{n} = n/\varepsilon$ AND LET $\tau = \varepsilon^2 t$ $\quad \dot{n} = dn/d\tau$.

SO

$$\varepsilon V_{\hat{n}} \dot{n} + V_s \dot{s} \varepsilon^2 = V_{\hat{n}\hat{n}} + g(U,V) - V - \varepsilon \kappa V_{\hat{n}} + O(\varepsilon^2)$$

$$\tau \left(\varepsilon U_{\hat{n}} \dot{n} + U_s \dot{s} \varepsilon^2 \right) = \frac{D}{\varepsilon^2} U_{\hat{n}\hat{n}} - \frac{D\kappa}{\varepsilon} U_{\hat{n}} + \frac{1}{\varepsilon} f(U,V) + O(1).$$

WE THEN EXPAND

$$U = \bar{U}_0 + \varepsilon \bar{U}_1 + \cdots \qquad V = \bar{V}_0 + \varepsilon \bar{V}_1 + \cdots$$

THEN

$$\bar{U}_{0\hat{n}\hat{n}} = 0 \qquad \bar{V}_{0\hat{n}\hat{n}} + \bar{V}_0 + g(\bar{U}_0, \bar{V}_0) = 0$$

$$\bar{U}_{1\hat{n}\hat{n}} = \kappa \bar{U}_{0\hat{n}} - f(\bar{U}_0, \bar{V}_0)/D, \qquad \bar{V}_{1\hat{n}\hat{n}} - \bar{V}_1 + g_{V_0}(\bar{U}_0, \bar{V}_0)\bar{V}_1$$

$$= \kappa V_{0n} - g_U(\bar{U}_0, \bar{V}_0)\bar{U}_1 + \bar{V}_{0\hat{n}} \dot{n}$$

WE HAVE $\bar{U}_0$ = CONSTANT AND $\bar{V}_0$ IS PULSE.

SINCE $\bar{U}_0$ = CONSTANT, WE HAVE

$$L\bar{V}_1 \equiv \bar{V}_{1\hat{\eta}\hat{\eta}} - \bar{V}_1 + g_V(\bar{U}_0,\bar{V}_0)\bar{V}_1 = \kappa V_{0\eta} - g_U(\bar{U}_0,\bar{V}_0)\bar{U}_1 + \bar{V}_{0\hat{\eta}}\dot{\eta}$$

$$\bar{U}_{1\hat{\eta}\hat{\eta}} = -f(\bar{U}_0,\bar{V}_0)/D \implies \bar{U}_{1\hat{\eta}}(\infty) - \bar{U}_{1\hat{\eta}}(-\infty) = -\frac{1}{D}\int_{-\infty}^{\infty} f(\bar{U}_0,\bar{V}_0)\,d\hat{\eta}.$$

NOW THE SOLVABILITY CONDITION FROM $L\bar{V}_{0\hat{\eta}} = 0$ GIVES

$$\kappa\int_{-\infty}^{\infty} \bar{V}_{0\hat{\eta}}^2\,d\eta - \int_{-\infty}^{\infty} g_U^0\,\bar{U}_1\bar{V}_{0\hat{\eta}}\,d\hat{\eta} + \int_{-\infty}^{\infty}\bar{V}_{0\hat{\eta}}^2\,\dot{\eta}\,d\hat{\eta} = 0.$$

THUS
$$\dot{\eta} = -\kappa + \frac{\int_{-\infty}^{\infty} g_U^0\,\bar{V}_{0\hat{\eta}}\bar{U}_1\,d\hat{\eta}}{\int_{-\infty}^{\infty}(\bar{V}_{0\hat{\eta}}^2)\,d\hat{\eta}}. \qquad (1)$$

NOW DEFINE $I \equiv \int_{-\infty}^{\infty} g_U^0\,\bar{V}_{0\hat{\eta}}\bar{U}_1\,d\hat{\eta}$.

NOW INTEGRATE BY PARTS: DEFINE $\mathcal{J}(\bar{V}_0) = \int_0^{V_0} g_U(U_0,\zeta)\,d\zeta$.

$\mathcal{J}$ IS EVEN IN $\hat{\eta}$.

THEN $\frac{d}{d\hat{\eta}}\mathcal{J}(\bar{V}_0) = \bar{V}_{0\hat{\eta}}\,g_U(U_0,\bar{V}_0)$.

