PATTERN FORMATION

$$(1)\quad \begin{cases} u_t = D_1 \Delta u + \lambda f(u,v) \\ v_t = D_2 \Delta v + \lambda g(u,v) \end{cases}$$

$(\lambda > 0)$ λ CORRESPONDS TO changing DOMAIN length $\lambda \uparrow$ domain is longer.

$u_x = v_x = 0$ AT $x = 0, 1$

NOW ASSUME THAT (u_e, v_e) IS A STEADY STATE SOLUTION.

DEFINE $$J_0 = \begin{pmatrix} f_u^e & f_v^e \\ g_u^e & g_v^e \end{pmatrix}$$

THEN IF $\det J_0 > 0$, $\operatorname{trace} J_0 < 0$ WE HAVE THAT (u_e, v_e) IS STABLE WRT ODE DYNAMICS, OR WRT SPATIALLY HOMOGENEOUS PERTURBATIONS.

NOW FOR (1) WE LINEARIZE

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_e \\ v_e \end{pmatrix} + \begin{pmatrix} \xi \\ \eta \end{pmatrix} e^{\sigma t} e^{iqx}$$

WE OBTAIN THAT

$$(1.5)\quad \sigma \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -D_1 q^2 + \lambda f_u^e & \lambda f_v^e \\ \lambda g_u^e & -D_2 q^2 + \lambda g_v^e \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

THUS WE GET

$$(2)\quad \sigma^2 - [\lambda \operatorname{trace} J_0 - (D_1 + D_2) q^2]\sigma + Q(q) = 0$$

WHERE $Q(q) = D_1 D_2 q^4 - q^2 \lambda [D_1 g_v^e + D_2 f_u^e] + \lambda^2 \det J_0$.

THEREFORE WE CONCLUDE THAT $\mathrm{RE}[\sigma] < 0$ $\forall q$ IFF

(i) $\lambda \operatorname{trace} J_0 - (D_1 + D_2) q^2 < 0$

(ii) $Q(q) > 0$. (NOTICE $\sigma_+ \sigma_- = Q$).

NOTICE THAT SINCE $\operatorname{trace} J_0 < 0$ THEN (i) IS ALWAYS SATISFIED.

THUS $\mathrm{RE}[\sigma] > 0$ FOR SOME q WHEN $Q(q) < 0$.

NOW IT IS CONVENIENT TO INTRODUCE $\hat{\lambda}$ by $\lambda = g^2 \hat{\lambda}$.

THEN $$Q(g) = g^4 \, F(\hat{\lambda})$$

WITH $$F(\hat{\lambda}) = \hat{\lambda}^2 \det J_0 - \hat{\lambda} \left[D_1 g_v^e + D_2 f_u^e \right] + D_1 D_2 .$$

NOW WE FACTOR

$$F(\hat{\lambda}) = \det J_0 \left[(\tilde{\lambda} - \hat{\lambda}^*)^2 + \frac{D_1 D_2}{\det J_0} - \left(\frac{D_1 g_v^e + D_2 f_u^e}{2 \det J_0} \right)^2 \right]$$

WHERE $$\hat{\lambda}^* = \frac{D_1 g_v^e + D_2 f_u^e}{2 \det J_0} .$$

NOW WE HAVE THAT $F(\hat{\lambda}) < 0$ FOR λ ON RANGE

$$\tilde{\lambda}_- \equiv \sqrt{\left(\frac{D_1 g_v^e + D_2 f_u^e}{2 \det J_0} \right)^2 - \frac{D_1 D_2}{\det J_0}} < \tilde{\lambda} - \hat{\lambda}^* < \sqrt{\left(\frac{D_1 g_v^e + D_2 f_u^e}{2 \det J_0} \right)^2 - \frac{D_1 D_2}{\det J_0}} \equiv \tilde{\lambda}_+$$

WHEN $$D_1 g_v^e + D_2 f_u^e > 0$$

AND $$(D_1 g_v^e + D_2 f_u^e)^2 > 4 D_1 D_2 \det J_0 .$$

THUS WE HAVE $f(\hat{\lambda}) < 0$ ON $\tilde{\lambda}_- < \hat{\lambda} - \tilde{\lambda}_* < \hat{\lambda}_+$

WHEN THE CONDITION

$$(3) \quad (D_1 g_v^e + D_2 f_u^e) > 2 \sqrt{D_1 D_2 \det J_0} \quad \text{HOLDS.}$$

RECALLING THAT $Q(g) = g^4 F(\hat{\lambda})$ WE THEN HAVE $Q < 0$

ON $$g^2 (\hat{\lambda}_* + \tilde{\lambda}_-) < \lambda < g^2 (\hat{\lambda}_* + \tilde{\lambda}_+)$$

