EXISTENCE AND STABILITY OF A PULSE OR SPIKE

WE BEGIN WITH

$$(1) \quad \begin{cases} V_t = \varepsilon^2 V_{xx} - V + V^3/U^2, \qquad \tau U_t = D U_{xx} - U + \dfrac{1}{\varepsilon} V^3 \qquad -1 < x < 1 \quad (1 > 0) \\ U_x = V_x = 0 \quad \text{AT} \quad x = \pm 1 \end{cases}$$

WE CONSTRUCT A SPIKE CENTERED AT $x=0$ FOR THE STEADY-STATE PROBLEM

IN THE <u>INNER REGION</u> WE PUT $y = \varepsilon^{-1} x$ AND $\mathcal{U}(y) = U[\varepsilon y]$, $\mathcal{V}(y) = V(\varepsilon y)$

SO THAT FOR (1)

$$(2) \qquad \mathcal{V}'' - \mathcal{V} + \mathcal{V}^3/\mathcal{U}^2 = 0, \quad -\infty < y < \infty$$

$$\frac{D\, \mathcal{U}_{yy}}{\varepsilon^2} - \mathcal{U} + \frac{1}{\varepsilon} \mathcal{V}^3 = 0.$$

TO LEADING ORDER $\mathcal{U}_{yy} = 0$ AND WE TAKE $\mathcal{U} = \mathcal{U}_0$ CONSTANT

THEN LET $\mathcal{V} = \mathcal{U}_0 w$ SO THAT FROM (2)

$$\left.\begin{aligned} & w'' - w + w^3 = 0, \quad -\infty < y < \infty \\ & w(0) > 0, \quad w'(0) = 0, \quad w \to 0 \text{ AS } |y| \to \infty \end{aligned}\right\} (3)$$

WE HAVE $w = \sqrt{2}\, \operatorname{sech} y$.

NOW WE RECALL THAT IF $f(\infty) = 0$ AND $\int_{-\infty}^{\infty} f(y)\, dy$ FINITE, THEN

$$\frac{1}{\varepsilon} f(x/\varepsilon) \xrightarrow{\varepsilon \to 0} \left(\int_{-\infty}^{\infty} f(y)\, dy \right) \delta(x)$$

IN THE SENSE OF DISTRIBUTIONS. THUS,

$$\frac{1}{\varepsilon} V^3 \xrightarrow{\varepsilon \to 0} \frac{1}{\varepsilon} \varepsilon \int_{-\infty}^{\infty} \mathcal{U}_0^3 w^3\, dy \;\; \delta(x) = \mathcal{U}_0^3 \int_{-\infty}^{\infty} w^3\, dy \;\; \delta(x)$$

WE CALCULATE $\displaystyle\int_{-\infty}^{\infty} w^3\, dy = \int_{-\infty}^{\infty} w\, dy = \sqrt{2} \int_{-\infty}^{\infty} \operatorname{sech} y\, dy = \sqrt{2}\, \pi$.

THUS IN THE OUTER REGION WE HAVE $V \sim 0$ AND

$$(4) \quad \begin{cases} D U_{xx} - U = -\mathcal{U}_0^3 \sqrt{2}\, \pi\, \delta(x), \quad -1 < x < 1 \\ U_x = 0 \quad \text{AT} \quad x = \pm 1 \end{cases}$$

WE NOW INTRODUCE THE 1-D GREEN'S FUNCTION $G(x)$ SATISFYING

$$\left.\begin{aligned} D G_{xx} - G &= -\delta(x) \quad \text{in } |x| \le 1 \\ G_x &= 0 \quad \text{at } x = \pm 1. \end{aligned}\right\} \qquad (5)$$

WE RECALL THAT G IS CONTINUOUS ACROSS $x=0$ AND THAT $D G_x\big|_+ - D G_x\big|_- = -1$ WHERE $G_x\big|_\pm = dG/dx\big|_{x=0^\pm}$.