THUS $$I = \int_{-\infty}^{\infty}\frac{d}{d\hat{\eta}}\mathcal{J}(\bar{V}_0)\bar{U}_1\,d\hat{\eta} = \underbrace{\bar{U}_1\mathcal{J}(\bar{V}_0)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}\mathcal{J}(\bar{V}_0)\bar{U}_{1\hat{\eta}}\,d\hat{\eta}.$$

SINCE $\mathcal{J}(0) = 0$.

NOW INTEGRATE BY PARTS AGAIN

LET $\hat{\mathcal{J}}(\hat{\eta}) \equiv \int_0^{\hat{\eta}}\mathcal{J}(V_0(\hat{\eta}))\,d\hat{\eta}$. NOTICE THAT $\hat{\mathcal{J}}$ IS ODD IN $\hat{\eta}$.

IN ADDITION $\bar{U}_{1\hat{\eta}\hat{\eta}}$ IS <u>EVEN</u> IN $\hat{\eta}$.

THUS $\quad I = -\hat{\mathcal{J}}(\hat{\eta})\,\bar{U}_{,\hat{\eta}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \underbrace{\hat{\mathcal{J}}(\hat{\eta})}_{\text{ODD}}\, \underbrace{\bar{U}_{,\hat{\eta}\hat{\eta}}}_{\text{EVEN}}\, d\hat{\eta} = -\hat{\mathcal{J}}(\infty)\,\bar{U}_{,\hat{\eta}}(\infty) + \hat{\mathcal{J}}(-\infty)\,\bar{U}_{,\hat{\eta}}(-\infty)$

THUS SINCE $\hat{\mathcal{J}}(-\infty) = -\hat{\mathcal{J}}(\infty)$

$$I = -\hat{\mathcal{J}}(\infty)\left[\bar{U}_{,\hat{\eta}}(\infty) + \bar{U}_{,\hat{\eta}}(-\infty)\right].$$

AND $\quad \hat{\mathcal{J}}(\infty) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \mathcal{J}(\bar{V}_0(\hat{\eta}))\, d\hat{\eta}.$

SUBSTITUTING INTO (1) WE OBTAIN

$$(2) \qquad \dot{\eta} = -\kappa + -\frac{\hat{\mathcal{J}}(\infty)}{\int_{-\infty}^{\infty} \bar{V}_{0\hat{\eta}}^{\,2}\, d\hat{\eta}} \left[\bar{U}_{,\hat{\eta}}(\infty) + \bar{U}_{,\hat{\eta}}(-\infty)\right]$$

WHERE κ IS CURVATURE OF Γ AND

$$(3) \qquad \hat{\mathcal{J}}(\infty) \equiv \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{J}(\bar{V}_0(\hat{\eta}))\, d\hat{\eta} \qquad \text{WITH} \qquad \mathcal{J}(\bar{V}_0) \equiv \int_0^{\bar{V}_0} g_u(U_0, \xi)\, d\xi .$$

<u>REMARK</u> (i) $\bar{V}_0$ IS SOLUTION OF THE HOMOCLINIC PROBLEM

$$\bar{V}_{0\hat{\eta}\hat{\eta}} - \bar{V}_0 + g(\bar{U}_0, \bar{V}_0) = 0 \qquad -\infty < \hat{\eta} < \infty$$

$$\bar{V}_0 \to 0 \quad \text{AS} \quad \hat{\eta} \to \pm\infty .$$

(ii) $\bar{U}_0$ IS A CONSTANT, INDEPENDENT OF $\hat{\eta}$, BUT DEPENDING ON ARCLENGTH S TO BE FOUND BY MATCHING.

(iii) IN (2), WE CALCULATE $\bar{U}_{,\hat{\eta}}(\infty) + \bar{U}_{,\hat{\eta}}(-\infty)$ IN TERMS OF OUTER SOLUTION FOR $\bar{U}$

(iv) IN (2) THE LAW OF EVOLUTION OF INTERFACE IS THAT THERE IS A <u>LOCAL PART</u> (THE κ) AND A GLOBAL PART (THE AVERAGE OF $U_{1\hat{\Lambda}}$) DRIVING THE MOTION.

