Strong Localized Perturbations: Theory and Applications

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1 Strong Localized Perturbations in 2-D Domains

In this section we continue our analysis of steady-state elliptic problems in a two-dimensional domain with multiple small traps.

1 Summing the Infinite Logarithmic Expansion: A Simple Model Problem

We first consider a simple problem to illustrate some main ideas for treating elliptic PDE problems with infinite logarithmic expansions. Consider a two-dimensional bounded domain Ω with a small trap Ω_ε of radius $\mathcal{O}(\varepsilon)$ centered at some $x_0 \in \Omega$. We assume that the boundary of the domain is also trapping. Then, the expected time $w(x)$ for a Brownian particle to be captured given that it starts from $x \in \Omega \backslash \Omega_\varepsilon$ satisfies Poisson's equation with a Dirichlet condition

$$\triangle w = -\beta \,, \quad x \in \Omega \backslash \Omega_\varepsilon \,, \tag{1.1 a}$$

$$w = 0 \,, \quad x \in \partial\Omega \,, \tag{1.1 b}$$

$$w = 0 \,, \quad x \in \partial\Omega_\varepsilon \,. \tag{1.1 c}$$

where $\beta = 1/D$ and D is the constant diffusivity. We assume that Ω_ε has radius $\mathcal{O}(\varepsilon)$ and that $\Omega_\varepsilon \to x_0$ uniformly as $\varepsilon \to 0$, where $x_0 \in \Omega$. We denote the scaled subdomain that results from an $\mathcal{O}(\varepsilon^{-1})$ magnification of the length scale of Ω_ε by $\Omega_1 \equiv \varepsilon^{-1}\Omega_\varepsilon$. Observe, that in contrast to the problems considered in 3-D, we first suppose that the outer boundary on $\partial\Omega$ is also absorbing.

The asymptotic solution to (1.1) is constructed in two different regions: an outer region defined at an $\mathcal{O}(1)$ distance from the localized trap, and an inner region defined in an $\mathcal{O}(\varepsilon)$ neighborhood of the trap Ω_ε. The analysis below will show how to calculate the sum of all the logarithmic terms for w in in the limit $\varepsilon \to 0$ of small core radius.

In the outer region we expand the solution to (1.1) as

$$w(x;\varepsilon) = W_0(x;\nu) + \sigma(\varepsilon) W_1(x;\nu) + \cdots \,. \tag{1.2}$$

Here $\nu = \mathcal{O}(1/\log\varepsilon)$ is a gauge function to be chosen, and we assume that $\sigma \ll \nu^k$ for any $k > 0$ as $\varepsilon \to 0$. Thus, W_0 contains all of the logarithmic terms in the expansion. Substituting (1.2) into (1.1 *a*) and (1.1 *b*), and letting $\Omega_\varepsilon \to x_0$

as $\varepsilon \to 0$, we get that W_0 satisfies

$$\triangle W_0 = -\beta\,, \quad x \in \Omega \backslash \{x_0\}\,, \tag{1.3 a}$$

$$W_0 = 0\,, \quad x \in \partial\Omega\,, \tag{1.3 b}$$

$$W_0 \quad \text{is singular as} \quad x \to x_0\,. \tag{1.3 c}$$

The matching of the outer and inner expansions will determine a singularity behavior for W_0 as $x \to x_0$.

In the inner region near Ω_ε we introduce the inner variables

$$y = \varepsilon^{-1}(x - x_0)\,, \qquad v(y;\varepsilon) = W(x_0 + \varepsilon y; \varepsilon)\,. \tag{1.4}$$