WE CAN CALCULATE $G(x)$ AS

$$\left.\begin{aligned} G(x) &= \frac{\cosh\left[(1-|x|)/\sqrt{D}\right]}{2\sqrt{D}\,\sinh\left[1/\sqrt{D}\right]} \\ \text{AND IN PARTICULAR} \quad G(0) &= \frac{1}{2\sqrt{D}}\coth\left(1/\sqrt{D}\right). \end{aligned}\right\} \qquad (6)$$

THE SOLUTION TO (4) IS SIMPLY

$$u = U_0^3\,\sqrt{2}\,\pi\,G(x). \qquad (7)$$

NOW EVALUATING AT $x=0$ WE OBTAIN THAT $u(0) = U_0$ FROM THE MATCHING CONDITION SO THAT

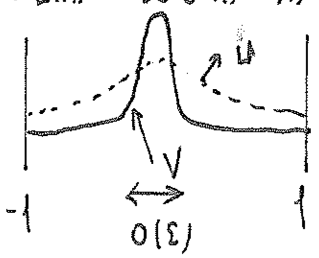
$$U_0 = \sqrt{2}\,\pi\,U_0^3\,G(0)$$

WHICH GIVES
$$\left. U_0 = \frac{1}{\left[G(0)\sqrt{2}\,\pi\right]^{1/2}} = \frac{1}{\left[G(0)\int_{-\infty}^{\infty} w^3\,dy\right]^{1/2}} \right\} \qquad (6.5)$$

IN SUMMARY, OUR CONSTRUCTION OF A 1-SPIKE SOLUTION ON $-1<x<1$ CENTERED AT $x=0$ GIVES

$$\left.\begin{aligned} v &\sim U_0\, w\left[x/\varepsilon\right] \qquad w(y) = \sqrt{2}\,\operatorname{sech} y \\ u &\sim U_0\, G(x)/G(0) \qquad \text{on } -1<x<1 \\ \text{WHERE} \quad U_0 &= \frac{1}{\left[G(0)\sqrt{2}\,\pi\right]^{1/2}} \end{aligned}\right\} \qquad (7)$$

THE PICTURE LOOKS AS FOLLOWS



<u>REMARKS</u> (i) IF YOU WANT N SPIKES ON A DOMAIN $-1 < x < 1$
WE SIMPLY LET $\ell = 1/N$ AND USE TRANSLATION AND
PERIODIC EXTENSION.

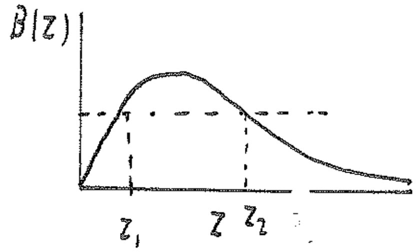
(ii) FOR THE INFINITE LINE PROBLEM WE CAN LET $\ell \to \infty$.

(iii) WE CAN ALSO CONSTRUCT ASYMMETRIC SPIKE PATTERNS AS FOLLOWS.

WE CALCULATE
$$u(\ell) = \mathrm{U}_0 \frac{G(\ell)}{G(0)} = C \, B\left(\ell/\sqrt{D}\right)$$

where $B(z) = \dfrac{[\tanh(z)]^{1/2}}{\cosh(z)}$, AND C IS A CONSTANT DEPENDING ON D.

WE PLOT $B(z)$ AS



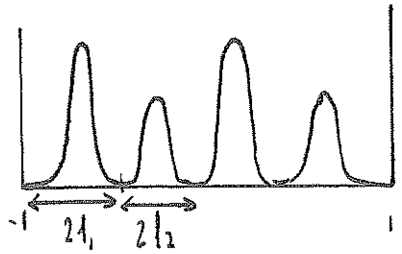
THUS $\exists$ TWO VALUES z_1 AND z_2 FOR WHICH $B(z_1) = B(z_2)$.

THE IMPLICATION IS THAT $\exists$ ℓ_1, ℓ_2 FOR WHICH $u(\ell_1) = u(\ell_2)$.

THUS TO CONSTRUCT A PATTERN WITH N_1 SPIKES OF "TYPE ℓ_1" AND N_2 SPIKES OF "TYPE ℓ_2" WE HAVE FOR $-1 < x < 1$ THAT

$$\left.\begin{aligned} &2N_1\ell_1 + 2N_2\ell_2 = 2 \quad \text{DOMAIN LENGTH} \\ &\text{AND} \quad B\left(\ell_1/\sqrt{D}\right) = B\left(\ell_2/\sqrt{D}\right) \end{aligned}\right\} \quad (8)$$

WHICH IS A COUPLED NONLINEAR ALGEBRAIC SYSTEM FOR ℓ_1 AND ℓ_2 FOR GIVEN $N_1 > 0$, $N_2 > 0$.