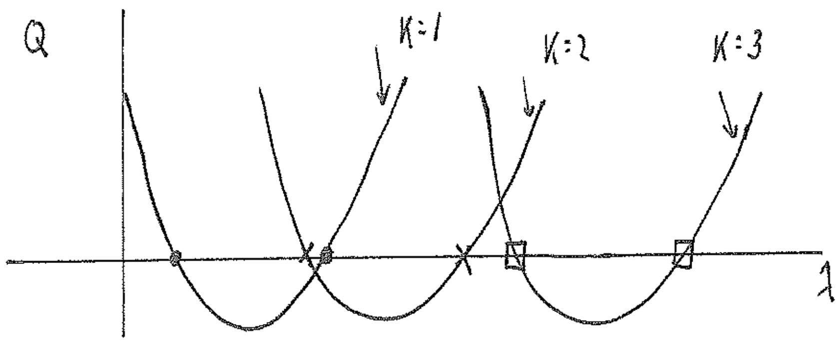
WHEN (3) HOLDS, THEN (1) IS SAID TO BE <u>TURING</u> <u>UNSTABLE</u>.

NOW WITH NEUMANN BOUNDARY CONDITIONS ON A DOMAIN OF LENGTH 1 WE HAVE THAT

$$q = q_K = K\pi \qquad \text{WITH} \qquad K = 1, 2, 3, \ldots$$

WE DO NOT WORRY ABOUT $K=0$ SINCE THIS MODE IS STABLE.

IN THIS WAY WE OBTAIN THE QUALITATIVE PLOT OF Q VERSUS λ FOR $q = q_1$, $q = q_2$, ETC.. UNDER CONDITION (3)



• points are $q_1^2(\hat{\lambda}_* + \hat{\lambda}_\pm)$

x points are $q_2^2(\tilde{\hat{\lambda}}_* + \hat{\lambda}_\pm)$

□ points are $q_3^2(\tilde{\hat{\lambda}}_* + \hat{\lambda}_\pm)$

THUS WE SEE THAT λ IS INCREASED THE MODE WITH $K=1$ i.e. $\cos(\pi x)$ GOES UNSTABLE FIRST. AS $\lambda \uparrow$ THE HIGHER MODES ARE UNSTABLE. NOW ANALYZE WHEN (3) HOLDS

<u>REMARKS</u> (i) NOW IF $D_1 = D_2$ THEN (3) <u>CANNOT</u> HOLD.

<u>PROOF</u> RECALL WE ASSUMED $\text{trace } J_0 < 0 \rightarrow f_u^e + g_v^e < 0$.

THUS (3) IS IMPOSSIBLE WHEN $D_1 = D_2$.

(ii) IF $D_1 \neq D_2$ THEN FOR $\text{trace } J_0 < 0$, $\det J_0 > 0$ AND (3) TO HOLD, WE MUST HAVE

(I) ONE OF f_u^e, g_v^e IS POSITIVE.

(II) J_0 CAN'T BE A SYMMETRIC MATRIX.

PROOF IF $D_1 \neq D_2$ THEN IF BOTH f_u^e, g_v^e WERE NEGATIVE, THEN (3) IS CLEARLY IMPOSSIBLE. THUS AT LEAST ONE OF f_u^e, g_v^e MUST BE POSITIVE. BUT SINCE $\text{trace } J_0 = f_u^e + g_v^e < 0 \rightarrow$ ONLY ONE OF f_u^e, g_v^e IS POSITIVE. TAKING WLOG

$$f_u^e > 0 , \quad g_v^e < 0$$

THEN (3) HOLDS ONLY IF D_2/D_1 IS SUFFICIENTLY LARGE.

THE u IS THE ACTIVATOR ($f_u^e > 0 \rightarrow$ AUTOCATALYSIS TERM) WHILE v IS THE INHIBITOR ($g_v^e < 0 \rightarrow$ INHIBITS ITS OWN PRODUCTION). THE CONDITION D_2/D_1 LARGE ENOUGH MEANS THAT WE MUST HAVE SHORT RANGE ACTIVATION AND LARGE RANGE INHIBITION TO GENERATE A TURING INSTABILITY. THIS PROVES (I).

FOR THE SECOND STATEMENT WE HAVE $\det J_0 = f_u^e g_v^e - f_v^e g_u^e > 0$ IS ASSUMED. BUT SINCE f_u^e, g_v^e HAVE DIFFERENT SIGNS, $f_u^e g_v^e < 0$ WHICH REQUIRES THAT $f_v^e g_u^e < 0$, i.e. f_v^e, g_u^e DIFFERENT SIGNS. THUS

$$J_0 = \begin{pmatrix} f_u^e & f_v^e \\ g_u^e & g_v^e \end{pmatrix}$$ IS NOT A SYMMETRIC MATRIX.

(iii) FOR $f_u^e > 0$, $g_v^e < 0$ AND $D_2/D_1 \gg 1$ THEN (3) REDUCES APPROXIMATELY TO

$$g_v^e + \frac{D_2}{D_1} f_u^e > 2\sqrt{\frac{D_2}{D_1} \det J_0} \implies \frac{D_2}{D_1}\left(f_u^e\right)^2 > 4 \det J_0 .$$

$$\text{OR} \quad D_2/D_1 \gtrsim \frac{4 \det J_0}{\left(f_u^e\right)^2} .$$

FOR OTHER SYSTEMS THE BIFURCATION PARAMETER b MAY APPEAR IN THE KINETICS ONLY

$$u_t = D_1 u_{xx} + f(u, v, b)$$

$$v_t = D_2 v_{xx} + g(u, v, b)$$

POSSIBLY also the spatially homogeneous state can have a HOPF BIFURCATION AFTER RELAXING conditions on f AND g.

