Strong Localized Perturbations: Theory and Applications

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1 Strong Localized Perturbations in 2-D Domains

In this section we extend the analysis in 3-D to treat some related steady-state elliptic problems in a two-dimensional domain with multiple small traps.

1.1 Some Fundamentals: Leading-Order Eigenvalue Asymptotics

We first recall a basic result from potential theory. Suppose that $\Delta u + p(x)u = \delta(x - x_0)$ for $x \in \Omega \in \mathbb{R}^2$. Then, the singularity has the form

$$u \sim \frac{1}{2\pi} \log |x - x_0| \,, \qquad \text{as} \quad x \to x_0 \,.$$

The derivation of this is simple, and proceeds as in the derivation of the corresponding 3-D result considered previously. The leading-order behavior of the singularity is independent of the lower order term $p(x)u$, provided that $p(x)$ is smooth.

To illustrate the asymptotic approach and scalings needed in the 2-D case, we consider the following simple eigenvalue problem posed in a domain with a small hole:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{for } x \in \Omega \backslash \Omega_\epsilon \\ u = 0 & \text{for } x \in \partial\Omega \\ u = 0 & \text{for } x \in \partial\Omega_\varepsilon \\ \int_{\Omega \backslash \Omega_\epsilon} u^2 \, dx = 1 \,. \end{cases} \tag{1.1}$$

Here Ω_ϵ is a small hole of "radius" $\mathcal{O}(\varepsilon)$, for which $\Omega_\epsilon \to \{x_0\}$ as $\epsilon \to 0$, where x_0 is an interior point of Ω. Let μ_0, ϕ_0 be the principal first eigenpair of the unperturbed problem, so that

$$\begin{cases} \Delta \phi_0 + \lambda \phi_0 = 0 & \text{for } x \in \Omega \\ \phi_0 = 0 & \text{for } x \in \partial\Omega \\ \int_{\Omega} \phi_0^2 \, dx = 1 \,. \end{cases} \tag{1.2}$$

Now we will expand the eigenvalue of (1.1) that is close to μ_0 as $\lambda \sim \mu_0 + \nu(\epsilon)\lambda_1 + \cdots$, with $\nu(\epsilon) \to 0$ as $\epsilon \to 0$. Here $\nu(\varepsilon)$ is an unknown gauge function to be determined. In the outer region away from the hole, we expand

[htbp]

Table 1. The logarithmic capacitance d for some cross-sectional shapes of $\Omega_0 = \varepsilon^{-1}\Omega_\varepsilon$.

Shape of $\Omega_0 \equiv \varepsilon^{-1}\Omega_\varepsilon$	Logarithmic Capacitance d
circle, radius a	$d = a$
ellipse, semi-axes a, b	$d = \frac{a+b}{2}$
equilateral triangle, side h	$d = \frac{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)^3 h}{8\pi^2} \approx 0.422h$
isosceles right triangle, short side h	$d = \frac{3^{3/4}\Gamma\left(\frac{1}{4}\right)^2 h}{2^{7/2}\pi^{3/2}} \approx 0.476h$
square, side h	$d = \frac{\Gamma\left(\frac{1}{4}\right)^2 h}{4\pi^{3/2}} \approx 0.5902h$

$u = \phi_0 + \nu u_1 + \cdots$. Upon substituting these expansions into (1.1) we obtain

$$\begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 & \text{for } x \in \Omega \backslash \{x_0\} \\ u_1 = 0 & \text{for } x \in \partial\Omega \\ \int_\Omega u_1 \phi_0 \, dx = 0 \,. \end{cases} \tag{1.3}$$

In addition, u_1 is to satisfy some singularity condition as $x \to x_0$ that will be determined after constructing the inner expansion and then matching the inner and outer expansions.