If we naively assume that $v = \mathcal{O}(1)$ in the inner region, we obtain the leading-order problem for v that $\triangle_y v = 0$ outside Ω_1, with $v = 0$ on $\partial\Omega_1$ and $v \to W_0(x_0)$ as $|y| \to \infty$, where $\triangle_y$ denotes the Laplacian in the y variable. This far-field condition as $|y| \to \infty$ is obtained by matching v to the outer solution. However, in two-dimensions there is no solution to this problem since the Green's function for the Laplacian grows logarithmically at infinity. To overcome this difficulty, we require that $v = \mathcal{O}(\nu)$ in the inner region and we allow v to be logarithmically unbounded as $|y| \to \infty$. Therefore, we expand v as

$$v(y;\varepsilon) = V_0(y;\nu) + \mu_0(\varepsilon)V_1(y) + \cdots\,, \tag{1.5 a}$$

where we write V_0 in the form

$$V_0(y;\nu) = \nu\gamma v_c(y)\,. \tag{1.5 b}$$

Here $\gamma = \gamma(\nu)$ is a constant to be determined with $\gamma = \mathcal{O}(1)$ as $\nu \to 0$, and we assume that $\mu_0 \ll \nu^k$ for any $k > 0$ as $\varepsilon \to 0$. Substituting (1.4) and (1.5) into (1.1 a) and (1.1 c), and allowing $v_c(y)$ to grow logarithmically at infinity, we obtain that $v_c(y)$ satisfies

$$\triangle_y v_c = 0\,, \quad y \notin \Omega_1\,; \qquad v_c = 0\,, \quad y \in \partial\Omega_1\,, \tag{1.6 a}$$

$$v_c \sim \log|y|\,, \quad \text{as} \quad |y| \to \infty\,. \tag{1.6 b}$$

The unique solution to (1.6) has the following far-field asymptotic behavior:

$$v_c(y) \sim \log|y| - \log d + \frac{p \cdot y}{|y|^2} + \cdots\,, \quad \text{as} \quad |y| \to \infty\,. \tag{1.6 c}$$

The constant $d > 0$ is the logarithmic capacitance of Ω_1, while the vector p is called the dipole vector.

The leading-order matching condition between the inner and outer solutions will determine the constant γ in (1.5 b). Upon writing (1.6 c) in outer variables and substituting into (1.5 b), we get the far-field behavior

$$v(y;\varepsilon) \sim \gamma\nu\left[\log|x - x_0| - \log(\varepsilon d)\right] + \cdots\,, \quad \text{as} \quad |y| \to \infty\,. \tag{1.7}$$

Choosing

$$\nu(\varepsilon) = -1/\log(\varepsilon d)\,, \tag{1.8}$$

and matching (1.7) to the outer expansion (1.2) for W, we obtain the singularity condition for W_0,

$$W_0 = \gamma + \gamma\nu \log|x - x_0| + o(1) \,, \quad \text{as} \quad x \to x_0 \,. \tag{1.9}$$

The singularity behavior in (1.9) specifies both the regular and singular part of a Coulomb singularity. As such, it provides one constraint for the determination of γ. More specifically, the solution to (1.3) together with (1.9) must determine γ, since for a singularity condition of the form $W_0 \sim S \log|x - x_0| + R$ for an elliptic equation, the constant R is not arbitrary but is determined as a function of S, x_0, and Ω.

The solution for W_0 is decomposed as

$$W_0(x;\nu) = W_{0H}(x) - 2\pi\gamma\nu G_d(x;x_0) \,. \tag{1.10}$$

Here $W_{0H}(x)$ is the smooth function satisfying the unperturbed problem

$$\triangle W_{0H} = -\beta \,, \quad x \in \Omega \,; \qquad W_{0H} = 0 \,, \quad x \in \partial\Omega \,. \tag{1.11}$$

In (1.10), $G_d(x;x_0)$ is the Dirichlet Green's function satisfying

$$\triangle G_d = -\delta(x - x_0) \,, \quad x \in \Omega \,; \qquad G_d = 0 \,, \quad x \in \partial\Omega \,, \tag{1.12 a}$$

$$G_d(x;x_0) = -\frac{1}{2\pi} \log|x - x_0| + R_d(x_0;x_0) + o(1) \,, \quad \text{as} \quad x \to x_0 \,. \tag{1.12 b}$$

Here $R_{d00} \equiv R_d(x_0;x_0)$ is the regular part of the Dirichlet Green's function $G_d(x;x_0)$ at $x = x_0$. This regular part is also known as either the self-interaction term or the Robin constant (cf. [**1**]).