<u>EXAMPLE</u> CONSIDER THE GIERER-MEINHARDT MODEL

$$u_t = \varepsilon^2 u_{xx} - u + u^p / v^r \qquad p > 1,\ r > 0,\ m > 0,\ s \geq 0,$$

$$\tau v_t = D v_{xx} - v + u^m / v^s \qquad D > 0,\ \tau > 0.$$

EITHER ON INFINITE LINE OR WITH NEUMANN BC.

THE STEADY-STATE IS $(u_e, v_e) = (1, 1)$.

WE HAVE $f(u,v) = -u + u^p/v^r$, $g(u,v) = \frac{1}{\tau}\left(-v + u^m/v^s\right)$

WE CALCULATE

$$(1)\left\{\begin{array}{ll} f_u^e = p-1 & f_v^e = -r \\ g_u^e = m/\tau & g_v^e = -(1+s)/\tau \end{array}\right.$$

THUS PUT $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{iqx + \sigma t} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$

WE GET UPON LINEARIZING

$$\begin{pmatrix} -\varepsilon^2 q^2 + (p-1) & -r \\ m/\tau & -\dfrac{[Dq^2 + (1+s)]}{\tau} \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \sigma \begin{pmatrix} \zeta \\ \eta \end{pmatrix}$$

set $\det\left((\ldots) - \sigma I \right) = 0$

NOW WE MULTIPLY TO FIND CHARACTERISTIC POLYNOMIAL

$$\begin{pmatrix} -\varepsilon^2 q^2 + (p-1) - \sigma & -\Gamma \\ M/\tau & -\dfrac{[Dq^2 + (1+s)]}{\tau} - \sigma \end{pmatrix} \begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \underset{\sim}{0}$$

$$\sigma^2 - \frac{[Dq^2 + (1+s)]}{\tau}[-\varepsilon^2 q^2 + (p-1)] + \sigma[\varepsilon^2 q^2 - (p-1)] + \sigma\frac{[Dq^2 + (1+s)]}{\tau} + \frac{\Gamma M}{\tau} = 0.$$

DEFINE $J_0 = \begin{pmatrix} p-1 & -\Gamma \\ M/\tau & -(1+s)/\tau \end{pmatrix}$

$$\text{trace } J_0 = p-1 - (1+s)/\tau$$

$$\det J_0 = -(p-1)\frac{(1+s)}{\tau} + \frac{\Gamma M}{\tau}.$$

WE OBTAIN

$$(2) \quad \begin{cases} \sigma^2 - \left[\text{trace } J_0 - \left(\dfrac{D}{\tau} + \varepsilon^2\right) q^2\right]\sigma + Q(q) = 0 \\ \text{WITH} \quad Q(q) = \varepsilon^2 \dfrac{D}{\tau} q^4 - q^2\left[-\varepsilon^2 (1+s)/\tau + D/\tau\,(p-1)\right] + \det J_0 \end{cases}$$

WE ASSUME THAT $\det J_0 > 0$, $\text{trace } J_0 < 0$,

THUS $\det J_0 > 0 \longrightarrow \dfrac{M\Gamma}{p-1} - (s+1) > 0$

$\text{trace } J_0 < 0 \longrightarrow 0 < \tau < \dfrac{(1+s)}{p-1}.$

NOTICE THAT IF WE LABEL

$$\left.\begin{aligned} \mathcal{J} &\equiv \frac{M\Gamma}{p-1} - (s+1) \\ \text{THEN} \quad \det J_0 &= \frac{(p-1)}{\tau}\mathcal{J}. \end{aligned}\right\} \quad (3)$$

WE CONCLUDE THAT $\mathrm{Re}\,\sigma < 0$ IFF

$$\left.\begin{array}{l} \mathrm{trace}\ J_0 - \left(\frac{D}{\tau} + \varepsilon^2\right) q^2 < 0 \\ \text{AND} \qquad Q(q) > 0. \end{array}\right\} \quad (4)$$

SINCE THE FIRST CONDITION ALWAYS HOLDS, WE HAVE INSTABILITY FOR SOME WAVENUMBERS q IFF

$$Q(q) < 0 \quad \text{FOR SOME} \quad q_- < q < q_+.$$

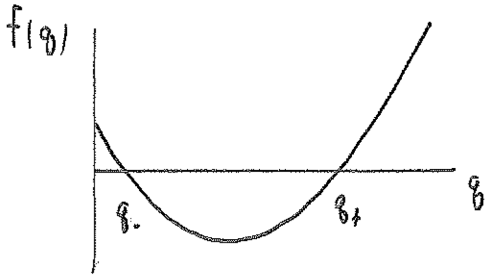
NOW WE CONSIDER

$$Q(q) = \frac{1}{\tau} \quad f(q) \qquad (5)$$

WITH $\quad f(q) \equiv \varepsilon^2 D q^4 - q^2\left[D(p-1) - \varepsilon^2(1+s)\right] + (p-1)s$

WE NOW LOOK FOR ROOTS OF $f(q) = 0$.