In the inner region near the hole, we let $y = \varepsilon^{-1}(x - x_0)$, and we expand $u = \nu(\epsilon) v_0(y) + \cdots$, where $\Delta_y v_0 = 0$. We want $v_0(y) \sim A_0 \log|y|$, as $|y| \to \infty$, and so we write $v_0(y) = A_0 v_c(y)$, where $v_c(y)$ satisfies the canonical inner problem

$$\begin{cases} \Delta_y v_c = 0 & \text{for } y \notin \Omega_0 \\ v_c = 0 & \text{for } y \in \partial\Omega_0 \\ v_c \sim \log|y| & \text{as } |y| \to \infty \,. \end{cases} \tag{1.4 a}$$

The problem (1.4 a) has a unique solution for $v_c(y)$, with the more refined far-field behavior

$$v_c(y) \sim \log|y| - \log d + \mathcal{O}(|y|^{-1}) \,, \quad \text{as} \quad |y| \to \infty \,. \tag{1.4 b}$$

Here d is a constant determined by the solution, and is called the "logarithmic capacitance" of Ω_0.

Notice that, in contrast to the 3-D case, we require that $u \ll \mathcal{O}(1)$ in the inner region. This **key point** results from the simple fact that for a prescribed value $C \neq 0$ there is no solution w to the following problem:

$$\Delta_y w = 0 \,, \quad y \notin \Omega_0 \,,$$

$$w = 0 \,, \quad y \in \partial\Omega_0 \,; \qquad w \sim C \,, \quad \text{as } |y| \to \infty \,.$$

Therefore, we cannot simply impose in the inner region that $v \sim w + o(1)$ with $w \to \phi_0(x_0)$ as $|y| \to \infty$.

The logarithmic capacitance d depends on the shape of Ω_0 and not its orientation within the domain. A table of numerical values for d for different shapes of Ω_0 are given in [**5**], and some of these are reproduced in Table 1. A boundary integral method to compute d for arbitrarily-shaped domains Ω_1 is described and implemented in [**2**]. We observe that if we map (1.4) conformally to the unit disk, the constant d can be determined by the far-field behavior of the mapping.

Next, we write inner expansion in terms of outer variables as

$$u \sim \nu(\epsilon) A_0 [\log |y| - \log d] \sim \nu(\epsilon) A_0 [-\log(\epsilon d) + \log |x - x_0|] \,,$$

so that the matching condition becomes

$$\phi_0(x_0) + \cdots + \nu(\epsilon) u_1 \sim (-\log(\epsilon d))\, A_0 \nu(\epsilon) + A_0 \nu(\epsilon) \log |x - x_0| + \cdots \,.$$

Therefore, we must take $\nu(\epsilon) = \frac{-1}{\log(\epsilon d)}$ and choose $A_0 = \phi_0(x_0)$. In addition, the matching condition gives the singularity condition $u_1(x) \to A_0 \log |x - x_0| = \phi_0(x_0) \log |x - x_0|$ as $x \to x_0$. Therefore, (1.3) becomes

$$\begin{cases} \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 & \text{for } x \in \Omega \backslash \{x_0\} \\ u_1 = 0 & \text{for } x \in \partial\Omega \\ u_1 \sim \phi_0(x_0) \log |x - x_0| & \text{as } x \to x_0 \\ \int_\Omega u_1 \phi_0 \, dx = 0 \,. & \end{cases}$$

This problem is equivalent to

$$\begin{cases} L u_1 := \Delta u_1 + \mu_0 u_1 = -\lambda_1 \phi_0 + 2\pi \phi_0(x_0) \delta(x - x_0) & \text{for } x \in \Omega \backslash \{x_0\} \\ u_1 = 0 & \text{for } x \in \partial\Omega \\ u_1 \sim \phi_0(x_0) \log |x - x_0| & \text{as } x \to x_0 \\ \int_\Omega u_1 \phi_0 \, dx = 0 \,. & \end{cases}$$