FOR FURTHER DETAILS ON THIS CONSTRUCTION SEE WARD, WEI EJAM 2002. $\square$

WE NOW RETURN TO OUR ONE-SPIKE SOLUTION ON $[-1,1]$

NOW WE ANALYZE THE STABILITY OF A 1-SPIKE SOLUTION TO (1) ON $-1<x<1$. WE WRITE

$$V = V_e + e^{\lambda t}\phi \quad , \quad U = U_e + e^{\lambda t}\Lambda$$

WITH V_e, U_e THE EQUILIBRIUM STATE. STABILITY IF $\mathrm{Re}\,\lambda < 0$.

WE OBTAIN THE LINEARIZED PROBLEM

$$\left.\begin{aligned} &\varepsilon^2\phi_{xx} - \phi + \frac{3V_e^2}{U_e^2}\phi - \frac{2V_e^3}{U_e^3}\Lambda = \lambda\phi \quad , \qquad \phi_x = 0 \,, \ x = \pm 1 \\ &D\Lambda_{xx} - (1+\tau\lambda)\Lambda = -\frac{1}{\varepsilon}\,3V_e^2\phi \quad , \qquad \Lambda_x = 0 \,, \ x = \pm 1. \end{aligned}\right\} \quad (9)$$

RECALL THAT THE COEFFICIENTS

V_e^2/U_e^2 AND V_e^3/U_e^3 ARE <u>LOCALIZED</u> NEAR $x=0$.

WE PUT $y = \varepsilon^{-1}x$ WE OBTAIN THAT Λ = CONSTANT TO LEADING ORDER AND $\phi = \Phi(x/\varepsilon)$. WE RECALL $V_e^2/U_e^2 \sim w^2$ AND $\dfrac{V_e^3}{U_e^3} \sim w^3$.

THUS WE OBTAIN

$$\left.\begin{aligned} &\Phi'' - \Phi + 3w^2\Phi - 2w^3\Lambda(0) = \lambda\Phi, \qquad -\infty < y < \infty \\ &\Phi \to 0 \ \text{ AS } \ |y| \to \infty. \end{aligned}\right\} \quad (10)$$

NOW WE WRITE THIS IN TERMS OF THE LOCAL OPERATOR L_0 AS

$$L_0\Phi \equiv \Phi'' - \Phi + 3w^2\Phi \qquad (11)$$

THUS WE HAVE

$$\left.\begin{aligned} L_0 \Phi - 2w^3 \Lambda(0) &= \lambda \Phi \qquad -\infty < y < \infty \\ \Phi &\to 0 \quad \text{AS} \quad |y| \to \infty \end{aligned}\right\} \qquad (12)$$

WE NOW MUST CALCULATE $\Lambda(0)$ FROM OUTER PROBLEM FOR Λ.

IN THE SENSE OF DISTRIBUTIONS WE HAVE

$$\frac{3}{\varepsilon} v_e^2 \phi \underset{\varepsilon \to 0}{\longrightarrow} 3 \bar{w}_0^2 \left(\int_{-\infty}^{\infty} w^2 \Phi \, dy \right) \delta(x).$$

THUS OUR PROBLEM FOR $\Lambda(x)$ IS

$$\left.\begin{aligned} D \Lambda_{xx} - (1+\tau\lambda)\Lambda &= -3 \bar{w}_0^2 \left(\int_{-\infty}^{\infty} w^2 \Phi \, dy \right) \delta(x) \\ \Lambda_x(\pm 1) &= 0 \end{aligned}\right\} \qquad (13)$$

WE DEFINE THE λ-DEPENDENT GREEN'S FUNCTION $G_\lambda(x)$ BY

$$\left.\begin{aligned} D G_{\lambda xx} - (1+\tau\lambda) G_\lambda &= -\delta(x) \qquad \text{IN} \quad -1 < x < 1 \\ G_{\lambda x} &= 0 \quad \text{AT} \quad x = \pm 1 \end{aligned}\right\} \qquad (14)$$

THE SOLUTION TO (13) IS SIMPLY

$$\Lambda(x) = 3 \bar{w}_0^2 \left(\int_{-\infty}^{\infty} w^2 \Phi \, dy \right) G_\lambda(x). \Big\} \qquad (15)$$

AS SUCH IF WE EVALUATE AT $x=0$ WE GET

$$\Lambda(0) = 3 \bar{w}_0^2 \int_{-\infty}^{\infty} w^2 \Phi \, dy \;\; G_\lambda(0).$$

WE RECALL THAT $\bar{w}_0^2 = \dfrac{1}{G(0) \int_{-\infty}^{\infty} w^3 \, dy}$ FROM (6.5).