NOTE: $\quad f(0) = (p-1)s > 0, \quad f \to +\infty$ AS $q \to \infty$. HOWEVER, IF $\varepsilon < 0$, WE HAVE $f(q) < 0$ FOR $O(1) < q < O(1/\varepsilon)$.



TO FIND THE EDGES OF THE INSTABILITY BAND FIND APPROXIMATE ROOTS OF (5) WHEN $\varepsilon \ll 1$.

<u>UPPER EDGE</u> BALANCE 1ST AND 2ND TERMS IN f IN (5). $\varepsilon^2 q^4 \sim q^2$

THUS PUT $q_+ = \frac{1}{\varepsilon} q_{+_0}$ SO $\quad f \sim \frac{1}{\varepsilon^2}\left[D q_{+_0}^4 - q_{+_0}^2 [D(p-1) \ldots]\right]$

$$\text{SO} \qquad q_{+_0} \simeq \sqrt{p-1}$$

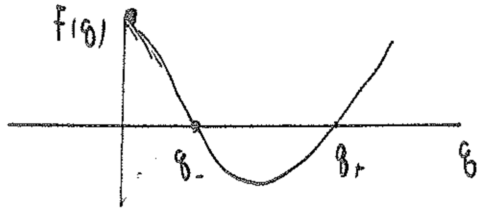
NOW TO FIND LOWER EDGE BALANCE 2ND AND 3RD TERMS IN

f TO OBTAIN

$$-q_-^2 \left[D(p-1) \ldots \right] + (p-1)\gamma + \text{small} = 0.$$

THUS $$q_- = \sqrt{\frac{\gamma}{D}} .$$

WE CONCLUDE THAT IN THE SINGULARLY PERTURBED LIMIT $\varepsilon \to 0$, THE GM MODEL IS TURING UNSTABLE TO A WIDE BAND OF MODES WITH



$$q_- < q < q_+$$

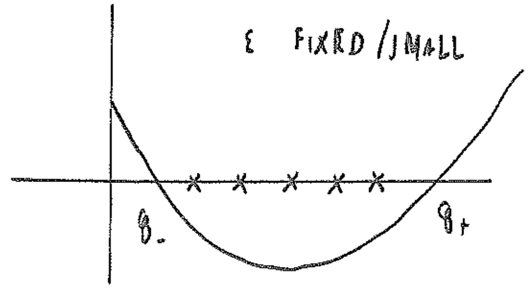
WITH $$q_- \sim \sqrt{\frac{\gamma}{D}} \quad \text{AS} \quad \varepsilon \to 0 \qquad \gamma \equiv \frac{m\Gamma}{p-1} - (s+1)$$

$$q_+ \sim \frac{\sqrt{p-1}}{\varepsilon} \quad \text{AS} \quad \varepsilon \to 0$$

ON A FINITE DOMAIN OF LENGTH 1 WITH NEUMANN BC $u_x = v_x = 0$ AT $x = 0, 1$, WE HAVE

$$q_k = k\pi . \qquad k = 1, 2, 3 \ldots$$

THUS FOR $\varepsilon \to 0$, $\exists$ MANY k FOR WHICH q_k LIES IN THE INTERVAL $q_- < q < q_+$

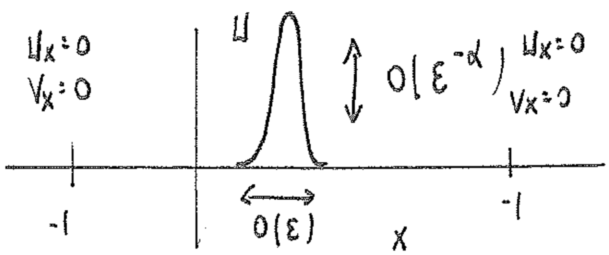


THE QUESTION IS WHAT IS THE "ATTRACTOR" THAT RESULTS FROM THIS WIDE BAND OF UNSTABLE MODES? IT TURNS OUT TO BE LOCALIZED SPIKE SOLUTIONS. FOR INSTANCE CONSIDER

$$U_t = \varepsilon^2 U_{xx} - U + U^3/V^2$$

$$\tau V_t = D V_{xx} - V + U^3 .$$

CAN WE CONSTRUCT A SOLUTION WHERE U IS LOCALIZED IN THE FORM OF SPIKES



WE PUT $U = \bar{U}\varepsilon^{-\alpha}$ AND $V = \bar{V}\varepsilon^{-\beta}$. WE WANT THAT THE U EQUATION IS INVARIANT: THUS $[U^2]/[V^2] = 1$ SO $\alpha = \beta$.