We then use Green's second identity

$$\int_\Omega (\phi_0 \partial_n u_1 - u_1 \partial_n \phi_0) \, dS = \int_\Omega (\phi_0 L u_1 - u_1 L \phi_0) \, dx \,,$$

with $\phi_0 = u_1 = 0$ on $\partial\Omega$ and $L\phi_0 = 0$. In this way, we get $\int_\Omega \phi_0 L u_1 \, dx = 0$, which can be written as

$$\int_\Omega \phi_0 \left(-\lambda_1 \phi_0 + 2\pi \phi_0(x_0) \delta(x - x_0) \right) \, dx = 0 \,.$$

This specifies λ_1 as

$$\lambda_1 = \frac{2\pi [\phi_0(x_0)]^2}{\int_\Omega \phi_0^2 \, dx} \,.$$

Therefore, we obtain a two-term expansion for the perturbation of the fundamental eigenvalue given by

$$\lambda \sim \mu_0 + \frac{2\pi \nu [\phi_0(x_0)]^2}{\int_\Omega \phi_0^2 \, dx} + \cdots \,, \qquad \nu = -\frac{1}{\log(\epsilon d)} \,. \tag{1.6}$$

Remarks:

(1) Further terms in the expansion have the form

$$\lambda \sim \mu_0 + A_1 \nu + A_2 \nu^2 + A_3 \nu^3 + \cdots \,,$$

which is an infinite-logarithmic expansion in powers of ν. Since $(\log(\epsilon d))^{-1}$ decreases only very slowly in ε, it would be preferable to find a method to "sum" the series. Such a method is developed and implemented below for a simple model problem. In particular, is the series convergent when ε is small, or only asymptotic? Our results below indicate that the series is in fact convergent for ε sufficiently small.

(2) If $u = 0$ on $\partial\Omega$ is replaced by $\partial_n u = 0$ on $\partial\Omega$, then $\mu_0 = 0$ and $\phi_0 = \frac{1}{\sqrt{|\Omega|}}$ so that $\int_\Omega \phi_0^2 dx = 1$. This yields the

leading-order result

$$\lambda \sim \frac{2\pi\nu}{|\Omega|}\,, \quad \text{as} \quad \epsilon \to 0\,.$$

Therefore, the leading-order asymptotics is independent of the location of the hole. Further terms in the expansion of the eigenvalue must be obtained to determine the effect of the location of the hole. This is done below for a problem where the leading-order asymptotics gives no information.

For an annular domain, we now confirm our two-term asymptotic result by comparing it with the result obtained by expanding the exact eigenvalue relation for small ε. The calculation below will be a bit tedious as it involves detailed calculations of special functions. The reader can skip this if he/she chooses.

The eigenvalue problem in an annular domain is

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \epsilon < r < 1 \\ u = 0 & \text{on } r = 1 \\ u = 0 & \text{on } r = \epsilon\,. \end{cases}$$

The unperturbed solution is $\phi_0 = J_0(\sqrt{\mu_0} r)$ where $J_0(\sqrt{\mu_0}) = 0$ and $\sqrt{\mu_0} = \tau_0$, with z_0 is the first zero of $J_0(z)$.

Using the perturbation formula we have $v_c(y) = \log|y|$, since $\Delta_y v_c = 0$, $v_c = 0$ on $|y| = 1$, so that $d = 1$. Then, $x_0 = 0$ and $\phi_0(x_0) = J_0(0) = 1$. Therefore, from (1.6), we obtain

$$\lambda \sim \mu_0 + \frac{2\pi\nu}{\int_\Omega \phi_0^2(x)\,dx} \sim \mu_0 + \frac{2\pi\nu}{2\pi \int_0^1 r J_0^2(\sqrt{\mu_0} r)\,dr}\,.$$

We recall the integral identity $\int_0^1 r J_0^2(\sqrt{\mu_0} r)\,dr = \frac{1}{2}(J_0'(\sqrt{\mu_0}))^2$, when $J_0(\sqrt{\mu_0}) = 0$, so that the expression above becomes

$$\lambda \sim \mu_0 + \left(-\frac{1}{\log(\epsilon)}\right)\left(\frac{2}{\left[J_0'(\sqrt{\mu_0})\right]^2}\right) + \cdots\,. \tag{1.7}$$