SO THAT $$\Lambda(0) = 3 \frac{G_\lambda(0)}{G(0)} \frac{\int w^2 \Phi \, dy}{\int w^3 \, dy} \qquad (16)$$

FROM (14) WE CALCULATE $$G_\lambda(0) = \frac{1}{2\sqrt{D(1+\tau\lambda)} \tanh\left[\sqrt{\frac{1+\tau\lambda}{D}}\,\right]} \qquad (17)$$

WE SUBSTITUTE INTO (12) TO OBTAIN OUR <u>NONLOCAL EIGENVALUE PROBLEM</u> NLEP

GIVEN BY

$$\left.\begin{aligned} & L_0 \Phi - \chi(\lambda)\, w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \lambda \Phi , \quad -\infty < y < \infty \\ & \Phi \to 0 \text{ AS } |y| \to \infty \end{aligned}\right\} \quad (18)_1$$

WHERE $\chi(\lambda) \equiv 6\, G_\lambda(0) / G(0)$. WE USE (17) AND (6) FOR

$G(0)$ TO CALCULATE

$$\chi(\lambda) = \frac{6}{\sqrt{1+\tau\lambda}} \, \frac{\tanh[\theta_0]}{\tanh[\theta_\lambda]} \quad \text{WHERE} \quad \theta_0 \equiv 1/\sqrt{D}, \quad \theta_\lambda = \sqrt{(1+\tau\lambda)/D}. \qquad (18)_2$$

<u>REMARKS</u>

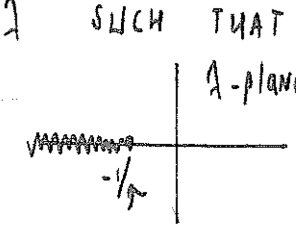
(i) $(18)_1$ WITH $(18)_2$ IS A NON-SELF ADJOINT PROBLEM. RECALL THAT IF L_0 IS SELF-ADJOINT, THEN ON $-1<x<1$, IF

$$L_0 \Phi + b(x) \int_{-1}^{1} a(x) \Phi(x)\, dx = \lambda \Phi$$

IT IS SELF-ADJOINT IFF $b(x) = K\, a(x)$ $\forall K$.

(ii) NOW NOTICE THAT THE MULTIPLIER OF THE NLEP $\chi(\lambda)$ DEPENDS ON THE EIGENVALUE PARAMETER.

(iii) WE MUST TAKE THE BRANCH OF $\sqrt{1+\tau\lambda}$ SUCH THAT $\mathrm{RE}\left[\sqrt{1+\tau\lambda}\right] > 0$, i.e. WITH A CUT AS SHOWN



THIS ENSURES THAT THE SOLUTION $G_\lambda(x)$ FOR THE CASE $D \to 0$ IS DECAYING AT ∞.

(iv) OUR QUESTION IS WHETHER $\exists$ ANY UNSTABLE EIGENVALUES OF (18)?

(V) IF $\chi \equiv 0$, THEN WE RECALL THAT

$$L_0 \Phi = \nu \Phi$$

HAS AN EIGENVALUE $\nu_0 = 3 > 0$ WITH $\Phi_0 = W^2$.

SO OUR QUESTION IS WHETHER THE NLEP CAN "STABILIZE" THIS SITUATION AT LEAST FOR SOME RANGE OF τ.

(Vi) WE NOTICE THAT $\Phi = w' \longleftrightarrow \lambda = 0$ FOR TRANSLATION MODE INDEPENDENT OF χ. THUS WE ARE ONLY INTERESTED IN EIGENFUNCTIONS FOR WHICH $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$.

(Vii) WE WILL SHOW THAT FOR EIGENFUNCTIONS FOR WHICH $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$, WE HAVE

$$\mathrm{RE}(\lambda) < 0 \quad \text{IF} \quad 0 \leq \tau < \tau_H$$

AND THAT A HOPF BIFURCATION OCCURS AT SOME $\tau = \tau_H > 0$.

AN INTERESTING SIMPLE LIMITING CASE IS THE <u>INFINITE LINE PROBLEM</u> WHERE WE REPLACE $X = \pm 1$ WITH $X = \pm \infty$.