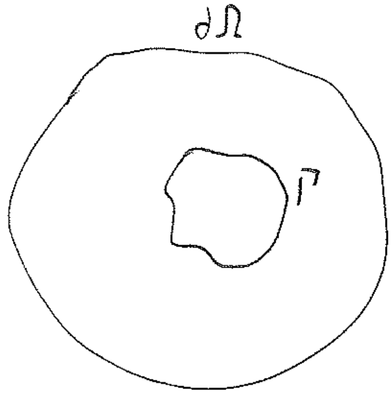
NOW WE CONSIDER THE OUTER SOLUTION. IN THE OUTER REGION WHERE WE ARE AT $\gg O(\varepsilon)$ DISTANCE FROM Γ, THEN $V \approx 0$ AND SINCE $F(\cdot\, U, 0) = 0$ WE HAVE THAT QUASI-STEADY CONDITIONS

$$\begin{cases} D\Delta U - U = 0 & \text{INSIDE } \Gamma \\ & \text{AND OUTSIDE } \Gamma \\ \text{WITH } \partial_n U = 0 \text{ ON } \partial\Omega. \end{cases}$$

NOW THE MATCHING CONDITION IS THAT $U = \bar{U}_0$ ON Γ.

THUS WE HAVE QUASI-STEADY PROBLEM

$$(4)\quad \begin{cases} D\Delta U - U = 0 & \text{INSIDE } \Gamma \\ & \text{OUTSIDE } \Gamma \\ U = \bar{U}_0 \text{ ON } \Gamma \\ \partial_n U = 0 \text{ ON } \partial\Omega \end{cases}$$



NOW THE HEIGHT OF THE PULSE IS DETERMINED BY U_0.

IF WE INTEGRATED THE FULL PDE FOR U OVER Ω, WE GET

$$\int_\Omega \left(D\Delta U - U + \frac{1}{\varepsilon} F(U, V) \right) d\underline{x} = -\int_\Omega U \, d\underline{x} + |\Gamma| \int_{-\infty}^{\infty} F(\bar{U}_0, \bar{V}_0)\, d\hat{\Lambda} = 0.$$

THE EQUATION $|\Gamma| \int_{-\infty}^{\infty} f(\bar{U}_0, \bar{V}_0)\, d\hat{\eta} = \int_{\Omega} u\, d\underline{x}$ IS AN EFFECTIVE EQUATION FOR $\bar{U}_0$. EQUIVALENTLY, WE CAN USE TOP OF PAGE (54) TO DETERMINE $\bar{U}_0$ AS

$$\bar{U}_{1\hat{\eta}}(\infty) - \bar{U}_{1\hat{\eta}}(-\infty) = -\frac{1}{D}\int_{-\infty}^{\infty} f(\bar{U}_0, \bar{V}_0)\, d\hat{\eta}.$$

THE FINAL STEP IN THE PROCESS IS AS FOLLOWS:

<u>STEP 1</u> SUPPOSE $\bar{U}_0$ IS REGARDED AS KNOWN ON Γ. THEN WE CAN SOLVE (4) NUMERICALLY FOR U_0 WITH Γ A GIVEN SHAPE. WE COMPUTE THE BEHAVIOR OF U AS WE APPROACH THE CURVE Γ. WE DERIVE FOR $\underline{x}_0$ ON Γ THAT

$$u \sim \bar{U}_0 + \nabla u \cdot (\underline{x} - \underline{x}_0) + \cdots$$

BUT NOW $\nabla u = \frac{u_s}{1-\kappa\eta}\hat{t} + u_\eta \hat{n}$ $\quad \underline{x} - \underline{x}_0 = \eta\, \hat{n}$.

THUS $u \sim \bar{U}_0 + \varepsilon\, u_{1\eta}\Big|_{\eta=0^\pm} \hat{\eta} + O(\varepsilon^2)$

NOW THE FAR FIELD FORM OF INNER SOLUTION IS

$$\bar{U}_{1\hat{\eta}}(\infty) - \bar{U}_{1\hat{\eta}}(-\infty) = -\frac{1}{D}\int_{-\infty}^{\infty} f(\bar{U}_0, \bar{V}_0)\, d\hat{\eta}.$$

AND $u \sim \bar{U}_0 + \varepsilon \bar{U}_1 + \cdots$ AS $\hat{\eta} \to \pm\infty$.