Now we compare (1.7) with the exact solution. In the class of radially symmetric eigenfunctions, we obtain

$$u = J_0(\sqrt{\lambda} r) - \frac{J_0(\sqrt{\lambda})}{Y_0(\sqrt{\lambda})} Y_0(\sqrt{\lambda} r)\,.$$

Setting $u(\varepsilon) = 0$ gives the eigenvalue relation as

$$J_0(\sqrt{\lambda}) = \frac{J_0(\sqrt{\lambda}\epsilon)}{Y_0(\sqrt{\lambda}\epsilon)} Y_0(\sqrt{\lambda})\,. \tag{1.8}$$

To solve this eigenvalue relation for $\varepsilon \ll 1$, we first recall that

$$J_0(z) \sim 1 + \mathcal{O}(z^2) \qquad Y_0(z) \sim \frac{2}{\pi}[\log(z) - \log 2 + \gamma] + \cdots\,, \qquad \text{as} \quad z \to 0\,,$$

where γ is Euler's constant. Therefore, with $z = \sqrt{\lambda}$ we obtain for $\varepsilon \ll 1$ that (1.8) becomes

$$J_0(z) \sim Y_0(z) \frac{\pi}{2} [\log(\epsilon z) - \log 2 + \gamma]^{-1}\,.$$

To find the root of this expression we expand

$$z = z_0 + \left(\frac{-1}{\log\varepsilon}\right) z_1 + \cdots\,,$$

Here $z_0 = \sqrt{\lambda_0}$ is the first root of $J_0(z_0) = 0$, so that $z_0 = \sqrt{\mu_0}$. Then, we use Taylor series to obtain

$$J_0(z_0) + \left(\frac{-1}{\log \varepsilon}\right) J_0'(z_0) z_1 + \ldots \sim \frac{\pi Y_0(z_0)}{2 \log \epsilon} + \cdots .$$

This yields that $z_1 = -\frac{\pi}{2} \frac{Y_0(z_0)}{J_0'(z_0)}$. Now we write $\sqrt{\lambda} = z = z_0 + \left(-\frac{1}{\log \epsilon}\right) z_1 + \cdots$. Hence, we get $\lambda \sim z_0^2 + \left(-\frac{1}{\log \epsilon}\right) 2 z_0 z_1 + \ldots$, which yields $\lambda \sim \mu_0 + \left(-\frac{1}{\log \epsilon}\right) 2\sqrt{\mu_0} z_1$. In summary, we obtain that

$$\lambda \sim \mu_0 + \left(-\frac{1}{\log \epsilon}\right)\lambda_1 + \cdots , \qquad \lambda_1 = 2\sqrt{\mu_0} z_1 = 2\sqrt{\mu_0}\left(-\frac{\pi}{2}\frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})}\right) = -\pi\sqrt{\mu_0}\frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})} . \tag{1.9}$$

To write this result in a form to compare with the result obtained above from the asymptotic theory, we need an identity that is based on the Wronskian relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} J_0(\sqrt{\lambda} r)\right) Y_0(\sqrt{\lambda} r) - \left(\frac{\mathrm{d}}{\mathrm{d}r} Y_0(\sqrt{\lambda} r)\right) J_0(\sqrt{\lambda} r) = -\frac{2}{\pi r} .$$

Now evaluating this identity at $r = 1$, and setting $\lambda = \mu_0$ where $J_0(\sqrt{\mu_0}) = 0$, we get

$$Y_0(\sqrt{\mu_0}) = \frac{2}{\pi\sqrt{\mu_0} J_0'(\sqrt{\mu_0})} .$$

Substituting this into the result of (1.9) we obtain

$$\lambda_1 = -\pi\sqrt{\mu_0}\left(\frac{Y_0(\sqrt{\mu_0})}{J_0'(\sqrt{\mu_0})}\right) = \frac{2}{(J_0'(\sqrt{\mu_0}))^2} ,$$

which gives the two-term expansion

$$\lambda \sim \mu_0 + \left(-\frac{1}{\log \epsilon}\right)\frac{2}{(J_0'(\sqrt{\mu_0}))^2} + \cdots ,$$

in agreement with the asymptotic result given in (1.7).