THEN WE HAVE IN PLACE OF (6) AND (14) THAT

$$\left\{ \begin{aligned} & D G_{XX} - G = -\delta(X) \\ & G \to 0 \text{ AS } |X| \to \infty \end{aligned} \right. \Longrightarrow \quad G(X) = \frac{1}{2\sqrt{D}} e^{-|X|/\sqrt{D}}$$

$$\left\{ \begin{aligned} & D G_{\lambda XX} - (1+\tau\lambda) G = -\delta(X) \\ & G_\lambda \to 0 \text{ AS } |X| \to \infty \end{aligned} \right. \Longrightarrow \quad G_\lambda(X) = \frac{1}{2\sqrt{D(1+\tau\lambda)}} e^{-\theta_\lambda |X|}$$

WITH $\theta_\lambda = \theta_0 \sqrt{1+\tau\lambda}$ AND $\theta_0 = 1/\sqrt{D}$. THUS

$$\left. \chi = 6\, G_\lambda(0) / G(0) = \frac{6}{\sqrt{1+\tau\lambda}} \right\} \quad (19) \qquad \text{FOR INFINITE LINE PROBLEM.}$$

WE NOW REFORMULATE THE NLEP:

$$\left.L_0 \Phi - \chi w^3 \frac{\int_{-\infty}^{\infty} w^2 \Phi \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} = \lambda \Phi \right\} \quad (20)$$

WE DEFINE $I \equiv \int_{-\infty}^{\infty} w^2 \Phi \, dy \Big/ \int_{-\infty}^{\infty} w^3 \, dy$ SO THAT

$$(L_0 - \lambda) \Phi = \chi w^3 I,$$

WHICH YIELDS $\Phi = \chi I (L_0 - \lambda)^{-1} w^3.$

NOW MULTIPLY BY w^2 AND INTEGRATE

$$\int_{-\infty}^{\infty} w^2 \Phi \, dy = \chi I \int_{-\infty}^{\infty} w^2 (L_0 - \lambda)^{-1} w^3 \, dy.$$

BUT $I = \int_{-\infty}^{\infty} w^2 \Phi \, dy \Big/ \int_{-\infty}^{\infty} w^3 \, dy$ SO THAT IF $\int_{-\infty}^{\infty} w^2 \Phi \, dy \neq 0$,

THEN $1 = \chi \int_{-\infty}^{\infty} w^2 (L_0 - \lambda)^{-1} w^3 \, dy \Big/ \int_{-\infty}^{\infty} w^3 \, dy$.

WE CONCLUDE THAT THE EIGENVALUES λ OF THE NLEP ARE THE ROOTS OF $g(\lambda) = 0$, WHERE

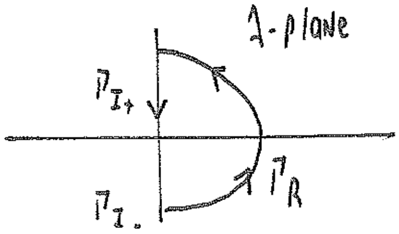
$$\left.\begin{aligned} g(\lambda) &\equiv C(\lambda) - \mathcal{F}(\lambda) \\ \text{WHERE} \quad C(\lambda) &= \frac{1}{\chi(\lambda)} \\ \mathcal{F}(\lambda) &= \frac{\int_{-\infty}^{\infty} w^2 (L_0 - \lambda)^{-1} w^3 \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} \end{aligned}\right\} \quad (21)$$

WE ARE LOOKING FOR ROOTS OF (21) IN $\mathrm{Re}\,\lambda > 0$.

RECALL FOR THE <u>INFINITE-LINE PROBLEM</u> THAT

$$g(\lambda) \equiv \frac{\sqrt{1+\tau\lambda}}{6}.$$

TO ANALYZE THE ROOTS TO $g(\lambda)=0$ WE CAN USE ALL THE TOOLS FROM COMPLEX ANALYSIS, I.E. THE WINDING NUMBER IN CALCULATING THE CHANGE IN THE ARGUMENT OF $g(\lambda)$ OVER A LARGE SEMI-CIRCLE.



$$\Gamma \equiv \Gamma_R \cup \Gamma_{I_+} \cup \Gamma_{I_-}$$

NOTICE THAT $\hat{f}(\lambda)$ HAS A SIMPLE POLE AT $\lambda = \nu_0 = 3$ IN $\mathrm{RE}\,\lambda > 0$ BUT IS OTHERWISE ANALYTIC. BY COMPLEX ANALYSIS

$$\lim_{R\to\infty} [\arg g]\Big|_{\Gamma} = 2\pi(N-P)$$

N: # zeroes of $g(\lambda)$ in $\mathrm{RE}\,\lambda > 0$

P: # poles of $g(\lambda)$ in $\mathrm{RE}\,\lambda > 0$.

WE WANT TO FIND N. WE HAVE $P=1$ SINCE $\hat{f}$ HAS A POLE AT $\lambda = \nu_0 = 3$.