Upon substituting (1.12 *b*) into (1.10) and letting $x \to x_0$, we compare the resulting expression with (1.9) to obtain that γ is given by

$$\gamma = \frac{W_{0H}(x_0)}{1 + 2\pi\nu R_{d00}} \,. \tag{1.13}$$

Therefore, for this problem, γ is determined as the sum of a geometric series in ν. The range of validity of (1.13) is limited to values of ε for which $2\pi\nu|R_{d00}| < 1$. This yields,

$$0 < \varepsilon < \varepsilon_c \,, \qquad \varepsilon_c \equiv \frac{1}{d} \exp\left[2\pi R_{d00}\right] \,. \tag{1.14}$$

We summarize our result as follows:

Principal Result 3.1: For $\varepsilon \ll 1$, the outer expansion for (1.1) is

$$w \sim W_0(x;\nu) = W_{0H}(x) - \frac{2\pi\nu W_{0H}(x_0)}{1 + 2\pi\nu R_{d00}} G_d(x;x_0) \,, \quad \text{for} \quad |x - x_0| = \mathcal{O}(1) \,, \tag{1.15 a}$$

and the inner expansion with $y = \varepsilon^{-1}(x - x_0)$ is

$$w \sim V_0(y;\nu) = \frac{\nu W_{0H}(x_0)}{1 + 2\pi\nu R_{d00}} v_c(y) \,, \quad \text{for} \quad |x - x_0| = \mathcal{O}(\varepsilon) \,. \tag{1.15 b}$$

Here $\nu = -1/\log(\varepsilon d)$, d is defined in (1.6 c), $v_c(y)$ satisfies (1.6), and W_{0H} satisfies the unperturbed problem (1.11). Also $G_d(x;x_0)$ and $R_{d00} \equiv R_d(x_0;x_0)$ are the Dirichlet Green's function and its regular part satisfying (1.12).

This formulation is referred to as a hybrid asymptotic-numerical method since it uses the asymptotic analysis as a means of reducing the original problem (1.1) with a hole to the simpler asymptotically related problem (1.3) with singularity behavior (1.9). This related problem does not have a boundary layer structure and so is easy to solve numerically. The numerics required for the hybrid problem involve the computation of the unperturbed solution W_{0H} and the Dirichlet Green's function $G_d(x;x_0)$. In terms of G_d we then identify its regular part $R_d(x_0;x_0)$ at the singular point. From the solution to the canonical inner problem (1.6) we then compute the logarithmic capacitance, d. The result (1.15 a) then shows that the asymptotic solution only depends on the product of εd and not on ε itself. This feature allows for an asymptotic equivalence between traps of different cross-sectional shape, based on an effective 'radius' of the trap. This equivalence is known as Kaplun's equivalence principle (cf. [**2**], [**5**]).

An advantage of the hybrid method over the traditional method of matched asymptotic expansions is that the hybrid formulation is able to sum the infinite logarithmic series and thereby provide an accurate approximate solution. From another viewpoint, the hybrid problem is much easier to solve numerically than the full singularly perturbed problem (1.1). For the hybrid method a change of the shape of Ω_1 requires us to only re-calculate the constant d. This simplification does not occur in a full numerical approach. An explicit example comparing the result of the hybrid method with a full numerical solution is given in [**8**].

We now outline how Principal Result 3.1 can be obtained by a direct summation of a conventional infinite-order logarithmic expansion for the outer solution given in the form

$$W \sim W_{0H}(x) + \sum_{j=1}^{\infty} \nu^j W_{0j}(x) + \mu_0(\varepsilon) W_1 + \cdots , \qquad (1.16)$$

with $\mu_0(\varepsilon) \ll \nu^k$ for any $k > 0$. By formulating a similar series for the inner solution, we will derive a recursive set of problems for the W_{0j} for $j \geq 0$ from the asymptotic matching of the inner and outer solutions. We will then sum this series to re-derive the result in Principal Result 3.1.