In the next section we consider a simple problem to illustrate the methodology used to sum infinite logarithmic expansions for singularly perturbed PDE problems in 2-D domains with holes.

2 Higher Order Theory

In this section, we will extend our leading order theory to the case of K small traps of a common shape in the domain. We consider the following eigenvalue problem in a **2-D domain** with K small holes:

$$\triangle u + \lambda u = 0 , \quad x \in \Omega \backslash \Omega_p ; \qquad \Omega_p \equiv \cup_{j=1}^{K} \Omega_{\varepsilon_j} , \tag{2.1 a}$$

$$\partial_n u = 0 , \quad x \in \partial\Omega ; \qquad \int_{\Omega\backslash\Omega_p} u^2 \, dx = 1 \tag{2.1 b}$$

$$u = 0 , \quad x \in \partial\Omega_{\varepsilon_j} , \quad j = 1, \ldots, K . \tag{2.1 c}$$

We assume that each hole Ω_{ε_j} is centered at $x_j \in \Omega$. Since the holes have a common shape they must have a common logarithmic capacitance $d \equiv d_1 = \ldots = d_K$.

We first derive a two-term expansion for the lowest eigenvalue λ_0 of this problem in the form

$$\lambda_0 \sim \lambda_{00}\nu + \lambda_{01}\nu^2 + O(\nu^3) , \tag{2.2}$$

where λ_{00} and λ_{01} are to be found, and $\nu \equiv -1/\log(\varepsilon d)$.

In the outer region, away from $\mathcal{O}(\varepsilon)$ neighbourhoods of the holes, we expand the outer solution for u as

$$u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots . \tag{2.3}$$

The leading-order term is

$$u_0 = |\Omega|^{-1/2} , \tag{2.4}$$

where $|\Omega|$ is the area of Ω. Upon substituting (2.2) and (2.3) into (2.1 a) and (2.1 b), and collecting powers of ν, we obtain that u_1 satisfies

$$\triangle u_1 = -\lambda_1 u_0 , \quad x \in \Omega \backslash \{x_1, \ldots, x_K\} ; \qquad \int_\Omega u_1 \, dx = 0 , \tag{2.5 a}$$

$$\partial_n u_1 = 0 , \quad x \in \partial\Omega ; \quad u_1 \text{ singular as } x \to x_j , \quad j = 1, \ldots, K , \tag{2.5 b}$$

while u_2 satisfies

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 , \quad x \in \Omega \backslash \{x_1, \ldots, x_K\} ; \qquad \int_\Omega \left(u_1^2 + 2u_0 u_2\right) \, dx = 0 , \tag{2.6 a}$$

$$\partial_n u_2 = 0 , \quad x \in \partial\Omega ; \quad u_2 \text{ singular as } x \to x_j , \quad j = 1, \ldots, K . \tag{2.6 b}$$

Now in the j^{th} inner region we introduce the new variables by

$$y = \varepsilon^{-1}(x - x_j) , \qquad v(y) = u(x_j + \varepsilon y) . \tag{2.7}$$

We then expand the inner solution as

$$v(y) = \nu A_{0j} v_{cj}(y) + \nu^2 A_{1j} v_{cj}(y) + \cdots . \tag{2.8}$$

Upon substituting (2.7) and (2.8) into (2.1 a) and (2.1 c), we obtain that v_{cj} satisfies

$$\triangle_y v_{cj} = 0 , \quad y \notin \Omega_j ; \quad v_{cj} = 0 , \quad y \in \partial\Omega_j , \tag{2.9 a}$$

$$v_{cj}(y) \sim \log|y| - \log d + o(1) , \quad \text{as} \quad |y| \to \infty . \tag{2.9 b}$$

Here $\triangle_y$ is the Laplacian in the y variable, and $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$ is the same shape, up to a rotation. Thus, d is independent of j.