ON Γ_R, WE HAVE SINCE $\hat{f}(\lambda) = O(1/|\lambda|)$ AS $|\lambda|\to\infty$, AND $g \sim \tau^{1/2}\sqrt{\lambda}$ THAT $\lim_{R\to\infty}[\arg g]\Big|_{\Gamma_R} = \frac{\pi}{2}$.

THUS $$\frac{\pi}{2} + \lim_{R\to\infty}[\arg g]\Big|_{\Gamma_{I_+}} + \lim_{R\to\infty}[\arg g]\Big|_{\Gamma_{I_-}} = 2\pi(N-1).$$

SINCE $\overline{g(i\lambda_I)} = g(\overline{i\lambda_I})$ WE HAVE $[\arg g]\Big|_{\Gamma_{I_+}} = [\arg g]\Big|_{\Gamma_{I_-}}$

WHICH YIELDS,

$$\frac{\pi}{2} + 2[\arg g]\Big|_{\Gamma_{I_+}} = 2\pi(N-1)$$

WHICH YIELDS THE KEY FORMULA $$N = 5/4 + \frac{1}{\pi}[\arg g]\Big|_{\Gamma_{I_+}}$$

IN SUMMARY, THE NUMBER OF ROOTS TO $g(\lambda) = 0$ IN $\mathrm{RE}\,\lambda > 0$ IS GIVEN BY

$$N \equiv \frac{5}{4} + \frac{1}{\pi}\left[\arg g\right]\Big|_{\Gamma_{I+}} \qquad (22)$$

WHERE Γ_{I+} IS WRITTEN AS $\lambda = i\lambda_I$ WITH $0 < \lambda_I < \infty$, TRAVERSED IN DOWNWARD DIRECTION.

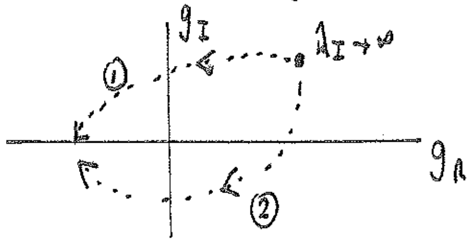
<u>REMARK</u> (i) WE NEED ONLY WRITE $\lambda = i\lambda_I$ AND SEPARATE

$$g(i\lambda_I) = g_R(\lambda_I) + i\, g_I(\lambda_I)$$

AND DETERMINE # WRAPPINGS AROUND ORIGIN IN THE (g_R, g_I) PLANE AS λ_I VARIES ON $0 < \lambda_I < \infty$.

AS $\lambda_I \to \infty$ WE HAVE $g_R \sim e^{\pi i/4}\sqrt{\gamma\lambda_I} \sim g_I$

SO THAT $\arg g \to \pi/4$ AS $\lambda_I \to \infty$



- IF WE HAD PATH 1 AS $\lambda_I \downarrow$ THEN $\left[\arg g\right]\Big|_{\Gamma_{I+}} = 3\pi/4$ SO $N = 2$
- IF WE HAD PATH 2 AS $\lambda_I \downarrow$ THEN $\left[\arg g\right]\Big|_{\Gamma_{I+}} = -5\pi/4$ AND SO $N = 0$.

(ii) THE KEY SIMPLIFICATION IN USING (22) IS THAT WE NEED ONLY LOOK ON IMAGINARY AXIS.

THIS DERIVATION AND REFORMULATION OF THE NLEP (18) TO (21) CAN BE DONE FOR ARBITRARY EXPONENT SETS WITH

$$V_t = \varepsilon^2 V_{xx} - V + V^p / U^q$$

$$\tau U_t = D U_{xx} - U + \frac{1}{\varepsilon} V^m / U^s .$$

WE NOW SHOW THAT OUR SPECIFIC CHOICE $p=3,\ q=2,\ m=3,\ S=0$ LEADS TO A MIRACULOUS SIMPLIFICATION OF (21).

LEMMA DEFINE $$f(\lambda) = \frac{\int_{-\infty}^{\infty} w^2 (L_0-\lambda)^{-1} w^3\, dy}{\int_{-\infty}^{\infty} w^3\, dy}$$

WITH $L_0 \Phi = \Phi'' - \Phi + 3w^2\Phi$. THEN $f(\lambda) = \dfrac{3}{2(3-\lambda)} \quad \forall \lambda.$ (23)

PROOF WE RECALL THE KEY IDENTITY $L_0(w^2) = 3(w^2)$.