NOW IN V-EQUATION WE INTEGRATE FROM $-1 < x < 1$ TO GET $\int_{-1}^{1} v\,dx = \int_{-1}^{1} u^3\,dx$ BUT SINCE $U = O(\varepsilon^{-\alpha})$ IN A SPIKE OF EXTENT $O(\varepsilon)$ IN X, WE MUST HAVE $-\beta = 1 - 3\alpha$ SO IF $\alpha = \beta \rightarrow \alpha = 1/2,\ \beta = 1/2$.

THUS WE RESCALE $U = \varepsilon^{-1/2}\bar{U}$, $V = \varepsilon^{-1/2}\bar{V}$ TO OBTAIN

$$\varepsilon^{-1/2}\bar{U}_t = \varepsilon^{-1/2}\left(\varepsilon^2 \bar{U}_{xx} - \bar{U} + \bar{U}^3/\bar{V}^2\right)$$

$$\varepsilon^{-1/2}\tau \bar{V}_t = D\varepsilon^{-1/2}\bar{V}_{xx} - \varepsilon^{-1/2}\bar{V} + \varepsilon^{-3/2}\bar{U}^3 .$$

THIS YIELDS THE RESCALED SYSTEM

$$(*)\quad \left\{ \begin{aligned} \bar{U}_t &= \varepsilon^2 \bar{U}_{xx} - \bar{U} + \bar{U}^3/\bar{V}^2 \\ \tau \bar{V}_t &= D\bar{V}_{xx} - \bar{V} + \varepsilon^{-1}\bar{U}^3 \end{aligned} \right.$$

IN INNER REGION FOR STEADY-STATE, WE PUT $y = x/\varepsilon$ AND GET $\bar{V} = \bar{V}_0$

THEN $\bar{U}_{yy} - \bar{U} + \bar{U}^3/\bar{V}_0^2 = 0$

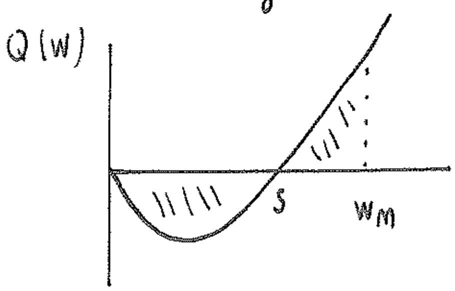
PRELIMINARY RESULTS ON HOMOCLINICS / SPIKES

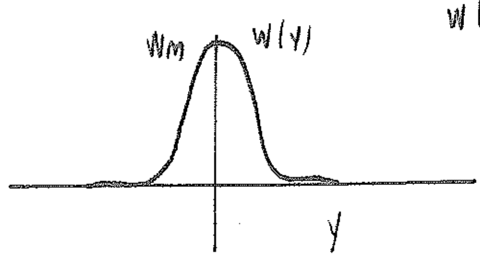
A HOMOCLINIC SOLUTION $w(y)$ EXISTS FOR

$$w'' + Q(w) = 0$$

WHEN $Q(0) = 0$, $Q'(0) < 0$, $Q(s) = 0$ FOR SOME $s > 0$, AND $\exists\ W_m > s$

SUCH THAT $\displaystyle\int_0^{W_m} Q(w)\, dw = 0$ WITH $Q(W_m) > 0$. UNIQUE IF $w'(0) = 0$.
$w(0) = W_m$





A SPECIAL CLASS OF SUCH HOMOCLINICS IS WHERE $Q(w) = -w + w^p$
WITH $p = 1$.

WE MULTIPLY BY w' TO GET $\quad w'^2/2 + V(w) = 0$ ON $0 < y < \infty$.

WITH $\displaystyle V(w) = \int_0^w Q(s)\, ds$. NOW $w'(0) = 0$ IF $w = W_m$.

WE CAN THEN WRITE $\quad w' = -\left[-2V(w)\right]^{1/2}$ WHICH IS SEPARABLE
AND CAN BE INTEGRATED TO GET $w(y)$ IMPLICITLY.

WE NOW FOCUS ON

$$(1)\quad \begin{cases} w'' - w + w^p = 0 & \text{WITH } p > 1. \\ w(0) > 0,\ w'(0) = 0,\ w \to 0 \text{ AS } |y| \to \infty. \end{cases}$$

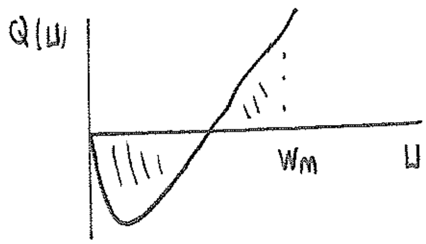
THE EXACT HOMOCLINIC IS

$$(2)\quad w(y) = \left(\frac{p+1}{2}\right)^{1/p-1} \left(\text{SECH}\left[\frac{(p-1)\,y}{2}\right]\right)^{2/p-1}$$

IF $p = 2$ WE HAVE $\quad w = \frac{3}{2}\ \text{sech}^2(y/2)$

IF $p = 3$ $\quad w = \sqrt{2}\ \text{SECH}\ y$.