In the outer region we expand the solution to (1.1) as in (1.16). In (1.16), $\nu = \mathcal{O}(1/\log\varepsilon)$ is a gauge function to be chosen, while the smooth function W_{0H} satisfies the unperturbed problem (1.11) in the unperturbed domain. By substituting (1.16) into (1.1 a) and (1.1 b), and letting $\Omega_\varepsilon \to x_0$ as $\varepsilon \to 0$, we get that W_{0j} for $j \geq 1$ satisfies

$$\triangle W_{0j} = 0\,, \quad x \in \Omega \backslash \{x_0\}\,, \qquad (1.17\,a)$$

$$W_{0j} = 0\,, \quad x \in \partial\Omega\,, \qquad (1.17\,b)$$

$$W_{0j} \quad \text{is singular as} \quad x \to x_0\,. \qquad (1.17\,c)$$

The matching of the outer and inner expansions will determine a singularity behavior for W_{0j} as $x \to x_0$ for each $j \geq 1$.

In the inner region near Ω_ε we introduce the inner variables

$$y = \varepsilon^{-1}(x - x_0)\,, \qquad v(y;\varepsilon) = W(x_0 + \varepsilon y;\varepsilon)\,. \qquad (1.18)$$

We then pose the explicit infinite-order logarithmic inner expansion

$$v(y;\varepsilon) = \sum_{j=0}^{\infty} \gamma_j \nu^{j+1} v_c(y)\,. \qquad (1.19)$$

Here γ_j are ε-independent coefficients to be determined. Substituting (1.19) and (1.1 *a*) and (1.1 *c*), and allowing $v_c(y)$ to grow logarithmically at infinity, we obtain that $v_c(y)$ satisfies (1.6) with far-field behavior (1.6 *c*).

Upon using the far-field behavior (1.6 *c*) in (1.19), and writing the resulting expression in terms of the outer variable $x - x_0 = \varepsilon y$, we obtain that

$$v \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1} \log|x - x_0| + \gamma_j\right] . \tag{1.20}$$

The matching condition between the infinite-order outer expansion (1.16) as $x \to x_0$ and the far-field behavior (1.20) of the inner expansion is that

$$W_{0H}(x_0) + \sum_{j=1}^{\infty} \nu^j W_{0j}(x) \sim \gamma_0 + \sum_{j=1}^{\infty} \nu^j \left[\gamma_{j-1} \log|x - x_0| + \gamma_j\right] . \tag{1.21}$$

The leading-order match yields that

$$\gamma_0 = W_{0H}(x_0) . \tag{1.22}$$

The higher-order matching condition, from (1.21), shows that the solution W_{0j} to (1.17) must have the singularity behavior

$$W_{0j} \sim \gamma_{j-1} \log|x - x_0| + \gamma_j \, , \quad \text{as} \quad x \to x_0 \, . \tag{1.23}$$

The unknown coefficients γ_j for $j \geq 1$, starting with $\gamma_0 = W_{0H}(x_0)$, are determined recursively from the infinite sequence of problems (1.17) and (1.23) for $j \geq 1$. The explicit solution to (1.17) with $W_{0j} \sim \gamma_{j-1} \log|x - x_0|$ as $x \to x_0$ is given explicitly in terms of $G_d(x; x_0)$ of (1.12) as

$$W_{0j}(x) = -2\pi \gamma_{j-1} G_d(x; x_0) . \tag{1.24}$$

Next, we expand (1.24) as $x \to x_0$ and compare it with the required singularity structure (1.23). This yields

$$-2\pi\gamma_{j-1} \left[-\frac{1}{2\pi} \log|x - x_0| + R_{d00} \right] \sim \gamma_{j-1} \log|x - x_0| + \gamma_j \, , \tag{1.25}$$

where $R_{d00} \equiv R_d(x_0; x_0)$. By comparing the non-singular parts of (1.25), we obtain a recursion relation for the γ_j, valid for $j \geq 1$, given by

$$\gamma_j = (-2\pi R_{d00})\, \gamma_{j-1} \, , \qquad \gamma_0 = W_{0H}(x_0) \, , \tag{1.26}$$

which has the explicit solution

$$\gamma_j = \left[-2\pi R_{d00}\right]^j W_{0H}(x_0) \, , \quad j \geq 0 \, . \tag{1.27}$$