DEFINE $$\mathcal{J} \equiv \int_{-\infty}^{\infty} w^2 (L_0-\lambda)^{-1} w^3\, dy.$$

WE WRITE $$\mathcal{J} = \frac{1}{3}\int_{-\infty}^{\infty} L_0(w^2)\left[(L_0-\lambda)^{-1} w^3\right] dy$$

$$= \frac{1}{3}\int_{-\infty}^{\infty}\Big(\left[(L_0-\lambda)+\lambda\right] w^2\Big)\Big((L_0-\lambda)^{-1} w^3\Big)\, dy$$

NOW INTEGRATE BY PARTS
$$= \frac{1}{3}\int_{-\infty}^{\infty}\Big((L_0-\lambda)^{-1}\left[(L_0-\lambda)+\lambda\right] w^2\Big)\, w^3\, dy$$

$$= \frac{1}{3}\int_{-\infty}^{\infty} w^5\, dy + \frac{\lambda}{3}\int_{-\infty}^{\infty}\left[(L_0-\lambda)^{-1} w^2\right] w^3\, dy$$

NOW INTEGRATE BY PARTS
$$\mathcal{J} = \frac{1}{3}\int_{-\infty}^{\infty} w^5\, dy + \frac{\lambda}{3}\int_{-\infty}^{\infty} w^2\left(\left[L_0-\lambda\right]^{-1} w^3\right) dy$$

$$\mathcal{J} = \frac{1}{3}\int_{-\infty}^{\infty} w^5\, dy + \frac{\lambda}{3}\mathcal{J}.$$

WE NOW SOLVE FOR $\mathcal{J}$.

THEREFORE, WE HAVE

$$\gamma = \frac{\int w^5 \, dy}{3-\lambda} .$$

THIS GIVES $\quad f(\lambda) = \dfrac{\gamma}{\int_{-\infty}^{\infty} w^3 \, dy} = \left(\dfrac{\int_{-\infty}^{\infty} w^5 \, dy}{\int_{-\infty}^{\infty} w^3 \, dy} \right) \dfrac{1}{3-\lambda} .$

IF $w = \sqrt{2} \, \mathrm{sech}\, y$ THEN $\int w^5 / \int w^3 = 3/2$. AS SUCH $f(\lambda) = \dfrac{3}{2(3-\lambda)}$.

WE HAVE IN SUMMARY THAT WE MUST FIND THE ROOTS IN $\mathrm{Re}\, \lambda > 0$ TO $g(\lambda) = 0$ WHERE

$$g(\lambda) = \frac{1}{\chi(\lambda)} - \frac{3}{2(3-\lambda)} . \qquad (24)$$

- FOR INFINITE-LINE PROBLEM (SEE (19))

$$1/\chi(\lambda) = \frac{\sqrt{1+\tau\lambda}}{6} \qquad (24)_1$$

- FOR FINITE-DOMAIN PROBLEM (SEE $(18)_2$) WE HAVE

$$\frac{1}{\chi(\lambda)} = \frac{\sqrt{1+\tau\lambda}}{6} \, \frac{\tanh(\varphi_\lambda)}{\tanh(\varphi_0)} \qquad \varphi_0 = 1/\sqrt{D}, \quad \varphi_\lambda = \sqrt{\frac{1+\tau\lambda}{D}} . \qquad (24)_2$$

<u>INFINITE LINE PROBLEM</u> WE WRITE

$$g(\lambda) = \frac{\sqrt{1+\tau\lambda}}{6} - \frac{3}{2(3-\lambda)}$$

WE OBSERVE THAT. $g(0) = 1/6 - 1/2 < 0$.

WE DECOMPOSE $g(i\lambda_I) = g_R(\lambda_I) + i g_I(\lambda_I)$ AND USE OUR WINDING NUMBER CRITERION (22).

WE CALCULATE

$$g(i\lambda_I) = \frac{1}{6}\sqrt{1+i\tau\lambda_I} - \frac{3}{2(3-i\lambda_I)}$$

$$= \frac{1}{6}\,\mathrm{RE}\left[\sqrt{1+i\tau\lambda_I}\right] + \frac{i}{6}\,\mathrm{IM}\left[\sqrt{1+i\tau\lambda_I}\right] - \frac{3(3+i\lambda_I)}{2(3-i\lambda_I)(3+i\lambda_I)}$$

$$= \frac{1}{6}\,\mathrm{RE}\left[\sqrt{1+i\tau\lambda_I}\right] + \frac{i}{6}\,\mathrm{IM}\left[\sqrt{1+i\tau\lambda_I}\right] - \frac{9}{2(9+\lambda_I^2)} - \frac{3i\lambda_I}{2(9+\lambda_I^2)}$$