NOW FOR THE EXISTENCE OF SPIKES WE CAN MODIFY PROPERTIES OF $Q(u)$ SLIGHTLY.



let $Q(u) = u \log u$

SO THAT $Q \to 0$ AS $u \to 0^+$

$Q'(0) \to -\infty$ AS $u \to 0^+$

THE EXACT SOLUTION TO

$$w'' + w \log w = 0, \quad w \to 0 \text{ as } |y| \to \infty$$
$$w(0) = w_m, \quad w'(0) = 0$$

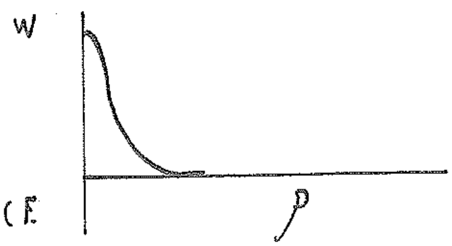
IS $w = e^{1/2} e^{-y^2/4}$ WHICH HAS FASTER THAN SIMPLE EXPONENTIAL DECAY $O(e^{-\sqrt{y}})$ AS $y \to \infty$.

<u>REMARK</u> IN 2-D WE HAVE TO CONSIDER FOR $p > 1$,

$$\Delta_\rho w - w + w^p = 0 \qquad \text{WITH} \qquad \Delta_\rho = w'' + \frac{1}{\rho} w'$$

WE WANT A RADIALLY SYMMETRIC SOLUTION $w(\rho)$ WITH

$$w \to 0 \text{ as } \rho \to \infty \qquad w = O(\rho^{-1/2} e^{-\rho}) \text{ as } \rho \to \infty$$
$$w(0) > 0, \quad w'(0) = 0.$$



WE WILL DISCUSS THE RIGOROUS PROPERTIES REGARDING THE EXISTENCE OF SUCH SOLUTIONS IN 2-D, AND UNIQUENESS.

RATHER CURIOUSLY, $\exists$ EXACT SOLUTION TO

$$\Delta_\rho w + w \log w = 0 \quad \text{IN} \quad \rho > 0$$
$$w'(0) = 0, \quad w(0) > 0, \quad w \to 0 \text{ as } \rho = |y| \to \infty.$$

ONE CAN VERIFY $w = e^{1 - \rho^2/4}$, WHICH IS A GAUSSIAN.

NOW WE RETURN TO 1-D AND CONSIDER SPECTRAL PROPERTIES OF THE LINEARIZED OPERATOR $L_0 \Phi \equiv \Phi'' + Q'(w)\Phi$. NOTICE L_0 IS SELF-ADJOINT

FOR $Q(w) = -w + w^p$ WE HAVE

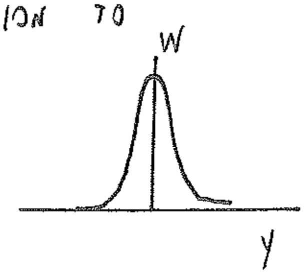
$$(3) \begin{cases} L_0 \Phi \equiv \Phi'' - \Phi + p\,w^{p-1}\Phi = \nu \Phi & \text{ON } -\infty < y < \infty \\ \Phi \to 0 \text{ AS } |y| \to \infty. \end{cases}$$

WHERE $w(y)$ GIVEN IN (2) IS UNIQUE SOLUTION TO

$$w'' - w + w^p = 0, \quad -\infty < y < \infty$$

$$w \to 0 \text{ AS } |y| \to \infty$$

$$w'(0) = 0, \quad w(0) > 0.$$



THE SPECTRUM OF (3) HAS BOTH A DISCRETE AND A CONTINUOUS SPECTRUM.

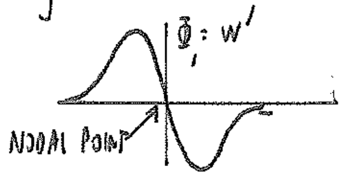
SIMPLE PROPERTIES

(i) CONTINUOUS SPECTRUM IS IN $\nu < -1$ REAL.

(ii) ANY DISCRETE EIGENVALUE MUST BE REAL SINCE L_0 SELF-ADJOINT.

(iii) $L_0 w' = 0$ SO $\nu_1 = 0$ IS Eigenvalue FOR $\Phi_1 = w'$

THIS IS TRANSLATION MODE

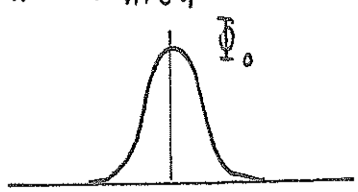


(iv) SINCE $\nu_1 = 0 \to$ THE EIGENFUNCTION HAS Φ_1 HAS ONE NODAL POINT.

SINCE THE PRINCIPAL OR LARGEST EIGENVALUE MUST HAVE NO NODAL POINTS, i.e. IS OF ONE SIGN, AND IS A SIMPLE EIGENVALUE,

THEN $\exists\ \nu_0 > 0$ AND $\Phi_0 > 0$ FOR WHICH

$$L_0 \Phi_0 = \nu_0 \Phi_0$$



i.e. $\exists$ ONE UNSTABLE EIGENVALUE.