Finally, to obtain the outer solution we substitute (1.24) and (1.27) into (1.16) to obtain

$$\begin{aligned} w - W_{0H}(x) &\sim \sum_{j=1}^{\infty} \nu^j \left(-2\pi\gamma_{j-1}\right) G_d(x; x_0) = -2\pi\nu G_d(x; x_0) \sum_{j=0}^{\infty} \nu^j \gamma_j \\ &\sim -2\pi\nu W_{0H}(x_0) G_d(x; x_0) \sum_{j=0}^{\infty} \left[-2\pi\nu R_{d00}\right]^j \\ &\sim -\frac{2\pi\nu W_{0H}(x_0)}{1 + 2\pi\nu R_{d00}} G_d(x_0; x_0) \, . \end{aligned} \tag{1.28 a}$$

Equation (1.28 a) agrees with equation (1.15 a) of Principal Result 3.1. Similarly, upon substituting (1.27) into the infinite-order inner expansion (1.19), we obtain

$$v(y;\varepsilon) = \nu W_{0H}(x_0) v_c(y) \sum_{j=0}^{\infty} [-2\pi R_{d00}\nu]^j = \frac{\nu W_{0H}(x_0)}{1+2\pi\nu R_{d00}} v_c(y)\,, \tag{1.29}$$

which recovers equation (1.15 b) of Principal Result 3.1. This derivation strongly suggests that infinite logarithmic expansions are not just asymptotic, but actually do converge when ε is sufficiently small.

We now consider an explicitly solvable example, for which we can check our asymptotic result in Principal Result 3.1. Consider the 2-D annular region $\epsilon < |x| < 1$. We readily calculate that the exact solution to (1.1) is

$$w = -\frac{\beta}{4}\left(r^2-1\right) - \frac{\beta}{4\log\varepsilon}\log r + \frac{\beta\varepsilon^2}{4\log\varepsilon}\log r\,. \tag{1.30}$$

Now in the outer region, we calculate the quantities that are needed in (1.15 a) of Principal Result 3.1. The solution W_{0H} and G_d to (1.11) and (1.12) is

$$W_{0H} = -\frac{\beta}{4}(r^2-1)\,, \qquad G_{0d}(x;0) = -\frac{1}{2\pi}\log r\,,$$

with $r=|x|$ and $R_d=0$. Thus, since $x_0=0$, we have $W_{0H}(0)=\beta/4$. Since the inner domain is a disk of radius ϵ, then $d=1$. In this way, (1.15 a) becomes in the outer region

$$w \sim W_0 = -\frac{\beta}{4}(r^2-1) - \frac{\beta}{4\log\varepsilon}\log r\,,$$

which agrees with (1.30), apart for the transcendentally small terms of order $\mathcal{O}(\varepsilon^2/\log\varepsilon)$.

2 The Mean First Passage Time in 2-D

In this section we use the method for summing logarithmic expansions to estimate the mean first passage time for a Brownian walker, starting from some point $x\in\Omega\backslash\Omega_a$, to become captured somewhere on an absorbing set Ω_a of small measure. We assume that the absorbing set consists of N non-overlapping regions Ω_{ε_j} centered at x_j for $j=1,\ldots,n$, each of radius $\mathcal{O}(\epsilon)$. It is well-known that the MFPT $u(x)$ satisfies

$$\begin{aligned} &\Delta u = -1/D\,, \quad x\in\Omega\setminus\cup_{j=1}^{N}\Omega_{\varepsilon_j}\,; \qquad \frac{\partial u}{\partial n}=0\,, \quad x\in\partial\Omega\,, \\ &u=0\,, \qquad x\in\partial\Omega_{\varepsilon_j}\,, \quad j=1\ldots,N\,. \end{aligned} \tag{2.1}$$

In the outer region we expand

$$u \sim U_0(x;\boldsymbol{\nu}) + \sigma(\varepsilon)U_1 + \cdots\,. \tag{2.2}$$