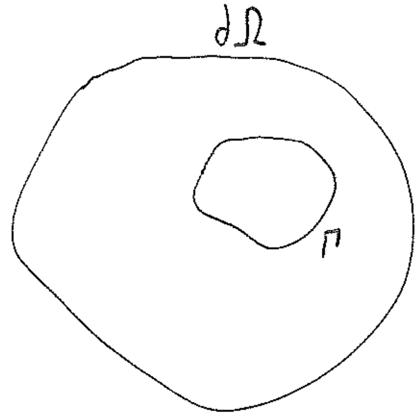
THUS
$$\bar{U}_1 \sim u_{1\eta}\Big|_{\eta=0^+} \hat{\eta} \quad \text{AS} \quad \hat{\eta} \to +\infty$$
$$\bar{U}_1 \sim u_{1\eta}\Big|_{\eta=0^-} \hat{\eta} \quad \text{AS} \quad \hat{\eta} \to -\infty.$$

WE CONCLUDE THAT

$$(5) \qquad \bar{U}_{\hat{\Lambda}}(\infty) - \bar{U}_{\hat{\Lambda}}(-\infty) = U_{\Lambda}\Big|_{\Lambda=0^+} - U_{\Lambda}\Big|_{\Lambda=0^-} .$$

IN SUMMARY, OUR FORMULATION IS AS FOLLOWS:

CONSIDER THE FREE BVP GIVEN BY



$$(6) \qquad \begin{cases} D\Delta U - U = 0, & \underline{x} \text{ INSIDE } \Gamma \\ & \underline{x} \text{ OUTSIDE } \Gamma \\ U = \bar{U}_0 & \text{ON } \Gamma \\ \partial_n U = 0 & \text{ON } \partial\Omega \end{cases}$$

WHERE $\bar{U}_0 = \bar{U}_0(s)$ ON Γ IS TO BE FOUND FROM

$$(7) \qquad U_{\Lambda}\Big|_{\Lambda=0^+} - U_{\Lambda}\Big|_{\Lambda=0^-} = -\frac{1}{D}\int_{-\infty}^{\infty} f(\bar{U}_0, \bar{V}_0)\, d\hat{\Lambda}$$

WHERE $\bar{V}_0 = \bar{V}_0(\hat{\Lambda})$ IS THE HOMOCLINIC ORBIT OF

$$(8) \qquad \begin{aligned} &\bar{V}_{0\hat{\Lambda}\hat{\Lambda}} - \bar{V}_0 + g(\bar{U}_0, \bar{V}_0) = 0 \quad \text{IN} \quad -\infty < \hat{\Lambda} < \infty \\ &\bar{V}_0 \to 0 \quad \text{AS} \quad \hat{\Lambda} \to \pm\infty \end{aligned}$$

NOTICE THAT $\bar{V}_0$ ALSO DEPENDS ON $\bar{U}_0$. IN PRINCIPLE WE CAN INTEGRATE THE RIGHT HAND SIDE IN (7), AND OBTAIN AN EQUATION FOR $\bar{U}_0(s)$. WE REMARK THAT $U_{\Lambda}\Big|_{\Lambda=0^{\pm}}$ DEPEND <u>NONLOCALLY</u> ON $\bar{U}_0$ AND SO (7) IS REALLY AN INTEGRAL EQUATION FOR $\bar{U}_0$.

THEN THE CURVE Γ IS EVOLVED ACCORDING TO

$$\dot{n} = -\kappa + B(s)\left(u_n\big|_{n=0^+} + u_n\big|_{n=0^-} \right)$$

WHERE

$$B(s) = \frac{\frac{1}{2}\int_{-\infty}^{\infty} f(\bar{V}_0, \bar{U}_0)\, d\hat{n}}{\int_{-\infty}^{\infty} (\bar{V}_{0\hat{n}})^2\, d\hat{n}}$$

AND

$$f(\bar{V}_0, U_0) = \int_0^{\bar{V}_0} g_u(U_0, \zeta)\, d\zeta .$$

SINCE $\bar{U}_0 = \bar{U}_0(s)$, THEN $B = B(s)$ (ALSO NOTE $\bar{V}_0 = \bar{V}_0(\hat{n})$ AND ALSO DEPENDS ON $\bar{U}_0$).