IV) WE CONCLUDE THAT THE HOMOCLINIC SOLUTION TO

$$U_t = U_{yy} - U + U^p$$

IS <u>UNSTABLE</u>. i.e. LET W BE HOMOCLINIC. LINEARIZE BY WRITING $U = W + e^{\nu t}\bar{\Phi}(y)$

TO OBTAIN $L_0 \Phi \equiv \Phi'' - \Phi + p W^{p-1}\Phi = \nu \Phi$.

BUT $\exists\ \nu_0 > 0$. $\longrightarrow$ UNSTABLE.

A MORE PRECISE RESULT FOR THE DISCRETE SPECTRUM OF L_0 IS GIVEN IN THE FOLLOWING PROPOSITION

<u>PROPOSITION</u> [DOELMAN ET. AL] LET J BE A POSITIVE INTEGER SATISFYING

$$J < \frac{p+1}{p-1} \leq J+1.$$

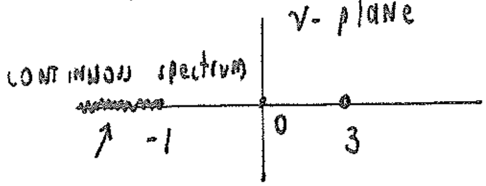
THEN (3) HAS $J+1$ DISCRETE EIGENVALUES ν_j FOR $j = 0,..,J$ GIVEN BY

$$\nu_j = \frac{1}{4}\Big[\,(p+1) - j(p-1)\Big]^2 - 1.$$

THE PROOF OF THIS IS DIFFICULT AND IS BASED ON OBSERVING THAT A DIFFERENTIAL OPERATOR WITH SECH^2 POTENTIAL CAN BE CONVERTED TO A PROBLEM WITH HYPERGEOMETRIC FUNCTIONS.

<u>EXAMPLE</u> LET $p = 3$, THEN $J < 2 \leq J+1$ SO $J = 1$. THUS $\exists$ TWO DISCRETE EIGENVALUES

$$\nu_0 = \frac{1}{4}\big[\,4\,\big]^2 - 1 = 3, \qquad \nu_1 = \frac{1}{4}\big[\,4\,\big] - 1 = 0.$$



$W = \sqrt{2}\,\text{sech}\,y$

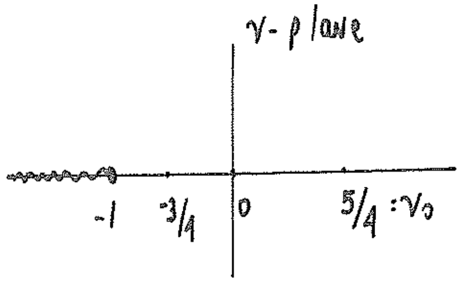
EXAMPLE LET $p=2$ SO THAT $J < 3 \leq J+1$ SO $J=2$.

THUS $\exists$ 3 DISCRETE EIGENVALUES

$$\nu_0 = \frac{1}{4}(3)^2 - 1 = 5/4$$

$$\nu_1 = 0$$

$$\nu_2 = 1/4\,(3-2)^2 - 1 = -3/4$$



$\square$

THE NEXT RESULT GIVES SOME PROPERTIES OF L_0 AND INTEGRALS OF w WHICH WE WILL USE LATER.

LEMMA THE FOLLOWING PROPERTIES HOLD IN 1-D FOR L_0 IN (3).

(i) $L_0 w = (p-1) w^p$

(ii) IF $p=3$ THEN $L_0(w^2) = 3(w^2)$.

(iii) $L_0\left(\dfrac{w}{p-1} + \dfrac{1}{2} y w'\right) = w$

(iv) $\dfrac{\int_{-\infty}^{\infty} w^2\,dy}{\int_{-\infty}^{\infty} w'^2\,dy} = \dfrac{2(p+1)}{(p-1)}$ AND $\dfrac{\int_{-\infty}^{\infty} w^{p+1}\,dy}{\int_{-\infty}^{\infty} w'^2\,dy} = \dfrac{p+3}{p-1}$

PROOF

(i) $L_0 w = (w)'' - (w) + p w^{p-1}(w) = w'' - w + p w^p$ BUT $w'' - w = -w^p$

SO THAT $L_0 w = -w^p + p w^p = (p-1) w^p$.

(ii) IF $p=3$ WE CALCULATE

$$\begin{aligned} L_0(w^2) &= (w^2)'' - (w^2) + 3w^2(w^2) \\ &= (2ww')' - w^2 + 3w^4 \\ &= 2w'^2 + 2ww'' - w^2 + 3w^4. \end{aligned}$$

BUT $w'' = w - w^3$ AND FROM A FIRST INTEGRAL $w'^2/2 = w^2/2 - w^4/4$.

WE SUBSTITUTE

$$L_0(w^2) = 2w\left(w - w^3\right) + 2\left(w^2 - w^4/2\right) - w^2 + 3w^4 = 3w^2. \quad \square$$

WE SHOWED EARLIER THAT $\nu_0 = 3$, THUS $\Phi_0 = w^2 = 2\,\mathrm{sech}^2(y)$.