Upon using the far-field form (2.9 b) in (2.8), and writing the resulting expression in outer variables, we get

$$v = A_{0j} + \nu \left[A_{0j} \log|x - x_j| + A_{1j}\right] + \nu^2 \left[A_{1j} \log|x - x_j| + A_{2j}\right] + \cdots . \tag{2.10}$$

The far-field behavior (2.10) must agree with the local behavior of the outer expansion (2.3). Therefore, we obtain that

$$A_{0j} = u_0 = |\Omega|^{-1/2} , \quad j = 1, \ldots K , \tag{2.11 a}$$

$$u_1 \sim u_0 \log|x - x_j| + A_{1j} , \quad \text{as} \quad x \to x_j , \quad j = 1, \ldots, K , \tag{2.11 b}$$

$$u_2 \sim A_{1j} \log|x - x_j| + A_{2j} , \quad \text{as} \quad x \to x_j , \quad j = 1, \ldots, K . \tag{2.11 c}$$

Equations (2.11 b) and (2.11 c) give the required singularity structure for u_1 and u_2 in (2.5) and (2.6), respectively.

The problem for u_1 with singular behavior (2.11 *b*) can be written in terms of the delta function as

$$\triangle u_1 = -\lambda_1 u_0 + 2\pi A_0 \sum_{j=1}^{K} \delta(x - x_j)\,, \quad x \in \Omega\,; \qquad \int_\Omega u_1\, dx = 0\,, \tag{2.12 a}$$

$$\partial_n u_1 = 0\,, \quad x \in \partial\Omega\,. \tag{2.12 b}$$

Upon using the divergence theorem we obtain that $-\lambda_1 u_0 \int_\Omega 1 dx + 2\pi A_0 K = 0$, so that with $u_0 = A_0$ from (2.11 *a*), we get

$$\lambda_1 = \frac{2\pi K}{|\Omega|}\,. \tag{2.13}$$

The solution to (2.12) can be written in terms of the Neumann Green's function as

$$u_1 = -2\pi u_0 \sum_{i=1}^{K} G_N(x; x_i)\,, \tag{2.14}$$

where the Neumann Green's function $G_N(x;\xi)$ satisfies

$$\triangle G_N = \frac{1}{|\Omega|} - \delta(x - \xi)\,, \quad x \in \Omega\,; \quad \partial_n G_N = 0\,, \quad x \in \partial\Omega\,, \tag{2.15 a}$$

$$G_N(x;\xi) \sim -\frac{1}{2\pi} \log|x - \xi| + R_N(\xi) + o(1)\,, \quad \text{as} \quad x \to \xi\,; \quad \int_\Omega G_N(x;\xi)\, dx = 0\,. \tag{2.15 b}$$

The constant $R_N(\xi)$ is the regular part of G_N at the singularity. Since G_N has a zero spatial average, it follows from (2.14) that $\int_\Omega u_1\, dx = 0$, as required in (2.12 *a*).

Next, we expand u_1 as $x \to x_j$. We use the local behavior for G_N, given in (2.15 *b*), to obtain from (2.14) that

$$u_1 \sim u_0 \log|x - x_j| - 2\pi u_0 \left[R_{Nj} + \sum_{\substack{i=1 \\ i \neq j}}^{K} G_{Nij} \right]\,, \quad x \to x_j\,, \tag{2.16}$$

where $G_{Nji} = G_N(x_j; x_i)$ and $R_{Nj} = R_N(x_j)$. Comparing (2.16) and the required singularity behavior (2.11 *b*), we obtain that

$$A_{1j} = -2\pi u_0 \left[R_{Nj} + \sum_{\substack{i=1 \\ i \neq j}}^{K} G_{Nij} \right]\,, \qquad j = 1, \ldots, N\,. \tag{2.17}$$