AS SUCH

$$g_R(\lambda_I) = \frac{1}{6}\,\mathrm{RE}\sqrt{1+i\tau\lambda_I} - \frac{9}{2(9+\lambda_I^2)}$$

$$g_I(\lambda_I) = \frac{1}{6}\,\mathrm{IM}\left[\sqrt{1+i\tau\lambda_I}\right] - \frac{3\lambda_I}{2(9+\lambda_I^2)}.$$

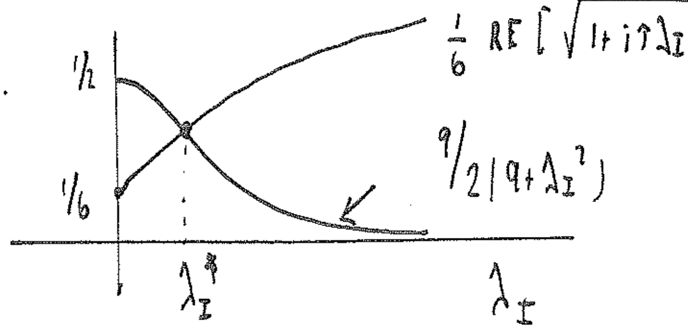
WE CAN CALCULATE

$$\mathrm{RE}\sqrt{1+i\tau\lambda_I} = \frac{1}{\sqrt{2}}\left[1+\sqrt{1+\tau^2\lambda_I^2}\right]^{1/2}$$

$$\mathrm{IM}\sqrt{1+i\tau\lambda_I} = \frac{1}{\sqrt{2}}\left[-1+\sqrt{1+\tau^2\lambda_I^2}\right]^{1/2}$$

WE LOOK FOR INTERSECTIONS WHERE $g_R = 0$ GRAPHICALLY

g_R plots.

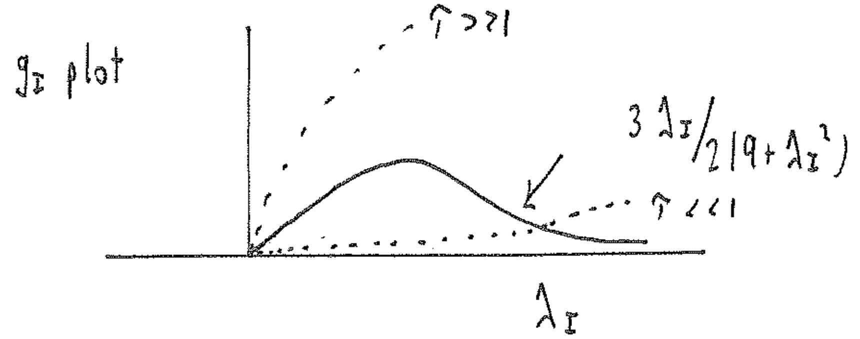


THUS, $\exists$ A UNIQUE POINT λ_I^* FOR WHICH $g_R(\lambda_I^*) = 0$.

FROM PAGE S10 WE CONCLUDE THAT EITHER $N=0$ OR $N=2$, $\forall \tau > 0$.

- $N=0$ IF $g_I(\lambda_I^*) < 0$. PATH 1 ON PAGE (S10).
- $N=2$ IF $g_I(\lambda_I^*) > 0$ PATH 2

NOW WE PLOT $g_I = 0$ GRAPHICALLY



DOTTED LINES ARE $\mathrm{IM}[\sqrt{1+i\tau\lambda_I}]$.

- WE CONCLUDE THAT IF $\tau << 1$, $\lambda_I^* = O(1)$ AND $g_I(\lambda_I^*) < 0$.

  THUS $N=0$.

- IF $\tau >> 1$ WE HAVE $\lambda_I^* = O(1/\tau)$ AND $g_I(\lambda_I^*) > 0$

  THUS $N=2$.

WE THEN HAVE $N=0$ IF $\tau << 1$

$N=2$ IF $\tau >> 1$

AS A RESULT OF THESE LIMITING BEHAVIORS AND THE CONTINUITY OF THE EIGENVALUE PATH AS A FUNCTION OF τ, WE MUST HAVE A HOPF BIFURCATION POINT AT SOME $\tau = \tau_H$ (possibly NON-UNIQUE) where $\lambda = i\lambda_{I_H}$ IS A ROOT TO $g(\lambda) = 0$.