(iii)

$$L_0\left(\frac{w}{p-1} + \frac{1}{2} y w'\right) = \left(\frac{w}{p-1} + \frac{1}{2} y w'\right)'' - \left(\frac{w}{p-1} + \frac{1}{2} y w'\right) + p w^{p-1}\left(\frac{w}{p-1} + \frac{1}{2} y w'\right)$$

$$= \frac{w''}{p-1} + w'' + \frac{1}{2} y w''' - \frac{w}{p-1} - \frac{1}{2} y w' + \frac{p w^p}{p-1} + \frac{1}{2} p y w^{p-1} w'$$

$$= \frac{p}{p-1}\left(w - w^p\right) + \frac{1}{2} y\left(w' - p w^{p-1} w'\right) - \frac{w}{p-1} - \frac{1}{2} y w' + \frac{p w^p}{p-1} + \frac{1}{2} p y w^{p-1} w'$$

ONCE WE USE $w'' = -w + w^p$ AND $w''' = -w' + p w^{p-1} w'$.

CANCELLING THE UNDERLINED TERMS GIVES

$$L_0\left(\frac{w}{p-1} + \frac{1}{2} y w'\right) = \frac{p}{p-1} w - \frac{w}{p-1} = w. \quad \square$$

(iv) TO CALCULATE THE INTEGRALS

$$w'' - w + w^p = 0.$$

MULTIPLY BY w' AND INTEGRATE

$$w'^2/2 = w^2/2 - w^{p+1}/p+1$$

THIS GIVES $\displaystyle\int_{-\infty}^{\infty} w'^2\, dy = \int_{-\infty}^{\infty} w^2\, dy - \frac{2}{p+1}\int_{-\infty}^{\infty} w^{p+1}\, dy.$

LET $\displaystyle I_1 = \frac{\int_{-\infty}^{\infty} w^2\, dy}{\int_{-\infty}^{\infty} w'^2\, dy} \qquad I_2 = \frac{\int_{-\infty}^{\infty} w^{p+1}\, dy}{\int_{-\infty}^{\infty} w'^2\, dy}.$

THEN $\displaystyle 1 = I_1 - \frac{2}{p+1} I_2. \qquad (*)$

TO GET A SECOND EQUATION MULTIPLY BY w AND USE INTEGRATION BY PARTS

$$\int_{-\infty}^{\infty} w w''\, dy = -\int_{-\infty}^{\infty} w'^2\, dy = \int_{-\infty}^{\infty} w^2\, dy - \int_{-\infty}^{\infty} w^{p+1}\, dy$$

THUS YIELDS $\quad -1 = I_1 - I_2 \quad (**)$

SOLVING $(*)$ AND $(**)$ FOR I_1 AND I_2 WE GET

$$I_2 = \frac{2(p+1)}{(p-1)} \qquad I_1 = \frac{p+3}{p-1}$$

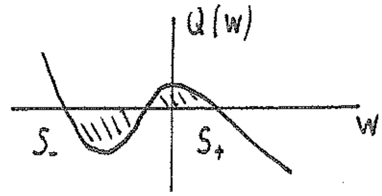
WE NOW BRIEFLY REMARK ON PROPERTIES OF HETEROCLINIC SOLUTIONS

HETEROCLINIC SOLUTIONS

CONSIDER $\quad w'' + Q(w) = 0$

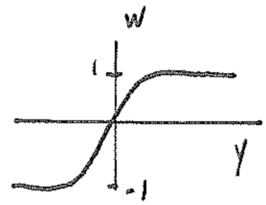
$$w \to S_+ \text{ AS } y \to +\infty, \quad w \to S_- \text{ AS } y \to -\infty, \text{ WITH } S_+ > S_-$$

WE PUT $\quad w'(0) > 0, \quad w(0) = 0$



WE REQUIRE $\quad Q(S_\pm) = 0, \quad Q'(S_\pm) < 0$ AND

$\int_{S_-}^{S_+} Q(w)\, dw = 0$ FOR EXISTENCE OF HETEROCLINIC



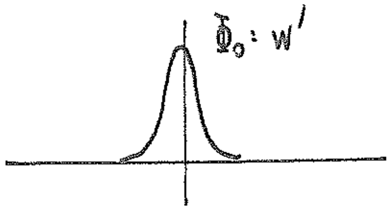
NOW IF $\quad Q(w) = 2(w - w^3)$ THEN $\quad S_\pm = \pm 1$ AND $w = \tanh y$

THE LINEARIZED OPERATOR IS

$$L_0 \Phi = \Phi'' + \Phi - 3w^2 \Phi = \nu \Phi$$

WE CLAIM $\quad \nu_0 = 0, \quad \Phi_0 = w'$ (WHICH HAS NO NODAL POINTS)

IS THE FIRST EIGENPAIR



THUS FOR THE PDE $\quad u_t = u_{yy} + 2(u - u^3)$ THEN $u_e = w(y)$

IS NEUTRALLY STABLE.