Next, we write the problem (2.6) in Ω as

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 + 2\pi \sum_{j=1}^{K} A_{1j} \delta(x - x_j)\,, \quad x \in \Omega\,; \quad \partial_n u_2 = 0\,, \quad x \in \partial\Omega\,. \tag{2.18}$$

Since $\int_\Omega u_1\, dx = 0$ and $u_0 = |\Omega|^{-1/2}$, the divergence theorem applied to (2.18) determines λ_2 as $\lambda_2 u_0 |\Omega| = 2\pi \sum_{j=1} A_{1j}$. Finally, we use (2.17) for A_{1j}, we get

$$\lambda_2 = -\frac{4\pi^2}{|\Omega|} p(x_1, \ldots, x_K)\,, \qquad p(x_1, \ldots, x_K) \equiv \sum_{j=1}^{N} \left(R_{Njj} + \sum_{\substack{i=1 \\ i \neq j}}^{K} G_{Nji} \right)\,. \tag{2.19}$$

By combining the leading and next order terms, we obtain the two-term expansion

$$\lambda_0(\varepsilon) \sim \frac{2\pi\nu K}{|\Omega|} - \frac{4\pi^2\nu^2}{|\Omega|} p(x_1, \ldots, x_K) + \cdots\,, \qquad \nu = -1/\log(\varepsilon d)\,. \tag{2.20}$$

Now recall that the first passage time $w(x)$ for Brownian motion in a 2-D domain starting a point $x \in \Omega$ in a domain with K traps, and with constant diffusivity D, satisfies

$$\triangle w = -\frac{1}{D}\,, \quad x \in \Omega \backslash \Omega_p\,; \qquad \Omega_p \equiv \cup_{j=1}^{K} \Omega_{\varepsilon_j}\,, \tag{2.21 a}$$

$$\partial_n w = 0\,, \quad x \in \partial\Omega\,; \qquad w = 0\,, \quad x \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1, \ldots, K\,. \tag{2.21 b}$$

From our result for λ_0 above, we will calculate a two-term asymptotic expansion for the average mean first passage time, defined by $\bar{w} = |\Omega\backslash\Omega_p|^{-1}\int_{\Omega\backslash\Omega_p} w\,dx$. Here $|\Omega\backslash\Omega_p| \sim |\Omega| + O(\varepsilon^2)$ denotes the area of the domain with the holes removed.

Let ϕ_j, λ_j be the eigenpairs of (2.1) for $j = 0, 1, 2\ldots$ ordered by $\lambda_0 < \lambda_1 < \lambda_2$.... We calculated an asymptotic expansion for the lowest eigenpair λ_0 and ϕ_0 above. We will normalize the eigenpairs by $\int_{\Omega\backslash\Omega_p} \phi_j^2\,dx = 1$, and we know that the eigenfunctions are orthogonal in the sense that $\int_\Omega \phi_j\phi_k\,dx = 0$ for $j \neq k$. We then expand the solution w of (2.21) in terms of ϕ_j as $w = \sum_{j=0} c_j\phi_j$. By orthogonality, we obtain that

$$c_j = \int_{\Omega\backslash\Omega_p} w\phi_j\,dx \tag{2.22}$$

Next, we multiply the equation in (2.21) by ϕ_j and use Green's second identity to obtain

$$\int_{\Omega\backslash\Omega_p} \phi_j \triangle w\,dx - \int_{\Omega\backslash\Omega_p} w\triangle\phi_j\,dx = 0$$
$$-\frac{1}{D}\int_{\Omega\backslash\Omega_p} \phi_j\,dx + \lambda_j \int_{\Omega\backslash\Omega_p} \phi_j w\,dx = 0$$

Thus, $c_j = (D\lambda_j)^{-1}\int_{\Omega\backslash\Omega_p} \phi_j\,dx$, so that from (2.22) we get

$$w = \frac{1}{D}\sum_{j=0}^{\infty} \frac{\phi_j}{\lambda_j}\int_{\Omega\backslash\Omega_p} \phi_j\,dx\,.$$