Here $\boldsymbol{\nu}\equiv(\nu_1,\ldots,\nu_N)$ is defined in terms of the logarithmic gauge functions $\nu_j\equiv -1/\log(\varepsilon d_j)$ for $j=1,\ldots,N$. As shown below, the constant d_j is obtained from a canonical logarithmic capacitance problem defined near the j-th trap. In the outer expansion, the correction term σ is assumed to satisfy $\sigma\ll\nu_j^k$ for each $j=1,\ldots,N$, and for any positive power k, so that the correction term induced by U_1 is **beyond-all-orders** or **transcendentally small** with respect to all of the logarithmic terms captured by U_0.

Upon substituting (2.2) into (2.1), we obtain that U_0 satisfies

$$\Delta U_0 = -1/D\,, \quad x\in\Omega\setminus\{x_1,\ldots,x_N\}\,; \qquad \partial_n U_0 = 0\,, \quad x\in\partial\Omega\,. \tag{2.3}$$

Since the traps shrink to the points $x_j \in \Omega$ as $\varepsilon \to 0$ for $j = 1, \ldots, N$, this outer problem for U_0 must be supplemented by appropriate singularity conditions as $x \to x_j$, for each $j = 1, \ldots, N$. These singularity conditions are derived below by matching the outer expansion to an inner expansion that is constructed near each of the N traps.

For the inner problem near the j-th trap, we define an inner variable $y = \varepsilon^{-1}(x - x_j)$ and the corresponding magnified trap Ω_j by $\Omega_j = \varepsilon^{-1}\Omega_{\varepsilon j}$. Near the j-th trap, we introduce the inner solution v_j by $v_j(y) = u(x_j + \varepsilon y)$, and we pose the inner expansion

$$v_j \sim \nu_j \gamma_j(\boldsymbol{\nu}) v_{cj}(y) + \alpha(\varepsilon, \boldsymbol{\nu}) v_{1j}(y) + \cdots . \tag{2.4}$$

Here γ_j is an unknown constant to be determined. The gauge function α is assumed to be beyond-all-orders with respect to the logarithmic terms, and so satisfies $\alpha \ll \nu_j^k$ for any positive integer k as $\varepsilon \to 0$, and for each $j = 1, \ldots, N$. We have that $v_{cj}(y)$ satisfies the logarithmic capacitance problem for the j-th trap:

$$\begin{gathered} \Delta_y v_{cj} = 0\,, \quad y \notin \Omega_j\,; \qquad v_{cj} = 0\,, \quad y \in \partial\Omega_j\,, \\ v_{cj} \sim \log|\mathbf{y}| - \log d_j + o(1) \quad \text{as} \quad |y| \to \infty\,. \end{gathered} \tag{2.5}$$

The constant d_j is known as the logarithmic capacitance of Ω_j.

Upon substituting the far-field behavior of v_{cj} as $|y| \to \infty$ into (2.4), and re-writing the result in terms of the outer variable, we obtain from the matching condition that the outer solution U_0 must have the following singularity structure:

$$U_0(x, \boldsymbol{\nu}) \sim \nu_j \gamma_j \log|x - x_j| + \gamma_j\,, \quad \text{as} \quad x \to x_j\,, \qquad j = 1, \ldots, N\,. \tag{2.6}$$

For each $j = 1, \ldots, N$, (2.6) specifies both the regular and singular part of the outer solution. As such, for each $j = 1, \ldots, N$, we have one constraint for the determination of the γ_j for $j = 1, \ldots, N$. Overall, these constraints will lead to a linear algebraic system for the unknown γ_j for $j = 1, \ldots, N$.