Now we calculate $\bar{w}$ to get

$$\bar{w} = \frac{1}{|\Omega\backslash\Omega_p|}\int_{|\Omega\backslash\Omega_p|} w\,dx = \frac{1}{D}\sum_{j=0}^{\infty}\frac{1}{\lambda_j}\left(\int_{\Omega\backslash\Omega_p} \phi_j\,dx\right)^2$$

Finally, we notice that $\lambda_0 \to 0$ as $\varepsilon \to 0$ and that $\int_{Omega\backslash\Omega_p} \phi_j\,dx \to 0$ as $\varepsilon \to 0$ for $j \geq 1$. This follows since for $\varepsilon \to 0$ the first eigenfunction satisfies $\phi_0 \sim |\Omega|^{-1/2}$, and the orthogonality of eigenfunction property holds. Thus only the $j = 0$ term above is retained, and with $\phi_0 \sim |\Omega|^{-1/2}$, we calculate

$$\bar{w} \sim \frac{1}{\lambda_0 D|\Omega|}\left(\int_\Omega |\Omega|^{-1/2}\,dx\right)^2 = \frac{1}{D\lambda_0}$$

Finally, we use our two-term estimate for λ_0 as given above in (2.20) to get the two-term expansion for the average mean first time

$$\bar{w} \sim \frac{|\Omega|}{2\pi\nu K D} + \frac{|\Omega| p(x_1, \ldots, x_K)}{K^2 D} + \cdots\,, \qquad \nu = -1/\log\varepsilon\,. \tag{2.23}$$

If we want to minimize $\bar{w}$ we must choose the trap locations to minimize $p(x_1, \ldots, x_L)$. When Ω is the unit disk, the Green's matrix can be derived explicitly since the Neumann Green's function is known. The problem of minimizing

p over choices of the trap locations is studied in [**3**]. Is an open problem to characterize the spatial configuration of the trap set that minimizes p when N is large. Does the optimizing pattern tend to a hexagonal structure as N increaes?

2.1 Small Application to Ecology

We now show how this result for λ_0 above immediately applies to determining a critical value of the diffusivity D for the extinction threshold of a population satisfying the diffuse logistic model

$$U_t = D\triangle U + \mu U\left(1 - U/\beta\right) \tag{2.24}$$

in a 2-D domain with reflecting outer boundary, and with localized regions where the population is extinct. Here μ and β are positive constants.

To non-dimensionalize this problem, assume that the localized patches of extinction, referred to as patches, have radius σ and that the length scale of the domain is L. If we assume that $\sigma \ll L$, then we define $\varepsilon = \sigma/L$. We scale U by the saturation constant $u = \beta U$, and obtain under steady-state conditions the nonlinear eigenvalue problem

$$\triangle u + \lambda u(1-u) = 0\,, \quad x \in \Omega\backslash\Omega_p\,; \qquad \Omega_p \equiv \cup_{j=1}^{K}\Omega_{\varepsilon_j}\,, \tag{2.25 a}$$

$$\partial_n u = 0\,, \quad x \in \partial\Omega\,; \qquad u = 0\,, \quad x \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1,\ldots,K\,. \tag{2.25 b}$$

Here $\lambda \equiv L^2\mu/D$ is a dimensionless parameter. Notice that $u = 0$ is a solution for all values of λ. This is the extinct fish solution. We want to know at what **minimum value** of λ will a branch of nontrivial solutions bifurcate from the $u = 0$ solution. Linearizing around $u = 0$, the local bifurcating branch is at the first eigenvalue $\lambda = \lambda_0$ of the Laplacian problem (2.1). Thus

$$\frac{L^2\mu}{D} = \lambda_0(\varepsilon) \sim \frac{2\pi\nu K}{|\Omega|} - \frac{4\pi^2\nu^2}{|\Omega|}p(x_1,\ldots,x_K) + \cdots\,, \qquad \nu = -1/\log(\varepsilon d)\,, \tag{2.26}$$

would give a threshold value of D for a bifurcating solution branch.

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