The outer problem (2.3) for U_0 can be defined in Ω by introducing singular Dirac delta function forces. We obtain that (2.3) can be re-written as

$$\Delta U_0 = -\frac{1}{D} + 2\pi \sum_{k=1}^{N} \nu_k \gamma_k \delta(x - x_k)\,, \quad x \in \Omega\,; \qquad \partial_n U_0 = 0\,, \quad x \in \partial\Omega\,, \tag{2.7}$$

where U_0 must satisfy the singularity behavior (2.6). By applying the divergence theorem to (2.7), we must have that

$$\sum_{k=1}^{N} \nu_k \gamma_k = \frac{|\Omega|}{2\pi D}\,. \tag{2.8}$$

Next, we write U_0 in terms of a sum of Neumann Green's functions of the form

$$U_0 = -2\pi \sum_{k=1}^{N} \nu_k \gamma_k G(\mathbf{x}; \mathbf{x}_k) + \chi\,, \tag{2.9}$$

where χ is an arbitrary constant to be determined below. Here the Green's function $G(x; \xi)$ is the unique solution to

$$\begin{gathered} \Delta G = \frac{1}{|\Omega|} - \delta(x - \xi)\,, \qquad x \in \Omega\,; \qquad \partial_n G = 0\,, \quad x \in \partial\Omega\,, \\ G \sim -\frac{1}{2\pi} \log|x - x_j| + R(\xi) \quad \text{as} \quad x \to \xi\,; \qquad \int_\Omega G\, dx = 0\,. \end{gathered} \tag{2.10}$$

Finally, in (2.9), we observe that the unknown constant χ is

$$\chi = \frac{1}{|\Omega|} \int_\Omega U_0 \, dx \, . \tag{2.11}$$

Therefore, χ has the interpretation as the asymptotic estimate for the **average** MFPT u_{ave}, where the initial point for the Brownian walk is uniformly distributed in the domain, i.e.

$$u_{\text{ave}} \sim \chi \, . \tag{2.12}$$

To determine the linear algebraic system for γ_j, we expand the solution in (2.9) as $x \to x_j$ and equate the resulting expression with the required singular behavior in (2.6). This leads, for each $j = 1, \ldots, N$, to

$$-2\pi \sum_{k=1, k\neq j}^{N} \nu_k \gamma_k G(x_j, x_k) + \nu_j \gamma_j \log|x - x_j| - 2\pi \gamma_j \nu_j R_j + \chi \sim \nu_j \gamma_j \log|x - x_j| + \gamma_j \, .$$

In this expression, the logarithmic terms in $|x - x_j|$ agree identically (as they should), and from the non-singular terms we obtain a linear algebraic system for the γ_j for $j = 1, \ldots, N$. We summarize our result in the following statement.

Principal Result 3.2: *For $\varepsilon \ll 1$, the asymptotic solution for the MFPT (2.1) in the outer region is given by*

$$u \sim -2\pi \sum_{j=1}^{N} \nu_j \gamma_j G(x; x_j) + \chi \, , \tag{2.13 a}$$

where the γ_j for $j = 1, \ldots, N$ and the constant χ are the solution to the $N+1$ dimensional linear algebraic system

$$\gamma_j + 2\pi \gamma_j \nu_j R_j + +2\pi \sum_{k=1, k\neq j}^{N} \nu_k \gamma_k G(x_j; x_k) = \chi \, , \quad j = 1, \ldots, N \, ; \qquad \sum_{j=1}^{N} \nu_j \gamma_j = \frac{|\Omega|}{2\pi D} \, . \tag{2.13 b}$$

For an arbitrary domain, G and R must be computed numerically. However, when Ω is the unit disk, these quantities are available and are given in [**4**].

The linear system in Principal Result 3.2 is asymptotically diagonally dominant when $\nu_{\max} = \max_j \nu_j$ is sufficiently small, and so is uniquely solvable when $\nu_{\max}$ is small enough. This system incorporates all of the logarithmic gauge functions in the asymptotic solution for the MFPT, leaving an error term that is beyond-all-orders in $(-1/\log(\varepsilon d_j))^k$. This error term, which we do not calculate here, arises from the local gradient behavior of G as $x \to x_j$ as well as from the dipole far-field behavior of the canonical inner solution. An advantage of the hybrid method over the traditional method of matched asymptotic expansions is that the hybrid formulation is able to effectively "sum" an infinite series of logarithmic gauge functions, thereby providing a highly accurate approximate solution.

Finally, we will use (2.13 b) to recover our two-term result from the last lecture on the principal eigenvalue $\lambda_0(\varepsilon)$ of the Neumann eigenvalue problem for the case of identical traps, where $d_j = d$ for all j. Therefore, $\nu_j = \nu = -1/\log(\varepsilon d)$ for $j = 1, \ldots, N$. We recall that u_{ave} is related to $\lambda_0(\varepsilon)$ by (2.12), so that

$$\lambda_0 \sim 1/(D\chi) \, . \tag{2.14}$$

where χ is the solution to (2.13 *b*). For this common trap case, (2.13 *b*) can be written in matrix form as

$$\gamma + 2\pi\nu\mathcal{G}\gamma = \chi\mathbf{e}\,, \qquad \mathbf{e}^T\gamma = \frac{|\Omega|}{2\pi D\nu}\,. \tag{2.15}$$

Here $\gamma = (\gamma_1, \ldots, \gamma_N)$, $\mathbf{e} = (1, \ldots, 1)^T$, and $\mathcal{G}$ is the symmetric Green's matrix with diagonal entries $R(x_j)$ and off-diagonal entries $G(x_i; x_j)$.

For $\nu \to 0$, it is clear that we must expand

$$\gamma = \nu^{-1}\gamma_{\mathbf{0}} + \gamma_{\mathbf{1}} + \cdots\,, \qquad \chi = \nu^{-1}\chi_0 + \chi_1 + \cdots\,,$$

Upon substituting these expansions into (2.15) and equating powers of ν, we get from the $\mathcal{O}(\nu^{-1})$ terms that

$$\gamma_{\mathbf{0}} = \chi_0\mathbf{e}\,, \qquad \mathbf{e}^T\gamma_{\mathbf{0}} = \frac{|\Omega|}{2\pi D}\,,$$

which readily yields

$$\chi_0 = \frac{|\Omega|}{2\pi DN}\,, \qquad \gamma_{\mathbf{0}} = \chi_0\mathbf{e}\,.$$

From the $\mathcal{O}(1)$ terms, we get

$$\gamma_{\mathbf{1}} + 2\pi\mathcal{G}\gamma_{\mathbf{0}} = \chi_1\mathbf{e}\,, \qquad \mathbf{e}^T\gamma_{\mathbf{1}} = 0\,.$$

By taking the dot product of the first expression with $\mathbf{e}^T$, and using the second expression, we get

$$\chi_1 = \frac{2\pi}{N}\mathbf{e}^T\mathcal{G}\gamma_{\mathbf{0}} = \frac{2\pi\chi_0}{N}\mathbf{e}^T\mathcal{G}\mathbf{e}\,.$$

This yields the two-term expansion

$$\chi \sim \frac{\chi_0}{\nu}\left[1 + \frac{2\pi}{N}\nu\mathbf{e}^T\mathcal{G}\mathbf{e}\right]\,.$$

Finally, by substituting this last expression into (2.14), and using $(1+z)^{-1} \sim 1 - z$ for $z \ll 1$, together with the result for χ_0, we recover the estimate

$$\lambda \sim \frac{2\pi N\nu}{|\Omega|}\left(1 - \frac{2\pi\nu}{N}p(x_1, \ldots, x_N)\right)\,, \qquad p(x_1, \ldots, x_N) \equiv \mathbf{e}^T\mathcal{G}\mathbf{e}\,, \tag{2.16}$$

derived in the previous lecture.

Finally, we remark that for the unit disk, optimal configurations of traps, in the context of the two-term result (2.16), were identified in [**4**] for relatively small values of N. In the context of the time needed for a predator to locate prey sites in spatial ecology, the MFPT and related statistical quantities were analyzed recently in [**6**] from the viewpoint of strong localized perturbation theory. In [**7**], the narrow escape problem in which there are no traps, but instead there are small absorbing segments on the boundary of an otherwise reflecting domain boundary was analyzed with a similar methodology. Finally, this method for summing logarithmic expansions originates from [**9**] where linear eigenvalue problems for the Dirichlet boundary conditions with small traps and some nonlinear problems were studied.

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